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S 318

Green's function methods in probability theory

Three lectures by Dr. J. Keilson

Summary compiled by W. Molenaar

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List of errata in Report S 318 (lectures by Dr. Keilson)

page	line	for:	read:
2	eq. (1.1)	X	X_D (twice)
5	eq. (3.4)	$\int_{-\infty}^{\infty}$	$\int_{0}^{\infty}$
6	14	mx,	m,x
	15	we can find for	it will be shown that
	16	process	process is
8	5	k 0	$k < 0$
9	4	1962	1961
11	6	design	designate
	eq. (2.2)	$\Gamma(z,t)$	$\gamma(z,t)$
	last line	$= \nu t$	$= \nu t$, and $\gamma(z,\nu^{-1})$ is a characteristic function
13	eq. (3.6)	$N^{\frac{3}{2}}$	$N^{-\frac{3}{2}}$
20	5	the integrand in (8.5)	$\frac{a^2(z)}{1-a(z)}$
23	14	each ε	each small ε
24	eq. (3.8)	S_{x_o}	S_x
25	6	$-K \leq$	$-K <$
	10	$\{-K,-K+1,$	$\{-K+1,$
	eq. (4.1)	$\sum_{j=-K}$	$\sum_{j=-K+1}$
26	2 from below	$\lvert m\rvert^{-1}$	$\lvert \nu m\rvert^{-1}$
27	14, 19	$\lvert m\rvert^{-1}$	$\lvert \nu m\rvert^{-1}$
	16	A(x)U(x)	$A^+(x)U(x)$
	19	$x \leq 0$	$x < 0$
28	5 from below	$\lvert m\rvert^{-1}$	$\lvert \nu m\rvert^{-1}$
29	2,3	delete: "; after this the process remains there"	
	eq. (6.8)	$P_1(\infty)$	$P_o(\infty)$
30	1	$\lvert m\rvert^{-1}$	$\lvert \nu m\rvert^{-1}$
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33	2	(6.16)	(6.17)
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Contents

GREEN'S FUNCTION METHODS IN PROBABILITY THEORY

Three lectures held at the Mathematical Centre

by

Dr. J. Keilson

First lecture (30.9.63): Green's functions

0. Notation

We consider time-homogeneous Markov processes $X_D(k)$ and $X(t)$ in discrete and continuous time respectively. The processes take values x in a state space $\mathfrak{X}$, which may be the continuum or the lattice of integers.

For the most general class of such processes, those in discrete time, we define

$$F_k(x) = P\left\{X_D(k) \leqslant x\right\}, \quad W_k(x) = \frac{d}{dx} F_k(x),$$
$$F_{tr}(x',x) = P\left\{X_D(k+1) \leqslant x \middle| X_D(k) = x'\right\}, \tag{0.1}$$
$$\mathcal{A}(x',x) = \frac{d}{dx} F_{tr}(x',x),$$

where differentiation is to be understood in Schwartz's generalized sense. From (0.1) follows

$$W_k(x) = \int W_{k-1}(x')\mathcal{A}(x',x)dx' . \tag{0.2}$$

The corresponding process in continuous time is $X(t) = X_D(K(t))$, where the auxiliary process $K(t)$ gives the number of increments that have occurred in $(0,t)$. We will assume that $K(t)$ is a Poisson process:

$$\alpha_k(t) \stackrel{\text{def}}{=} P\left\{K(t) = k\right\} = \frac{(\nu t)^k}{k!} e^{-\nu t} . \tag{0.3}$$

Corresponding to $W_k(x)$ we now have the density

$$W(x,t) = \sum \alpha_k(t)\, W_k(x), \tag{0.4}$$

which is easily seen to be a solution of the continuity equation

$$\frac{\partial W(x,t)}{\partial t} = -\nu W(x,t) + \nu \int W(x',t)\, \mathcal{A}(x',x)\,dx'. \tag{0.5}$$

1. Homogeneous processes

A process X(k) is called spatially homogeneous (or simply: homogeneous) if

$$X(k) = X(0) + \sum_{1}^{k} \xi_j , \tag{1.1}$$

where ξ_j are mutually independent and have density function A(x). Now

$$\mathcal{A}(x',x) = A(x-x'), \tag{1.2}$$

and

$$\frac{\partial W(x,t)}{\partial t} = -\nu W(x,t) + \nu \int W(x',t)\; A(x-x')\,dx'. \tag{1.3}$$

We denote characteristic functions by corresponding lower case letters, e.g.

$$w_k(z) = \int_{-\infty}^{\infty} e^{izx}\; W_k(x)\,dx. \tag{1.4}$$

By (0.2) and (1.2) we have

$$w_k(z) = a^k(z)\; w_o(z), \tag{1.5}$$

and by (0.3) and (0.4)

$$w(z,t) = w(z,0)\; \exp(-\nu t\left[1-a(z)\right]). \tag{1.6}$$

The homogeneous process with drift in continuous time, $X_v(t)$, is now defined by

$$X_v(t) = X(t) + vt \tag{1.7}$$

Thus $X_o(t) = X(t)$ and the density function $W_v(\xi,t)$ is given by $W_v(\xi,t) = W_o(x-vt,t)$, from which

$$w_v(z,t) = w(z,0)\ \exp(izvt - \nu t\left[1-a(z)\right]). \qquad (1.8)$$

$W_v(x,t)$ is a solution of

$$\frac{\partial W_v}{\partial t} + v\frac{\partial W_v}{\partial x} = -\nu W_v + \nu \int W_v(x',t)\ A(x-x')dx'. \qquad (1.9)$$

2. Green's functions, simple cases

From the linearity of (1.3) it follows that we may write

$$W(x,t) = \int W(x',0)\ \Gamma(x-x',t)dx'. \qquad (2.1)$$

The function $\Gamma(x,t)$, called the Green's function of the process, may be interpreted as the probability density of a transition from 0 to x in time t. From (1.6) and (2.1) we deduce

$$\gamma(z,t) = \exp(-\nu t\left[1-a(z)\right]). \qquad (2.2)$$

When

$$A(x) = \sum_{-\infty}^{\infty} \varepsilon_n\ \delta(x-n)\ , \qquad (2.3)$$

(Dirac's delta-function), and X(0) is an integer, the process is confined to the lattice, and

$$W(x,t) = \sum_{-\infty}^{\infty} P_n(t)\ \delta(x-n)\ . \qquad (2.4)$$

The continuity equation (1.3) now takes the simple form

$$\frac{dP_n}{dt} = -\nu P_n(t) + \nu \sum_{-\infty}^{\infty} P_j(t)\ \varepsilon_{n-j}\ , \qquad (2.5)$$

and to (2.2) corresponds

$$\gamma(u,t) \overset{def}{=} \sum_{-\infty}^{\infty} \Gamma_n(t)u^n = \exp(-\nu t\left[1-\varepsilon(u)\right]), \qquad (2.6)$$

where $\varepsilon(u) \overset{def}{=} \sum_{-\infty}^{\infty} \varepsilon_n u^n$. Integrating along a contour about zero inside the unit circle we have

$$\Gamma_n(t) = \frac{1}{2\pi i} \oint u^{-n-1} \exp(-\nu t[1-\varepsilon(u)])\,du. \qquad (2.7)$$

Example 1: exponential increments. If U(x) is the unit step function, $a(z) = \frac{\lambda}{\lambda - iz}$

$$A(x) = \lambda e^{-\lambda x} U(x) \text{ and } A^{(k)}(x) = \frac{\lambda^k x^{k-1}}{(k-1)!} e^{-\lambda x} U(x), \qquad (2.8)$$

$\mathcal{X}(z,t) = \exp\left(\frac{i\nu t z}{\lambda - iz}\right)$

we can express the Green's function for the process with drift in a modified Bessel function I_1:

$$\Gamma(x,t) = \delta(x-vt)e^{-\nu t} + \lambda\nu t\, U(x-vt) e^{-\lambda\nu t - \lambda(x-vt)} \frac{I_1(2\sqrt{\lambda\nu t(x-vt)})}{\sqrt{\lambda\nu t(x-vt)}}. \qquad (2.9)$$

Example 2: diffusion approximation. If we have for each m a process with increments ξ_m at a rate ν_m, such that as $m \to \infty$

$$\bar{\xi}_m \to 0, \quad \nu_m \to \infty, \quad \nu_m \bar{\xi}_m \to v \text{ and } \nu_m \overline{\xi_m^2} \to \beta \qquad (2.10)$$

then X(t) becomes "continuous" in the limit,

$$\lim_{n \to \infty} \Gamma_m(x,t) = \frac{\exp[-(x-vt)^2/2\beta t]}{\sqrt{2\pi\beta t}}, \qquad (2.11)$$

and the equation of motion tends to the equation

$$\frac{\partial w}{\partial t} = \frac{1}{2}\beta \frac{\partial^2 w}{\partial x^2} - v \frac{\partial w}{\partial x}. \qquad (2.12)$$

Example 3: single step walk on the lattice. Here $\varepsilon_1 = \lambda\nu^{-1}$, $\varepsilon_{-1} = \eta\nu^{-1}$, $\lambda + \eta = \nu$, and one obtains

$$\Gamma_n(t) = \left(\frac{\lambda}{\eta}\right)^{n/2} e^{-(\lambda+\eta)t} I_{|n|}(2t\sqrt{\lambda\eta}). \qquad (2.13)$$

3. Inhomogeneous source terms

We now consider the inhomogeneous equation with source term $\rho(x,t)$:

$$\frac{\partial W}{\partial t} + \nu W(x,t) - \nu \int W(x',t) A(x-x')dx' = \rho(x,t). \tag{3.1}$$

If $\rho(x,t) = \delta(x)\,\delta(t-0^{+})$ we have for $W(x,0) = 0$ that $W(x,0^{++}) = \delta(x)$ and $W(x,t) = \Gamma(x,t)$ for $t > 0^{++}$. For general $W(x,0) = W_0(x)$ and $\rho(x,t)$ we have

$$W(x,t) = \int_{-\infty}^{\infty} W_0(x')\,\Gamma(x-x',t)dx' + \int_{-\infty}^{\infty}\int_0^t \rho(x',t')\,\Gamma(x-x',t-t')dt'dx'. \tag{3.2}$$

For $t \to \infty$, x fixed, we have $\Gamma(x,t) \to 0$. So if $\rho(x,t)$ is such that $\lim_{t\to\infty} \rho(x,t) = \rho(x)$, we find

$$\lim_{t\to\infty} W(x,t) = \int_{-\infty}^{\infty} \rho(x')\,\Gamma_{\infty}(x-x')dx', \tag{3.3}$$

where Γ_{∞} is the steady state Green's function:

$$\Gamma_{\infty}(x) \overset{\text{def}}{=} \int_0^{\infty} \Gamma(x,t)dt = \nu^{-1}\left[\delta(x) + \sum_{j=1}^{\infty} A^{(j)}(x)\right]. \tag{3.4}$$

Convergence conditions for Γ_{∞}, essentially $\int x\,A(x)dx \neq 0$, are given by FELLER and OREY (1961) and W.L. SMITH (1962). Analogously, for the motion on the lattice, the solution of

$$\frac{dP_n}{dt} + \nu P_n(t) - \nu \sum_j P_j(t)\,\varepsilon_{n-j} = \rho_n(t) \tag{3.5}$$

is

$$P_n(t) = \sum_j P_j(0)\,\Gamma_{n-j}(t) + \sum_j \int_0^t \rho_j(t')\,\Gamma_{n-j}(t-t')dt'. \tag{3.6}$$

If $\rho_n(t)$ is such that $\lim_{t\to\infty} \rho_n(t) = \rho_n$ we have

$$\lim_{t\to\infty} P_n(t) = \sum_j \rho_j\,\Gamma_{\infty(n-j)} \quad \text{with} \quad \Gamma_{\infty n} \overset{\text{def}}{=} \int_0^{\infty} \Gamma_n(t)dt. \tag{3.7}$$

4. Bounded homogeneous processes

We now consider a process with state space $\mathcal{X} = \{ x | 0 < x < \infty \}$. We call it a bounded homogeneous process if it is spatially homogeneous apart from a certain zone near the boundary, i.e. if there exists an $x_b > 0$ such that, for $\mathcal{A}$ defined in (0.1), $\mathcal{A}(x',x) = A(x-x')$ holds for $x' > x_b$ and all $x \in \mathcal{X}$ and for $x > x_b$ and all $x' \in \mathcal{X}$, while for $x < 0$ $\mathcal{A}(x',x) = 0$. The process is irreducible if any state x can be reached from any state x' in a finite number of transitions, it is ergodic (stationary) if it is irreducible and $\int xA(x)dx < 0$. For a process of this type starting at $n_o > 0$ the first passage time density (probability density of the time t that elapses before the process reaches the boundary x=0) may be shown to be

$$S_{n_o}(t) = \frac{n_o}{t} \Gamma_{-n_o}(t) . \tag{4.1}$$

If in the lattice case $\mathcal{A}(m,x) = \sum_n \vartheta_{mn} \delta(x-n)$, with $\vartheta_{mn} = \varepsilon_{n-m}$ for $m,n > n_b$ and $\vartheta_{mn} = 0$ for $n < 0$, we can find for the limit probability of the ergodic process a finite sum

$$\lim_{t \to \infty} P_n(t) = \sum_{j=b_-}^{b_+} \beta_j \Gamma_{\infty}(n-j) . \tag{4.2}$$

5. Generalized image method

Differentiating trivial identities like

$$\oint \frac{d}{du} \exp\left[\nu t\, \varepsilon(u)\right] du = 0 \tag{5.1}$$

R times with respect to t and using (2.7), one can calculate for every $R \geq 0$ the linear recurrence relations for the Green's function elements:

$$\sum_{-\infty}^{\infty} k\, \varepsilon_k^{(R+1)} \Gamma_{-k}(t) = 0 . \tag{5.2}$$

Suppose we have $\varepsilon_1 \neq 0$, $\varepsilon_n = 0$ for $n > 1$ (single step property in the positive direction). Let N(t) be the first passage time process from $n_o > 0$ to state zero or any negative state. Then

$$P_n(t) \overset{def}{=} P\left\{N(t) = n \middle| N(0) = n_o\right\}$$

obeys

$$\begin{cases} \dfrac{dP_n}{dt} = -\nu P_n + \nu \varepsilon_1 P_{n-1}(1-\delta_{n,1}) + \nu \sum\limits_1^{\infty} \varepsilon_{-k} P_{n+k}, & (n \geqslant 1) \\ P_n(0) = \delta_{n,n_o}. \end{cases} \quad (5.3)$$

We compare this with an unrestricted process $N^*(t)$ with probabilities $P_n^*(t)$ and equations

$$\begin{cases} \dfrac{dP_n^*}{dt} = -\nu P_n + \nu \sum\limits_{-1}^{\infty} \varepsilon_{-k} P_{n+k}^* & (\text{all } n) \\ P_n^*(0) = \delta_{n,n_o} + \sum\limits_{-\infty}^{-1} q_k(n_o)\, \delta_{n,k} \end{cases} \quad (5.4)$$

We want to <u>compensate</u> the boundedness of our process N(t) by a proper choice of the <u>image coefficients</u> $q_k(n_o)$. The equations (5.3) and (5.4) are identical for $n > 0$ if $P_o^*(t) = 0$. Because of (5.2) with R=0, this will be the case, for $n_o = 1$, if

$$q_k(1) = \frac{-k\varepsilon_{-k}}{\varepsilon_1}. \quad (5.5)$$

If $n_o > 1$, we may use (5.2) for $R = 0,1,\ldots,n_o-1$ and eliminate the unknown $q_k(n_o)$ for this case too, using a relation between Γ_{-n_o} and Γ_j for $j > 0$. Thus we may always add compensating terms to our unbounded process $N^*(t)$ such that it behaves exactly like the bounded process N(t) in the region where N(t) is defined.

6. <u>Bounded homogeneous lattice processes</u>

In the state space $\mathcal{N} = \{0,1,2,\ldots\}$ we have the Markov process N(t) in continuous time with <u>transition matrix</u> $\{\vartheta_{jk}\}$ governing transitions from j to k ($\vartheta_{jj} = 0$). We assume that a) for all $j \geqslant n_b$, $\vartheta_{jk} = \varepsilon_{k-j}$ and $\vartheta_{kj} = \varepsilon_{j-k}$ (the process is "homogeneous"); b) negative increments are bounded, i.e. there is a $K < 0$ such that $\varepsilon_k = 0$ for $k < K$. We define

$$\nu_{jk} = \nu \vartheta_{jk} \; ; \quad \nu_j = \sum_k \nu_{jk} \; ; \quad P_j(t) = P\{N(t) = j\}. \tag{6.1}$$

The equations of motion are

$$\frac{dP_j(t)}{dt} = - \nu_j P_j(t) + \sum_{k=0}^{\infty} P_k(t) \nu_{kj} \qquad (j=0,1,\ldots) \tag{6.2}$$

We introduce a set of dummy states with indices $j = -1,-2,\ldots$ and put $\nu_{jk} = 0$ for $k < 0$. Then the dummy states are always empty, and (6.2) is also valid for negative j.

Now let $\nu'_{jk} \overset{def}{=} \nu \varepsilon_{k-j}$ for all j,k be the conditional transition frequencies for the associated homogeneous process. Then

$$\frac{dP_j(t)}{dt} = -\nu'_j P_j(t) + \sum_{k=-\infty}^{\infty} P_k(t)\nu'_{kj} + \rho_j(t), \text{ for all } j, \tag{6.3}$$

where

$$\rho_j(t) \overset{def}{=} (\nu'_j - \nu_j) P_j(t) + \sum_{-\infty}^{\infty} (\nu_{kj} - \nu'_{kj}) P_k(t) \tag{6.4}$$

may be regarded as source terms. From assumption a) we have for $j \geq n_b$ $\rho_j(t) = 0$, and from (6.4)

$$\rho_j(t) = - \sum_{0}^{\infty} P_k(t) \nu'_{kj} \qquad \text{for } j < 0. \tag{6.5}$$

Because of assumption b) $\rho_j(t) = 0$ for $j \leq K$. Thus $\rho_j(t) = 0$ except for $K \leq j \leq n_b$. If $N(0) = n_o$, the solution of (6.3) will therefore be

$$P_j(t) = \sum_{K+1}^{n_b-1} \int_0^t \rho_k(t') \Gamma_{j-k}(t-t')dt' + \Gamma_{n-n_o}(t). \tag{6.6}$$

If the process is ergodic

$$P_j(\infty) \overset{def}{=} \lim_{t \to \infty} P_j(t) = \sum_{K+1}^{n_b-1} \rho_k(\infty) \Gamma_{\infty(j-k)} \tag{6.7}$$

is again a finite sum of Green's function elements. In most cases

$\Gamma_{\infty n}$ will be asymptotically geometric for large n, i.e. $\Gamma_{\infty n} \sim \alpha \beta^n$. Thus

$$P_j(\infty) \sim \alpha \Big(\sum_k \rho_k(\infty) \beta^{-k} \Big) \beta^j \qquad (j \to \infty) \qquad (6.8)$$

has the same geometric decay constant. Other random walks are treated in KEILSON (1962) by a related method.

Second lecture (7.10.63) : Asymptotic properties of Green's functions

1. Characteristic functions and their convergence strips

If

$$a(z) \overset{\text{def}}{=} \int_{-\infty}^{\infty} e^{izx}\, dF(x) = a^{+}(z) + a^{-}(z), \qquad (1.1)$$

with

$$a^{+}(z) \overset{\text{def}}{=} \int_{0^+}^{\infty} e^{izx}\, dF(x) \text{ and } a^{-}(z) \overset{\text{def}}{=} \int_{-\infty}^{0^+} e^{izx}\, dF(x) \qquad (1.2)$$

then $a^{+}(z)$ is convergent and regular at least in the upper half plane, and the same applies to $a^{-}(z)$ in the lower half plane. For absolutely continuous F we have $a^{+}(z) \longrightarrow 0$ or $a^{-}(z) \longrightarrow 0$, as $|z| \longrightarrow \infty$ in all directions in the upper and lower half plane respectively.
If a^{+} or a^{-} converges also for some z in the other half plane, there will be a common convergenge strip for $a^{+}(z)$ and $a^{-}(z)$, where a(z) is regular. The strip will terminate at singularities on the imaginary axis which may be poles, essential singularities or branch-points.

For our one-dimensional random walks with increment distribution function F(x) and (generalized) density A(x), we will assume in this lecture that it is not degenerate, that the first two moments exist, and that there exists a convergence strip of positive (may be infinite) length. Now the moment generating function

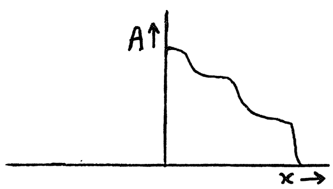
$$M(\vartheta) \overset{\text{def}}{=} a(i\vartheta) = \int e^{-\vartheta x}\, dF(x), \qquad (1.3)$$

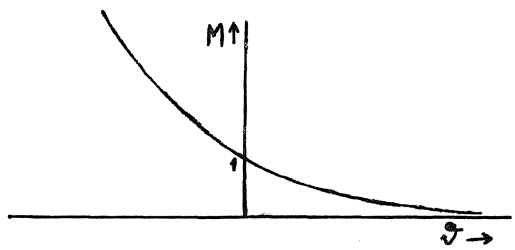
having

$$M''(\vartheta) = \int e^{-\vartheta x} x^2 dF(x) > 0, \qquad (1.4)$$

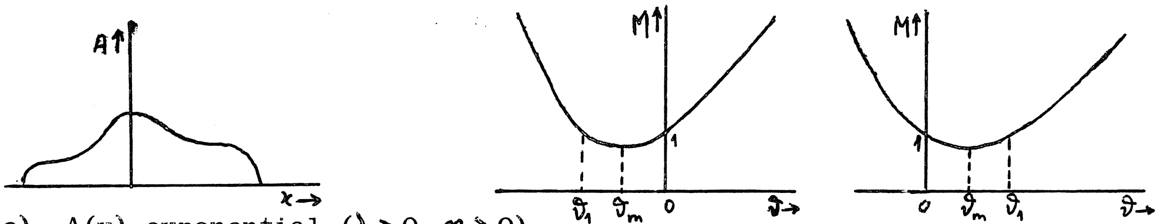
is a convex function of ϑ. Several possible graphs of this function are roughly sketched below.

a) $A(x) = 0$ unless $0 \leq x < K^{+}$

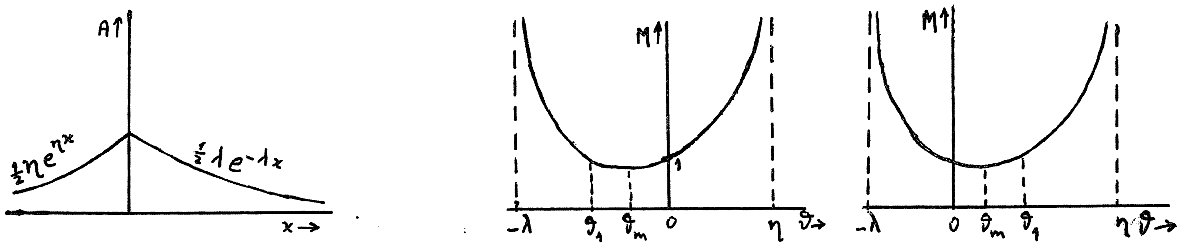




b) $A(x) = 0$ unless $K^- < x < K^+$: $M'(0) = -A_1 > 0$ $M'(0) = -A_1 < 0$



c) $A(x)$ exponential $(\lambda > 0,\ \eta > 0)$



In cases a) and b) $M(\vartheta)$ is an entire function, while in case c) it has poles at $\vartheta = -\lambda$ and $\vartheta = \eta$. The sign of the value ϑ_m where $M(\vartheta)$ is minimal is the sign of the first moment $A_1 \overset{\text{def}}{=} \int xA(x)dx$. If there exists a $\vartheta \neq 0$ with $M(\vartheta) = 1$ we shall design this value by ϑ_1 ($M(0) = 1$ is trivial). If the convergence strip ends with a branch-point η, we have in general a finite value $M(\eta)$ instead of the behaviour shown in figure c).

2. Central limit behavior of the Green's function

For the process in discrete time we found in the first lecture

$$\Gamma_N(x) = A^{(N)}(x) \text{ and } \gamma_N(z) = a^N(z). \qquad (2.1)$$

The Central Limit Theorem implies that $\Gamma_N(x)$ will tend to the Gaussian law for $N \to \infty$.

The process in continuous time, which is of greater interest to us, is given by

$$\Gamma(x,t) = \sum_{k=0}^{\infty} \frac{(\nu t)^k}{k!} e^{-\nu t} A^{(k)}(x) \text{ and}$$

$$\gamma(z,t) = \exp(-\nu t\left[1-a(z)\right]) = \left[\gamma(z,\nu^{-1})\right]^N , \qquad (2.2)$$

where $N \overset{\text{def}}{=} \nu t$. If we put $A_n \overset{\text{def}}{=} \int x^n A(x)dx$, we can show that

$$\overline{X(t)} = NA_1 \text{ and } \overline{X^2(t)} = NA_2 , \tag{2.3}$$

and the Central Limit Theorem gives now

$$\Gamma(x,t) \sim \frac{\exp\left[-(x-NA_1)^2/2NA_2\right]}{\sqrt{2\pi NA_2}} \qquad (N = \nu t \to \infty). \tag{2.4}$$

For given x and t, this central limit approximation (2.4) may be rather coarse. In the next section we will discuss the saddlepoint approximation, introduced into statistics by DANIELS (1954). This will turn out to be a more refined approximation, while it will also permit us to give a "rule of thumb" for the domain of values (x,t) for which (2.4) can be safely used.

3. Saddlepoint approximation

$\Gamma(x,t)$ is the inverse Fourier transform of $\gamma(z,t)$, so by (2.2)

$$\Gamma(x,t) = \frac{1}{2\pi} \int_{-\infty+i\vartheta}^{\infty+i\vartheta} \exp(-\nu t[1-a(z)] - izx)dz \tag{3.1}$$

for any (real) ϑ in the convergence strip. Substituting z=iv we get

$$\Gamma(x,t) = \frac{1}{2\pi i} \int_{\vartheta-i\infty}^{\vartheta+i\infty} \exp(N[M(v)-1 + \frac{vx}{N}])dv. \tag{3.2}$$

The saddlepoint v is found, after differentiating the expression in square brackets, from

$$M'(v) + \frac{x}{N} = 0 . \tag{3.3}$$

By a simple argument the relevant solutions of (3.3) must be real values. $M'(\vartheta)$ is monotonic because of $M''(\vartheta) > 0$. If we know that $M'(\vartheta)$ ranges from $-\infty$ to $+\infty$ as ϑ ranges through the convergence strip (this is true in most applications; cf. the discussion for discrete time in DANIELS, p.637-639) then (3.3) has exactly one real solution for v. What we have done amounts to solving

$$\zeta(x,N) \overset{\text{def}}{=} \frac{x-NA_1}{N} = \int xA(x)(e^{-x\vartheta}-1)\,dx = M'(0) - M'(\vartheta) \qquad (3.4)$$

for ϑ, and we will denote the unique solution by ϑ_ζ. The number ϑ_ζ, which depends on x and N, can be obtained without great difficulties, analytically or if necessary numerically.

Switching the contour in (3.2) to Re $v = \vartheta_\zeta$, we have

$$\Gamma(x,t) = \frac{1}{2\pi} e^{-N[1-M(\vartheta_\zeta)]+x\vartheta_\zeta} \int_{-\infty}^{\infty} e^{N[M(\vartheta_\zeta+iy)-M(\vartheta_\zeta)-iy\,M'(\vartheta_\zeta)]}\,dy. \qquad (3.5)$$

Let $y\sqrt{N} = s$. The integrand can be expanded about s=0 and becomes

$$\exp\left[-\tfrac{1}{2}M''(\vartheta_\zeta)s^2\right] \cdot \left[1 + N^{-\frac{1}{2}}O_1(s) + N^{-1}E_2(s) + N^{-\frac{3}{2}}O_3(s) + \ldots\right], \qquad (3.6)$$

where O_j and E_j denote odd and even power series functions respectively. The expansion is valid for $|s| < R_\zeta\sqrt{N}$, where R_ζ is the distance from the saddlepoint $z = i\vartheta_\zeta$ to the nearest singularity of a(z). We can integrate (3.6) and obtain the saddlepoint approximation:

$$\Gamma(x,t) \sim \frac{e^{-N+NM(\vartheta_\zeta)+x\vartheta_\zeta}}{\sqrt{2\pi NM''(\vartheta_\zeta)}} \left[1 + \frac{f_1(\vartheta_\zeta)}{N} + \frac{f_2(\vartheta_\zeta)}{N^2} + \ldots\right]. \qquad (3.7)$$
$$(N \to \infty)$$

Remarks:

a) (3.7) is an asymptotic expansion in powers of N^{-1} (the odd functions vanish when integrated).

b) the answer depends only on ζ and N, i.e. on x and N.

c) the coefficients $f_i(\vartheta_\zeta)$ are available from the Bürmann-Lagrange inversion formula for power series.

d) If a(z) is an entire function the expansion (3.7) may converge.

e) The approximation is good when $f_1(\vartheta_\zeta) \ll N$.

f) In the domain of (x,t) values or (ζ,N) values that are really interesting, ϑ_ζ is small and f_1 is of the order of magnitude of unity. Then e) reduces to $N \gg 1$.

4. Domain of validity of the Central Limit Theorem

We have seen in section 2 that the Central Limit Theorem is valid for $\Gamma(x,t)$ as $t \to \infty$. Thus the probability mass of the Green's function will concentrate for $N = \nu t \to \infty$ in the interval

$$\frac{|x-NA_1|}{\sqrt{NA_2}} = O(1) \ , \tag{4.1}$$

which implies by (3.4) that $\zeta(x,N) \to 0$. We want to solve ϑ_ζ from

$$\varphi(\vartheta_\zeta) \overset{\text{def}}{=} \int xA(x)\left[\exp(-x\vartheta_\zeta)-1\right] dx = \zeta \ . \tag{4.2}$$

Now $\varphi(\vartheta)$ is analytic in ϑ_ζ as ϑ_ζ lies in the convergence strip, and

$$\varphi'(\vartheta) = -\int x^2 A(x) \ \exp(-x\vartheta_\zeta)\, dx \tag{4.3}$$

is $\neq 0$ for $\vartheta=0$, so that ϑ_ζ is a regular function of ζ in some region about $\zeta=0$. As $\vartheta_o = 0$ the Bürmann-Lagrange formula gives us

$$\vartheta_\zeta = \sum_1^\infty \beta_n \zeta^n \ , \tag{4.4}$$

an expansion which converges for $|\zeta| < \zeta_M$ if ζ_M is the point (if any) nearest to the origin for which $\varphi'(\vartheta_{\zeta_M}) = 0$. As $\beta_1 = -1/A_2$, for large N (small ζ) we have approximately

$$\vartheta_\zeta \approx -\zeta/A_2 \ . \tag{4.5}$$

As ϑ_ζ will be small too we have

$$M(\vartheta_\zeta) \approx 1 - A_1\vartheta_\zeta + \tfrac{1}{2}A_2\vartheta_\zeta^{\ 2}. \tag{4.6}$$

Substituting (4.5) and (4.6) into (3.7) we may use

$$N(1-M(\vartheta_\zeta) - \frac{x}{N}\vartheta_\zeta) \approx \frac{N\zeta^2}{2A_2} \ , \tag{4.7}$$

and the dominant term of (3.7) becomes the right hand member of the central limit approximation (2.4). A practical criterion (rule of thumb) for the applicability of (4.5) and (4.6) is $|\vartheta_\zeta| \ll A_2^{-\frac{1}{2}}$, where $\sqrt{A_2}$ is a characteristic length of the distribution, about x=0, not the mean. By (4.5) this means $|\zeta| \ll \sqrt{A_2}$. The central limit approximation (2.4) will thus be useful, when

$$N \gg 1 \quad \text{and} \quad |x - NA_1| \ll N\sqrt{A_2} \ . \tag{4.8}$$

Other characteristic lengths may be given by postulating that the ratios of subsequent terms in (4.6) are $\ll 1$.

5. Conjugate transformations of Khinchin

Studying the first passage time density

$$S_x(t) = \frac{x}{t}\Gamma(-x,t) \tag{5.1}$$

which is the continuous analogue of (4.1) in the first lecture (page 6), we want to know the asymptotic behavior of $\Gamma(x,t)$ for fixed x. Now in the special case $A_1 = 0$, we have for $N \gg 1$

$$\frac{\zeta}{\sqrt{A_2}} = \frac{x}{N\sqrt{A_2}} \ll 1 \ , \tag{5.2}$$

and (4.8) reduces to $N \gg 1$. But if $A_1 \neq 0$ we cannot use the central limit theorem for fixed x, as the zone of validity given by (4.8) wanders off to infinity. The Khinchin transformations are a tool to overcome this difficulty and to employ the central limit theorem in different "zones of convergence". Together with the given density

$$W(x,t) = \int W_o(x')\,\Gamma(x-x',t)\,dx' \tag{5.3}$$

we have a family of conjugate densities

$$W^*_\vartheta(x,t) \overset{\text{def}}{=} \frac{W(x,t)\exp(-\vartheta x)}{\int W(x,t)\exp(-\vartheta x)\,dx} \tag{5.4}$$

where the parameter ϑ may take any real value in the common convergence strip of $M(\vartheta) = a(i\vartheta)$ and $w_o(i\vartheta)$. Now

$$w^*_\vartheta(z,t) = \frac{w(z+i\vartheta,t)}{w(i\vartheta,t)}, \tag{5.5}$$

and as $w(z,t) = w_o(z)\,\gamma(z,t) = w_o(z)\exp(-\nu t\,[1-a(z)])$, this gives

$$w^*_\vartheta(z,t) = w^*_{o\vartheta}(z)\exp(-\nu^*_\vartheta t[1-a^*_\vartheta(z)]), \tag{5.6}$$

with $\nu^*_\vartheta \overset{\text{def}}{=} \nu\, a(i\vartheta)$, $a^*_\vartheta(z) \overset{\text{def}}{=} \dfrac{a(z+i\vartheta)}{a(i\vartheta)}$ and $w^*_o(z) \overset{\text{def}}{=} \dfrac{w_o(z+i\vartheta)}{w_o(i\vartheta)}$. (5.7)

This gives us a <u>family of conjugate processes</u> $X^*_\vartheta(t)$ governed by ν^*_ϑ, $A^*_\vartheta(x)$ and $W^*_{o\vartheta}(x)$. In the special case $W_o(x) = \delta(x-x_o)$ the choice of the parameter ϑ is only restricted by the convergence strip of a(z). For the moments

$$A^*_{\vartheta n} \overset{\text{def}}{=} \int x^n A^*_\vartheta(x)\,dx \tag{5.8}$$

we find by differentiating $a^*_\vartheta(z)$ in z=0

$$A^*_{\vartheta_1} = \frac{-M'(\vartheta)}{M(\vartheta)} \quad\text{and}\quad A^*_{\vartheta 2} = \frac{M''(\vartheta)}{M(\vartheta)}. \tag{5.9}$$

Thus, when $N^*_\vartheta \overset{\text{def}}{=} \nu^*_\vartheta t \gg 1$ and $\dfrac{|x-N^*_\vartheta A^*_{\vartheta 1}|}{N^*_\vartheta\sqrt{A^*_{\vartheta 2}}} \ll 1$, (5.10)

we have

$$\Gamma^*_\vartheta(x;t) \sim \frac{\exp\left[-(x-N^*_\vartheta A^*_{\vartheta 1})^2/2N^*_\vartheta A^*_{\vartheta 2}\right]}{\sqrt{2\pi N^*_\vartheta A^*_{\vartheta 2}}} \tag{5.11}$$

For the validity of (5.10) for fixed x we need $A^*_{\vartheta 1} = 0$, which by (5.9) amounts to $M'(\vartheta) = 0$ i.e. to $\vartheta = \vartheta_m$ (minimal value, see section 1). We note that

$$\frac{d}{d\vartheta}A^*_{\vartheta 1} = \frac{-MM''+M'^2}{M^2} = -(A^*_{\vartheta 2}-A^2_{\vartheta 1}) = -\sigma^{*2}_\vartheta, \tag{5.12}$$

minus the variance of the conjugate density A^*_ϑ. So the conjugate mean $A^*_{\vartheta 1}$ is monotonic decreasing with ϑ. From (5.4) and (5.7) follows

$$\Gamma(x,t) = \exp(\nu^*_\vartheta t - \nu t + \vartheta x)\,\Gamma^*_\vartheta(x,t), \tag{5.13}$$

so that we can study the asymptotic behavior of $\Gamma(x,t)$ for fixed x by applying (5.11) to the conjugate process with $\vartheta = \vartheta_m$. This is always available when $|M'(\vartheta)|$ becomes infinitely large at both boundaries of the convergence strip (figures b) and c) of section 1).

6. Corresponding results for discrete time

Under conditions (4.8), the central limit approximation (2.4) is also valid for the density

$$\Gamma_N(x) = A^{(N)}(x) \tag{6.1}$$

in discrete time, but now the variance is $N\sigma^2 = N(A_2 - A_1^2)$ instead of NA_2. If necessary we employ a Khinchine transformation for a ϑ such that the equivalent of (5.10) is fulfilled and find for $\vartheta = \vartheta_m$

$$\Gamma_N(x) \sim M^N(\vartheta)\exp(\vartheta x)\,\frac{\exp(-x^2/2N\sigma_\vartheta^{*2})}{\sigma_\vartheta^* \sqrt{2\pi N}}\,, \tag{6.2}$$

valid if

$$N \gg 1 \quad \text{and} \quad |x| \ll N\sigma_\vartheta^* . \tag{6.3}$$

7. Lattice considerations

We can rewrite (2.7) of lecture one, with $N = \nu t$, as

$$\Gamma_n(t) = \frac{e^{-N}}{2\pi i} \oint \exp(N\left[\varepsilon(u) - \frac{(n+1)\log u}{N}\right])du. \tag{7.1}$$

If the generating function $\varepsilon(u)$ of the increments has a convergence annulus bounded by poles, there will exist for every value of $\alpha \overset{\text{def}}{=} (n+1)N^{-1}$ a saddlepoint ϑ_α on the positive real axis. We may integrate (7.1) along the circle about the origin through this saddlepoint, instead of the unit circle, and obtain

$$\Gamma_n(t) \sim \frac{\vartheta_\alpha^{-n-1}\exp\left[-N+N\varepsilon(\vartheta_\alpha)\right]}{2\pi N(\varepsilon''(\vartheta_\alpha)+\alpha\vartheta_\alpha^2)}\left[1+\frac{f_1(\alpha)}{N}+\frac{f_2(\alpha)}{N^2}+\ldots\right]$$

$$(N = \nu t \to \infty) \qquad (7.2)$$

The Gaussian approximation and the conjugate transformations can be obtained in a similar way to that of sections 4 and 5.

8. Asymptotic behavior of the steady state Green's function

In the first lecture we have introduced the steady state Green's function $\Gamma_\infty(x)$ on page 5, equation (3.4). For the "extended renewal density"

$$T(x) \overset{\text{def}}{=} \sum_{j=1}^{\infty} A^{(j)}(x) \qquad (8.1)$$

we shall give a theorem that is an extension of the renewal theorems in which A(x) has only positive support. An elaborate treatment has been given by FELLER and OREY (1961) and SMITH (1962). In the simplified form given here, geared to our own requirements, we state only the case $A_1 \overset{\text{def}}{=} \int xA(x)dx > 0$; for $A_1 < 0$ the asymptotic behavior of T(x) at plus and minus infinity is reversed.

Theorem. If (1) A(x) is of bounded variation,
(2) A(x) is continuous for all $x \neq 0$,
(3) $A_1 > 0$,

then T(x) is (A) convergent; (B) continuous for all $x \neq 0$; (c) bounded for all x; (D) $T(-\infty) = 0$; (E) $T(\infty) = A_1^{-1}$.

Proof. First suppose that a(z) is a rational function; this corresponds to $A^+(x)$ and $A^-(x)$ of general exponential form. Because of assumption (1) the degree must be negative. We define for $0 \leq |u| < 1$

$$t(u,z) = \frac{u\ a(z)}{1-u\ a(z)} \qquad (8.2)$$

and consider $M(v) = a(iv)$ for $0 < v < \vartheta_1$ (cf. section 1). In this region $|a(iv)| < 1$, so for $|u| < 1$ the function t(u,z) is analytic, it is a rational function in z and $\lim_{|z|\to\infty} t(u,z) = 0$. Now we have

$$T(u,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{u\ a(z)}{1-u\ a(z)}\, dz\ , \tag{8.3}$$

where we may replace the contour by the line $(-\infty+i\vartheta_m, \infty+i\vartheta_m)$. We may now permit $u \to 1$. The function $T(u,x)$ is then seen to be analytic at $u=1$ and we get

$$T(x) = T(1,x) = \frac{1}{2\pi} \int_{i\vartheta_m-\infty}^{i\vartheta_m+\infty} \frac{a(z)}{1-a(z)}\, dz, \tag{8.4}$$

from which the convergence (A) is straightforward. For the function

$$T(1,x) - A(x) = e^{\vartheta_m x} \int_{-\infty}^{\infty} \frac{a^2(\tau+i\vartheta_m)}{1-a(\tau+i\vartheta_m)}\, e^{-i\tau x}\, d\tau \tag{8.5}$$

we remark that the integrand is absolutely integrable because of the rationality of $a(z)$. This proves (B). We can now close the contour in the upper or lower half plane, depending on the sign of x, and prove (D) and (E): there are only a finite number of singularities, and the residues give only exponentially decaying contributions except the residue in 0 which gives A_1^{-1}. The boundedness (c) of $T(x)$ follows from (8.5).

For a more general $a(z)$ the proof can be reduced to the rational case. Let $B(x)$ be a second density obeying assumptions (1) (2) (3) and having a rational characteristic function $b(z)$. If $\mathcal{F}$ denotes a Fourier transform, we can write

$$\mathcal{F}(T(u,x)) = \frac{u\ a(z)}{1-u\ a(z)} = \frac{u\ b(z)}{1-u\ b(z)} \cdot \left[\frac{u(a(z)-b(z))}{1-u\ a(z)} + 1 \right] \cdot u\ a(z) \tag{8.6}$$

It is clear that we can choose B such that $B_1 = A_1$, i.e. $a'(0) = b'(0)$. If we put

$$T_B(u,x) = \mathcal{F}^{-1} \frac{u\ b(z)}{1-u\ b(z)} \quad \text{and} \quad D(u,x) = \mathcal{F}^{-1} \frac{u(a(z)-b(z))}{1-u\ a(z)} \tag{8.7}$$

then $T_B(u,x)$ and $D(u,x)$ are well behaved as $u \to 1$ and

$$T(x) = A(x) + D(1,x) * A(x) + T_B(x) * D(1,x) * A(x) + T_B(x) * A(x). \tag{8.8}$$

As $\int D(1,x)dx = 0$, the behavior of $T(x)$ for large x is dominated by $T_B(x) * A(x)$, and the asymptotic behavior of $T_B(x)$ is already proved. This proves (D) and (E). For the other parts we need the assumed bounded variation property of $A(x)$, which gives $\int |A'(x)| dx < \infty$, to prove the absolute integrability of the integrand in (8.5). We will not give the details of this proof.

Third lecture (14.10.63): Random walk problems

A. PASSAGE TIMES

1. Passage from the quadrant

Consider a process $\{X(t), Y(t)\}$ in the Euclidean plane such that $(X(0), Y(0)) = (x_o, y_o)$ is a point in the first quadrant I, and that $\{X(t)\}$ and $\{Y(t)\}$ are independent homogeneous Markov processes. Let τ be the time at which the process leaves the first quadrant I for the first time. We are interested in the passage time density $S_{x_o y_o}(\tau)$. Define

$$P_I(t) = P\{X(t') > 0, \ Y(t') > 0 \quad \text{for } 0 \leqslant t' \leqslant t \,|\, x_o, y_o\}. \qquad (1.1)$$

Then

$$P_I(t) = P_X\{X(t') > 0 \text{ for } 0 \leqslant t' \leqslant t \,|\, x_o\} P_Y\{Y(t') > 0 \quad \text{for } 0 \leqslant t' \leqslant t \,|\, y_o\} \qquad (1.2)$$

and

$$\frac{dP_I}{dt} = \frac{dP_X}{dt} P_Y + \frac{dP_Y}{dt} P_X , \qquad (1.3)$$

so that

$$S_{x_o y_o}(\tau) = - \frac{dP_I}{dt}(\tau) =$$

$$= S_{Xx_o}(\tau)\left(1 - \int_0^\tau S_{Yy_o}(t)dt\right) + S_{Yy_o}(\tau)\left(1 - \int_0^\tau S_{Xx_o}(t)dt\right). \qquad (1.4)$$

We may now employ the results for the one-dimensional passage time densities S_X and S_Y (cf. (5.1) of lecture 2, p.15).

2. Point to point passage

On an R-dimensional lattice of points $\vec{n} = (n_1, n_2, \ldots, n_R)$ we consider a process $\{\vec{N}(t)\} = \{N_1(t), N_2(t), \ldots, N_R(t)\}$ such that $\vec{N}(0) = \vec{n}_o$ and that the processes $\{N_i(t)\}$ $(i=1,\ldots,R)$ are independent homogeneous Markov processes. We want to know the passage time density $S_{\vec{n}_o}(\tau)$ of the first time τ at which the process reaches $\vec{0}$. Now a classical renewal argument says that, in order to be at $\vec{0}$ at time t, one must have arrived there for the first time at some τ, $0 \leqslant \tau \leqslant t$, and then be found again at $\vec{0}$ after time $t-\tau$. Hence

$$P\{\vec{N}(t) = \vec{0} | \vec{N}(0) = \vec{n}_0\} = S_{\vec{n}_0}(t) * P\{\vec{N}(t) = 0 | \vec{N}(0) = \vec{0}\} . \tag{2.1}$$

With the R-dimensional Green's function, which is the obvious generalization of the one introduced on page 3, we can write this as

$$\Gamma_{-\vec{n}_0}(t) = S_{\vec{n}_0}(t) * \Gamma_{\vec{0}}(t); \tag{2.2}$$

for the Laplace transforms

$$\gamma_{\vec{n}}(s) \stackrel{\text{def}}{=} \int_0^\infty e^{-st} \Gamma_{\vec{n}}(t)\,dt \text{ and } \sigma_{\vec{n}}(s) \stackrel{\text{def}}{=} \int_0^\infty e^{-st} S_{\vec{n}}(t)\,dt \tag{2.3}$$

this means

$$\sigma_{\vec{n}}(s) = \frac{\gamma_{-\vec{n}}(s)}{\gamma_{\vec{0}}(s)} . \tag{2.4}$$

The independence of $N_i(t)$ implies that with probability one only one coordinate changes at a time; this implies

$$\gamma_{\vec{n}}(s) = \int_0^\infty e^{-st} \prod_{i=1}^{R} \Gamma_{(i)n_i}(t)\,dt, \tag{2.5}$$

where $\Gamma_{(i)}$ denotes the Green's function associated with the movement in the direction of the i-th coordinate axis. In the special case where for each i this movement is only allowed to be a single step, each $\Gamma_{(i)}$ is a modified Bessel function (see example 3 on page 4).

From (2.4) and (2.5) one can calculate explicit formulae for the probability of ultimate arrival (in general less than unity), the means and variances of arrival times, etc.

Suppose we have a walk $\{N_D(k)\}$ in discrete time where only one coordinate changes at a time. If we change over to $\{N(t) = N_D(K(t))\}$ in continuous time (cf. page 1) the probability of ultimate arrival is unmodified. But one introduces at this step the independence of the coordinate motions, which is excluded in the discrete case by the assumption of only one changing coordinate at a time.

3. A passage problem for the continuum with exponentially distributed negative increments

Suppose we have the process

$$X_D(k) = X_D(k-1) + \xi_k, \tag{3.1}$$

and $\{X(t)\} = \{X_D(K(t))\}$ where the auxiliary process K(t) is again a Poisson process with parameter ν. Furthermore suppose that

$$\begin{aligned} &\text{a) } F(x) = P\{\xi_k \leqslant x\} \text{ is absolutely continuous;} \\ &\text{b) } F'(x) = q\eta \ \exp(\eta x) \quad \text{for } x < 0; \\ &\text{c) } X(0) = x_o > 0. \end{aligned} \tag{3.2}$$

If τ is the first time at which $X(\tau)$ is negative, we want to know its passage time density $S_{x_o}(\tau)$.

Lemma. The times τ_j at which the process X(t) has a traversal from positive to negative (i.e. $X(\tau_j-\varepsilon) > 0$, $X(\tau_j+\varepsilon) < 0$ for each $\varepsilon > 0$) generate a renewal process, i.e. the random variables

$$\vartheta_j \overset{\text{def}}{=} \tau_j - \tau_{j-1} \quad (\tau_o \overset{\text{def}}{=} 0) \tag{3.3}$$

are independent and identically distributed.

Proof. If it is known that there has been a positive-to-negative traversal at time τ_j, then the density in x at time $\tau_j + \varepsilon$ $(\varepsilon \searrow 0)$ is $\eta \exp(\eta x)U(-x)$, no matter what the state of the system was before τ_j. By this special property of the exponential distribution the distribution of the interval ϑ_{j+1} will be identical with, and independent of, the distribution of the previous interval ϑ_j.

For these renewal processes, the mean renewal time is infinite for any value of $m \overset{\text{def}}{=} \int \xi \, dF(\xi)$. The renewal process will terminate since there is a non-zero probability of being lost at infinity. In the special case m=0 there will always be another positive-to-negative traversal, but the expectation of ϑ is ∞. We define

$$
\begin{aligned}
Q(\tau) &= \text{density of renewal times} \\
R_{x_0}(t) &= \text{renewal density when } X(0) = x_0 \\
R(t) &= \text{renewal density when a renewal event has occurred at } t=0.
\end{aligned}
\tag{3.4}
$$

Then

$$
\begin{aligned}
R_{x_0}(t) &= S_{x_0}(t) + S_{x_0}(t) * \left[Q(t) + Q^{(2)}(t) + \ldots\right] = \\
&= S_{x_0}(t) + S_{x_0}(t) * R(t).
\end{aligned}
\tag{3.5}
$$

When $m \neq 0$ we have $\int_0^\infty Q(\tau)d\tau < 1$. The expression of $R_{x_0}(t)$ and $R(t)$ in terms of the Green's functions, with an obvious probabilistic interpretation, is

$$
\begin{aligned}
R_{x_0}(t) &= \nu q \int_{-\infty}^{0} \left[\int_0^\infty W(x',t)\eta\, e^{\eta(x-x')} dx'\right] dx = \\
&= \nu q \int_0^\infty \Gamma(x'-x_0,t) e^{-\eta x'} dx' ;
\end{aligned}
\tag{3.6}
$$

$$
R(t) = \nu q \int_0^\infty dx\, e^{-\eta x} \int_{-\infty}^{0} dy\, \eta e^{\eta y}\, \Gamma(x-y,t). \tag{3.7}
$$

From (3.5), (3.6) and (3.7) we can obtain $S_{x_0}(t)$ by Laplace transformation and inversion. The calculation can be found in KEILSON (1963).. We give only the answer, which is simple:

$$
S_{x_0}(\tau) = \left[1 + \eta^{-1}\frac{d}{dx}\right]\left[\frac{x}{\tau}\Gamma(-x,\tau)\right]. \tag{3.8}
$$

In this case the same answer can be reached by Wiener-Hopf methods, since explicit factorization is possible. Our method, which proceeds by the real characterization (3.5), can be slightly generalized to a case where the Wiener-Hopf theory fails because of factorization difficulties (see next section).

4. Passage time for the lattice with bounded negative increments

Suppose we have a homogeneous process $\{N(t)\}$ on the lattice of integers, with $N(0) = n_o > 0$, and $\varepsilon_k = 0$ for $k < -K$. Once more we ask for the passage density $S_{n_o}(\tau)$ of the first time τ at which $N(t)$ becomes non-positive.

For $-K \leq i \leq 0$ we define an "i-event" as a transition from a positive state to state i. The succession of all such events forms a multi-state semi-renewal process (associated with a semi-Markov process). We now have a matrix of densities of renewal times: for $i,j \in \{-K,-K+1,\ldots,0\}$ and real positive τ, $Q_{ij}(\tau)$ is the probability , given an i-event at time zero, that the next event will be a j-event and will take place at time τ . When the mean increment m is not zero there may be no subsequent j-event at all:

$$\sum_{j=-K}^{0} \int_0^{\infty} Q_{ij}(\tau)d\tau < 1 \qquad (4.1)$$

with a non-zero probability of being lost at $n=\infty$ or $n=-\infty$. We define:

$$\begin{aligned} S_{n_oj}(t) &= \text{first passage time density from } n_o > 0 \text{ to } j \leq 0; \\ R_{n_oj}(t) &= \text{renewal density for j-events when } N(0)=n_o; \\ R_{ij}(t) &= \text{renewal density for j-events after an i-event at } t=0. \end{aligned} \qquad (4.2)$$

Now

$$\begin{aligned} \vec{R}_{n_o}(t) &= \vec{S}_{n_o}(t) + \vec{S}_{n_o}(t) * [Q(t) + Q^{(2)}(t) + \ldots] = \\ &= \vec{S}_{n_o}(t) + \vec{S}_{n_o}(t) * R(t), \end{aligned} \qquad (4.3)$$

where the convolution of the vector $\vec{S}_{n_o}(t)$ and the matrix $R(t)$ means a vector with j-th component

$$\sum_i \int_0^t S_{n_oi}(t') R_{ij}(t-t')dt' . \qquad (4.4)$$

The renewal densities $R_{n_o j}(t)$ and $R_{ij}(t)$ are again available from the Green's functions:

$$R_{n_o j}(t) = \nu \sum_{k=1}^{j+K} \varepsilon_{j-k} \Gamma_{k-n_o}(t), \qquad (4.5)$$

$$R_{ij}(t) = \nu \sum_{k=1}^{j+K} \varepsilon_{j-k} \Gamma_{k-i}(t). \qquad (4.6)$$

The vector Volterra integral equation (4.3) has a formal Neumann series solution. When $|m|$ is far from zero, the convergence of the series is rapid and we have approximately

$$\vec{S}_{n_o}(t) \approx \vec{R}_{n_o}(t) - \vec{R}_{n_o}(t) * R(t). \qquad (4.7)$$

The Laplace transforms of $R_{ij}(t)$ form a matrix $r(s)$, for which we note that for $m \neq 0$, with I the unit matrix,

$$I + r(s) = I + q(s)\left[I-q(s)\right]^{-1} = \left[I-q(s)\right]^{-1}$$

has an inverse because of (4.1).

B. ERGODIC DISTRIBUTIONS

5. A structural property of the steady state Green's function

The steady state Green's function $\Gamma_{\infty}(x)$, or $\Gamma_{\infty n}$ for the lattice, was defined on page 5 by (3.4) and (3.7). For many simple but important processes, it is constant for negative x. Some examples and proofs are given here.

I. Lattice with skip-free property in one direction

Theorem. If (a) $\varepsilon_j = 0$ for $j < -1$ and $\varepsilon_{-1} \neq 0$; (b) $m = \sum_{-\infty}^{\infty} k\,\varepsilon_k < 0$, (c) $\varepsilon(u) = \sum_{-\infty}^{\infty} \varepsilon_k u^k$ is analytic at $u=1$, then $\Gamma_{\infty n} = |m|^{-1}$ for all $n \leq 0$.

Proof. Define for $0 \leqslant |z| < 1$

$$\Gamma_{\infty n}(z) = \frac{1}{2\pi i \nu} \oint \frac{u^{-n-1}du}{1-z\varepsilon(u)} , \tag{5.1}$$

with integration in positive direction along the unit circle. From $m < 0$ we know $\varepsilon'(1) < 0$, and together with the analyticity of $\varepsilon(u)$ at $u=1$ this implies that the integration contour may be replaced by a circle with slightly larger radius. We then have analyticity at $z=1$ and we may permit $z \to 1$ (cf. second lecture, top of page 19). Because of assumptions (a) and (c) we know that $1-\varepsilon(u)$ has only one singularity inside the new contour, namely a simple pole at $u=0$. By Rouché's theorem or by the principle of the argument there can be only one zero inside the contour, and we know this zero to be at $u=1$. We can now use the method of residues to find $\Gamma_{\infty n} = \lim_{z \to 1} \Gamma_{\infty n}(z)$ for $n \leqslant 0$. As the residue at $u=1$ is independent of n we find the same value for any $n \leqslant 0$, and this is the limiting value $|m|^{-1}$ found for $n \to -\infty$.

II. Continuum with exponentially distributed negative increments

Theorem. If (a) $A(x) = q\eta e^{\eta x}\, U(-x) + A(x)\, U(x)$;

(b) $m = \int xA(x)dx < 0$;

(c) $a(z)$ is analytic at $z=0$;

then $\Gamma_{\infty}(x) = |m|^{-1}$ for all $x \leqslant 0$.

Proof. As in the preceding proof we may show that

$$\int \frac{a(z)}{1-a(z)}\, e^{-izx} dz \tag{5.2}$$

is independent of x for $x \leqslant 0$.

III. Continuum with skip-free property in one direction

We consider a process with drift $v < 0$ (cf. (1.9) on page 3) and only positive increments governed by $A^{+}(x)$ at Poisson epochs with mean ν. An example is the Takács process for the virtual waiting time in queueing theory. The Fourier transform (with respect to space) of the

Green's function is (see (1.8) on page 3):

$$\gamma(z,t) = \exp(izvt- \nu t[1-a^+(z)]). \qquad (5.3)$$

Its Laplace transform (with respect to time) becomes

$$\tilde{\gamma}(z,s) = \frac{1}{s+\nu-izv-\nu a^+(z)} = \frac{1}{s+\nu-izv} \cdot \frac{1}{1-b^-(s,z)a^+(z)}, \qquad (5.4)$$

where

$$b^-(s,z) \overset{def}{=} \frac{\nu}{s+\nu-izv}. \qquad (5.5)$$

Now

$$m \overset{def}{=} -i \frac{d}{dz}\left[b^-(0,z)a^+(z)\right]_{z=0} = \frac{v}{\nu} + \int xA^+(x)dx. \qquad (5.6)$$

If $a^+(z)$ is analytic at z=0, and $m < 0$, we may shift the contour for the inverse Fourier transform into the lower half plane and then let $s \to 0$. With a reversion of the order of integration one finds that

$$\Gamma_\infty(x) \overset{def}{=} \int_0^\infty \Gamma(x,t)dt = \frac{1}{2\pi i}\int \tilde{\gamma}(z,0)e^{-izx}dz. \qquad (5.7)$$

Closing the contour we find one pole at $z= -\frac{i\nu}{v}$ and one zero at z=0 for the function $1-b^-(0,z)a^+(z)$. One can now proceed as in the preceding proofs and find the following:

Theorem. If, for the process defined above, (a) $v < 0$; (b) $m < 0$; and (c) $a^+(z)$ is analytic at z=0, then $\Gamma_\infty(x) = |m|^{-1}$ for all $x \leq 0$.

6. Elementary applications

Example 1.

Consider a process $\{N_D(k)\}$ in discrete time on the finite lattice $\mathcal{N}=\{0,1,\ldots,K\}$ with sticking boundaries at 0 and K, which means that any increment step which would overshoot a boundary is modified into a step ending at the boundary; after this step the process remains there. Suppose the increment ξ_k is a lattice random variable which can only take the values -1 and L ($L \leq K$):

$$\varepsilon_n \overset{\text{def}}{=} P(\xi_k=n) = \varepsilon_{-1}\,\delta_{n,-1} + \varepsilon_L\,\delta_{n,L} \quad (\varepsilon_{-1}+\varepsilon_L=1), \tag{6.1}$$

and suppose that $m = L\,\varepsilon_L - \varepsilon_{-1} < 0$. We have

$$N_D(k) = \begin{cases} \max(0, N_D(k-1)+\xi_k) & \text{for } N_D(k-1)+\xi_k \leq K; \\ \min(K, N_D(k-1)+\xi_k) & \text{for } N_D(k-1)+\xi_k \geq 0. \end{cases} \tag{6.2}$$

With the auxiliary Poisson process K(t) we may change over to continuous time:

$$N(t) = N_D(K(t)). \tag{6.3}$$

Again we introduce fictitious states with negative indices and with indices K+1, K+2,... . By the method described on page 8 we have at once for $-\infty < n < \infty$:

$$\frac{dP_n(t)}{dt} = -\nu P_n(t) + \nu \sum_{-\infty}^{\infty} P_m(t)\,\varepsilon_{n-m} + \rho_n(t), \tag{6.4}$$

where

$$\begin{cases} \rho_{-1}(t) = -\nu\varepsilon_{-1}\,P_o(t) \\ \rho_o(t) = \nu\varepsilon_{-1}P_o(t) \end{cases} \tag{6.5}$$

$$\begin{cases} \rho_K(t) = \sum_{j=K+1}^{K+L} \nu\varepsilon_L\,P_{j-L}(t) \\ \rho_j(t) = -\nu\varepsilon_L\,P_{j-L}(t) \text{ for } j=K+1,\dots,K+L; \end{cases} \tag{6.6}$$

$$\rho_j(t) = 0 \text{ for all other } j. \tag{6.7}$$

Hence (cf. lecture one (6.7) on page 8)

$$P_n(\infty) = \nu\varepsilon_{-1}P_1(\infty)(\Gamma_{\infty n} - \Gamma_{\infty\,(n+1)}) + \sum_{j=K}^{K+L} \Gamma_{\infty\,(n-j)}\,\rho_j(\infty). \tag{6.8}$$

If we restrict ourselves to $0 \leqslant n \leqslant K$, then $\Gamma_{\infty(n-j)} = |m|^{-1}$ and the second term becomes

$$|m|^{-1} \sum_{j=K}^{K+L} \rho_j(\infty), \qquad (6.9)$$

which is zero because of the definition (6.6) for $\rho_j(t)$. We see that the second boundary does not influence the limit distributions. We have taken care that the fictitious states can never be reached, so that we have by (6.8)

$$1 = \sum_{n=0}^{K} P_n(\infty) =$$

$$= \nu\varepsilon_{-1} P_o(\infty) \left\{ (\Gamma_{\infty 0} - \Gamma_{\infty 1}) + (\Gamma_{\infty 1} - \Gamma_{\infty 2}) + \ldots + \right.$$
$$\left. + (\Gamma_{\infty K} - \Gamma_{\infty K+1}) \right\}. \qquad (6.10)$$

Solve $P_o(\infty)$ from this normalization condition and insert in (6.8):

$$P_n(\infty) = \frac{\Gamma_{\infty n} - \Gamma_{\infty(n+1)}}{\Gamma_{\infty 0} - \Gamma_{\infty(K+1)}} \qquad (n=0,1,\ldots,K). \qquad (6.11)$$

Example 2.

If a skip-free walk on the continuum as described in III of the previous section has two sticking boundaries at 0 and at 1 (now a real positive number), one can obtain either by a limiting process or from the theorem at the end of section 5 III that

$$P_\infty(x) = \frac{-\frac{d}{dx}\Gamma_\infty(x)}{\Gamma_\infty(0^+) - \Gamma_\infty(1)} \qquad (6.12)$$

We shall not give the proof here.

Example 3.

For Lindley's process with sticking boundaries at 0 and 1 and exponentially distributed negative increments, the discrete walk is given by

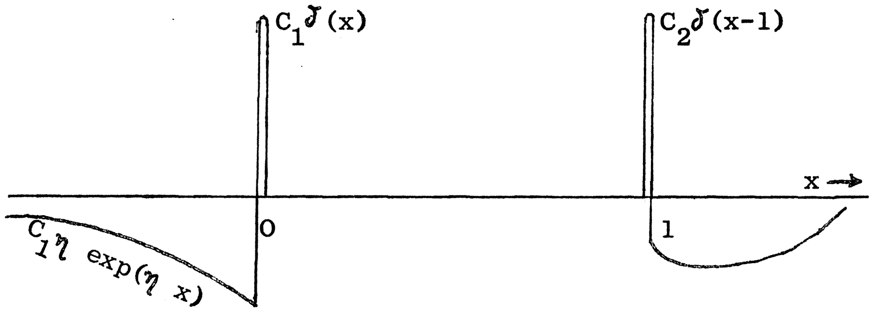
$$X_D(k) = \begin{cases} \max(0, X_D(k-1) + \xi_k) & \text{if } X_D(k-1) + \xi_k \leq 1, \\ \min(1, X_D(k-1) + \xi_k) & \text{if } X_D(k-1) + \xi_k \geq 0, \end{cases} \quad (6.13)$$

and the density of ξ_k is

$$A(x) = q\eta\, e^{\eta x} U(-x) + A^+(x) U(x);$$

once more suppose $m = \int xA(x)\,dx < 0$.

Like in example 1 we introduce fictitious states x ($x < 0$ and $x > 1$) and compensating source terms $\rho(x,t)$. On the negative half axis we compensate overshootings of the boundary with exponential lengths, while for $x > 1$ there are overshootings due to the positive increment density $A^+(x)$. The graph of the limit function $\rho(x,\infty)$ is sketched below; the probability mass placed at zero is equal to the surface between the exponential distribution and the axis, while the mass at $x=1$ cancels the contributions for $x > 1$.



In analogy with (6.7) on page 8 and with (6.8) of this section we find for $0 \leq x \leq 1$

$$W_{eq}(x) = \lim_{t \to \infty} \left(\int_{-\infty}^{0^+} + \int_{1^-}^{\infty} \right) \int_0^t \rho(y,t')\, \Gamma(x-y, t-t')\,dt\ \ dy =$$

$$= - \int_{-\infty}^{0^-} C_1\eta\, e^{\eta y}\, \Gamma_\infty(x-y)\,dy + C_1\Gamma_\infty(x) + \int_{1^-}^{\infty} \rho(y,\infty)\, \Gamma_\infty(x-y)\,dy. \quad (6.15)$$

Apart from the factor $(-C_1)$ the first term is a convolution of $\eta \exp(\eta x)U(-x)$ and $\Gamma_\infty(x)$. For $x \leqslant 1$ the third term is equal to

$$|\nu m|^{-1} \int_{1^-}^{\infty} \rho(y,\infty)\,dy = 0. \qquad (6.16)$$

It is obvious from the interpretation that $W_{eq}(x)$ will have mass concentrations at x=0 and x=1 ; this can also be verified analytically. Then (6.15) becomes, for $0 \leqslant x \leqslant 1$,

$$W_{eq}(x) = C_1 \left[\Gamma_\infty(x) - \Gamma_\infty(x) * \left\{ \eta e^{\eta x} U(-x) \right\} \right] + \\ + C_3\, \delta(x-1), \qquad (6.17)$$

where the first term is proportional to the Pollaczek distribution truncated at 1. For if

$$A(x) = \left\{ \eta e^{\eta x} U(-x) \right\} * B^+(x), \qquad (6.18)$$

where $B^+(x)$ is the service-time density, then (6.17) with $C_3=0$ implies

$$W_{eq}(z) = \frac{C_1}{1-a(z)} \left(1 - \frac{\eta}{\eta + iz}\right) = \frac{C_1}{1 - \frac{\eta}{\eta+iz} b^+(z)} \cdot \frac{iz}{\eta + iz}, \qquad (6.19)$$

where we have used

$$\Gamma_\infty(x) = \delta(x) + \sum_{j=1}^{\infty} A^{(j)}(x). \qquad (6.20)$$

The unknown constants C_1 and C_3 of (6.17) can be determined from the normalization condition

$$\int_{0^-}^{1^+} W_{eq}(x)\,dx = 1 \qquad (6.21)$$

and from

$$W_{eq}(0) = C_1 = \int_{-\infty}^{0^-} dx \int_{0^-}^{1^+} W_{eq}(x') q \eta\, e^{\eta(x-x')} dx' = \\ = q \int_{0^-}^{1^+} W_{eq}(x') e^{-\eta x'} dx'. \qquad (6.22)$$

7. Relation to the Hilbert problem

In the solution (6.16) for Lindley's process we have for $C_1=1$, $C_2=0$, a source term $\rho(x)$ of the form

$$q(\delta(x)-B^-(x)) \tag{7.1}$$

where $B^-(x)$ is a generalized density with support only on the negative half-axis. $W_{eq}(x)$ has the form $P_o\delta(x)+(1-P_o)B^+(x)$, and thus we may write

$$\left[P_o+(1-P_o)b^+(z)\right] = \frac{1}{1-a(z)} \cdot q \cdot (1-b^-(z)). \tag{7.2}$$

Writing this as

$$\frac{1}{1-a(z)} = \left\{P_o+(1-P_o)b^+(z)\right\} q^{-1} \cdot \frac{1}{1-b(z)} \tag{7.3}$$

we have found a factorization of the Hilbert kernel $\frac{1}{1-a(z)}$.

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