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MATHEMATICS

ON SLIPPAGE TESTS [1])

I. A GENERAL TYPE OF SLIPPAGE TEST AND A SLIPPAGE TEST FOR NORMAL VARIATES

BY

R. DOORNBOS AND H. J. PRINS

(Communicated by Prof. D. VAN DANTZIG at the meeting of September 28, 1957)

1. *Summary*

In this paper slippage tests for variates following various specified distributions, viz the normal, the Poisson, the binomial and the negative binomial, as well as a slippage test for the method of m rankings and a distributionfree k-sample slippage test, are discussed. These tests are all of the general type discussed in section 2. The choice of a test criterion for this type is a plausible one, but in some cases the tests can be proved to be optimal in a sense as described by a theorem of WALD.

For discrete variates the tests are derived as special cases of a slippage test for a general class of distribution functions. The class of distribution functions consists of all distribution functions, for which a close approximation to the true significance levels using a specified method is possible.

In the case of a test for Poisson variates it is possible to give the powerfunctions of the test in very good approximation, using the same method.

The same techniques were used previously for obtaining slippage tests for gamma variates by W. G. COCHRAN (1941), R. DOORNBOS (1956), and R. DOORNBOS and H. J. PRINS (1956) and for normal variates by E. PAULSON (1952). The slippage test for normal variates given here is a generalization of the one given by PAULSON. H. A. DAVID (1956) applied the same principle, without proof however, in two other cases.

2. *Introduction*

The general type of slippage test considered in this paper serves to decide whether one variate (or a group of variates if the variates occur in groups) slipped or no slippage occurred. These tests arise from the demands of a practical problem which is of a more general type, than the tests describe. For instance in industrial quality control in investigating a process one does not want to decide whether one variate slipped but one wants to decide if variates slipped and if so, how many and which ones.

Thus the tests described here have a restricted practical usefullness, as under the hypotheses considered at most one variate slipped. Still

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[1]) Report SP 62 of the Statistical Department of the Mathematical Centre.

until a more general solution is found to the practical problem, these tests may serve their purpose.

MOSTELLER (1948) and PEARSON and CHANDRA SEKAR (1936) already pointed out this difficulty and TUKEY (without date) and ROSE and ROY (1953) tried to find a solution for the general problem for normal variates.

The tests dealt with in this paper are of the following general type. Suppose

$$\vec{\mathbf{y}}_1, \ldots, \vec{\mathbf{y}}_k \text{ [2]}$$

are k random vectors. Thus

$$\vec{\mathbf{y}}_i = (\mathbf{y}_{i1}, \ldots, \mathbf{y}_{in_i}) \qquad (i = 1, \ldots, k)$$

The variates $\mathbf{y}_{ij}$ are distributed independently and have all the same type of distribution function. These distribution functions contain an unknown parameter θ_i as well as other unknown parameters. The test serves to decide whether one of the θ_i has slipped.

The simultaneous distribution of the $\mathbf{y}_{ij}$ is

$$F(\vec{y}_1, \ldots, \vec{y}_k | \vec{\theta}, \vec{\theta}'),$$

where

$$\vec{\theta} = (\theta_1, \ldots, \theta_k)$$

and $\vec{\theta}'$ is the vector for the other unknown parameters.

We want to test

$$H_0 : \theta_1 = \ldots = \theta_k$$

with the k alternatives

$$H_i : \theta_i \text{ slipped to the right} \quad (i = 1, \ldots, k)$$

or we want to test H_0 with the k alternatives

$$H_i : \theta_i \text{ slipped to the left} \quad (i = 1, \ldots, k).$$

In order to get rid of the unknown parameters in all but the distributionfree cases sufficient estimates are used.

This sometimes implies using new, one-dimensional, variates, which are functions of the original variates and which have a simultaneous distribution function (in the discrete case a conditional distribution) which does not contain the unknown parameters.

We state the test criterion in terms of the new variates. These variates are

$$\mathbf{x}_1, \ldots, \mathbf{x}_k. \tag{2.1}$$

which are, under H_0, the hypothesis tested, distributed simultaneously with some distribution function $F(x_1, \ldots, x_k)$, which may be continuous or not.

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[2]) Symbols printed in bold type denote random variables.

Suppose the observed values of $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are $x_1, \ldots, x_k$ respectively. When testing against slippage to the right we determine the right hand tail probabilities

$$d_j \overset{\text{def}}{=} P[\mathbf{x}_j \geqq x_j], \qquad (j = 1, \ldots, k)\ ^{3)} \tag{2.2}$$

We reject H_0 and decide that the m-th population has slipped to the right if

$$d_m = \min_j d_j \leqq \alpha/k. \tag{2.3}$$

Testing against slippage to the right requires computing

$$e_j = P[\mathbf{x}_j \leqq x_j], \qquad (j = 1, \ldots, k). \tag{2.4}$$

Now H_0 is rejected and it is concluded that the m-th population has slipped to the left if

$$e_m = \min e_j \leqq \alpha/k. \tag{2.5}$$

The probability that an error of the first kind occurs when this procedure is applied, is derived along the following general lines. Consider a set of k real numbers $g_1, \ldots, g_k$ and the probabilities defined by

$$\begin{cases} p_i \overset{\text{def}}{=} P[\mathbf{x}_i \leqq g_i], \\ p_{i,j} \overset{\text{def}}{=} P[\mathbf{x}_i \leqq g_i \text{ and } \mathbf{x}_j \leqq g_j], & (i \neq j) \\ q_i \overset{\text{def}}{=} P[\mathbf{x}_i > g_i], \\ q_{i,j} \overset{\text{def}}{=} P[\mathbf{x}_i > g_i \text{ and } \mathbf{x}_j > g_j], & (i \neq j) \end{cases} \tag{2.6}$$

all computed under H_0.

Denoting by P the probability that at least one of the $\mathbf{x}_i$ does not exceed the corresponding value g_i, it follows from Bonferroni's inequality (cf. W. Feller (1950), chapter 4) that

$$\sum_i p_i - \sum_{i<j} p_{i,j} \leqq P \leqq \sum_i p_i. \tag{2.7}$$

For Q, i.e. the probability that at least one $\mathbf{x}_i$ exceeds g_i, we have

$$\sum_i q_i - \sum_{i<j} q_{i,j} \leqq Q \leqq \sum_i q_i. \tag{2.8}$$

Then in each case separately we proceed to prove the inequality

$$p_{i,j} \leqq p_i p_j, \tag{2.9}$$

or

$$q_{i,j} \leqq q_i q_j, \tag{2.10}$$

which is equivalent with (2.9) (cf. R. Doornbos and H. J. Prins (1956)). Of course, (2.9) and (2.10) to be only hold for a class of distribution functions $F(x_1, \ldots, x_k)$. The problem of finding general conditions to be

[3]) The symbol $\overset{\text{def}}{=}$ denotes an equality, defining the left hand member.

imposed on $F(x_1, \ldots, x_k)$, sufficient for the validity of (2.9) has only partly been solved in this paper. Besides in some cases (2.9) only holds for some sets $g_1, \ldots, g_k$, for instance for all $g_i \geqq 0$.

Assuming that (2.9) and (2.10) are true we get immediately from (2.7) and (2.8) respectively

$$\sum_i p_i - \sum_{i<j} p_i p_j \leqq P \leqq \sum_i p_i \tag{2.11}$$

and

$$\sum_i q_i - \sum_{i<j} q_i q_j \leqq Q \leqq \sum_i q_i \tag{2.12}$$

respectively. Denoting $\sum_i p_i$ (p needs not be $\leqq 1$) we have

$$p^2 = (\sum_i p_i)^2 = 2 \sum_{i<j} p_i p_j + \sum_i p_i^2 \geqq 2 \sum_{i<j} p_i p_j,$$

where the equality sign only holds if all p_i vanish, or

$$\sum_{i<j} p_i p_j \leqq \tfrac{1}{2} p^2.$$

Thus

$$p - \tfrac{1}{2}p^2 \leqq P \leqq p \tag{2.13}$$

and similarly

$$q - \tfrac{1}{2}q^2 \leqq Q \leqq q, \tag{2.14}$$

where $\sum_i q_i = q$.

Now, when testing H_0 against slippage to the left of one of the k variables the critical region is of the form $\{x_1 \leqq g_{1\alpha}$ or $\ldots$ or $x_k \leqq g_{k\alpha}\}$.

The values $g_{i\alpha}$ are determined so as to make all p_i equal to α/k where α is the prescribed level of significance. In the discontinuous case this will in general not be possible; there $g_{i\alpha}$ is the largest value which can be attained by $\mathbf{x}_i$ with a positive probability, satisfying

$$\alpha_i' \overset{\text{def}}{=} P[\mathbf{x}_i \leqq g_{i\alpha}] \leqq \alpha/k. \tag{2.15}$$

So from (2.13) it follows that the probability P_α of rejecting H_0, if H_0 is true, satisfies

$$\alpha - \alpha^2/2 \leqq P_\alpha \leqq \alpha \tag{2.16}$$

or

$$\alpha' - (\alpha')^2/2 \leqq P_\alpha \leqq \alpha' \qquad (\alpha' = \sum_i \alpha_i') \tag{2.17}$$

respectively, according to whether the continuous or the discontinuous case is considered.

Testing slippage to the right we get similar bounds for the probability of rejecting H_0 when H_0 is true.

3. *The slippage test for normal distributions*

We consider k normal distributions with unknown means $\mu_1, \mu_2, \ldots, \mu_k$ and common unknown variance σ^2. From these distributions we have samples of $n_1, n_2, \ldots, n_k$ independent observations respectively.

We want to test the hypothesis

$$H_0 : \mu_1 = \ldots = \mu_k = \mu \text{ say,} \tag{3.1}$$

against the alternatives

$$H_{1i} : \begin{cases} \mu_1 = \ldots = \mu_{i-1} = \mu_{i+1} = \ldots = \mu_k = \mu \\ \mu_i = \mu + \Delta \quad (\Delta > 0), \end{cases} \tag{3.2}$$

for one value of i, which, however, is not known, or

$$H_{2i} : \begin{cases} \mu_1 = \ldots = \mu_{i-1} = \mu_{i+1} = \ldots = \mu_k = \mu \\ \mu_i = \mu - \Delta \quad (\Delta > 0), \end{cases} \tag{3.3}$$

for one unknown value of i. From the observations

$$\begin{cases} \mathbf{y}_{11}, \ldots, \mathbf{y}_{1n_1}, \\ \mathbf{y}_{21}, \ldots, \mathbf{y}_{2n_2}, \\ \vdots \\ \mathbf{y}_{k_1}, \ldots, \mathbf{y}_{kn_k}, \end{cases} \tag{3.4}$$

the variables

$$\mathbf{b}_i = \frac{\sqrt{n_i}(\mathbf{y}_i - \mathbf{y})}{\sqrt{\sum_{j,l} (\mathbf{y}_{jl} - \mathbf{y})^2}}, \qquad (i = 1, \ldots, k) \tag{3.5}$$

are formed, where

$$\begin{cases} \mathbf{y}_i \stackrel{\text{def}}{=} \frac{1}{n_i} \sum_l \mathbf{y}_{il}, \\ \mathbf{y} \stackrel{\text{def}}{=} \frac{1}{N} \sum_{j,l} \mathbf{y}_{jl}, \end{cases} \tag{3.6}$$

and where N is defined by

$$N \stackrel{\text{def}}{=} \sum_j n_j. \tag{3.7}$$

The $\mathbf{b}_i$ take the place of the variables $\mathbf{x}_i$ in (2.1).

In the following section we shall prove the inequality corresponding to (2.9) if g_i and g_j have the same sign and it will be proved that

$$\mathbf{u}_i = \tfrac{1}{2}\left(1 + \sqrt{\frac{N}{N-n_i}}\, \mathbf{b}_i\right) \tag{3.8}$$

has a $\mathbf{B}$-distribution with parameters $(N-2)/2$ and $(N-2)/2$ or, that

$$\mathbf{t}_i = \sqrt{N-2}\, \frac{\sqrt{\frac{N}{N-n_i}}\, \mathbf{b}_i}{\sqrt{1 - \frac{N}{N-n_i}\, \mathbf{b}_i^2}}, \tag{3.9}$$

has a Students' t-distribution with $N-2$ degrees of freedom, for $i = 1, \ldots, k$.

Thus the procedure described in section 2 can be applied and the d_j and e_j values as defined by (2.2) and (2.4) may be obtained for instance by means of (3.8) and the methods described in section 6 of R. DOORNBOS and H. J. PRINS (1956).

In the present case the determination of the minimal d and e values is much simpler, however, because these minimum values correspond to the largest and the smallest of the u_i respectively and thus of

$$\sqrt{\frac{n_i}{N-n_i}}\,(y_i - y)$$

and consequently only one incomplete $\boldsymbol{B}$-integral has to be computed. The critical values $g_{i\alpha}$ for the $\boldsymbol{b}_i$ are determined from

$$g_{i\alpha} = \sqrt{\frac{N-n_i}{N}}\,(2\,u_{\alpha/k} - 1), \tag{3.10}$$

where $u_{\alpha/k}$ is defined by

$$P[\boldsymbol{u}_i \leqq u_{\alpha/k}] = \alpha/k. \tag{3.11}$$

Because of the symmetry of the distribution of u_i with respect to the point $\frac{1}{2}$, the critical values $G_{i\alpha}$ for the test against slippage to the right are

$$G_{i\alpha} = \sqrt{\frac{N-n_i}{N}}\,(2\,u_{1-\alpha/k} - 1) = -\,g_{i\alpha}. \tag{3.12}$$

In the simplest case, i.e. $n_1 = \ldots = n_k = 1$ our test-statistic reduces to the one suggested already by E.S. PEARSON and C. CHANDRA SEKAR (1936), but for a constant factor. Using previous work of W. R. THOMPSON (1935), who derived in this special case the distribution of $\boldsymbol{t}_i$ as defined by (3.9), PEARSON and CHANDRA SEKAR were able to derive certain percentage points of $\max_i \boldsymbol{b}_i$ and $\min_i \boldsymbol{b}_i$ without deriving the exact distribution. They used the same approximation as is done here, but only up to

$$g_{1\alpha} = \ldots = g_{k\alpha} = g_\alpha \leqq -\sqrt{\frac{k-2}{2\,k}} \left(\text{or } G_\alpha \geqq \sqrt{\frac{k-2}{2\,k}}\right),$$

because, if all n_i are equal, in that region the probability that two of the variables, e.g. $\boldsymbol{b}_i$ and $\boldsymbol{b}_j$, both do not exceed g_α or exceed G_α is equal to zero. Thus the level of significance is then exactly equal to α.

The exact distribution for $n_1 = \ldots = n_k = 1$ has been computed numerically by F. E. GRUBBS (1950), who gave tables of exact percentage points up to $k = 25$ for $\varepsilon = 0.10$, 0.05, 0.025 and 0.01.

E. PAULSON (1952) proposed the same test statistic (but for a constant factor) for slippage to the right and the same approximation as suggested here in the special case $n_1 = \ldots = n_k = n$ but he gives no bounds for the corresponding level of significance. PAULSON proved that in this case the use of $\max \boldsymbol{b}_i$ as test statistic has the following optimum property. Let

D_0 denote the decision that the k means are equal and let D_j ($j=1, \ldots, k$) denote the decision that D_0 is incorrect and the $\mu_j = \max(\mu_1, \ldots, \mu_k)$. Now the procedure:

$$(3.13) \qquad \begin{cases} \text{If } \boldsymbol{b}_m > \lambda_\alpha \text{ select } D_m, \\ \text{if } \boldsymbol{b}_m \leqq \lambda_\alpha \text{ select } D_0, \end{cases}$$

where m is the index of the maximum $\boldsymbol{b}$-value maximizes the probability of making a correct decision, subject to the following restrictions.

(a) when all means are equal, D_0 should be selected with probability $1-\alpha$,

(b) the decision procedure must be invariant if a constant is added to the observations,

(c) the decision procedure must be invariant when all the observations are multiplied by a positive constant, and

(d) the decision procedure must be symmetric in the sense that the probability of making a correct decision when the i-th mean has slipped to the right by an amount Δ must be the same for $i=1, 2, \ldots, k$.

The constant λ_α in (3.13) is determined by requirement (a). Our critical value G_α is an approximation of λ_α.

The case of slippage to the left, although not mentioned explicitely by Paulson is completely analogous and the same optimum property holds there.

4. *Outline of a proof of the results stated in* 3

In this section we merely sketch the proof of the inequality

$$(4.1) \quad P[\boldsymbol{b}_i \leqq g_i \text{ and } \boldsymbol{b}_j \leqq g_j] \leqq P[\boldsymbol{b}_i \leqq g_i] \cdot P[\boldsymbol{b}_j \leqq g_j], \text{ provided } g_i g_j \geqq 0,$$

where $\boldsymbol{b}_i$ and $\boldsymbol{b}_j$ are defined by (3.5) for all pairs i, j ($i \neq j$; $i, j = 1, \ldots, k$). Giving all details would require too much space.

First the marginal distributions of $\boldsymbol{b}_i$ and $\boldsymbol{b}_j$ and their simultaneous distribution have to be derived. These are

$$(4.2) \quad f(b_i) = \sqrt{\frac{N}{N-n_i}} \frac{\Gamma\left(\frac{N-1}{2}\right)}{\Gamma\left(\frac{N-2}{2}\right)} \frac{1}{\sqrt{\pi}} \left\{1 - \frac{N}{N-n_i} b_i^2\right\}^{\frac{N-4}{2}}, \qquad (i = 1, \ldots, k)$$

and

$$(4.3) \quad \begin{cases} g(b_i, b_j) = \\ \sqrt{\frac{N}{N-n_i-n_j}} \frac{N-3}{2\pi} \left\{1 - \frac{N-n_j}{N-n_i-n_j} b_i^2 - \frac{2\sqrt{n_i n_j}}{N-n_i-n_j} b_i b_j - \frac{N-n_i}{N-n_i-n_j} b_j^2\right\}^{\frac{N-5}{2}}. \end{cases}$$

Both formulae are valid in the regions where the expressions between braces are positive, outside these regions the respective density functions are zero. The region where $g(b, b_j)$ differs from zero is bounded by an ellipse in the (b_i, b_j) plane, with principle axes of length 1 and $\sqrt{\frac{N-n_i-n_j}{N}}$ and equations

$$(4.4) \qquad \begin{cases} b_i + \sqrt{\frac{n_j}{n_i}}\, b_j = 0, \\ b_i - \sqrt{\frac{n_i}{n_j}}\, b_j = 0. \end{cases}$$

When proving (4.1) we may obviously assume that the point (g_i, g_j) lies within this ellipse, because otherwise $P[\boldsymbol{b}_i \leqq g_i \text{ and } \boldsymbol{b}_j \leqq g_j] = 0$. Further we suppose that both g_i and g_j are $\leqq 0$. This is no restriction, for, when (4.1) holds for a pair of values g_i and g_j, the inequality $P[\boldsymbol{b}_i > -g_i \text{ and } \boldsymbol{b}_j > -g_j] \leqq P[\boldsymbol{b}_i > -g_i] \cdot P[\boldsymbol{b}_j > -g_j]$ also holds for reasons of symmetry. Consequently (4.1) is also true for $-g_i$ and $-g_j$ because of the equivalence of (2.9) and (2.10). We shall see that in the (g_i, g_j) region considered (4.1) holds with the $<$ sign. We have to prove

$$(4.5) \qquad \phi(g_i, g_j) \stackrel{\text{def}}{=} P[\boldsymbol{b}_i \leqq g_i] \cdot P[\boldsymbol{b}_j \leqq g_j] - P[\boldsymbol{b}_i \leqq g_i \text{ and } \boldsymbol{b}_j \leqq g_j] > 0.$$

The proof consists of showing consecutively

$$(4.6) \qquad \phi\left(-\sqrt{\frac{N-n_i-n_j}{N-n_j}},\, g_j\right) > 0,$$

$$(4.7) \qquad \phi\left(0,\, -\sqrt{\frac{N-n_i-n_j}{N-n_i}}\right) > 0$$

and

$$(4.8) \qquad \frac{d\,\phi(0, g_j)}{d g_j} \geqq 0 \qquad \left(-\sqrt{\frac{N-n_i-n_j}{N-n_i}} \leqq g_j \leqq 0\right).$$

From (4.7) and (4.8) it follows that

$$(4.9) \qquad \phi(0, g_j) > 0 \qquad \left(-\sqrt{\frac{N-n_i-n_j}{N-n_i}} \leqq g_j \leqq 0\right).$$

Further we can derive

$$(4.10) \qquad \frac{\partial\,\phi(g_i, g_j)}{\partial g_i} = \sqrt{\frac{N}{N-n_i}}\, \frac{\Gamma\left(\frac{N-1}{2}\right)}{\Gamma\left(\frac{N-2}{2}\right)}\, \frac{1}{\sqrt{\pi}} \left(1 - \frac{N}{N-n_i}\, g_i^2\right)^{\frac{N-4}{2}} \phi, (g_i, g_j),$$

where $\phi_1(g_i, g_j)$ is a decreasing function of g_δ if $g_i g_j \geqq 0$, thus $\frac{\partial\,\phi(g_i, g_j)}{\partial g_i}$ is everywhere positive, everywhere negative, or positive up to a certain point g_{0i} (depending upon g_j), say, and negative thereafter. So in virtue of (4.6) and (4.9) we may conclude

$$(4.11) \qquad \begin{cases} \phi(g_i, g_j) > 0, \text{ if } g_i \leqq 0, \\ \qquad\qquad g_j \leqq 0 \text{ and} \\ \qquad\qquad 1 - \frac{N-n_j}{N-n_i-n_j}\, g_i^2 - \frac{2\sqrt{n_i n_j}}{N-n_i-n_j}\, g_i g_j - \frac{N-n_i}{N-n_i-n_j}\, g_j^2 \geqq 0. \end{cases}$$

A detailed proof can be found in R. Doornbos, H. Kesten and H. J. Prins (1956).

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MATHEMATICS

ON SLIPPAGE TESTS

II. SLIPPAGE TESTS FOR DISCRETE VARIATES

BY

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5. *Slippage tests for some discrete variables*

In this section slippage tests will be discussed for variates which follow the Poisson, the binomial or the negative binomial law. These are special cases of a general class of variates determined by the condition of theorem 5.1. First we shall consider the *Poisson* case in some detail. Suppose we have a set of independent random variables

$$(5.1)\qquad \mathbf{z}_1, \ldots, \mathbf{z}_k$$

distributed according to Poisson distributions, i.e.:

$$(5.2)\qquad P[\mathbf{z}_i = z_i] = \frac{e^{-\mu_i}\mu_i^{z_i}}{z_i!}, \qquad (i = 1, \ldots, k), \mu_i > 0.$$

Now we want to test the hypothesis H_0 that the means μ_i have known ratios

$$(5.3)\qquad H_0: \frac{\mu_i}{\sum_j \mu_j} = p_i, \qquad (i = 1, \ldots, k).$$

This situation occurs for instance if from k Poisson-populations with, under H_0, equal means, known unequal numbers of observations are present and $z_1, \ldots, z_k$ represent the sums of the values obtained in these observations. In this case the p_i are proportional to the numbers of observations. Also k Poisson processes with the same parameter may be observed during different lengths of time. Then the p_i are proportional to these lengths of time.

We want to test H_0 against the alternatives

$$(5.4)\quad H_{1i}: \frac{\mu_i}{\sum_j \mu_j} = c\,p_i, \ \frac{\mu_l}{\sum_j \mu_j} = \frac{1-c p_i}{1-p_i}\, p_l \quad (l \neq i), 1 < c < \frac{1}{p_i}, \ c \text{ unknown},$$

for one unknown value of i or

$$(5.5)\quad H_{2i}: \frac{\mu_i}{\sum_j \mu_j} = c p_i, \ \frac{\mu_l}{\sum_j \mu_j} = \frac{1-cp_j}{1-p_i} p_l \quad (l \neq i), 0 < c < 1, c \text{ unknown},$$

for one unknown of i.

A well known property of Poisson-variates is: If $\mathbf{z}_1, \ldots, \mathbf{z}_k$ are independent Poisson-variates with means $\mu_1, \ldots, \mu_k$, then the simultaneous conditional distribution of $\mathbf{z}_1, \ldots, \mathbf{z}_k$ given their sum (i.e. $\sum \mathbf{z}_i = N$, N a constant), is a multinomial distribution with probabilities $p_i = \mu_i / \sum \mu_j$ and number of trials $\sum \mathbf{z}_i = N$. As the hypotheses (5.3), (5.4) and (5.5) only contain the ratios p_i it seems natural to use a conditional test for H_0, using only the multinomial distribution

$$(5.6)\quad P[\mathbf{z}_1 = z_1, \ldots, \mathbf{z}_k = z_k | \sum \mathbf{z}_i = N] = \frac{N!}{\prod z_i!} \prod p_i^{z_i}, \text{ if } \sum z_i = N \text{ and } 0 \text{ otherwise.}$$

From this it is clear that a test against slippage for Poisson vairates is closely related to a similar test for a multinomial distribution. The reader may easily translate the tests stated here into tests for the multinomial case.

In the next section the following theorem will be proved.

Theorem 5.1. *Suppose the discrete, random variables*

$$(5.7)\qquad \mathbf{u}_1, \ldots, \mathbf{u}_k$$

are distributed independently and can take integer values only (the latter assumption is not essential but gives a much simpler notation).

If

$$(5.8)\qquad \frac{P[\sum \mathbf{u}_l - \mathbf{u}_i - \mathbf{u}_j = a]}{P[\sum \mathbf{u}_l - \mathbf{u}_i - \mathbf{u}_j = a+1]}$$

where a is an integer, is a non decreasing function of a, then

$$(5.9)\quad P[\mathbf{u}_i \geqq u_i \text{ and } \mathbf{u}_j \geqq u_j | \sum \mathbf{u}_l = N] \leqq P[\mathbf{u}_i \geqq u_i | \sum \mathbf{u}_l = N] \cdot P[\mathbf{u}_j \geqq \mathbf{u}_j | \sum \mathbf{u}_l = N],$$

for every pair of integers u_i and u_j and for every non-negative integer N.

In the special case where $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are distributed according to the same type of distribution and this distribution has the property that a sum of k independent variates has again the same type of distribution, it is easy to verify whether condition (5.8) holds or not.

In our case the sum of $k-2$ of the variables $\mathbf{z}_i$ (given by (5.2)) has a Poisson-distribution with mean μ, say. So condition (5.8) states that

$$(5.10)\qquad \frac{e^{-\mu}\mu^a}{a!} \cdot \frac{(a+1)!}{e^{-\mu}\mu^{a+1}} = \frac{a+1}{\mu},$$

is non decreasing in a, which is clearly true.

Thus the inequality (5.9) holds for every pair $\mathbf{z}_i$, $\mathbf{z}_j$ and the procedure described in section 2 may be applied to the variables $\mathbf{z}_1, \ldots, \mathbf{z}_k$ under the condition $\sum z_i = N$ [1]). Now the marginal distribution of $\mathbf{z}_i$ under the condition $\sum \mathbf{z}_i = N$ is a binomial one, so when testing H_0 against H_{1i} $(i=1, \ldots, k)$ we compute, if $z_1, \ldots, z_k$ are the observed values and $\sum z_i = N$,

$$(5.11)\quad r_i \overset{\text{def}}{=} P[\mathbf{z}_i \geqq z_i | \sum \mathbf{z}_l = N] = \sum_{x=z_i}^{N} \binom{N}{x} p_i^x (1-p_i)^{N-x} = I_{p_i}(z_i, N-z_i+1),$$

where $I_{p_i}(z_i, N-z_i+1)$ stands for the incomplete **B**-function

$$\frac{N!}{(z_i-1)!(N-z_i)!} \int_0^{p_i} u^{z_i-1}(1-u)^{N-z_i}\,du.$$

Now H_0 is rejected if

$$(5.12)\qquad \min_i r_i \leqq \alpha/k$$

and then we decide that $\mu_j/\sum \mu_i > p_j$ if j is the smallest integer for which $r_j = \min r_i$.

If under $H_0: \mu_1 = \ldots = \mu_k$, all p_i are equal and the smallest r_i corresponds to the largest value z_i.

The test for slippage to the left is completely analogous.

A table of critical values for max z_i is given in section 11 for the case $p_1 = \ldots = p_k$.

Along the same lines as followed by R. Doornbos and H. J. Prins (1956) in the case of Γ-variates it can be shown that the probability Q_j of making the correct decision when the j-th population has slipped to the right (i.e. H_{1i} is true with $i=j$) satisfies the inequality

$$(5.13)\quad \begin{cases} I_{cp_j}(G_{j,\alpha}, N-G_{j,\alpha}+1)\,[1-\sum_{i\neq j} I_{\frac{1-cp_j}{1-p_j}p_i}(G_{i,\alpha}, N-G_{i,\alpha}+1)] \leqq \\ \qquad\qquad \leqq Q_j \leqq I_{cp_j}(G_{j,\alpha}, N-G_{j,\alpha}+1). \end{cases}$$

---

[1]) The validity of (5.9) in the case of Poisson-variates can also be proved in the following way, using their relation with Γ-variates. The well known relation

$$(1)\quad \begin{cases} P[\mathbf{z}_1 \geqq z_1 | \sum z_i = N] = \sum_{x=z_1}^{N} \binom{N}{x} p_1^x (1-p_1)^{N-x} = \\ \qquad = \frac{N!}{(z_1-1)!\,(N-z_1)!} \int_0^{p_1} u^{z_1-1}(1-u)^{N-z_1}\,du, \end{cases}$$

can be generalized to

$$(2)\quad \begin{cases} \sum_{x_1=z_{i_1}}^{N} \cdots \sum_{x_r=z_{i_r}}^{N} \frac{N!}{x_1!\ldots x_r!(N-x_1\ldots-x_r)!} p_{i_1}^{x_1} \cdots p_{i_r}^{x_r} (1-p_{i_1}\cdots-p_{i_r})^{N-x_1\ldots-x_r} = \\ \qquad = \frac{N!}{(z_{i_1}-1)!\ldots(z_{i_r}-1)!\,(N-z_{i_1}\ldots-z_{i_r})!} \int_0^{p_{i_1}} \cdots \\ \qquad \cdots \int_0^{p_{i_r}} u_1^{z_{i_1}-1}\ldots u_r^{z_{i_r}-1}(1-u_1\ldots-u^r)^{N-z_{i_1}\ldots-z_{i_r}}\,du_1\ldots du_r \\ \qquad (r \leqq k-1, (i_1, \ldots, i_r) \in (1, \ldots, k)), \end{cases}$$

which may be proved by induction or otherwise. Using (2) for $r=2$ it is seen immediately that inequality (4.10) in R. Doornbos and H. J. Prins (1956) is the same as (5.9) for Poisson variates.

Here $G_{l,\alpha}$ $(l=1, \ldots, k)$ is the smallest number which satisfies

$$P[\mathbf{z}_l \geqq G_{l,\alpha} | \sum \mathbf{z}_i = N, H_0] \leqq \alpha/k \tag{5.14}$$

or

$$I_{p_l}(G_{l,\alpha}, N - G_{l,\alpha} + 1) \leqq \alpha/k. \tag{5.15}$$

Clearly Q_j converges towards its upper bound when $c \to 1/p_j$ and for each $c \geqq 1$ the factor between square brackets is larger than $1-(k-1)\alpha/k$ according to (5.15).

In the case of slippage to the left we have analogously

$$\left\{\begin{aligned} &[1-I_{cp_j}(g_{j,\alpha}, N-g_{j,\alpha}+1)]\,(1-\alpha) \leqq \\ &[1-I_{cp_j}(g_{j,\alpha}, N-g_{j,\alpha}+1)]\,[1-\sum_{i \neq j}\{1-I_{\frac{1-cp_j}{1-p_j}p_i}(g_{i,\alpha}, N-g_{i,\alpha}+1)\}] \leqq \\ &\leqq P_j \leqq 1-I_{cp_j}(g_{j,\alpha}, N-g_{j,\alpha}+1), \end{aligned}\right. \tag{5.16}$$

where $g_{l,\alpha}$ $(l=1, \ldots, k)$ is the largest number satisfying

$$1 - I_{p_l}(g_{l,\alpha}+1, N-g_{l,\alpha}) \leqq \alpha/k. \tag{5.17}$$

We can apply theorem 5.1 also to the case of independent variables

$$\mathbf{v}_1, \ldots, \mathbf{v}_k \tag{5.18}$$

which are distributed according to *binomial* laws with numbers of trials $n_1, \ldots, n_k$ and probabilities of success $p_1, \ldots, p_k$. Now the hypothesis H_0 is

$$H_0 : p_1 = \ldots = p_k = p, \text{ say} \tag{5.19}$$

and the alternatives are

$$\left\{\begin{aligned} H_{1i} : p_1 = \ldots = p_{i-1} = p_{i+1} = \ldots = p_k = p, \\ p_i = c \cdot p \quad (1 \leqq c \leqq 1/p), \end{aligned}\right. \tag{5.20}$$

for one unknown value of i and

$$\left\{\begin{aligned} H_{2i} : p_1 = \ldots = p_{i-1} = p_{i+1} = \ldots = p_k = p, \\ p_i = c \cdot p \quad (0 \leqq c \leqq 1), \end{aligned}\right. \tag{5.21}$$

for one unknown value of i.

Because, under H_0, the sum of $(k-2)$ of the variates (5.18) has again a binomial distribution with number of trials, n say, and probability of a success in each trial p, the condition (5.8) of theorem 5.1 reads:

$$\frac{\binom{n}{a} p^a (1-p)^{n-a}}{\binom{n}{a+1} p^{a+1}(1-p)^{n-a-1}} = \frac{a+1}{n-a} \cdot \frac{1-p}{p} \tag{5.22}$$

is a non decreasing function of a, which is true. So in this case also the approximation procedure described in section 2 can be applied to obtain a conditional test for slippage under the condition that the sum of the variates $\sum \mathbf{v}_i$ has a constant value N. The conditional distribution of $\mathbf{v}_i$ is a hypergeometrical one

$$(5.23) \qquad P[\mathbf{v}_i = v_i | \sum \mathbf{v}_i = N] = \binom{n_i}{v_i} \binom{\sum n_j - n_i}{N - v_i} \binom{\sum n_j}{N}^{-1} \qquad (\mathbf{v}_i \geqq 0),$$

so with help of this distribution critical values for the tests with prescribed level of significance may be obtained, in the same way as was done with the Poisson variates.

Provided that none of the values n_i, $\sum n_j - n_i$, N and $\sum n_j - N$ are very small, a good approximation to the sum of the tail terms of the hypergeometric series of equation (5.23) may be obtained from the integral under a normal curve, having the mean $n_i N / \sum n_j$ and variance

$$\frac{n_i(\sum n_j - n_i) N (\sum n_j - N)}{(\sum n_j)^2 (\sum n_j - 1)}.$$

In the special case $n_1 = \ldots = n_k = n$, the test procedure for slippage to the right reduces to comparing the largest variate $\mathbf{v}_m$ with a constant v_0 determined by the level of significance α, such that v_0 is the largest value satisfying

$$P[\mathbf{v}_i \geqq v_0 | \sum \mathbf{v}_i = N] \leqq \alpha / k.$$

The same holds for the variates

$$(5.24) \qquad \mathbf{w}_1, \ldots, \mathbf{w}_k,$$

which are independently distributed according to *negative binomial* laws, with parameters $r_1, \ldots, r_k$ and probabilities $p_1, \ldots, p_k$, i.e.

$$(5.25) \qquad P[\mathbf{w}_i = w_i] = \binom{w_i + r_i - 1}{r_i - 1} p_i^{r_i} q_i^{w_i}$$

where r_i is an integer $\geqq 1$ and $0 \leqq p_i \leqq 1$, whilst $p_i + q_i = 1$.

The hypothesis H_0 is

$$(5.26) \qquad H_0 : q_1 = \ldots = q_k = q, \text{ say}$$

and the alternatives are

$$(5.27) \qquad \begin{cases} H_{1i} : q_1 = \ldots = q_{i-1} = q_{i+1} = \ldots = q_k = q, \\ \qquad q_i = c \cdot q \quad (1 \leqq c \leqq 1/q), \end{cases}$$

for one unknown value of i or

$$(5.28) \qquad \begin{cases} H_{2i} : q_1 = \ldots = q_{i-1} = q_{i+1} = \ldots = q_k = q, \\ \qquad q_i = c \cdot q \quad (0 \leqq c \leqq 1), \end{cases}$$

for one unknown value of i.

The hypotheses are stated in terms of the q_i and not in terms of the p_i in order to obtain that slippage to the right of the i-th population corresponds to a large value of $\mathbf{w}_i$.

Under H_0, the sum of a set of independent negative binomial variates has again a negative binomial distribution with the same probability p (or q) and a parameter r, say, which is the sum of the r_i of the individual variates. So condition (5.8) gives here

$$\frac{\binom{a+r-1}{r-1} p^r q^a}{\binom{a+r}{r-1} p^r q^{a+1}} = \frac{(a+1)}{(a+r)} \cdot \frac{1}{q} \tag{5.29}$$

is a non decreasing function of a, which is true if $r \geqq 1$. Thus again the method of section 2 may be applied. The conditional distribution of $\mathbf{w}_i$ under the condition $\sum \mathbf{w}_j = N$, has the form

$$P[\mathbf{w}_i = w_i | \sum \mathbf{w}_j = N] = \frac{\binom{w_i + r_i - 1}{r_i - 1}\binom{N + \sum r_j - w_i - r_i - 1}{\sum r_j - r_i - 1}}{\binom{N + \sum r_j - 1}{\sum r_j - 1}}, \quad (w_i = 0, \ldots, N). \tag{5.30}$$

The critical region for the test against H_{1i} $(i = 1, \ldots, k)$ (5.27) consists of large values of the variables $\mathbf{w}_i$. In the case where $r_1 = \ldots = r_k$ the test statistic is the largest variate $\mathbf{w}_m$, when testing against slippage to the right and the smallest when testing against slippage to the left.

If in the Poisson case (5.1) $p_1 = \ldots = p_k$, then the following optimum property can be proved [2]). As in the case of the normal distribution we denote by D_0 the decision that H_0 is true and by Di $(i = 1, \ldots, k)$ the decision that H_{1i} is true, i.e. that H_{1i} is true and that the i-th population has slipped to the right. Now the procedure:

$$\begin{cases} \text{if } \mathbf{z}_m > \lambda_{\alpha, N} \text{ select } D_m, \\ \text{if } \mathbf{z}_m \leqq \lambda_{\alpha, N} \text{ select } D_0, \end{cases} \tag{5.31}$$

under the condition that $\sum \mathbf{z}_i = N$, where m is the index of the maximum $\mathbf{z}$ value, maximizes the probability of making a correct decision when H_{1m} is true subject to the following restrictions:

(a) When H_0 is true, D_0 should be selected with probability $\geqq 1 - \alpha$.

(b) The probability of making a correct decision when the i-th population has slipped by an amount c must be the same for $i = 1, \ldots, k$.

The constant $\lambda_{\alpha, N}$ in (5.31) is determined by the level of significance α and depends on N, the sum of the variates. A proof will be given in the next section.

6. *Proofs of the results stated in section* 5

Starting with the proof of theorem 5.1 we have that

$$\frac{P[\mathbf{u}_i = y] \cdot P[\mathbf{u}_j = x] \cdot P[\sum \mathbf{u}_l - \mathbf{u}_i - \mathbf{u}_j = N - x - y]}{P[\mathbf{u}_i = y] \cdot P[\mathbf{u}_j = x + 1] \cdot P[\sum \mathbf{u}_l - \mathbf{u}_i - \mathbf{u}_j = N - x - y - 1]} \tag{6.1}$$

[2]) In the sequel only the case of slippage to the right is considered but all statements may be easily translated for the other case.

is non-increasing in y, according to (5.8). Dividing (6.1) by the factor

$$\text{(6.2)} \qquad \frac{P[\sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = x+1]}{P[\sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = x]}$$

which does not depend on y, (6.1) changes into

$$\text{(6.3)} \qquad \frac{P[\mathbf{u}_i = y | \sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = x]}{P[\mathbf{u}_i = y | \sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = x+1]}.$$

Thus also (6.3) is non increasing in y for all values of x. This means that there exists a value y_0, which may depend on x, which has the property that

$$\text{(6.4)} \begin{cases} P[\mathbf{u}_i = y | \sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = x] \geqq P[\mathbf{u}_i = y | \sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = x+1], \text{if } y \geqq y_0 \\ P[\mathbf{u}_i = y | \sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = x] \leqq P[\mathbf{u}_i = y | \sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = x+1], \text{if } y < y_0 \end{cases}.$$

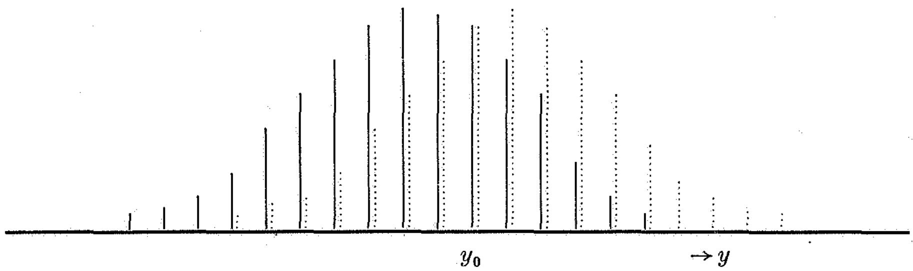

Fig. 6.1. $P[\mathbf{u}_i = y | \sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = x]$ (dotted lines), and $P[\mathbf{u}_i = y | \sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = x+1]$ (full lines).

This situation is sketched in figure 6.1. It follows that for each value of u_i

$$\text{(6.5)} \qquad P(x) \stackrel{\text{def}}{=} \sum_{y=u_i}^{\infty} P[\mathbf{u}_i = y | \sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = x]$$

is a non increasing function of x. Now

$$\text{(6.6)} \begin{cases} \dfrac{P[\mathbf{u}_i \geqq u_i \text{ and } \mathbf{u}_j \geqq u_j | \sum \mathbf{u}_l = N]}{P[\mathbf{u}_j \geqq u_j | \sum \mathbf{u}_l = N]} = \\ = \dfrac{\sum_{x=u_j}^{\infty} P[\mathbf{u}_j = x | \sum \mathbf{u}_l = N] \sum_{y=u_i}^{\infty} P[\mathbf{u}_i = y | \sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = x]}{\sum_{x=u_j}^{\infty} P[\mathbf{u}_j = x | \sum \mathbf{u}_l = N]} \leqq \\ \leqq \sum_{y=u_i}^{\infty} P[\mathbf{u}_i = y | \sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = u_j]. \end{cases}$$

In the same way we have

$$\text{(6.7)} \qquad \frac{P[\mathbf{u}_i \geqq u_i \text{ and } \mathbf{u}_j < u_j | \sum \mathbf{u}_l = N]}{P[\mathbf{u}_j < u_j | \sum \mathbf{u}_l = N]} \geqq P[\mathbf{u}_i = y | \sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = u_j].$$

From (6.6) and (6.7) it follows that, in the notation of (2.6), where $u_i = g_i + 1$ and $u_j = g_j + 1$, whilst $\mathbf{u}_i$ under the condition $\sum \mathbf{u}_l = N$ stands for $\mathbf{x}_i$ and $\mathbf{u}_j$ under the condition $\sum \mathbf{u}_l = N$ for $\mathbf{x}_j$

(6.8) $$\frac{q_{i,j}}{q_j} \leqq \frac{q_i - q_{i,j}}{1 - q_j}$$

or

(6.9) $$q_{i,j} \leqq q_i \cdot q_j$$

which proves the theorem, because (6.9) is the same as (5.9).

The proof of the optimality of our procedure in the Poisson case is a straightforward application of the theory of A. WALD (1950). It consists mainly in showing that for any c and N there exists a set of non zero a priori probabilities $g_0, \ldots, g_k$, which are functions of N so that, when g_i is the probability that D_i is the correct decision the decision procedure described in section 5 maximizes the probability of making the correct decision. Assuming that this has been demonstrated, it follows easily that (5.31) is the optimum solution. For suppose there exists an allowable decision procedure, which for some c and N has a greater probability than (5.31) of making the correct decision when some category has slipped to the right by an amount c. Then this procedure will have a greater probability than (5.31) of making a correct decision (for these values of c and N) with respect to any set of a priori probabilities, with $\max_i g_i > 0$, which would be a contradiction.

According to A. WALD (1950), pp. 127–128 the optimum solution is given by the rule: "For each j ($j = 0, \ldots, k$) decide D_j for all points in the sample space where j is the smallest integer for which $g_j f_j = \max\{g_0 f_0, \ldots, g_k f_k\}$, where f_j is the joint elementary probability law of $\mathbf{z}_1, \ldots, \mathbf{z}_k$ under the hypothesis H_{1j}."

We consider the special a priori distribution $g_0 = 1 - g\ k$, $g_1 = \ldots = g_k = g$. For instance the region where D_1 is selected is given by the points in the sample space where $f_1 > f_i$ ($i = 2, \ldots, k$) and $g f_1 > (1 - gk) f_0$.

Here we have

(6.10) $$\begin{cases} f_0(z_1, \ldots, z_k \mid \sum \mathbf{z}_l = N) = \dfrac{N!}{\prod z_l!} \left(\dfrac{1}{k}\right)^N \\ f_i(z_1, \ldots, z_k \mid \sum \mathbf{z}_l = N) = \dfrac{N!}{\prod z_l!} \left(\dfrac{1}{k}\right)^N c^{z_i} \left(\dfrac{k-c}{k-1}\right)^{N - z_i}, \qquad (1 < c < k). \end{cases}$$

As $c^{z_i}\left(\dfrac{k-c}{k-1}\right)^{N-z_i}$ is monotonously increasing in z_i for $1 < c < k$ the region where $f_1 > f_i$ is given by $z_1 > z_i$ and the region where $g f_1 > (1 - gk) f_0$ by $z_1 > L$, L depending on c and N.

Thus the Bayes solution is: if z_m is the maximum of $z_1, \ldots, z_k$ select D_m if $z\ \ > L$, otherwise select D_0. Define the function $F(g)$ by the equation

(6.11) $$F(g) = c^{\lambda_{\alpha,N}} \left(\frac{k-c}{k-1}\right)^{N - \lambda_{\alpha,N}} - \frac{1 - gk}{g}$$

where $\lambda_{\alpha, N}$ is the constant used in (5.31). It is obvious that $F(g)$ is a continuous function of g, with $F(1/k) > 0$ and that there exists a δ with $0 < \delta < 1/k$ such that $F(\delta) < 0$. Hence there exists a value g^* with $0 < \delta < g^* < 1/k$ such that $F(g^*) = 0$. To get the Bayes solution relative to $(1 - kg^*, g^*, \ldots, g^*)$ it is only necessary in the solution given above to replace L by $\lambda_{\alpha, N}$. Thus the procedures (5.31) is the Bayes solution relative to $(1 - kg^*, g^*, \ldots, g^*)$ which proves that it is an optimum one.

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(To be continued).

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MATHEMATICS

ON SLIPPAGE TESTS

III. TWO DISTRIBUTIONFREE SLIPPAGE TESTS AND TWO TABLES [1)]

BY

R. DOORNBOS AND H. J. PRINS

(Communicated by Prof. D. VAN DANTZIG at the meeting of May 31, 1958)

7. *Slippage tests for the method of m rankings*

In the well known method of m rankings due to M. FRIEDMAN (1937) (cf. M. G. KENDALL (1955), chapters 6 and 7) m "observers" are considered. Each observer ranks k "objects". The method of m rankings enables us to investigate whether the observers agree in their opinion about the objects. For that reason one tests the hypothesis H_0, which states that the rankings are chosen at random from the collection of all permutations of the numbers 1, ..., k and that they are independent.

Here we present tests which are powerful especially against the alternative that one of the objects has larger probability than the other ones of being ranked high (or low), whilst the other $(k-1)$ objects are ranked in a random order. We denote the sums of the m ranks of each object by

$$\mathbf{s}_1, \ldots, \mathbf{s}_k, \qquad (m \leqq \mathbf{s}_i \leqq km). \tag{7.1}$$

Obviously we have

$$\sum_{i=1}^{k} \mathbf{s}_i = \tfrac{1}{2} mk(k+1). \tag{7.2}$$

In section 8 the following theorem will be proved.

Theorem 7.1. *For each pair* $\mathbf{s}_i, \mathbf{s}_j$ *of the variates* (7.1) *and for every pair of integers* s_i, s_j *the following inequality holds under* H_0

$$P[\mathbf{s}_i \leqq s_i \text{ and } \mathbf{s}_j \leqq s_j] \leqq P[\mathbf{s}_i \leqq s_i] \cdot P[\mathbf{s}_j \leqq s_j]. \tag{7.3}$$

So we can apply our approximation method of section 2 for obtaining slippage tests for $\mathbf{s}_1, \ldots, \mathbf{s}_k$. Because the marginal distributions of the $\mathbf{s}_i$ are all equal under H_0, the test statistic for the test against slippage to the right is max $\mathbf{s}_i$ and for testing against slippage to the left min $\mathbf{s}_i$. The critical values are determined by the smallest integer S_α satisfying

$$P[\mathbf{s}_i \geqq S_\alpha] \leqq \alpha/k \tag{7.4}$$

and the largest integer s_α satisfying

$$P[\mathbf{s}_i \leqq s_\alpha] \leqq \alpha/k, \tag{7.5}$$

respectively.

[1)] Parts I and II in Indagationes Mathematicae, 20, 38–55 (1958) and Proc. Kon. Ned. Ak. van Wetensch., 61, Series A, 38–55 (1958).

The distribution of $\mathbf{s}_i$ is easily seen to be symmetric with respect to the mean value $\frac{1}{2}m(k+1)$, so we have

$$s_\alpha = m(k+1) - S_\alpha. \tag{7.6}$$

In section 8 it will be shown that the distribution of $\mathbf{s}_i$, under H_0, reads

$$\left\{ \begin{array}{l} P[\mathbf{s}_i = n] = \sum_{x=0}^{\infty} I_{n-kx-m} \binom{m}{x} \binom{n-kx-1}{m-1} (-1)^x k^{-m}, \\ \qquad\qquad (i = 1, \ldots, k;\ m \leqq n \leqq km) \end{array} \right. \tag{7.7}$$ [2]

where I_y is defined by

$$\left\{ \begin{array}{l} I_y = 0 \text{ if } y < 0, \\ I_y = 1 \text{ if } y \geqq 0. \end{array} \right. \tag{7.8}$$

The tables of critical values s_α, presented in section 11, are based on this formula.

8. *Proofs of the results of section* 7

First we shall prove theorem 7.1. We suppose that both s_i and s_j are lying between m and km, because otherwise (7.3) obviously holds with the equality sign. For $m=1$ we have

$$\left\{ \begin{array}{l} P[\mathbf{s}_i \leqq s_i \text{ and } \mathbf{s}_j \leqq s_j | m=1] = \dfrac{s_i s_j - \min(s_i, s_j)}{k(k-1)}, \\ P[\mathbf{s}_i \leqq s_i | m=1] = \dfrac{s_i}{k}, \\ P[\mathbf{s}_j \leqq s_j | m=1] = \dfrac{s_j}{k}, \end{array} \right. \tag{8.1}$$

so in that case (7.3) is true. Now let us suppose that (7.3) is true for m observers, then we have

$$\left\{ \begin{array}{l} P[\mathbf{s}_i \leqq s_i \text{ and } \mathbf{s}_j \leqq s_j | m+1] = \\ = \sum_{a \neq b} P[\mathbf{s}_i \leqq s_i - a \text{ and } \mathbf{s}_j \leqq s_j - b | m] \cdot P[\text{the } i\text{-th object has rank } a \text{ and the } j\text{-th object rank } b \text{ in the } (m+1)\text{-st ranking}] = \\ = \sum_{a \neq b} P[\mathbf{s}_i \leqq s_i - a \text{ and } \mathbf{s}_j \leqq s_j - b | m] \cdot \dfrac{1}{k(k-1)} \leqq \\ \leqq \sum_{a \neq b} P[\mathbf{s}_i \leqq s_i - a | m] \cdot P[\mathbf{s}_j \leqq s_j - b | m] \cdot \dfrac{1}{k(k-1)} = \\ = \sum_{a=1}^{k} P[\mathbf{s}_i \leqq s_i - a | m] \cdot \dfrac{1}{k} \cdot \sum_{b=1}^{k} P[\mathbf{s}_j \leqq s_j - b | m] \cdot \dfrac{1}{k} + \\ + \dfrac{1}{k^2(k-1)} \sum_{a=1}^{k} P[\mathbf{s}_i \leqq s_i - a | m] \cdot \sum_{b=1}^{k} P[\mathbf{s}_j \leqq s_j - b | m] + \end{array} \right. \tag{8.2}$$

[2] We owe this formula to Mr. A. BENARD, Statistical Department of the Mathematical Centre.

$$(8.2)\quad \begin{cases} -\frac{1}{k(k-1)}\sum_{a=1}^{k} P[\mathbf{s}_i \leqq s_i - a|m]\cdot P[\mathbf{s}_j \leqq s_j - a|m] = \\ = P[\mathbf{s}_i \leqq s_i|m+1]\cdot P[\mathbf{s}_j \leqq s_j|m+1] + \\ -\frac{1}{k(k-1)}\sum_{a=1}^{k}\left\{P[\mathbf{s}_i \leqq s_i - a|m] - \frac{\sum_{b=1}^{k} P[\mathbf{s}_i \leqq s_i - b|m]}{k}\right\}\cdot \\ \cdot\left\{P[\mathbf{s}_j \leqq s_j - a|m] - \frac{\sum_{b=1}^{k} P[\mathbf{s}_j \leqq s_j - b|m]}{k}\right\} \leqq \\ \leqq P[\mathbf{s}_i \leqq s_i|m+1]\cdot P[\mathbf{s}_j \leqq s_j|m+1]. \end{cases}$$

So theorem 7.1 is proved by induction.

Formula 7.7 can be proved in the following way:

$k^m P[\mathbf{s}_i = n|m]$ = the number of partitions of n into m positive integers, no one being larger than k (different permutations of the same integers are counted as different partitions).

Thus

$k^m P[\mathbf{s}_i = n|m]$ = coefficient of z^n in $(z+z^2+\ldots+z^k)^m$ = coefficient of z^{n-m} in $\left(\frac{1-z^k}{1-z}\right)^m$ = coefficient of z^{n-m} in

$$\sum_{x=0}^{\infty}\binom{m}{x}(-1)^x z^{kx}\sum_{r=0}^{\infty}\binom{m+r-1}{r}z^r =$$

$$= \sum_{x=0}^{\infty} I_{n-kx-m}\binom{m}{x}\binom{n-kx-1}{m-1}(-1)^x,$$

which proves (7.7).

9. *A distribution free k-sample slippage test*

We consider the independent variates

$$(9.1)\qquad \mathbf{u}_1, \ldots, \mathbf{u}_k,$$

which have, under H_0, the same continuous distribution function. From the i^{th} population we have t_i independent observations $\mathbf{u}_{ij}$ $(j = 1, \ldots, t_i)$. We want to test H_0 against the alternatives

$$(9.2)\qquad H_{1i}\begin{cases} P[\mathbf{u}_i > \mathbf{u}_j] > \frac{1}{2} \quad (j \neq i), \\ \mathbf{u}_j \quad (j = 1, \ldots, i-1, i+1, \ldots, k) \quad \text{follow the same distribution,} \end{cases}$$

for one unknown value of i and

$$(9.3)\qquad H_{2i}\begin{cases} P[\mathbf{u}_i > \mathbf{u}_j] < \frac{1}{2} \quad (j \neq i), \\ \mathbf{u}_j \quad (j = 1, \ldots, i-1, i+1, \ldots, k) \quad \text{follow the same distribution.} \end{cases}$$

Now the following test procedure is proposed. If all observations $\mathbf{u}_{ij}$ $(i = 1, \ldots, k;\ j = 1, \ldots, t_i)$ are ranked, we denote by $\mathbf{T}_i$ the sum of the ranks of the observations $\mathbf{u}_{ij}$ $(j = 1, \ldots, t_i)$. As $\mathbf{T}_i$ is a linear function of WILCOXON's test statistic applied to the i^{th} sample and the other $k-1$

samples together, its distribution function under H_0 is known (cf. H. B. MANN and D. R. WHITNEY (1947)). So for each set of values $T_1, \ldots, T_k$ we can, under H_0, compute

$$q_i = P[\mathbf{T}_i \geqq T_i]. \tag{9.4}$$

Now, when testing H_0 against H_{1i}, H_0 is rejected when $\min q_i \leqq \alpha/k$. A similar procedure is followed for slippage to the left. In the next section we shall prove the inequality

$$P[\mathbf{T}_i \geqq T_i \text{ and } \mathbf{T}_j \geqq T_j] \leqq P[\mathbf{T}_i \geqq T_i] \cdot P[\mathbf{T}_j \geqq T_j], \tag{9.5}$$

so the limits, between which the level of significance may vary, are known also in this case.

Let now for every fixed i the hypothesis $H_{1,i}$ be

$$\begin{cases} P[\mathbf{u}_i > \mathbf{u}_j] > \frac{1}{2} \quad (j \neq i), \\ \mathbf{u}_j \quad (j=1, \ldots, i-1, i+1, \ldots, k), \quad \text{follow the same distribution.} \end{cases}$$

Put

$$P[T_i | H_0] \stackrel{\text{def}}{=} P[\mathbf{T}_i \geqq T_i | H_0].$$

This probability still depends on $t_1, \ldots, t_k$.

In the same way as in sections 3 and 5 we consider the decision procedure δ:

Decide that H_0 is true if

$$P[T_j | H_0] > \frac{\alpha}{k} \text{ for } j=1, \ldots, k.$$

Decide that $H_{1,j}$ is true if j is the smallest integer such that

$$P[T_j | H_0] \leqq \frac{\alpha}{k} \text{ and } P[T_l | H_0] \geqq P[T_j | H_0],\ l \neq j.$$

We prove in the next section

Theorem 9.1. *If $H_{1,j}$ is true, the probability of a correct decision with the procedure δ tends to* 1 *if $t_1 \to \infty, \ldots, t_k \to \infty$ such that*

$$\liminf \frac{t_i}{\sum t_l} > 0 \quad (i=1, \ldots, k).$$

Another test for the k-sample slippage problem was proposed by F. MOSTELLER (1948) (cf. also F. MOSTELLER and J. W. TUKEY (1950)) who uses as test statistic the number of observations of the sample with the largest observation which exceed all observations of all other samples. A comparison of the power of both tests with respect to some alternatives of practical interest seems desirable.

10. *Proof of the inequality* (9.5) *and of theorem* 9.1 [3])

For definiteness we take in (9.5) $i=1$, $j=2$. We also take $k=3$. This

[3]) The proofs in this section were found by Mr. H. KESTEN, then working in the Statistical Department of the Mathematical Centre.

is no restriction on the generality as pooling of the 3rd, 4th, ... and kth sample does not affect

$$P[T_1|H_0],\ P[T_2|H_0] \text{ or } P[T_1,T_2|H_0] \overset{\text{def}}{=} P[\mathbf{T}_1 \geqq T_1 \text{ and } \mathbf{T}_2 \geqq T_2|H_0].$$

Put now

$$t \overset{\text{def}}{=} t_1+t_2+t_3 \tag{10.1}$$

and define

$P_{n_1,n_2,n_3}[T_i] \overset{\text{def}}{=} P[T_i|H_0]$ if $t_1=n_1, t_2=n_2, t_3=n_3$.

$P_{n_1,n_2,n_3}[T_i,\ 1] \overset{\text{def}}{=} P[\mathbf{T}_i \geqq T_i$ and the largest element belongs to sample number $1|H_0]$ if $t_1=n_1,\ t_2=n_2,\ t_3=n_3$.

$P_{n_1,n_2,n_3}[T_i|1] \overset{\text{def}}{=}$ the conditional probability of $\mathbf{T}_i \geqq T_i$ under H_0, given that the largest element belongs to sample number 1 if $t_1=n_1,\ t_2=n_2,\ t_3=n_3$.

In the same way we define

$$P_{n_1,n_2,n_3}[T_i,T_j],\ P_{n_1,n_2,n_3}[T_i,T_j,\ 1] \text{ and } P_{n_1,n_2,n_3}[T_i,T_j|1]$$

for the events $\{\mathbf{T}_i \geqq T_i$ and $\mathbf{T}_j \geqq T_j\}$.

We shall prove (9.5) by induction with respect to $n_1+n_2+n_3$. So we have to prove

$$P_{n_1,n_2,n_3}[T_1,T_2] \leqq P_{n_1,n_2,n_3}[T_1]\cdot P_{n_1,n_2,n_3}[T_2]. \tag{10.2}$$

Clearly (10.2) holds for $n_1+n_2+n_3=2$ (we take $\mathbf{T}_i=0$ with probability 1 when $t_i=0$). Now suppose (10.2) holds if $n_1+n_2+n_3 \leqq t-1$. We have

$$P_{t_1,t_2,t_3}[T_1,T_2] = \sum_{i=1}^{3} \frac{t_i}{t} P_{t_1,t_2,t_3}[T_1,T_2|i]. \tag{10.3}$$

For the first term of the sum in the right hand member we get

$$\left\{\begin{aligned} P_{t_1,t_2,t_3}[T_1,T_2|1] &= P_{t_1-1,t_2,t_3}[T_1-t,T_2] \leqq \\ &\qquad \text{(according to our assumption)} \\ &\leqq P_{t_1-1,t_2,t_3}[T_1-t]\cdot P_{t_1-1,t_2,t_3}[T_2] = P_{t_1,t_2,t_3}[T_1|1]\cdot P_{t_1,t_2,t_3}[T_2|1]. \end{aligned}\right. \tag{10.4}$$

In the same way, it can be derived that

$$P_{t_1,t_2,t_3}[T_1,T_2|2] \leqq P_{t_1,t_2,t_3}[T_1|2]\cdot P_{t_1,t_2,t_3}[T_2|2]. \tag{10.5}$$

Further

$$\left\{\begin{aligned} P_{t_1,t_2,t_3}[T_1,T_2|3] &= P_{t_1,t_2,t_3-1}[T_1,T_2] \leqq P_{t_1,t_2,t_3-1}[T_1]\cdot P_{t_1,t_2,t_3-1}[T_2] = \\ &= P_{t_1,t_2,t_3}[T_1|3]\cdot P_{t_1,t_2,t_3}[T_2|3]. \end{aligned}\right. \tag{10.6}$$

So, combining (10.3), (10.4), (10.5) and (10.6) we find, dropping the subscripts

$$P[T_1,T_2] \leqq \sum_{i=1}^{3} \frac{t_i}{t} P[T_1|i]\cdot P[T_2|i] = \sum_{i=1}^{3} P[T_1|i]\cdot P[T_2,i]. \tag{10.7}$$

We have

$$P[T_1|2] = P[T_1|3] = P[T_1|2 \text{ or } 3] \tag{10.8}$$

and similarly with 1 and 2 interchanged, and

$$\begin{cases} P[T_1]\cdot P[T_2] = \left\{\frac{t_1}{t}P[T_1|1]+\frac{t_2+t_3}{t}P[T_1|2 \text{ or } 3]\right\}\cdot \\ \qquad\qquad \cdot\{P[T_2,1]+P[T_2,2 \text{ or } 3]\}. \end{cases} \tag{10.9}$$

From (10.7), (10.8) and (10.9) we see that it is sufficient to prove

$$\begin{cases} \sum_{i=1}^{3} P[T_1|i]\cdot P[T_2,i] = P[T_1|1]\cdot P[T_2,1]+P[T_1|2]\cdot P[T_2,2 \text{ or } 3] \leqq \\ \qquad \leqq \left\{\frac{t_1}{t}P[T_1|1]+\frac{t_2+t_3}{t}P[T_1|2 \text{ or } 3]\right\}\{P[T_2,1]+P[T_2,2 \text{ or } 3]\} \end{cases} \tag{10.10}$$

or its equivalent

$$\{P[T_1|1]-P[T_1|2]\}\left\{\frac{t_2+t_3}{t}P[T_2,1]-\frac{t_1}{t}P[T_2,2 \text{ or } 3]\right\} \leqq 0. \tag{10.11}$$

But the inequality

$$P[T_1|1] \geqq P[T_1|2] \tag{10.12}$$

holds as can be seen in the following way. (10.12) is equivalent to

$$t_1 P[T_1,2] \leqq t_2 P[T_1,1]. \tag{10.13}$$

Consider now a ranking which gives T_1 and 2 (i.e. the largest element belongs to the 2nd sample and $\mathbf{T}_1 \geqq T_1$) and interchange the last element with every element of the first sample. This gives t_1 different rankings with T_1 and 1. In this way we get each ranking with T_1 and 1 at most t_2 times, because in a ranking with T_1 and 1 the last element can be interchanged with at most t_2 different elements of the second sample. This proves (10.13) and thus (10.12). Interchanging 1 and 2 in (10.12) we find

$$P[T_2|2] \geqq P[T_2|1]. \tag{10.14}$$

(10.11) and thus (10.2) is an immediate consequence of (10.12) and (10.14). This completes the proof of (9.5).

We now turn to the proof of theorem 9.1. Let $H_{1,1}$ be true. If $t_i \to \infty$ $(i=1, \ldots, k)$ such that

$$\liminf \frac{t_1}{\sum_{i=1}^{k} t_i} > 0 \text{ and } \liminf \frac{\sum_{i=1}^{k} t_i - t_1}{\sum t_i} > 0,$$

we know that Wilcoxon's test comparing sample 1 with the other samples pooled is consistent. This means

$$\lim_{t_i\to\infty} P[P[\mathbf{T}_1] \leqq \eta|H_{1,1}] = 1 \quad \text{for every } \eta(0 \leqq \eta \leqq 1) \tag{10.15}$$

or the exceedance probability found in the first sample converges to 0 in probability (cf. D. VAN DANTZIG (1951)).

In a similar way as in D. VAN DANTZIG (1951) we find, if

$$p \overset{\text{def}}{=} P[\mathbf{u}_1 > u_j | H_{1,1}] > \tfrac{1}{2}$$

(10.16) $$E(\mathbf{T}_j|H_0) = \tfrac{1}{2}t_j(\sum t_i - t_j) + \tfrac{1}{2}t_j(t_j+1)$$

and

(10.17) $$E(\mathbf{T}_j|H_{1,1}) = \tfrac{1}{2}t_j(\sum t_i - t_j - t_1) + (1-p)t_j t_1 + \tfrac{1}{2}t_j(t_j+1) < E(\mathbf{T}_j|H_0).$$

Further

(10.18) $$\sigma^2(\mathbf{T}_j|H_{1,1}) \leqq 3\sigma^2(\mathbf{T}_j|H_0).$$

From (10.15) we have

(10.19) $$\lim_{t_i\to\infty} P[P[\mathbf{T}_j] \leqq P[\mathbf{T}_1]|H_{1,1}] \leqq \lim_{t_i\to\infty} P[P[\mathbf{T}_j] \leqq \eta | H_{1,1}]$$

for every $\eta (0 \leqq \eta \leqq 1)$.

As the limit distribution under H_0 of $\frac{\mathbf{T}_j - E(\mathbf{T}_j|H_0)}{\sigma(\mathbf{T}_j|H_0)}$ is normal with mean 0 and unit variance (10.19) leads to

(10.20) $$\begin{cases} \lim_{t_i\to\infty} P[P[\mathbf{T}_j] \leqq \eta|H_{1,1}] = \lim_{t_i\to\infty} P\left[\frac{\mathbf{T}_j - E(\mathbf{T}_j|H_0)}{\sigma(\mathbf{T}_j|H_0)} \geqq \xi_\eta | H_{11}\right] \leqq \\ \leqq \lim_{t_i\to\infty} P\left[\frac{\mathbf{T}_j - E(\mathbf{T}_j|H_{1,1})}{\sigma(\mathbf{T}_j|H_{1,1})} \geqq \sqrt{3}\,\xi_\eta | H_{1,1}\right] \leqq \frac{1}{3\xi_\eta^2} \end{cases}$$

where ξ_η is defined by

$$\frac{1}{\sqrt{2\pi}} \int_{\xi_\eta}^{\infty} e^{-\frac{x^2}{2}}\, dx = \eta.$$

(10.20) is valid for every $\eta (0 \leqq \eta \leqq 1)$ and as $\xi_\eta \to \infty$ $(\eta \to 0)$ (10.19) combined with (10.20) gives

(10.21) $$\lim_{t_i\to\infty} P[P[\mathbf{T}_j] \leqq P[\mathbf{T}_1]|H_{1,1}] = 0.$$

If $\mathrm{H}_{1,1}$ is true the probability of correct decision is

(10.22) $$\begin{cases} P[P[\mathbf{T}_1] \leqq \frac{\alpha}{k} \text{ and } P[\mathbf{T}_1] < P[\mathbf{T}_j] \text{ for } j \neq 1 | H_{1,1}] \geqq \\ \geqq P[P[\mathbf{T}_1] \leqq \frac{\alpha}{k} | H_{1,1}] - \sum_{j=2}^{k} P[P[\mathbf{T}_j] \leqq P[\mathbf{T}_1] | H_{1,1}]. \end{cases}$$

(10.15) and (10.21) show that the probability of a correct decision converges to 1, which proves theorem 9.1.

11. *Tables of critical values for the Poisson distribution and for the method of m rankings*

Table 11.1 gives critical values for the test for Poisson variates against slippage to the right if H_0 is: $p_1 = p_2 = \ldots = p_k$. The critical values for max z_i as test statistic are given for the values 0.05 (the upper numbers) and 0.01 (the lower numbers) of α. Owing to the discontinuous character of the binomial distribution the true level of significance will generally be less, and very often considerably less, than α. Therefore approximated true levels of significance (i.e. α' cf. (2.17)) are shown also. The exact values satisfy inequality (2.13). The table was constructed with the help of a table of the binomial distribution. This can also be done for critical values for the test against slippage to the left.

Table 11.2 gives critical values for specified α for the method of m rankings, when testing against slippage to the left with min s_i as test statistic. If this critical value is equal to 1, the critical value r at the same level of significance for the test against slippage to the right is given by $r = m(k+1) - 1$. As in table 11.1 the approximated true levels of significance (α') are also given.

12. *Acknowledgements*

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TABLE 11.1

Critical values for the slippage test to the right in the Poisson-case with $H_0: \mu_1 = \mu_2 = \ldots = \mu_k$. Test statistic: max z_i. Approximate significance level 0.05 (upper values) and 0.01 (lower values). The approximated true level of significance is written behind the critical value. Number of observations k, sum of the observations n

n \ k	2	3	4	5	6	7	8	9	10
2	— — — —	— — — —	— — — —	— — — —	— — — —	— — — —	— — — —	— — — —	— — — —
3	— — — —	— — — —	— — — —	3 0.040 — —	3 0.028 — —	3 0.020 — —	3 0.016 — —	3 0.012 — —	3 0.010 3 0.010
4	— — — —	4 0.037 — —	4 0.016 — —	4 0.008 4 0.008	4 0.005 4 0.005	4 0.003 4 0.003	4 0.002 4 0.002	3 0.045 4 0.001	3 0.037 4 0.001
5	— — — —	5 0.012 — —	5 0.004 5 0.004	4 0.034 5 0.002	4 0.020 5 0.001	4 0.013 5 0.000	4 0.009 4 0.009	4 0.006 4 0.006	4 0.005 4 0.005
6	6 0.031 — —	6 0.004 6 0.004	5 0.019 6 0.001	5 0.008 5 0.008	5 0.004 5 0.004	4 0.035 5 0.002	4 0.024 5 0.001	4 0.017 5 0.001	4 0.013 5 0.001
7	7 0.016 — —	6 0.021 7 0.001	6 0.005 6 0.005	5 0.023 6 0.002	5 0.012 6 0.001	5 0.007 5 0.007	5 0.004 5 0.004	4 0.037 5 0.003	4 0.027 5 0.002
8	8 0.008 8 0.008	7 0.008 7 0.008	6 0.017 7 0.002	6 0.006 6 0.006	5 0.028 6 0.003	5 0.016 6 0.001	5 0.010 5 0.010	5 0.006 5 0.006	5 0.004 5 0.004
9	8 0.039 9 0.004	7 0.025 8 0.003	6 0.040 7 0.005	6 0.015 7 0.002	6 0.007 6 0.007	5 0.032 6 0.003	5 0.020 6 0.002	5 0.013 6 0.001	5 0.009 5 0.009
10	9 0.021 10 0.002	8 0.010 9 0.001	7 0.014 8 0.002	6 0.032 7 0.004	6 0.015 7 0.002	6 0.008 6 0.008	5 0.036 6 0.004	5 0.024 6 0.002	5 0.016 6 0.001
11	10 0.012 11 0.001	8 0.027 9 0.004	7 0.030 8 0.005	7 0.010 7 0.010	6 0.028 7 0.004	6 0.015 7 0.002	6 0.008 6 0.008	5 0.040 6 0.005	5 0.028 6 0.003
12	10 0.039 11 0.006	9 0.012 10 0.002	8 0.011 9 0.002	7 0.020 8 0.003	6 0.048 7 0.008	6 0.026 7 0.004	6 0.015 7 0.002	6 0.009 6 0.009	5 0.043 6 0.005
13	11 0.022 12 0.003	9 0.027 10 0.005	8 0.023 9 0.004	7 0.035 8 0.006	7 0.015 8 0.002	6 0.042 7 0.007	6 0.024 7 0.003	6 0.015 7 0.002	6 0.009 6 0.009
14	12 0.013 13 0.002	10 0.012 11 0.002	8 0.041 9 0.009	8 0.012 9 0.002	7 0.025 8 0.004	7 0.012 8 0.002	6 0.038 7 0.006	6 0.023 7 0.003	6 0.015 7 0.002
15	12 0.035 13 0.007	10 0.026 11 0.005	9 0.017 10 0.003	8 0.021 9 0.004	7 0.040 8 0.008	7 0.019 8 0.003	7 0.010 8 0.001	6 0.035 7 0.005	6 0.022 7 0.003
16	13 0.021 14 0.004	10 0.048 12 0.002	9 0.030 10 0.007	8 0.035 9 0.007	8 0.013 9 0.002	7 0.030 8 0.005	7 0.016 8 0.002	7 0.009 7 0.009	6 0.033 7 0.005
17	13 0.049 15 0.002	11 0.024 12 0.006	9 0.050 11 0.002	9 0.013 10 0.002	8 0.021 9 0.004	7 0.045 8 0.009	7 0.024 8 0.004	7 0.013 8 0.002	6 0.047 7 0.008
18	14 0.031 15 0.008	11 0.044 13 0.003	10 0.022 11 0.005	9 0.021 10 0.005	8 0.032 9 0.007	8 0.014 9 0.003	7 0.035 8 0.007	7 0.020 8 0.003	7 0.012 8 0.002
19	15 0.019 16 0.004	12 0.022 13 0.006	10 0.036 11 0.009	9 0.033 10 0.008	8 0.048 10 0.002	8 0.021 9 0.004	7 0.050 9 0.002	7 0.028 8 0.005	7 0.017 8 0.003
20	15 0.041 17 0.003	12 0.039 14 0.003	11 0.016 12 0.004	9 0.050 11 0.003	9 0.017 10 0.004	8 0.031 9 0.007	8 0.015 9 0.003	7 0.040 8 0.008	7 0.024 8 0.004
21	16 0.027 17 0.007	13 0.021 14 0.006	11 0.026 12 0.007	10 0.020 11 0.005	9 0.026 10 0.006	8 0.044 10 0.002	8 0.022 9 0.004	8 0.011 9 0.002	7 0.033 8 0.006
22	17 0.017 18 0.004	13 0.035 15 0.003	11 0.040 13 0.003	10 0.031 11 0.008	9 0.037 10 0.009	9 0.015 10 0.003	8 0.031 9 0.007	8 0.016 9 0.003	7 0.044 8 0.009
23	17 0.035 19 0.003	14 0.019 15 0.005	12 0.019 13 0.005	10 0.045 12 0.003	10 0.014 11 0.003	9 0.022 10 0.005	8 0.042 9 0.010	8 0.022 9 0.004	8 0.012 9 0.002
24	18 0.023 19 0.007	14 0.031 15 0.010	12 0.029 13 0.008	11 0.019 12 0.005	10 0.020 11 0.005	9 0.030 10 0.007	9 0.014 10 0.003	8 0.030 9 0.006	8 0.017 9 0.003
25	18 0.043 20 0.004	14 0.049 16 0.005	12 0.043 14 0.004	11 0.028 12 0.008	10 0.029 11 0.008	9 0.041 11 0.002	9 0.019 10 0.004	8 0.040 9 0.009	8 0.023 9 0.005

TABLE 11.2

Critical values s_α of the test statistic $\min s_i$ for the slippage test to the left for the method of m rankings. Level of significance α, number of rankings m, number of ranked objects k. The approximated true levels of significance are written behind the corresponding critical values

k	α \ m	3	4	5	6	7	8	9
2	0.05	— —	— —	— —	6 0.031	7 0.016	8 0.008	10 0.039
	0.025	— —	— —	— —	— —	7 0.016	8 0.008	9 0.004
	0.01	— —	— —	— —	— —	— —	8 0.008	9 0.004
3	0.05	— —	4 0.037	5 0.012	7 0.029	9 0.049	10 0.021	12 0.032
	0.025	— —	— —	5 0.012	6 0.004	8 0.011	10 0.021	11 0.008
	0.01	— —	— —	— —	6 0.004	7 0.001	9 0.004	11 0.008
4	0.05	— —	4 0.016	6 0.023	8 0.027	10 0.029	12 0.030	14 0.029
	0.025	— —	4 0.016	6 0.023	7 0.007	9 0.009	11 0.010	13 0.011
	0.01	— —	— —	5 0.004	7 0.007	9 0.009	10 0.003	12 0.003
5	0.05	3 0.040	5 0.040	7 0.034	9 0.027	11 0.021	14 0.038	16 0.028
	0.025	— —	4 0.008	6 0.010	8 0.009	11 0.021	13 0.016	15 0.013
	0.01	— —	4 0.008	6 0.010	8 0.009	10 0.008	12 0.006	14 0.005
6	0.05	3 0.028	5 0.023	8 0.043	10 0.027	13 0.037	16 0.045	18 0.028
	0.025	— —	5 0.023	7 0.016	9 0.011	12 0.017	15 0.023	17 0.014
	0.01	— —	4 0.005	6 0.005	8 0.004	11 0.007	13 0.005	16 0.007
7	0.05	3 0.020	6 0.044	8 0.023	11 0.027	14 0.029	17 0.029	21 0.048
	0.025	3 0.020	5 0.014	8 0.023	10 0.012	13 0.015	16 0.016	19 0.016
	0.01	— —	4 0.003	7 0.009	9 0.005	12 0.007	15 0.008	18 0.008
8	0.05	3 0.016	6 0.029	9 0.031	12 0.028	16 0.043	19 0.035	23 0.046
	0.025	3 0.016	5 0.010	8 0.014	11 0.014	15 0.025	18 0.021	21 0.017
	0.01	— —	5 0.010	7 0.005	10 0.006	13 0.007	16 0.006	20 0.010
9	0.05	4 0.049	7 0.048	10 0.038	13 0.029	17 0.036	21 0.042	25 0.045
	0.025	3 0.012	6 0.021	9 0.019	12 0.016	16 0.022	19 0.016	23 0.019
	0.01	— —	5 0.007	8 0.009	11 0.008	14 0.006	18 0.009	21 0.007
10	0.05	4 0.040	7 0.035	11 0.046	14 0.030	18 0.032	23 0.048	27 0.045
	0.025	3 0.010	6 0.015	9 0.013	13 0.017	17 0.019	21 0.020	25 0.020
	0.01	3 0.010	5 0.005	8 0.006	12 0.009	15 0.006	19 0.008	23 0.008