MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

STATISTISCHE AFDELING

Leiding: Prof. Dr D. van Dantzig
Chef van de Statistische Consultatie: Prof. Dr J. Hemelrijk

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Maximum likelihood estimation of partially or completely ordered parameters

by

Constance van Eeden

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1. Introduction

The problem treated in this report concerns the maximum likelihood estimation of partially or completely ordered parameters of probability distributions. A special case of this problem, the maximum likelihood estimation of ordered probabilities, has been treated in [2] .

The problem will be formulated in section 2; in section 4 and 5 methods will be given by means of which the estimates may be found. For the proofs of the theorems we need some lemma's which will be proved in section 3 and in section 6 some examples will be given.

2. The problem

Consider k independent random variables $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_k$ 1) and n_i independent observations $x_{i,1}, x_{i,2}, \ldots, x_{i,n_i}$ of $\underline{x}_i$ $(i = 1, 2, \ldots, k)$. The distribution of $\underline{x}_i$ contains one unknown parameter θ_i $(i = 1, 2, \ldots, k)$ and its distribution function is

$$(2.1) \qquad F_i(x_i \mid \theta_i) \stackrel{\text{def}}{=} P[\underline{x}_i \leq x_i \mid \theta_i] \qquad (i = 1, 2, \ldots k).$$

Two types of restrictions are imposed on the parameters $\theta_1, \theta_2, \ldots, \theta_k$. First let $\mathcal{J}_i$ be a closed interval such that $F_i(x_i \mid y_i)$ is a distributionfunction for each value of $y_i \in \mathcal{J}_i$ $(i = 1, 2, \ldots, k)$. By meansof the choice of $\mathcal{J}_i$ restrictions of the type $c_i \leq \theta_i \leq d_i$ may be imposed. The second type of restrictions consists of a partial or complete ordering of the parameters, which may be described as follows. Let $\alpha_{i,j}$ $(i, j = 1, 2, \ldots, k)$ be numbers satisfying the conditions

$$(2.2) \qquad \begin{cases} 1. \ \alpha_{i,j} = -\alpha_{j,i}, \\ 2. \ \alpha_{i,j} = 0 \text{ if the intersection } \mathcal{J}_i \cap \mathcal{J}_j \text{ contains at most} \\ \quad \text{one point,} \\ 3. \ \alpha_{i,j} = 0, +1 \text{ or } -1 \text{ in all other cases} \end{cases}$$

and

$$(2.3) \qquad \alpha_{i,j} = 1 \text{ if } \qquad \alpha_{i,h} = \alpha_{h,j} = 1 \quad \text{for any } h .$$

The restrictions imposed on $\theta_1, \theta_2, \ldots, \theta_k$ are then

$$(2.4) \qquad \begin{cases} 1. \ \alpha_{i,j}(\theta_i - \theta_j) \leq 0 \\ 2. \ \theta_i \in \mathcal{J}_i \end{cases} \qquad (i, j = 1, 2, \ldots, k).$$

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1) Random variables will be distinguished from numbers (e.g. from the values they take in an experiment) by underlining their symbols.

and it will be supposed that the parameters $\theta_1, \theta_2, \ldots, \theta_k$ are numbered in such a way that

(2.5) $\alpha_{i,j} \geqq 0$ for each pair of values (i,j).

No other restrictions on $\theta_1, \theta_2, \ldots, \theta_k$ are admitted, such that all points $y_1, y_2, \ldots, y_k$ of the Cartesian product

(2.6) $$G \overset{\text{def}}{=} \prod_{i=1}^{k} \vartheta_i ,$$

satisfying

(2.7) $$\alpha_{i,j}(y_i - y_j) \leqq 0 \qquad (i, j = 1, 2, \ldots, k)$$

belong to the parameterspace, which thus is a convex subdomain of G. This subdomain will be denoted by D.

Let

(2.8) $$\begin{cases} 1. & \alpha_{i,j} = 0 \text{ for } \varkappa_0 \text{ pairs of values } (i,j) \text{ with } i < j, \\ 2. & \alpha_{i,j} = 1 \text{ for } \varkappa_1 \text{ pairs of values } (i,j) \text{ with } i < j, \end{cases}$$

then

(2.9) $$\varkappa_0 + \varkappa_1 = \binom{k}{2} .$$

Let further $f_i(x_i \mid \theta_i)$ denote the density function of $\underline{x}_i$ if $\underline{x}_i$ possesses a continuous probability distribution and $P[\underline{x}_i = x_i \mid \theta_i]$ if $\underline{x}_i$ possesses a discrete probability distribution and let

(2.10) $$\begin{cases} 1. & L_i = L_i(y_i) \overset{\text{def}}{=} \sum_{\gamma=1}^{n_i} \lg f_i(x_{i,\gamma} \mid y_i) \qquad (i = 1, 2, \ldots, k), \\ 2. & L = L(y_1, y_2, \ldots, y_k) \overset{\text{def}}{=} \sum_{i=1}^{k} L_i(y_i) . \end{cases}$$

Then the maximum likelihood estimates of $\theta_1, \theta_2, \ldots, \theta_k$ are the values of $y_1, y_2, \ldots, y_k$ which maximize L in the domain D.
<u>Unless explicitely stated otherwise L will only be considered in this domain D</u>; the maximum likelihood estimates will throughout this paper be denoted by $t_1, t_2, \ldots, t_k$.
Further the restrictions $\theta_i \leqq \theta_j$ (i.e. $\alpha_{i,j} = 1$) satisfying

(2.11) $$\alpha_{i,h} \cdot \alpha_{h,j} = 0 \quad \text{for each } h \text{ between } i \text{ and } j$$

will be denoted by $R_1, R_2, \ldots, R_s$. Each R_λ thus corresponds with one pair (i,j); this pair will be denoted by (i_λ, j_λ).
Because of the transitivity relations (2.3) the system $R_1, R_2, \ldots, R_s$ is equivalent to (2.4.1) and uniquely determined by (2.4.1).
The restrictions $R_1, R_2, \ldots, R_s$ will be called the <u>essential restrictions</u>.

<u>Remark 1</u>: H.D. BRUNK [1] described a method by means of which the estimates of $\theta_1, \theta_2, \ldots, \theta_k$ may be found if the distribution of x_i belongs to the "exponential family" $(i=1,2,\ldots,k)$ and if moreover $\mathcal{Y}_i$ is the set of <u>all</u> values of y_i for which $F_i(x_i \mid y_i)$ is a distribution function $(i=1,2,\ldots k)$.
His method however leads to much more complicated computations than ours.

3. <u>Lemma's</u>

<u>Definition: A function $\varphi(y)$ of a variable y will be called strictly unimodal in an interval $\mathcal{Y}$ if there exists a value $y^* \in \mathcal{Y}$ such that</u>

$$(3.1) \qquad \varphi(y) < \varphi(z) < \varphi(y^*)$$

<u>for each pair of values $(y,z) \in \mathcal{Y}$ with</u>

$$(3.2) \qquad y < z < y^*$$

<u>and for each pair of values $(y,z) \in \mathcal{Y}$ with</u>

$$(3.3) \qquad y^* < z < y.$$

It follows at once from this definition that a strictly unimodal function $\varphi(y)$ is bounded in every closed subdomain of $\mathcal{Y}$ not containing y^*.

Now let $\varphi_\kappa(y_\kappa)$ be a strictly unimodal function of y_κ in the interval $\mathcal{Y}_\kappa$ $(\kappa=1,2,\ldots,k)$ and let further

$$(3.4) \qquad \Phi(y_1, y_2, \ldots, y_k) \overset{\text{def}}{=} \sum_{\kappa=1}^{k} \varphi_\kappa(y_\kappa),$$

then

<u>Lemma I:</u> <u>$\Phi(y_1, y_2, \ldots, y_k)$ possesses a unique maximum in</u>

$$(3.5) \qquad \Gamma \overset{\text{def}}{=} \prod_{\kappa=1}^{k} \mathcal{Y}_\kappa.$$

<u>Proof:</u> Let $\varphi_\kappa(y_\kappa)$ attain its maximum in $\mathcal{Y}_\kappa$ for $y_\kappa = y_\kappa^*$ $(\kappa=1,2,\ldots,k)$. Then it follows from the fact that $\Phi(y_1, y_2, \ldots, y_k)$ is the sum of the k functions $\varphi_\kappa(y_\kappa)$ and that Γ is the Cartesian product of the k intervals $\mathcal{Y}_\kappa$, that $\Phi(y_1, y_2, \ldots, y_k)$ possesses a unique maximum in Γ and attain this maximum for $y_\kappa = y_\kappa^*$ $(\kappa = 1, 2, \ldots, k)$.

We now define a function V as follows.

Let $y_1^\circ, y_2^\circ, \ldots, y_k^\circ$ be a given point in Γ with $y_\varkappa^\circ \neq y_\varkappa^*$ for at least one value of $\varkappa$ and let

(3.6) $$\begin{cases} y_\varkappa(\beta) \overset{\text{def}}{=} (1-\beta)\, y_\varkappa^\circ + \beta\, y_\varkappa^* \quad (\varkappa = 1, 2, \ldots, k), \\ 0 \leqq \beta \leqq 1. \end{cases}$$

Then $\{ y_1(\beta), y_2(\beta), \ldots, y_k(\beta) \}$ is a point in Γ and V is defined by

(3.7) $$V(\beta) \overset{\text{def}}{=} \Phi\{ y_1(\beta), y_2(\beta), \ldots, y_k(\beta) \}.$$

Lemma II: $V(\beta)$ is a monotone increasing function of β in the interval $0 \leqq \beta \leqq 1$.

Proof: Consider a value of $\varkappa$ with

(3.8) $$y_\varkappa^\circ = y_\varkappa^*$$

then

(3.9) $$y_\varkappa(\beta) = y_\varkappa^* \quad \text{for each } \beta \text{ with } 0 \leqq \beta \leqq 1.$$

Thus in this case we have

(3.10) $$\varphi_\varkappa(y_\varkappa^\circ) = \varphi_\varkappa\{ y_\varkappa(\beta) \} = \varphi_\varkappa(y_\varkappa^*) \quad \text{for each } \beta \text{ with } 0 \leqq \beta \leqq 1.$$

Now consider a value of $\varkappa$ with

(3.11) $$y_\varkappa^\circ \neq y_\varkappa^*,$$

then it follows from the fact that $\varphi_\varkappa(y_\varkappa)$ is, in the interval $\mathcal{J}_\varkappa$, a strictly unimodal function of $y_\varkappa$ and attain its maximum in $\mathcal{J}_\varkappa$ for $y_\varkappa = y_\varkappa^*$ that

(3.12) $$\varphi_\varkappa(y_\varkappa^\circ) < \varphi_\varkappa\{ y_\varkappa(\beta_1) \} < \varphi_\varkappa\{ y_\varkappa(\beta_2) \} < \varphi_\varkappa(y_\varkappa^*)$$

for each pair of values (β_1, β_2) with $0 < \beta_1 < \beta_2 < 1$.
From (3.4) and the fact that there exists at least one value of $\varkappa$ with (3.11) it follows then that

(3.13) $$V(0) < V(\beta_1) < V(\beta_2) < V(1)$$

for each pair of values (β_1, β_2) with $0 < \beta_1 < \beta_2 < 1$.

Lemma III: If C is a closed convex subdomain of Γ, not containing the point $(y_1^*, y_2^*, \ldots, y_k^*)$, then $\Phi(y_1, y_2, \ldots, y_k)$ attains its maximum in C only in one or more points on its border.

Proof: Consider any inner point $y_1^\circ, y_2^\circ, \ldots, y_k^\circ$ of C and let $y_\varkappa(\beta)$ be defined by (3.6) $(\varkappa = 1, 2, \ldots, k)$. Then, C being a closed convex domain not containing the point $(y_1^*, y_2^*, \ldots, y_k^*)$ there exists a value of β in the interval $0 < \beta < 1$, say β_0, such that $\{y_1(\beta_0), y_2(\beta_0), \ldots, y_k(\beta_0)\}$ is a border point of C. Further it follows from Lemma II that

$$\Phi\{y_1(\beta_0), y_2(\beta_0), \ldots, y_k(\beta_0)\} > \Phi(y_1^\circ, y_2^\circ, \ldots, y_k^\circ). \tag{3.14}$$

Thus for each inner point $(y_1^\circ, y_2^\circ, \ldots, y_k^\circ)$ of C there exists a border point $(y_1, y_2, \ldots, y_k)$ of C with a larger value of Φ. Moreover Φ is bounded in C, because the point $(y_1^*, y_2^*, \ldots, y_k^*)$ is not contained in C. Thus Φ has a maximum in C, which can evidently only be attained in border points.

4. <u>The maximum likelihood estimates of $\theta_1, \theta_2, \ldots, \theta_k$</u>

Let M be a subset of the numbers $1, 2, \ldots, k$; let further

$$\vartheta_M \stackrel{\text{def}}{=} \bigcap_{i \in M} \vartheta_i \tag{4.1}$$

and if $\vartheta_M \neq 0$

$$L_M(z) \stackrel{\text{def}}{=} \sum_{i \in M} L_i(z) \qquad z \in \vartheta_M. \tag{4.2}$$

Throughout this report it will be supposed that the following condition is satisfied

(4.3) <u>Condition: For each M with $\vartheta_M \neq 0$ the function $L_M(z)$ is strictly unimodal in the interval ϑ_M</u>.

Now let M_ν $(\nu = 1, 2, \ldots, N)$ be subsets of the numbers $1, 2, \ldots, k$ with

$$\text{(4.4)} \quad \begin{cases} 1. \ \bigcup_{\nu=1}^{N} M_\nu = \{1, 2, \ldots, k\}, \\ 2. \ M_{\nu_1} \cap M_{\nu_2} = 0 \text{ for each pair of values } \nu_1, \nu_2 = 1, 2, \ldots, N \text{ with } \nu_1 \neq \nu_2, \\ 3 \ \ \vartheta_{M_\nu} \neq 0 \text{ for each } \nu = 1, 2, \ldots, N, \end{cases}$$

where

$$\vartheta_{M_\nu} \stackrel{\text{def}}{=} \bigcap_{i \in M_\nu} \vartheta_i \qquad (\nu = 1, 2, \ldots, N). \tag{4.5}$$

Let further

$$G_N \stackrel{\text{def}}{=} \prod_{\nu=1}^{N} \vartheta_{M_\nu} \tag{4.6}$$

and

(4.7) $$L_{M_\nu}(z_\nu) \overset{def}{=} \sum_{i \in M_\nu} L_i(z_\nu) \qquad z_\nu \in \mathcal{J}_{M_\nu} \quad (\nu = 1, 2, \ldots, N).$$

Then for all points in G_N $L(y_1, y_2, \ldots, y_k)$ reduces to a function of N variables $z_1, z_2, \ldots, z_N$; we denote this function by $L'(z_1, z_2, \ldots, z_N)$ and thus have

(4.8) $$L'(z_1, z_2, \ldots, z_N) = \sum_{\nu=1}^{N} L_{M_\nu}(z_\nu),$$

which is according to (4.3), a sum of strictly unimodal functions.

<u>Theorem I: L possesses a unique maximum in D</u>

<u>Proof:</u> This theorem will be proved by induction.
Let $M_1, M_2, \ldots, M_N$ be an arbitrary set of subsets of the numbers $1, 2, \ldots, k$ satisfying (4.4) and let

(4.9) $$D_{N,s} \overset{def}{=} D \cap G_N,$$

where s denotes the number of essential restrictions defining D and where G_N is defined by (4.6). Then $D_{N,s}$ is convex and:

for $N = k$ we have $\mathcal{J}_{M_\nu} = \mathcal{J}_\nu$ $(\nu = 1, 2, \ldots, N)$, thus $G_N = G$ and $D_{N,s} = D$
for $s = 0$ we have $D = G$ thus $D_{N,0} = G_N$.

We shall say that the function $L'(z_1, z_2, \ldots, z_N)$ can be monotonously traced to its maximum in $D_{N,s}$ if

(4.10)
1. $L'(z_1, z_2, \ldots, z_N)$ possesses a unique maximum in $D_{N,s}$,
2. every point of $D_{N,s}$ can be connected with the point in $D_{N,s}$ where L' assumes its maximum by means of a line in $D_{N,s}$ such that L' increases monotonously along this line. (Such a line will be called a trace)

For $s = 0$ $L'(z_1, z_2, \ldots, z_N)$ has this property for every set $M_1, M_2, \ldots, M_N$ satisfying (4.4) and every N. This follows from the fact that L' is the sum of strictly unimodal functions and that $D_{N,0}$ is the Cartesian product of the intervals $\mathcal{J}_{M_\nu}$ $(\nu = 1, 2, \ldots, N)$, so that the Lemma's I and II may be applied.

Let us now suppose that it has been proved that L' can be monotonously traced to its maximum for all values of $s \leq s_0$ for every set $M_1, M_2, \ldots, M_N$ satisfying (4.4) and for every N . We then prove that the same holds for $s_0 + 1$ essential restrictions.

Consider, for a given set $M_1, M_2, \ldots, M_N$, satisfying

(4.4), a domain D_{N,s_0+1} and the domain D_{N,s_0} which is obtained by omitting one of the essential restrictions defining D_{N,s_0+1}. Let this be the restriction $R_\lambda : z_{i_\lambda} \leqq z_{j_\lambda}$. Then clearly

$$(4.11) \qquad D_{N,s_0+1} \subset D_{N,s_0}.$$

Now L' has a unique maximum in D_{N,s_0}, attained in (say) the point T. We first consider the case that T is outside D_{N,s_0+1}. Then an arbitrary point P of D_{N,s_0+1} with $z_{i_\lambda} < z_{j_\lambda}$ can be connected with T by means of a trace in D_{N,s_0} and this trace must contain at least one border point of D_{N,s_0+1} with $z_{i_\lambda} = z_{j_\lambda}$, because within D_{N,s_0+1} we have: $z_{i_\lambda} < z_{j_\lambda}$ and outside D_{N,s_0+1}: $z_{i_\lambda} > z_{j_\lambda}$. The first of these points when following the trace be denoted by U; then L' assumes a larger value in U than in P. Now U lies in a domain $D_{N',s_0'}$, where $N' = N-1$ and $s_0' \leqq s_0$ and L' can thus monotonously be traced from U to its unique maximum in $D_{N',s_0'}$ by means of a trace within $D_{N',s_0'}$. The trace from P to U in D_{N,s_0+1} and from U to the maximum of L' in $D_{N',s_0'}$ together form a trace from P to the maximum of L' in D_{N,s_0+1}.

Consider next the case where T is a point of D_{N,s_0+1}. Then L' attains a unique maximum in D_{N,s_0+1} in T. If T is the maximum of L' in G_N then, according to Lemma II, L' can be monotonously traced to its maximum from every point of D_{N,s_0+1} by means of a straight line, connecting this point with T. If T is not the maximum of L' in G_N then it follows from Lemma III that T is a border point of D_{N,s_0+1} where at least two z_ν from $z_1, z_2, \ldots, z_N$ corresponding to an essential restriction for D_{N,s_0+1} are equal. Let this pair be

$$(4.12) \qquad z_{i_\mu} = z_{j_\mu},$$

then we consider the domain D'_{N,s_0} which is obtained from D_{N,s_0+1} by omitting the restriction $R_\mu : z_{i_\mu} \leqq z_{j_\mu}$ from the essential restrictions defining D_{N,s_0+1}. The maximum of L' in D'_{N,s_0} then exists and the point where it is attained is a point of D'_{N,s_0} with $z_{i_\mu} \geqq z_{j_\mu}$. The rest of the proof for this case is then the same as for the first case considered.

Thus L' can be monotonously traced to its maximum in every $D_{N,s}$, one of which is D.

<u>Remark 2:</u> For $s=0$ and $N=k$ we have $D_{N,s} = G$. Thus L attains a unique maximum in G in a point which will be denoted by $v_1, v_2, \ldots, v_k$.

<u>Theorem II: If $t'_1, t'_2, \ldots, t'_k$ are the values of $y_1, y_2, \ldots, y_k$ which maximixe L in G and under the restrictions $R_1, \ldots, R_{\lambda-1}, R_{\lambda+1}, \ldots, R_s$ then</u>

$$(4.13)\quad \begin{cases} 1.\ t_i = t'_i \quad (i = 1, 2, \ldots, k) \quad \text{if} \quad t'_{i_\lambda} \leqq t'_{j_\lambda}, \\ 2.\ t_{i_\lambda} = t_{j_\lambda} \quad \text{if} \quad t'_{i_\lambda} > t'_{j_\lambda}. \end{cases}$$

<u>Proof</u>: The R_λ have not been arranged in a special order, thus we may take without any loss of generality $\lambda = s$. First consider the case that $t'_{i_s} \leqq t'_{j_s}$; then $t'_1, t'_2, \ldots, t'_k$ satisfy all restrictions $R_1, R_2, \ldots, R_s$; thus in this case we have

$$(4.14)\qquad t_i = t'_i \quad (i = 1, 2, \ldots, k).$$

If $t'_{i_s} > t'_{j_s}$ then (4.13.2) may be proved as follows. The domain defined by the essential restrictions $R_1, R_2, \ldots, R_{s-1}$ will be denoted by D' . Then for each point $(y_1, y_2, \ldots, y_k)$ in D with $y_{i_s} < y_{j_s}$ there exists a trace in D' from the point $(y_1, y_2, \ldots, y_k)$ to the point $(t'_1, t'_2, \ldots, t'_k)$ and this trace contains a point $(y'_1, y'_2, \ldots, y'_k)$ with

$$(4.15)\quad \begin{cases} 1.\quad y'_{i_s} = y'_{j_s}, \\ 2.\ L(y'_1, y'_2, \ldots, y'_k) > L(y_1, y_2, \ldots, y_k). \end{cases}$$

Thus, if $t'_{i_s} > t'_{j_s}$, then $L(y_1, y_2, \ldots, y_k)$ attains its maximum in D for $y_{i_s} = y_{j_s}$; (4.13.2) then follows from the uniqueness of this maximum

<u>Remark 3</u>:

If

$$(4.16)\qquad P[x_i = 1] = \theta_i \quad , \quad P[x_i = 0] = 1 - \theta_i \quad (i = 1, 2, \ldots, k)$$

and

$$(4.17)\qquad a_i \overset{\text{def}}{=} \sum_{\gamma=1}^{n_i} x_{i,\gamma} \quad , \quad b_i \overset{\text{def}}{=} n_i - a_i \quad (i = 1, 2, \ldots, k)$$

then

$$(4.18)\qquad L(y_1, y_2, \ldots, y_k) = \sum_{i=1}^{k} \{a_i \lg y_i + b_i \lg(1 - y_i).$$

In [2] it has been proved that, if G_i is the interval (0,1), this function L satisfies the following condition.

(4.19) <u>Condition: If $(y_1, y_2, \ldots, y_k)$ and $(z_1, z_2, \ldots, z_k)$ are any two points in G with $y_i \neq z_i$ for at least one value of i and if</u>

$$y_i(\beta) \overset{\text{def}}{=} (1-\beta)y_i + \beta z_i \quad (i = 1, 2, \ldots, k)$$

then $L\{y_1(\beta), y_2(\beta), \ldots, y_k(\beta)\}$ is a strictly unimodal function of β in the interval $0 \leq \beta \leq 1$.

This condition is stronger than condition (4.3) and the theorems I and II of this report have been proved in [2] by using condition (4.19).

Further if condition (4.19) is satisfied then theorem I of this report may be proved in a more simple way than we did in [2] as follows. Consider any two points $(y_1, y_2, \ldots, y_k)$ and $(z_1, z_2, \ldots, z_k)$ in D with $y_i \neq z_i$ for at least one value of i and

(4.20) $$L(y_1, y_2, \ldots, y_k) = L(z_1, z_2, \ldots, z_k).$$

Then it follows from condition (4.19) that there exists a point $(\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_k)$ in D with

(4.21) $$L(\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_k) > L(y_1, y_2, \ldots, y_k).$$

Thus L possesses a unique maximum in D.

The maximum likelihood estimates of $\theta_1, \theta_2, \ldots, \theta_k$ may always be found by repeatedly applying theorem II. This follows from the fact that $L'(z_1, z_2, \ldots, z_N)$ is a sum of strictly unimodal functions and that $D_{N,s}$ is a convex subdomain of the Cartesian product of the intervals $\mathcal{Y}_{M_\nu}$ $(\nu = 1, 2, \ldots, N)$ for each set $M_1, M_2, \ldots, M_N$ and each N.

This leads however to a rather complicated procedure which may often be simplified by using one of the theorems of the following section.

5. Some special theorems

The theorems III-VI in this section may be proved in precisely the same way as the theorems II-V in [2].

Theorem III: If $\alpha_{i,j}(v_i - v_j) \leq 0$ for each pair of values (i,j) then

(5.1) $$t_i = v_i \quad (i = 1, 2, \ldots, k).$$

Theorem IV: If $\ell_1, \ell_2, \ldots, \ell_m$ is a set of values satisfying

(5.2) $$\alpha_{i,\ell_1} = \alpha_{i,\ell_2} = \ldots = \alpha_{i,\ell_m} = 0 \quad \text{for each } i \neq \ell_1, \ell_2, \ldots, \ell_m$$

then the maximum likelihood estimates of $\theta_{\ell_1}, \theta_{\ell_2}, \ldots, \theta_{\ell_m}$ are the values of $y_{\ell_1}, y_{\ell_2}, \ldots, y_{\ell_m}$ which maximize $L_{\ell_1} + L_{\ell_2} + \ldots + L_{\ell_m}$ in the domain

$$(5.3) \qquad D_1 \begin{cases} \alpha_{i,j}(y_i - y_j) \leqq 0 \\ y_i \in \vartheta_i \end{cases} \qquad (i,j = \ell_1, \ell_2, \ldots, \ell_m).$$

<u>Theorem V</u>: <u>If for some pair of values (i,j) with $i<j$</u>

$$(5.4) \qquad \alpha_{i,j}(v_i - v_j) > 0$$

<u>and</u>

$$(5.5) \qquad \begin{cases} 1. \ \alpha_{i,h} = \alpha_{h,j} = 0 & \text{for each } h \text{ between } i \text{ and } j, \\ 2. \ \alpha_{h,i} = \alpha_{h,j} & \text{for each } h < i, \\ 3. \ \alpha_{i,h} = \alpha_{j,h} & \text{for each } h > j, \end{cases}$$

<u>then</u>

$$(5.6) \qquad t_i = t_j .$$

<u>Theorem VI</u>: <u>If (i,j) is a pair of values satisfying</u>

$$(5.7) \qquad v_i \leqq v_j$$

<u>and</u>

$$(5.8) \qquad \begin{cases} 1. \ \alpha_{i,j} = 0, \\ 2. \ \alpha_{h,i} \leqq \alpha_{h,j} & \text{for each } h < i, \\ 3. \ \alpha_{i,h} \geqq \alpha_{j,h} & \text{for each } h > j, \end{cases}$$

<u>then</u>

$$(5.9) \qquad t_i \leqq t_j .$$

<u>Theorem VII</u>: <u>If (i,j) is a pair of values with</u>

$$(5.10) \qquad \alpha_{i,j} = 0,$$

<u>if D' is the subdomain of D where $y_i \leqq y_j$ and if $(t'_1, t'_2, \ldots, t'_k)$ is the point where L assume its maximum in D' then</u>

$$(5.11) \qquad \begin{cases} 1. \ t_1 = t'_1, t_2 = t'_2, \ldots, t_k = t'_k & t'_i < t'_j, \\ 2. \ t_i \geqq t_j & t'_i = t'_j . \end{cases}$$

<u>Proof</u>: The proof of this theorem differs from the one given for theorem VI in [2] only in the form of the trace from a point in D' to the maximum in D. This trace which is a straight l line in [2], need not be straight now(cf. the proof of theorem II of the present report).

6. <u>Examples</u>

In this section the pooled samples of $\underline{x}_i$ and $\underline{x}_j$ will be denoted by $x'_{i,\gamma}$ $(\gamma = 1, 2, \ldots, n'_i)$, where $n'_i = n_i + n_j$.

6.1 <u>$\underline{x}_i$ possesses a normal distribution with mean θ_i and known variance $(i = 1, 2, \ldots, k)$.</u>

Without any loss of generality we may suppose that $\sigma^2\{\underline{x}_i\} = 1$ for all i; then

(6.1.1) $$L_i(y_i) = -\frac{1}{2} n_i \lg 2\pi - \frac{1}{2}\sum_{\gamma=1}^{n_i}(x_{i,\gamma} - y_i)^2 \qquad (i = 1, 2, \ldots, k).$$

From (6.1.1) it follows that

(6.1.2) $$\frac{dL_i(y_i)}{dy_i} = \sum_{\gamma=1}^{n_i}(x_{i,\gamma} - y_i) \qquad (i = 1, 2, \ldots, k),$$

thus, if

(6.1.3) $$m_i \overset{\text{def}}{=} \frac{1}{n_i}\sum_{\gamma=1}^{n_i} x_{i,\gamma} \qquad (i = 1, 2, \ldots, k),$$

then

(6.1.4) $$\frac{dL_i(y_i)}{dy_i} \begin{cases} > 0 & \text{if} \quad y_i < m_i, \\ = 0 & \text{if} \quad y_i = m_i, \\ < 0 & \text{if} \quad y_i > m_i. \end{cases} \qquad (i = 1, 2, \ldots, k).$$

From (6.1.4) it follows that $L_i(y_i)$ is a strictly unimodal function of y_i in the interval $(-\infty, +\infty)$, thus $L_i(y_i)$ is a strictly unimodal function of y_i in each closed subinterval $\mathcal{Y}_i$ of the interval $(-\infty, +\infty)$ $(i = 1, 2, \ldots, k)$.
Further if $y_i = y_j$ then $L_i(y_i) + L_j(y_j)$ reduces to one term of the form

(6.1.5) $$L_i(y_i) + L_j(y_i) = -\frac{1}{2} n_i' \lg 2\pi - \frac{1}{2}\sum_{\gamma=1}^{n_i'}(x'_{i,\gamma} - y_i)^2$$

and analogous relations hold if more than two of the y_i are equal. Thus L satisfies condition (4.3).
From (6.1.5) it follows further that if L attain its maximum for $y_i = y_j$ then the two samples of $\underline{x}_i$ and $\underline{x}_j$ are to be pooled.

The procedure will now be illustrated by means of the following example.
Suppose $k = 4$, $r_0 = 2$, $r_1 = 4$ and

(6.1.6) $$\alpha_{1,2} = \alpha_{1,3} = \alpha_{3,4} = 1.$$

Let further

(6.1.7)

i	1	2	3	4
$x_{i,\gamma}$	-0,40	1,43	-0,70	0,29
	2,56	1,86	2,61	0
	0,25	0,06	0,79	1,31
	2,87	0,07	0,86	0,15
		1,14	0,14	2,53
		0,29		1,86
		2,57		
		0,85		
		1,21		
$n_i m_i$	5,28	9,48	3,70	6,14
n_i	4	9	5	6
m_i	1,32	1,05	0,74	1,02

and let $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$ be the intervals

(6.18)

i	1	2	3	4
ϑ_i	$(-\infty, 1)$	$(-\infty, +\infty)$	$(0{,}5, \infty)$	$(-\infty, +\infty)$

Then it follows from (6.1.7) and (6.1.8) that the coordinates of the maximum in G are

(6.1.9)

i	1	2	3	4
v_i	1	1,05	0,74	1,02

From (6.1.6) and (6.1.9) it then follows that the pairs $i = 3$, $j = 2$ and $i = 4, j = 2$ satisfy (5.7) and (5.8). Thus according to theoremVI L attains its maximum in D for

(6.1.10) $$y_1 \leqq y_3 \leqq y_4 \leqq y_2 .$$

From (6.1.9), (6.1.10) and theorem V then follows

(6.1.11) $$t_1 = t_3 .$$

In this way the problem is reduced to the case of 3 samples with $\alpha_0' = 0$,

(6.1.12)

i	1(+3)	4	2
$x'_{i,\gamma}$	-0,40	0,29	1,43
	2,56	0	1,86
	0,25	1,31	0,06
	2,87	0,15	0,07
	-0,70	2,53	1,14
	2,61	1,86	0,29
	0,79		2,57
	0,86		0,85
	0,14		1,21
$n'_i m'_i$	8,98	6,14	9,48
n'_i	9	6	9
m'_i	0,998	1,02	1,05
ϑ'_i	(0,5 , 1)	$(-\infty, +\infty)$	$(-\infty, +\infty)$
v'_i	0,998	1,02	1,05

and

(6.1.13) $\alpha'_{1,4} = \alpha'_{4,2} = 1.$

From (6.1.11), (6.1.12) and (6.1.13) then follows

(6.1.14) $t_1 = t_3 = 0{,}998$, $t_2 = 1{,}05$, $t_4 = 1{,}02.$

6.2. $\underline{x}_i$ possesses a normal distribution with known mean and variance θ_i $(i = 1, 2, \ldots, k)$.

We suppose without loss of generality $\mathcal{E}\underline{x}_i = 0 \; (i = 1, 2, \ldots, k)$; then

(6.2.1) $$L_i(y_i) = -\frac{1}{2} n_i \lg 2\pi - \frac{1}{2} n_i \lg y_i - \frac{1}{2} \frac{\sum_{\gamma=1}^{n_i} x_{i,\gamma}^2}{y_i} \qquad (i = 1, 2, \ldots, k).$$

From (6.2.1) it follows, if

(6.2.2) $$s_i^2 \overset{\text{def}}{=} \frac{1}{n_i} \sum_{\gamma=1}^{n_i} x_{i,\gamma}^2 \qquad (i = 1, 2, \ldots, k),$$

that

(6.2.3) $$\frac{d L_i(y_i)}{d y_i} \begin{cases} > 0 & \text{if} \quad 0 \leqq y_i < s_i^2 , \\ = 0 & \text{if} \quad y_i = s_i^2 \\ < 0 & \text{if} \quad y_i > s_i^2 \end{cases} \qquad (i = 1, 2, \ldots, k);$$

thus $L_i(y_i)$ is a strictly unimodal function of y_i in the interval $(0, \infty)$.

Further if $y_i = y_j$ then $L_i(y_i) + L_j(y_j)$ reduces to

$$(6.2.4) \qquad L_i(y_i)+L_j(y_i) = -\tfrac{1}{2} n_i \lg 2\pi - \tfrac{1}{2} n_i \lg y_i - \tfrac{1}{2}\frac{\sum_{\gamma=1}^{n_i} x_{i,\gamma}'^2}{y_i}$$

and analogously for more than two of the y_i equal.
Thus L satisfies condition (4.3) and if L attains its maximum for $y_i = y_j$ then the two samples of $\underline{x}_i$ and $\underline{x}_j$ are to be pooled. Numerically the method is thus precisely the same as in 6.1, with s_i^2 in stead of m_i .

6.3 $\underline{x}_i$ possesses a Poisson distribution with parameter θ_i $(i=1,2,\dots,k)$

In this case we have

$$(6.3.1) \quad L_i(y_i) = -n_i y_i + \sum_{\gamma=1}^{n_i} x_{i,\gamma} \lg y_i - \sum_{\gamma=1}^{n_i} \lg x_{i,\gamma}! \quad (i=1,2,\dots,k).$$

From (6.3.1) it follows that, if

$$(6.3.2) \qquad m_i \overset{\text{def}}{=} \frac{1}{n_i}\sum_{\gamma=1}^{n_i} x_{i,\gamma} \qquad (i=1,2,\dots,k),$$

then

$$(6.3.3) \qquad \frac{d L_i(y_i)}{d y_i} \begin{cases} >0 & \text{if } 0 \leqq y_i < m_i, \\ =0 & \text{if } y_i = m_i, \\ <0 & \text{if } y_i > m_i; \end{cases} \qquad (i=1,2,\dots,k)$$

thus $L_i(y_i)$ is a strictly unimodal function of y_i the interval $(0,\infty)$ $(i=1,2,\dots,k)$.
Further if $y_i = y_j$ then $L_i(y_i)+L_j(y_j)$ reduces to

$$(6.3.4) \qquad L_i(y_i)+L_j(y_i) = -n_i y_i + \sum_{\gamma=1}^{n_i} x_{i,\gamma}' \lg y_i - \sum_{\gamma=1}^{n_i} x_{i,\gamma}'! \, ;$$

thus L satisfies condition (4.3) and if L attains its maximum for $y_i = y_j$ then the two samples of $\underline{x}_i$ and $\underline{x}_j$ are to be pooled.

The theorems of the foregoing sections may e.g. also be applied in the following case.

6.4. $\underline{x}_i$ possesses a normal distribution with mean θ_i and known variance for $i=\ell_1,\ell_2,\dots,\ell_g$ and a Poisson distribution with parameter θ_i for $i \neq \ell_1,\ell_2,\dots,\ell_g$.

Taking $\sigma^2\{\underline{x}_i\}=1$ for $i=\ell_1,\ell_2,\dots,\ell_g$ we have

$$(6.4.1) \quad \begin{cases} L_i(y_i) = -\tfrac{1}{2} n_i \lg 2\pi - \tfrac{1}{2}\sum_{\gamma=1}^{n_i}(x_{i,\gamma}-y_i)^2 & (i=\ell_1,\ell_2,\dots,\ell_g), \\ L_i(y_i) = -n_i y_i + \sum_{\gamma=1}^{n_i} x_{i,\gamma}\lg y_i - \sum_{\gamma=1}^{n_i}\lg x_{i,\gamma}! & (i\neq \ell_1,\ell_2,\dots,\ell_g). \end{cases}$$

From the sections 6.1 and 6.3 it follows that $L_i(y_i)$ is a strictly unimodal function of y_i in the interval $(-\infty, +\infty)$ for $i = \ell_1, \ell_2, \ldots, \ell_g$ and in the interval $(0, \infty)$ for $i \neq \ell_1, \ell_2, \ldots, \ell_g$. Further, if $y_i = y_j$, where $\underline{x}_i$ possesses a normal and $\underline{x}_j$ a Poisson distribution then $L_i(y_i) + L_j(y_j)$ reduces to

$$(6.4.3) \qquad L_i(y_i) + L_j(y_i) = -\tfrac{1}{2} n_i \lg 2\pi - \tfrac{1}{2} \sum_{\gamma=1}^{n_i} (x_{i,\gamma} - y_i)^2 - n_j y_i + \sum_{\gamma=1}^{n_j} x_{j,\gamma} \lg y_i - \sum_{\gamma=1}^{n_j} \lg x_{j,\gamma}!.$$

It may be proved as follows that $L_{i,j}(y_i) \overset{\text{def}}{=} L_i(y_i) + L_j(y_i)$ is a strictly unimodal function of y_i in the interval $(0, \infty)$. We have

$$(6.4.4) \qquad \frac{d L_{i,j}(y_i)}{d y_i} = n_i (m_i - y_i) - n_j + \frac{n_j m_j}{y_i}.$$

Thus if $m_i - \frac{n_j}{n_i} \leqq 0$ and $m_j = 0$ then

$$(6.4.5) \qquad \frac{d L_{i,j}(y_i)}{d y_i} < 0 \text{ for each } y_i > 0$$

and in all other cases

$$(6.4.6) \qquad \frac{d L_{i,j}(y_i)}{d y_i} \begin{cases} > 0 & \text{if } 0 \leqq y_i < m_i' \overset{\text{def}}{=} \frac{1}{2}\left\{m_i - \frac{n_j}{n_i} + \sqrt{\left(m_i - \frac{n_j}{n_i}\right)^2 + 4\frac{n_j m_j}{n_i}}\right\}, \\ = 0 & \text{if } y_i = m_i', \\ < 0 & \text{if } y_i > m_i'. \end{cases}$$

Analogous relations hold if more than two of the y_i are equal. Thus L satisfies condition (4.3).

This case will be illustrated by means of the following example. Suppose $k = 4$, $r_0 = r_1 = 3$,

$$(6.4.7) \qquad \alpha_{1,2} = \alpha_{1,4} = \alpha_{3,4} = 1$$

and $\ell_1 = 1$, $\ell_2 = 2$, $g = 2$. Further

(6.4.8)

i	1	2	3	4
$x_{i,\gamma}$	5,38 3,88 4,14 5,36 5,48	4,84 3,56 4,40 4,77	4 5 3 3 4	2 7 5 4
$n_i m_i$	24,24	17,57	19	18
n_i	5	4	5	4
m_i	4,85	4,39	3,8	4,5
$\mathcal{Y}_i$	$(-\infty, 5)$	$(-\infty, +\infty)$	$(0, \infty)$	$(0, 4)$
v_i	4,85	4,39	3,8	4

Then the pairs $i=3,\ j=2;\ i=4, j=2$ and $i=3, j=1$ satisfy (5.7) and (5.8). Thus the problem is reduced to the case of the 4 samples (6.4.8) with $\alpha'_0 = 0$ and

(6.4.9) $$\alpha'_{3,1} = \alpha'_{1,4} = \alpha'_{4,2} = 1.$$

From (6.4.3), (6.4.9) and theorem V then follows

(6.4.10) $$t_1 = t_4.$$

In this way the problem is reduced to the problem of maximizing the function

(6.4.11) $$L'(y_1, y_2, y_3) \stackrel{def}{=} L(y_1, y_2, y_3, y_1)$$

in the domain

(6.4.12) $$D' \begin{cases} 0 \leqq y_3 \leqq y_1 \leqq y_2, \\ y_1 \leqq 4. \end{cases}$$

From (6.4.5) and (6.4.6) it follows that

(6.4.13)

i	3	1	2
m'_i	3,8	4,8	4,39
$\mathcal{Y}'_i$	$(0, \infty)$	$(0, 4)$	$(-\infty, +\infty)$
v'_i	3,8	4	4,39

Thus

(6.4.14) $$t_1 = t_4 = 4\ ,\ t_2 = 4{,}39\ ,\ t_3 = 3{,}8.$$

References

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[2] van Eeden, Constance, Maximum likelihood estimation of ordered probabilities, Proc. Kon. Ned. Akad. v. Wet. A 59 (1956), Indagationes Mathematicae 18 (1956)