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TESTING WITH REPLACEMENT AND THE PRODUCT LIMIT ESTIMATOR

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Testing with Replacement and the Product Limit Estimator *)

by

R.D. Gill

ABSTRACT

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. strictly positive r.v.'s with d.f. F. Define $\tilde{N}(t) = \#\{j: X_1 + \ldots + X_j \le t\}$. In "testing with replacement" n independent copies of $\tilde{N}$ are observed each over the time interval $[0,T]$ and we are interested in nonparametric estimation of F based on these observations. We prove consistency of the product limit estimator as $n \to \infty$ for arbitrary F, and weak convergence for two cases: integer valued X_i; and X_i with a continuous d.f. We use the theory of stochastic integrals to do this, and explain the similarity of our results with those for the product limit estimator in the model of "random censorship".

KEY WORDS & PHRASES: *testing with replacement, product limit estimator, censuring, martingale, stochastic integral*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

Let $X_1, X_2, \ldots$ be strictly positive i.i.d. random variables (r.v.'s) with distribution function (d.f.) F. Define $S_0 = 0$, $S_j = \sum_{i=1}^{j} X_i$, $j = 1,2,\ldots$ and $\tilde{N}(t) = \#\{j \geq 1 : S_j \leq t\}$, $t \geq 0$. We consider nonparametric estimation of F based on the first n of an infinite sequence of independent realisations of $\tilde{N}$, each observed over the fixed time interval $[0,T]$, $T < \infty$. This situation might arise when light bulbs are lifetested in a large number n of sockets, failed bulbs being replaced immediately by new ones.

We call an X_j such that $S_j \leq T$ an uncensored observation; if $S_{j-1} < T < S_j$ we call X_j censored, for we only observe in this case that X_j takes an unknown value strictly greater than $T - S_{j-1}$. If $n \to \infty$ the empirical d.f. based on the uncensored observations is inconsistent. An obvious alternative estimator of F is the product limit estimator of KAPLAN and MEIER (1958) which also takes account of the censored observations; it is introduced in section 4 of this paper.

Using some results summarized in section 2 from the theory of stochastic integrals, we establish (in section 3) certain key equalities involving the 1st and 2nd moments of processes related to the empirical d.f.'s of the censored and uncensored observations. We also indicate that the same relationships hold in the model of "random censorship" (BRESLOW and CROWLEY (1974)) which explains the striking similarity between the results obtained in both models. In section 4 the equalities are exploited to prove strong consistency of the product limit estimator with arbitrary F, in section 5 we consider weak convergence for integer valued X_j, and in section 6 for X_j with a continuous d.f.

In appendix 1 some propositions are proved on counting processes which are possibly of independent interest, because (for a univariate process at least) they extend results previously only available for the continuous case, while the proofs are more elementary than previous ones.

2. RESULTS FROM THE THEORY OF STOCHASTIC INTEGRALS

We state here only those results needed in this paper; definitions and notations are adapted to our purpose and are by no means standard. Surveys of the theory aimed at applications in the theory of counting processes, of which renewal processes are special cases, are contained in AALEN (1977) and AALEN (1978). The theory here has been compiled chiefly from MEYER (1976).

We start with a fixed complete probability space (Ω,F,P) and an increasing right continuous family $\{F_t\}_{t\in[0,\infty)}$ of sub σ-algebras of F such that F_0 contains all P-null sets of F. For the sake of brevity we omit the word "almost" in such phrases as "almost all sample paths" and likewise omit mention of P-null sets when we construct σ-algebras in section 3.

A stochastic process $\{X(t)\}_{t\in[0,\infty)}$ is often denoted simply as X. A process X is called *adapted* (to $\{F_t\}$) if X(t) is F_t measurable for all t, and *integrable* if $E|X(t)| < \infty$ for all t. We write A^+ for the family of adapted processes whose sample paths are right continuous (rt.cts.), non-decreasing, and zero at time zero. $A = A^+ - A^+$ is the collection of processes which are the difference of two processes in A^+, A^+_{int} is the family of integrable processes in A^+, and $A_{int} = A^+_{int} - A^+_{int}$. If $X \in A$, X_- is the process defined by $X_-(t) = X(t-)$ and ΔX is the process $X - X_-$. We write ΔX^p for the process $(\Delta X)^p$ for given $p > 0$. This Δ notation is also used for functions on $[0,\infty)$. Other points of notation: $\wedge$ means minimum and $\vee$ maximum, and χ denotes an indicator function.

Given a process $X \in A$ and another process Y whose paths are measurable functions on $[0,\infty)$ the *stochastic integrals* $\int_0^t |Y(s)|\ |dX|(s)$ and $\int_0^t Y(s)dX(s)$ are the ordinary pathwise Lebesgue-Stieltjes integrals, the latter being defined for $\omega \in \Omega$ for which the former is finite. We write $\int YdX$ for the process $\{\int_0^t Y(s)dX(s)\}_{t\in[0,\infty)}$.

Also we shall come across martingales, predictable processes and counting processes. We write M for the collection of martingales which are members of A. Note that if $M \in M$, $EM(t) = 0$ for all t. A process is called *predictable* if, considered as a function of (t,ω) on $[0,\infty)\times\Omega$, it is measurable with respect to the σ-algebra on that space generated by the left continuous (lt.cts.) adapted processes. So the latter are predictable; and if X,Y are predictable and $f: \mathbb{R} \to \mathbb{R}$ is a measurable function, then the processes $X + Y$, X/Y, $f(X)$ etc. are predictable too. Also a process all of whose paths are equal to a single Borel measurable function on $[0,\infty)$ is predictable. Finally a *counting process* is a member of A^+ whose paths are integer valued functions on $[0,\infty)$ with jumps of size +1 only.

We need the following theorem, which can be easily derived from chap. I n°12 and chap. II n° 18,25 and 30 of MEYER (1976).

THEOREM 1. (a) *Let* $M \in \mathcal{M} \cap \mathcal{A}_{int}$ *and let* H *be a bounded predictable process. Then* $\int HdM \in \mathcal{M} \cap \mathcal{A}_{int}$.
(b) *Let* $M_i \in \mathcal{M} \cap \mathcal{A}_{int}$ *be such that* M_i^2 *is integrable*; $i = 1,2$. *Then there exists a unique predictable process* $\langle M_1,M_2\rangle \in \mathcal{A}_{int}$ *such that* $M_1M_2 - \langle M_1,M_2\rangle \in \mathcal{M}$. *If* H_1 *and* H_2 *are bounded predictable processes, then* $(\int H_i dM_i)^2$ *is integrable,* $i = 1,2$, *and* $\langle \int H_1 dM_1, \int H_2 dM_2\rangle = \int H_1H_2 d\langle M_1,M_2\rangle$.

COROLLARY. *Let* $M \in \mathcal{M} \cap \mathcal{A}_{int}$ *be such that* M^2 *is integrable, and let* H,H' *be bounded predictable processes. Then for all* t, $E \int_0^t H(s)dM(s) = 0$ *and*

$$E\left(\left(\int_0^t H(s)dM(s)\right)\left(\int_0^t H'(s)dM(s)\right)\right) = E \int_0^t H(s)H'(s)d\langle M,M\rangle(s).$$

3. MARTINGALES IN RENEWAL THEORY

Let $\tilde{N}$ be the renewal process defined in section 1. Letting $\mathcal{F}_t$ be the σ-algebra generated by $\tilde{N}(s)$, $s \le t$, we see that $\tilde{N}$ is a counting process. Furthermore $E\tilde{N}(t) < \infty$ for all t (FELLER (1971) XI. 1, lemma). Define $L(t) = t - S_{\tilde{N}(t-)}$; $L(t)$ is the time between the last jump of $\tilde{N}$ in $[0,t)$ and t. Let $G(t) = \int_0^t (1-F(s-))^{-1}dF(s)$ and let $A \in \mathcal{A}^+$ be the process

$$A(t) = \sum_{i=1}^{\tilde{N}(t-)} G(X_i) + G(L(t))$$

(an empty sum is zero)

In appendix 1 we prove

THEOREM 2. A *is predictable, and* $A \in \mathcal{A}^+_{int}$. *Let* $M = \tilde{N} - A$. *Then* $M \in \mathcal{M} \cap \mathcal{A}_{int}$, M^2 *is integrable and*

$$\langle M,M\rangle = \int (1-\Delta A)dA.$$

We now introduce two new processes N and Y which together record the censored and uncensored observations which result when $\tilde{N}$ is observed on $[0,T]$:

$$N(t) \stackrel{\text{def}}{=} \#\{j: X_j \le t \text{ and } S_j \le T\},$$

the number of uncensored observations less than or equal to t.

$$Y(t) \stackrel{\text{def}}{=} \#\{j: X_j \ge t \text{ and } S_{j-1} \le T - t\},$$

for $t > 0$ the number of censored or uncensored observations which are known to take a value greater than or equal to t. N and Y are nonnegative integer valued, N nondecreasing and rt.cts. and Y nonincreasing and lt.cts. $N(0) = 0$ and $N(t) = \tilde{N}(T)$ for $t \ge T$. $Y(0) = \tilde{N}(T) + 1$ and $Y(t) = 0$ for $t > T$.

<u>LEMMA 3</u>. *For any bounded measurable function* f *on* $[0,\infty)$,

(1) $$\int_0^T f(L(s))d\tilde{N}(s) = \int_0^\infty f(t)dN(t)$$

(2) $$\int_0^T f(L(s))dA(s) = \int_0^\infty f(t)Y(t)dG(t)$$

(3) $$\int_0^T f(L(s))(1-\Delta A(s))dA(s) = \int_0^\infty f(t)Y(t)(1-\Delta G(t))dG(t).$$

<u>PROOF</u>.

$$\int_0^T f(L(s))d\tilde{N}(s) = \sum_{j=1}^{\tilde{N}(T)} f(L(S_j)) = \sum_{j=1}^{N(T)} f(X_j) = \int_0^T f(t)dN(t)$$

which gives (1). To prove (2) and (3) we need some more notation. Define $J = \tilde{N}(T-)$ and redefine in this proof $S_{J+1} = T$. Then $(0,T] = \cup_{j=0}^{J} (S_j,S_{j+1}]$ where each subinterval is nonempty. For $j = 0,\dots,J$ and $s \in (S_j,S_{j+1}]$ we have $\tilde{N}(s-) = j$, $L(s) = s - S_j$, $A(s) = A(S_j) + G(s-S_j)$ and $\Delta A(s) = \Delta G(s-S_j)$. Furthermore $Y(t) = \#\{j = 0,\dots,J: S_j+t \le S_{j+1}\}$, $t > 0$. So

$$\int_0^T f(L(s))dA(s) = \sum_{j=0}^{J} \int_{(S_j,S_{j+1}]} f(L(s))dA(s)$$

$$= \sum_{j=0}^{J} \int_{(0,S_{j+1}-S_j]} f(t)dG(t)$$

$$= \int_0^T f(t)Y(t)dG(t),$$

which gives (2). Finally (3) follows from (2) because $\Delta A(s) = \Delta G(L(s))$. □

Define

(4) $$W = N - \int YdG.$$

We now combine the previous results to get

PROPOSITION 4. *Let* f,f' *be bounded nonnegative measurable functions. Then*

(5) $$0 = E\int_0^\infty f(t)dW(t) = E\int_0^\infty f(t)dN(t) - \int_0^\infty f(t)EY(t)dG(t)$$

and

(6) $$E\left(\left(\int_0^\infty f(t)dW(t)\right)\left(\int_0^\infty f'(t)dW(t)\right)\right) = \int_0^\infty f(t)f'(t)EY(t)(1-\Delta G(t))dG(t).$$

PROOF. By (1), (2) and (4), noting that $M = \tilde{N} - A$,

$$\int_0^T f(L(s))dM(s) = \int_0^\infty f(t)dN(t) - \int_0^\infty f(t)Y(t)dG(t) = \int_0^\infty f(t)dW(t).$$

L is lt.cts. and adapted, and therefore f(L) is a bounded predictable process. By Theorem 2 and the corollary to Theorem 1, the first part of (5) holds. It is easy to show that $E(\int_0^\infty f(t)dN(t)) < \infty$, and by Fubini's theorem

$$E\left(\int_0^\infty f(t)Y(t)dG(t)\right) = \int_0^\infty f(t)EY(t)dG(t),$$

which completes the proof of (5).

Using (3) and (Theorem 2) $\langle M,M\rangle = \int(1-\Delta A)dA$ we find

$$\int_0^T f(L(s))f'(L(s))d\langle M,M\rangle(s) = \int_0^\infty f(t)f'(t)Y(t)(1-\Delta G(t))dG(t).$$

The corollary to Theorem 1 and Fubini's theorem now yield (6). □

In the sequel we exploit these two formulae, so it is useful at this point to indicate how the same relationships hold in the model of random censorship (among other models for censuring). Let X be a single r.v. with d.f. F, let U be an independent r.v., and let F_t^* be the σ-algebra generated by U and $\chi_{[0,s]}(X)$, $s \le t$; $t \in [0,\infty]$. One observes $X^* = X \wedge U$ and $\chi_{\{X^*=X\}}$. Defining N^* by $N^*(t) = \chi_{\{X^*\le t \text{ and } X^*=X\}}$ and Y^* by $Y^*(t) = \chi_{\{X^*\ge t\}}$ it turns out, combining the results of appendix 1 with AALEN (1976) chapter 5C, that $W^* = N^* - \int Y^* dG \in \mathcal{M} \cap A_{int}$, W^{*2} is integrable, and $\langle W^*,W^*\rangle = \int Y^*(1-\Delta G)dG$. The measurable functions f,f' are, as processes, predictable and so by the corollary to Theorem 1 with $t = \infty$, (5) and (6) continue to hold with W,N and Y replaced by W^*,N^* and Y^*. This explains why such similar results can be obtained for testing with replacement and for random censorship. In fact one can obtain stronger results for random censorship by exploiting other consequences of the fact that W^* is a martingale (W is not); however (5) and (6) are sufficient for our purposes.

4. STRONG CONSISTENCY OF THE PRODUCT LIMIT ESTIMATOR

Throughout T is fixed, $0 < T < \infty$. The d.f. F satisfies $F(0) = 0$ but is otherwise arbitrary. Consider an infinite sequence of independent copies of the renewal process $\tilde{N}$, and let N_n be the sum of the first n independent realisations of N, similarly for Y_n. So given n renewal processes observed on [0,T], $N_n(t)$ is the number of uncensored observations less than or equal to t, and $Y_n(t)$ the number of censored or uncensored observations known to be greater than or equal to t. (In "random censorship", we would consider the sum of n independent copies of N^*,Y^*.)

The product limit estimator of F is defined by

$$\hat{F}_n(t) = 1 - \prod_{s\le t}\left(1 - \frac{\Delta N_n(s)}{Y_n(s)}\right)$$

where by convention 0/0 = 0, so only s for which $Y_n(s) \geq \Delta N_n(s) > 0$ give rise to a factor not equal to 1 in the product.

We shall consider the product limit estimator as a function of the random functions $n^{-1}N_n$ and $n^{-1}Y_n$. Using the techniques of the Glivenko-Cantelli theorem it is easy to show that a.s. $n^{-1}N_n(t)$ and $n^{-1}Y_n(t)$ converge uniformly in t to their respective expectations $EN(t)$ and $EY(t)$; note that $N(t), Y(t) \leq \tilde{N}(T) + 1$ for all t, and $E\tilde{N}(T) < \infty$. We shall extend the definition of $\hat{F}_n$ in a continuous way so that it is also defined as a function of (EN, EY).

Suppose G_1 and G_2 are bounded, nondecreasing, rt.cts. functions on $[0,\infty)$ such that $G_i(0) = 0$. Think of $G_1(t)$ and $G_2(t)$ as being the number of uncensored and censored observations less then or equal to t. Define $\bar{G}_i(t) = G_i(\infty) - G_i(t-)$ and $\bar{G}(t) = \bar{G}_1(t) + \bar{G}_2(t)$. G_1 plays the role of N and $\bar{G}$ that of Y. Denote the space of such pairs $(G_1,\bar{G})$ as $\mathcal{G}$. For $(G_1,\bar{G}) \in \mathcal{G}$ define

$$\phi(G_1,\bar{G})(t) = 1 - \prod_{s\leq t} \left(1 - \frac{\Delta G_1(s)}{\bar{G}(s)}\right) \exp\left(-\int_0^t \frac{dG_{1c}(s)}{\bar{G}(s)}\right)$$

where G_{1c} is the continuous part of G_1 and by convention $\exp(-\infty) = 0$. Note that $\hat{F}_n = \phi(n^{-1}N_n, n^{-1}Y_n)$, and that $\phi(G_1,\bar{G})$ is a rt.cts. nondecreasing function on $[0,\infty)$, zero at time zero, and bounded by 1. Our definition extends that of PETERSON (1975) who stated but did not prove that ϕ is continuous on $\mathcal{G}$, which is the content of the following lemma. A proof is given in appendix 2.

<u>LEMMA 5</u>. *Let* ρ_τ *be the supremum metric on* $[0,\tau]$. *Let* $(G_1,\bar{G}) \in \mathcal{G}$ *be fixed and let* $\tau > 0$ *satisfy* $\bar{G}(\tau) > 0$. *Then* $\rho_\tau(\phi(G_1,\bar{G}),\phi(G_1',\bar{G}')) \to 0$ *as* $\max(\rho_\tau(G_1,G_1'),\rho_\tau(\bar{G},\bar{G}')) \to 0$.

<u>COROLLARY</u>.

$$\phi(F,1-F_-)(t) = 1 - \prod_{s\leq t} \left(1 - \frac{\Delta F(s)}{1-F(s-)}\right) \exp\left(-\int_0^t \frac{dF_c(s)}{1-F(s-)}\right) = F(t).$$

<u>PROOF OF COROLLARY</u>. A d.f. F can be arbitrarily well approximated by a step (distribution) function making a finite number of jumps; and for such a d.f.

the result is trivial. So by Lemma 5 the result holds for t such that $1 - F(t-) > 0$. If for some s, $\Delta F(s) = 1 - F(s-) > 0$, it is easy to check that it now holds for all t; and otherwise it is easy to check by taking limits as $t_n \uparrow s$ where $1 - F(t_n-) > 0$ for all n but $1 - F(s-) = 1 - F(s) = 0$. □

Now $(EN,EY) \in G$. Fix a $\tau > 0$ such that $EY(\tau) > 0$; this condition is equivalent to $\tau \leq T$ and $1 - F(\tau-) > 0$. Combining the continuity of ϕ with the Glivenko-Cantelli theorem for $n^{-1}N_n, n^{-1}Y_n$ gives

$$\sup_{t\in[0,\tau]} |\hat{F}_n(t) - \phi(EN,EY)(t)| \to 0 \quad \text{as } n \to \infty \quad \text{a.s.}$$

It remains to evaluate $\phi(EN,EY)$. Put $f = \chi_{[0,t]}$ in (5); this gives

$$EN(t) = \int_0^t EY(s) \frac{dF(s)}{1-F(s-)} . \tag{7}$$

So $\Delta EN(s) = EY(s) \dfrac{\Delta F(s)}{1-F(s-)}$ and $d(EN)_c(s) = \dfrac{EY(s)}{1-F(s-)} dF_c(s)$;

$$\phi(EN,EY)(t) = 1 - \prod_{s\leq t} \left(1 - \frac{\Delta F(s)}{1-F(s-)}\right) \exp\left(-\int_0^t \frac{dF_c(s)}{1-F(s-)}\right) \text{ for } t \leq \tau$$

$$= F(t) \quad \text{by the corollary.}$$

Define $\sigma = \sup\{t \leq T: F(t) < 1\}$. If $F(\sigma-) < 1$, consistency has been proved on $[0,\sigma]$ and so in effect on $[0,T]$. Otherwise it has been proved on $[0,\tau]$ for all $\tau < \sigma$ and $F(\tau) \uparrow 1$ as $\tau \uparrow \sigma$. Because $\hat{F}_n$ is increasing and bounded above by 1 it is easy to extend consistency to $[0,\sigma]$ and so in effect to $[0,T]$; this proves

THEOREM 6.

$$\sup_{t\in[0,T]} |\hat{F}_n(t) - F(t)| \to 0 \quad \textit{as} \quad n \to \infty \quad \text{a.s.}$$

which was conjectured by KAPLAN and MEIER (1958).

5. WEAK CONVERGENCE: DISCRETE CASE

In this section we suppose the X_i take values in $\mathbb{N}$. All time variables s,t etc. and the fixed T are supposed to be in $\mathbb{N}$. Hence

$$\hat{F}_n(t) = 1 - \prod_{s=1}^{t} \left(1 - \frac{\Delta N_n(s)}{Y_n(s)}\right).$$

Note that

$$\Delta G(t) = \begin{cases} P(X_i = t \mid X_i \geq t) & \text{if } P(X_i \geq t) > 0 \\ 0 & \text{otherwise} \end{cases}$$

and that

$$F(t) = 1 - \prod_{s=1}^{t} (1 - \Delta G(s)).$$

Put $f = \chi_{\{t\}}$ and $f' = \chi_{\{t'\}}$. Then (5) gives

$$\mathcal{E}(\Delta N(t) - \Delta G(t) Y(t)) = 0.$$

Define

$$\sigma_{tt'} = \mathrm{cov}(\Delta N(t) - \Delta G(t)Y(t), \Delta N(t') - \Delta G(t')Y(t')).$$

Then (6) gives

$$\sigma_{tt'} = \begin{cases} 0 & t \neq t' \\ \Delta G(t)(1 - \Delta G(t))\mathcal{E}Y(t) & t = t' \end{cases}$$

and therefore by the central limit theorem

$$\{n^{-\frac{1}{2}}(\Delta N_n(t) - \Delta G(t) Y_n(t))\}_{t=1,\ldots,T} \xrightarrow{\mathcal{D}} N(0, (\sigma_{tt'})).$$

Let $\tau = \max\{t \le T: 1 - F(t-) > 0\}$. For $t \le \tau$, $EY(t) > 0$ and $nY_n(t)^{-1} \xrightarrow{P} (EY(t))^{-1}$; for $\tau < t \le T$, $EY(t) = 0$ and $\Delta G(t) = 0$. So recalling the convention $0/0 = 0$

$$\{n^{\frac{1}{2}}(\Delta N_n(t)/Y_n(t) - \Delta G(t))\}_{t=1,\ldots,T} \xrightarrow{\mathcal{D}} N(0,(\sigma^*_{tt'}))$$

where

$$\sigma^*_{tt'} = \begin{cases} 0 & t \ne t' \\ \Delta G(t)(1-\Delta G(t))/EY(t) & t = t' \end{cases} .$$

In fact $\{\Delta N_n(t)/Y_n(t)\}_{t=1,\ldots,T}$ is the maximum likelihood estimator of $\{\Delta G(t)\}_{t=1,\ldots,T}$. As the likelihood function (which given a single observation of $\tilde{N}$ on $[0,T]$ equals $\prod_{t=1}^{T} \Delta G(t)^{\Delta N(t)}(1-\Delta G(t))^{Y(t)-\Delta N(t)}$) possesses the needed regularity properties, the above result could have been obtained with maximum likelihood theory, but we prefer to present a unified approach here.

Using two Taylor expansions we have the following theorem.

<u>THEOREM 7</u>. *Suppose the* X_i *are positive integer valued r.v.'s and let* $\tau' = \max\{t \le T: 1 - F(t) > 0\}$. *Then*

$$\{n^{\frac{1}{2}}(-\log(1-\hat{F}_n(t)) + \log(1-F(t))\}_{t=1,\ldots,\tau'} \xrightarrow{\mathcal{D}} N(0,(\gamma_{tt'}))$$

where

$$\gamma_{tt'} = \sum_{s=1}^{t\wedge t'} \frac{\Delta G(s)}{(1-\Delta G(s))EY(s)}$$

and

$$\{n^{\frac{1}{2}}(\hat{F}_n(t)-F(t))\}_{t=1,\ldots,T} \xrightarrow{\mathcal{D}} N(0,(\gamma^*_{tt'}))$$

where

$$\gamma^*_{tt'} = \begin{cases} (1-F(t))(1-F(t'))\gamma_{tt'}, & t \text{ and } t' \le \tau' \\ 0 & \text{otherwise} \end{cases} .$$

6. WEAK CONVERGENCE: CONTINUOUS CASE

In this section we prove a result (Theorem 11), analogous to Theorem 7, for the case that F is continuous. To start with however we make no restrictions on F. Let $\tau \in (0,T]$ be fixed and satisfy $1 - F(\tau-) > 0$; let f be a bounded nonnegative measurable function on $[0,\tau]$. Define the following processes on $[0,\tau]$:

$$(8) \qquad \begin{cases} W_n(t) = n^{-\frac{1}{2}}\left(\int_0^t dN_n(s) - \int_0^t Y_n(s)dG(s)\right), \\ Z_n(t) = \int_0^t f(s)dW_n(s), \\ Z = Z_1, \quad W = W_1. \end{cases}$$

Replacing f by $f\chi_{[0,t]}$, f' by $f\chi_{[0,t']}$ in (5) and (6) shows that Z_n has zero mean and finite covariance function

$$(9) \qquad \operatorname{cov}(Z_n(t),Z_n(t')) = \int_0^{t\wedge t'} f(s)^2 EY(s)(1-\Delta G(s))dG(s)$$

which does not depend on n.

PROPOSITION 8. *Let* τ, f, W_n, Z_n *be as above. Then* $Z_n \xrightarrow{\mathcal{D}} Z_\infty$ *on* $D[0,\tau]$ *where* Z_∞ *is a zero mean Gaussian process with the same covariance function as in* (9). *In particular* $W_n \xrightarrow{\mathcal{D}} W_\infty$ *where* W_∞ *is* Z_∞ *with* $f \equiv 1$.

PROOF. We suppose for convenience that $f \le 1$. All time variables are restricted to the interval $[0,\tau]$.

By the central limit theorem the finite dimensional distributions of Z_n converge to the multivariate normal distribution with the same first and second moments. We still have to show tightness of Z_n. Let I_1 and I_2 be the intervals $(t_1,t]$, $(t,t_2]$ for time instants $t_1 < t < t_2$. Write $\Delta_i X$ for $\int_{I_i} dX$ for a process or function X of bounded variation; $\Delta_i X^p$ will always mean $(\Delta_i X)^p$. We shall need the well known fact that N(t) has finite moments of all orders; see for instance BARLOW and PROSCHAN (1975) p.178 exercise 14 for an elegant line of proof, using finiteness of the first moment.

We show that given $\alpha \in (0,1)$ there exists $C > 0$ such that

$$E(\Delta_1 Z_n^2 \Delta_2 Z_n^2) \le C\Delta_1 F^\alpha \Delta_2 F^\alpha. \tag{10}$$

Writing $(\Delta_1 Z_n, \Delta_2 Z_n)$ as a sum of n i.i.d. random variables with zero means each distributed as $(n^{-\frac{1}{2}}\Delta_1 Z, n^{-\frac{1}{2}}\Delta_2 Z)$, we find

$$n^2 E(\Delta_1 Z_n^2 \Delta_2 Z_n^2) = nE(\Delta_1 Z^2 \Delta_2 Z^2) + n(n-1)(E\Delta_1 Z^2)(E\Delta_2 Z^2)$$
$$+ 2n(n-1)(E\Delta_1 Z \Delta_2 Z)^2.$$

Replacing f by $f\chi_{I_1}$ and f' by $f\chi_{I_2}$ in (6) shows

$$E(\Delta_1 Z \Delta_2 Z) = 0;$$

similarly we find

$$E(\Delta_i Z^2) = \int_{I_i} f^2 E\, Y(1-\Delta G)(1-F_-)^{-1} dF \le C\Delta_i F^\alpha$$

since $(1-F_-)^{-1}$ is bounded on $[0,\tau]$ and $\alpha < 1$. The constant C will be a different one on each appearance!

This leaves the term $E(\Delta_1 Z^2 \Delta_2 Z^2)$ to be dealt with. Now $\Delta_i Z =$ $= \int_{I_i} f dN - \int_{I_i} fYdG$; so expanding, dropping negative terms, and using $f \le 1$ gives

$$E(\Delta_1 Z^2 \Delta_2 Z^2) \le E(\Delta_1 N^2 \Delta_2 N^2) + E(\Delta_1 N^2 (\int_{I_2} YdG)^2$$
$$+ E(\Delta_2 N^2 (\int_{I_1} YdG)^2) + 4E(\int_{I_1} YdG \int_{I_2} YdG \Delta_1 N \Delta_2 N)$$
$$+ E((\int_{I_1} YdG)^2 (\int_{I_2} YdG)^2) .$$

Applying $\int_{I_i} YdG \le CY(0)\Delta_i F$ we look at the general term $E(\Delta_1 N^a \Delta_2 N^b Y(0)^c)$. Suppose $a > 0$ and $b > 0$; the case $a = 0$ or $b = 0$ is similar but easier.

$$E(\Delta_1 N^a \Delta_2 N^b Y(0)^c) \le E(\chi_{\{\Delta_1 N>0, \Delta_2 N>0\}} Y(0)^{a+b+c})$$

$$\le \{P(\Delta_1 N>0, \Delta_2 N>0)\}^\alpha \{EY(0)^{(a+b+c)/\beta}\}^\beta \qquad \alpha = 1 - \beta \in (0,1)$$

$$\le C.P(\Delta_1 N>0, \Delta_2 N>0)^\alpha$$

where we have used Hölder's inequality and the fact $Y(0) = \tilde{N}(T) + 1$. Now

$$P(\Delta_1 N>0, \Delta_2 N>0) \le \sum_{1\le j\ne k\le \ell<\infty} P(X_j \in I_1, X_k \in I_2, \sum_{i=1}^{\ell} X_i \le T, \sum_{i=1}^{\ell+1} X_i > T)$$

$$= \sum_{\ell=2}^{\infty} \ell(\ell-1) P(X_1 \in I_1, X_2 \in I_2, \sum_{i=1}^{\ell} X_i \le T, \sum_{i=1}^{\ell+1} X_i > T)$$

$$\le \sum_{\ell=2}^{\infty} \ell(\ell-1) P(X_1 \in I_1, X_2 \in I_2, \sum_{i=3}^{\ell} X_i \le T)$$

$$= \Delta_1 F \Delta_2 F \sum_{\ell=2}^{\infty} \ell(\ell-1) F^{(\ell-2)*}(T)$$

where $*$ denotes convolution: $F^{0*} \equiv 1$, $F^{1*} = F$, and $F^{\ell *}(t) = \int_0^t F^{(\ell-1)*}(t-s)dF^{1*}(s)$. But putting $m(t) = \sum_{\ell=0}^\infty F^{\ell *}(t) = E(\tilde{N}(t))+1$ we find

$$\sum_{\ell=0}^{\infty} (\ell+1) F^{\ell *}(t) = m^{2*}(t)$$

$$\sum_{\ell=0}^{\infty} \tfrac{1}{2}(\ell+1)(\ell+2) F^{\ell *}(t) = m^{3*}(t) \le m(t)^3 < \infty.$$

Substituting back, $E(\Delta_1 N^a \Delta_2 N^b Y(0)^c) \le C\Delta_1 F^\alpha \Delta_2 F^\alpha$; similarly $E(\Delta_i N^a Y(0)^c) \le C\Delta_i F^\alpha$. This shows that $E(\Delta_1 Z^2 \Delta_2 Z^2) \le C\Delta_1 F^\alpha \Delta_2 F^\alpha$ and so (10) has been proved. Choosing $\alpha > \frac{1}{2}$ proves tightness of Z_n on $D[0,\tau]$ in view of thm. 15.6 and the remarks on page 133 of BILLINGSLEY (1968). □

The following two lemmas, analogous to results of BRESLOW and CROWLEY

(1974), link Z_n with the product limit estimator in the case that F is continuous.

LEMMA 9. *Let* $\tau > 0$ *satisfy* $EY(\tau) > 0$, *let* F *be continuous. Then*

$$\sup_{t\in[0,\tau]} |n^{\frac{1}{2}}[-\log(1-\hat{F}_n(t))+\log(1-F(t))]-Z_n(t)| \overset{P}{\to} 0$$

where Z_n *is given by* (8) *with* $f(s) = (EY(s))^{-1}$.

PROOF. The proof is given in two steps.

Step 1. (c.f. BRESLOW and CROWLEY (1974) Lemma 1).
Let A_n be the event $\{Y_n(\tau) > 0\}$, note that $P(A_n) \to 1$ as $n \to \infty$. Substituting $x = \Delta N_n(s)/Y_n(s)$ in the elementary inequality $0 \le -\log(1-x)-x \le x^2(1-x)^{-1}$ for $x \in [0,1)$ we obtain a.s. on A_n for all $s \le \tau$

$$0 \le -\log\left(1-\frac{\Delta N_n(s)}{Y_n(s)}\right) - \frac{\Delta N_n(s)}{Y_n(s)} \le \frac{\Delta N_n(s)^2}{Y_n(s)(Y_n(s)-\Delta N_n(s))} \le Y_n(\tau)^{-2}$$

because a.s. N_n makes jumps of size 1 only and has no jump at τ. Adding over s and inserting the equality

$$-\log(1-F(t)) = \int_0^t dG(t) = \int_0^t \frac{dEN(s)}{EY(s)}$$

(corollary to Lemma 5 and (7)) we obtain a.s. on A_n: simultaneously for all $t \le \tau$

$$0 \le -\log(1-\hat{F}_n(t)) + \log(1-F(t)) - \int_0^t \frac{dN_n(s)}{Y_n(s)} + \int_0^t \frac{dEN(s)}{EY(s)} \le \frac{N_n(\tau)}{Y_n(\tau)^2}.$$

This implies

$$\sup_{t\in[0,\tau]} |n^{\frac{1}{2}}[-\log(1-\hat{F}_n(t))+\log(1-F(t))]-n^{\frac{1}{2}}[\int_0^t \frac{dN_n(s)}{Y_n(s)} - \int_0^t \frac{dEN(s)}{EY(s)}]|$$
$$\le n^{-\frac{1}{2}} \frac{n^{-1}N_n(\tau)}{n^{-2}Y_n(\tau)^2} \quad \text{a.s. on } A_n$$

and therefore the left hand member of this inequality converges in probability to zero.

Step 2. We symbollically use the equality

$$\frac{a+u}{b+v} - \frac{a}{b} - \frac{ub-av}{b^2} = - \frac{v(ub-av)}{b^2(b+v)}$$

for b, b+v $\neq 0$ with $a = d(EN)(s)$, $a + u = n^{-1}dN_n(s)$, $b = EY(s)$ and $b + v = n^{-1}Y_n(s)$. So,

$$\begin{aligned} ub - av &= EY(s)(n^{-1}dN_n(s) - dEN(s)) - (n^{-1}Y_n(s) - EY(s))dEN(s) \\ &= EY(s)n^{-1}dN_n(s) - EY(s)n^{-1}Y_n(s)dG(s) \\ &= n^{-\frac{1}{2}}EY(s)dW_n(s). \end{aligned}$$

This gives on A_n for $t \leq \tau$,

$$\begin{aligned} R_n(t) &\overset{\text{def}}{=} n^{\frac{1}{2}}\left[\int_0^t \frac{dN_n(s)}{Y_n(s)} - \int_0^t \frac{dEN(s)}{EY(s)}\right] - \int_0^t \frac{dW_n(s)}{EY(s)} = \\ &= -\int_0^t \frac{n^{-1}Y_n(s) - EY(s)}{EY(s)n^{-1}Y_n(s)}\, dW_n(s) = \int_0^t \left(\frac{1}{n^{-1}Y_n(s)} - \frac{1}{EY(s)}\right)dW_n(s) \\ &= W_n(t)(H_n(t) - H(t)) - \int_0^t W_n(s-)d(H_n(s) - H(s)) \end{aligned}$$

where

$$H_n(t) = (n^{-1}Y_n(t+))^{-1}$$

and

$$H(t) = (EY(t+))^{-1}.$$

In theorem 4 of BRESLOW and CROWLEY (1974) it is shown for a similar expression that $\sup_{t\in[0,\tau]} |R_n(t)| \overset{P}{\to} 0$ as $n \to \infty$ using the facts:

$\sup_{t\in[0,\tau]} |H_n(t)-H(t)| \overset{P}{\to} 0$; H and H_n nondecreasing, rt.cts. on $[0,\tau]$; and $W_n \overset{\mathcal{D}}{\to} W_\infty$ on $D[0,\tau]$ where W_∞ a.s. has continuous sample paths (in our case this property follows from Proposition 8 and BILLINGSLEY (1968) thm. 12.4, and uses the fact that F is continuous).

Combining steps 1 and 2 proves the lemma. □

LEMMA 10. *Let* $\tau > 0$ *satisfy* $EY(\tau) > 0$ *and let* F *be continuous. Then*

$$R_n = \sup_{t\in[0,\tau]} |n^{\frac{1}{2}}(\hat{F}_n(t)-F(t))-(1-F(t))n^{\frac{1}{2}}[-\log(1-\hat{F}_n(t)) + \\ + \log(1-F(t))]| \overset{P}{\to} 0.$$

PROOF. By Taylor expansion $|(-e^{\hat{x}}+e^{x})+e^{x}(\hat{x}-x)| \le e^{x}(\hat{x}-x)^2 e^{|\hat{x}-x|}$. Put $x = \log(1-F(t))$, $\hat{x} = \log(1-\hat{F}_n(t))$, where we restrict attention to $A_n = \{Y_n(\tau) > 0\}$ and $t \le \tau$. Writing $Z_n^*(t) = n^{\frac{1}{2}}(-\log(1-\hat{F}_n(t))+\log(1-F(t)))$ we obtain on A_n

$$R_n \le n^{-\frac{1}{2}} \sup_{t\in[0,\tau]} (1-F(t))Z_n^*(t)^2\left(\frac{1-F(t)}{1-\hat{F}_n(t)} \vee \frac{1-\hat{F}_n(t)}{1-F(t)}\right).$$

By Proposition 8 and Lemma 9 Z_n^* is tight and so $\sup_t |Z_n^*(t)|$ is bounded in probability; together with Theorem 6 (consistency of $\hat{F}_n$) this proves $R_n \overset{P}{\to} 0$. □

THEOREM 11. *Let* F *be continuous and* $\tau > 0$ *be such that* $EY(\tau) > 0$ *(equivalently* $\tau \le T$ *and* $F(\tau) < 1$*). Then*

$$\{n^{\frac{1}{2}}[-\log(1-\hat{F}_n(t))+\log(1-F(t))]\}_{t\in[0,\tau]} \overset{\mathcal{D}}{\to} Z_\infty$$

and

$$\{n^{\frac{1}{2}}[\hat{F}_n(t)-F(t)]\}_{t\in[0,\tau]} \overset{\mathcal{D}}{\to} \tilde{Z}_\infty$$

in $D[0,\tau]$, *where* Z_∞ *is a zero mean Gaussian process with independent increments:*

$$\mathrm{cov}(Z_\infty(t),Z_\infty(t')) = \int_0^{t\wedge t'} \frac{dG(s)}{EY(s)} = \int_0^{t\wedge t'} \frac{dEN(s)}{(EY(s))^2}$$

and

$$\tilde{Z}_\infty(t) = (1-F(t))Z_\infty(t) \quad \textit{for all} \quad t \in [0,\tau].$$

PROOF. The theorem is an immediate consequence of Proposition 8 and Lemmas 9 and 10. □

REMARKS. If $\sigma \le T$ is such that $F(\tau) < F(\sigma) = 1\ \forall\tau < \sigma$, extension of weak convergence to $\tilde{Z}_\infty$ on $[0,\tau]$ for each $\tau < \sigma$ to weak convergence on $[0,\sigma]$ is an open problem, whose solution is important for some applications. This is also the case in "random censorship".

The covariance structure of Z_∞ can be consistently estimated in the obvious way leading to the construction of asymptotic confidence bands for F, based on the fact that

$$\sup_{t\in[0,\tau]} \left|- \log(1-\hat{F}_n(t)+ \log(1-F(t))\right| \left(\int_0^\tau \frac{dN_n(t)}{Y_n(t)^2}\right)^{-\frac{1}{2}}$$

is asymptotically distributed as the supremum over $[0,1]$ of the absolute value of a standard Brownian motion (for this distribution see FELLER (1971) p.343).

APPENDIX 1: PROOF OF THEOREM 2

We present two propositions of some independent interest; theorem 2 then easily follows. The proofs can be used to generalise Lemma 3.1 of BOEL, VARAIYA and WONG (1975) to the discontinuous case. The conventions and definitions of section 2 are still in force.

PROPOSITION 12. *Let* N *be a counting process such that* $N(t) \le n$ *for all* t. *Then there exists a unique predictable process* $A \in A^+_{int}$ *such that* $M \overset{def}{=} N-A \in \mathcal{M}$. *The process* M *satisfies*

$$\sup_{t\in[0,\infty)} EM^2(t) < \infty \quad \textit{and} \quad \langle M,M\rangle = \int (1-\Delta A)dA.$$

PROOF. We work on the time interval $[0,\infty]$ defining $N(\infty) = \lim_{t\to\infty} N(t)$. The first part of the proposition follows if we let A be the dual predictable projection of N (MEYER (1976) chap. I n° 9). Define $H(t) = \chi_{\{\Delta A(t)>1\}}$. A_- is lt. cts., therefore predictable; so ΔA and finally H is predictable. H is bounded on $[0,\infty]$ and $EN(\infty) = EA(\infty) \le n$, so $\int H dM \in M$ on $[0,\infty]$ by Theorem 1. Taking the expectation at time ∞, we see $E(\sum_{s:\Delta A(s)>1}(\Delta N(s)-\Delta A(s))) = 0$. But a.s., $\Delta N(s) \le 1$ for all s, so a.s. $\Delta A(s) \le 1$ for all s.

Next we note that $A(\infty)^2 \le 2\int_0^\infty A(s)dA(s)$ and since A is nonnegative and predictable $E\int_0^\infty A(s)dA(s) = E\int_0^\infty A(s)dN(s)$ by definition of the dual predictable projection. The latter term is bounded by $nEA(\infty) < \infty$. This proves that $\sup_{t\in[0,\infty]} EM^2(t) < \infty$ and so $\langle M,M\rangle$ exists.

Now

$$M(t)^2 - \int_0^t (1-\Delta A(s))dA(s) = 2\int_0^t M(s-)dM(s) + \sum_{s\le t} \Delta M(s)^2 - A(t) + \sum_{s\le t} \Delta A(s)^2$$

$$= 2\int_0^t M(s-)dM(s) + \sum_{s\le t} \Delta N(s)^2 - 2\sum_{s\le t} \Delta N(s)\Delta A(s) - A(t) + 2\sum_{s\le t} \Delta A(s)^2$$

$$= 2\int_0^t M(s-)dM(s) + M(t) - 2\int_0^t \Delta A(s)dM(s).$$

Since $E\int_0^\infty N(s-)dN(s)$, $E\int_0^\infty N(s-)dA(s)$, $E\int_0^\infty A(s-)dN(s)$ and $E\int_0^\infty A(s-)dA(s)$ are all finite (use the obvious inequalities and for the last term the fact that $EA(\infty)^2 < \infty$), $\int M_- dM$ is a martingale (MEYER (1976) chap. I, n° 12). By boundedness of ΔA, $\int \Delta A dM$ is one too. Therefore $M(t)^2 - \int_0^t (1-\Delta A(s))dA(s)$ is a martingale. By the uniqueness of $\langle M,M\rangle$ it remains to show that $\int(1-\Delta A)dA \in A^+_{int}$ and is predictable. Membership of A^+_{int} is no problem; by the decomposition (3.1) of a predictable member of A^+_{int} given in MEYER (1976) chap. I, the process $\{\sum_{s\le t}\Delta A(s)^2\}_{t\in[0,\infty]}$ is predictable, and so $\int(1-\Delta A)dA$ is too. □

PROPOSITION 13. *Let* N *be a counting process such that for all* t, $EN(t) < \infty$. *Then there exists a unique predictable* $A \in A^+_{int}$ *such that* $M = N - A \in M$.

The process M^2 *is integrable and* $\langle M,M\rangle = \int(1-\Delta A)dA$.

PROOF. Let T_n be the time of the nth jump of N, a stopping time, and define $N_n(t) = t \wedge T_n$. $T_n \to \infty$ as $n \to \infty$ a.s. Let $A_n \in A^+_{int}$ be the predictable process whose existence is proved in Proposition 12; let $M_n = N_n - A_n$. By MEYER (1976) chap. I, n° 9 there exists a unique $A \in A^+_{int}$ such that $M = N - A$ is a martingale. So by uniqueness and optimal stopping, $A_n(t) = A(t\wedge T_n)$ on each bounded interval of $[0,\infty)$ and so on $[0,\infty)$ itself.

We also know $EM_n(t)^2 = E\int_0^t(1-\Delta A_n(s))dA_n(s) \le EA_n(t) = EN_n(t) \le EN(t)$. Now because $\{t\wedge T_n\}_{n\in\mathbb{N}}$ are increasing bounded stopping times $E(M_n(t) \mid F_{t\wedge T_m}) = M_n(t\wedge T_m) = M_m(t)$ for $m \le n$, so $\{M_n(t)^2\}_{n\in\mathbb{N}}$ is a positive submartingale w.r.t. $\{F_{t\wedge T_n}\}_{n\in\mathbb{N}}$, bounded in L_1. Therefore by DOOB (1953) VII Thm. 4.1 $M_n(t)^2$ converges a.s. *and* in L_1 as $n \to \infty$. But $M_n(t)^2 \to M(t)^2$ a.s.; so $EM(t)^2 < \infty$. Therefore $\langle M,M\rangle$ exists; but using again uniqueness and optional stopping we see $\langle M,M\rangle\,(t\wedge T_n) = \langle M_n,M_n\rangle(t)$ which proves $\langle M,M\rangle =$
$= \int(1-\Delta A)dA$. □

PROOF OF THEOREM 2.

By renewal theory $E\tilde{N}(t) < \infty$ for all t and so Proposition 12 is applicable. But by prop. 3.1 of JACOD (1975) or Prop.3 of CHOU and MEYER (1975), the process B defined by

$$B(t) = \sum_{i=1}^{\tilde{N}(t-)} G(X_i) + G(t-S_{\tilde{N}(t-)})$$

is such that, in the notation of the proof of Proposition 12,

$$A(t\wedge T_n) = A_n(t) = B(t\wedge T_n).$$

Because $T_n \to \infty$ a.s. this implies $A = B$. □

APPENDIX 2: PROOF OF LEMMA 5

Let $H(t) = \int_0^t \frac{dG_1(s)}{\bar{G}(s)}$. H is rt.cts., nondecreasing, zero at time zero. $H(\tau) < \infty$ and for $s < \tau$, $0 \le \Delta H(s) < 1$; so $\sup_{s\in[0,\tau)} \Delta H(s) < 1$ (it is possible that $\Delta H(\tau) = 1$). Then

$$\phi(G_1,\bar{G})(t) = 1 - \exp\left[-H(t) + \sum_{s\le t} \{\Delta H(s) + \log(1-\Delta H(s))\}\right] \tag{11}$$

and

$$\phi(G_1,\bar{G})(\tau) = 1 - (1-\phi(G_1,\bar{G})(\tau-))(1-\Delta H(\tau)). \tag{12}$$

Absolute convergence of the sum in (11) is proved below.

We first show that the mapping $(G_1,\bar{G}) \to H$ is continuous.

$$\begin{aligned} H(t)-H'(t) &= \int_0^t \frac{dG_1(s)}{\bar{G}(s)} - \int_0^t \frac{dG_1'(s)}{\bar{G}'(s)} = \int_0^t \frac{dG_1(s)-dG_1'(s)}{\bar{G}(s)} + \\ &+ \int_0^t \left(\frac{1}{\bar{G}(s)} - \frac{1}{\bar{G}'(s)}\right) dG_1'(s) = \\ &= \left[\frac{G_1(s)-G_1'(s)}{\bar{G}(s)}\right]_0^t - \int_0^{t-} (G_1(s)-G_1'(s))\,d\left(\frac{1}{\bar{G}_+(s)}\right) + \\ &+ \int_0^t \frac{\bar{G}'(s)-\bar{G}(s)}{\bar{G}(s)\bar{G}'(s)}\, dG_1'(s). \end{aligned}$$

So

$$\rho_\tau(H,H') \le 2\,\frac{\rho_\tau(G_1,G_1')}{\bar{G}(\tau)} + \frac{\rho_\tau(\bar{G},\bar{G}')(G_1(\tau)+\rho_\tau(G_1,G_1'))}{\bar{G}(\tau)(\bar{G}(\tau)-\rho_\tau(\bar{G},\bar{G}'))}$$

which proves continuity.

It remains to show that ϕ is continuous as a function of H. For fixed H define $T = T(\delta) = \{t < \tau: \Delta H(t) > \delta\}$, $\delta > 0$. So $\#T(\delta) < \delta^{-1}H(\tau)$. Let $\varepsilon > 0$ satisfy $2\varepsilon < 1 - \sup_{s<\tau} \Delta H(s)$ and suppose $\rho_\tau(H,H') < \varepsilon$.

$s \in [0,\tau)\backslash T(\delta) \Rightarrow \Delta H'(s) \le \delta + 2\varepsilon$. For all s, $|\Delta H(s)-\Delta H'(s)| < 2\varepsilon$. By Taylor expansion $0 \le -\{\Delta H(s)+\log(1-\Delta H(s))\} \le \frac{1}{2}\frac{\Delta H(s)^2}{(1-\Delta H(s))^2}$ so for $t < \tau$

(13) $$\Big|\sum_{s\in[0,t]\backslash T}\{\Delta H(s) + \log(1-\Delta H(s))\}\Big| \le \frac{1}{2}\sum_{s\in[0,\tau)}\frac{\Delta H(s)^2}{(1-\Delta H(s))^2}$$

$$\le \frac{1}{2}\,\delta H(\tau)(1-\sup_{s<\tau}\Delta H(s))^{-2} = R_1(\delta)$$

(Putting $\delta = 1$ proves absolute convergence of the sum in (11).)
Similarly

(14) $$\Big|\sum_{s\in[0,t]\backslash T}\{\Delta H'(s)+\log(1-\Delta H'(s))\}\Big|$$
$$\le \frac{1}{2}(\delta+2\varepsilon)(H(\tau)+\varepsilon)(1-\sup_{s<\tau}\Delta H(s)-2\varepsilon)^{-2} = R_2(\delta,\varepsilon).$$

By another Taylor expansion

$$|\{\Delta H(s) + \log(1-\Delta H(s))\}-\{\Delta H'(s) + \log(1-\Delta H'(s))\}|$$

$$\le |\Delta H(s) - \Delta H'(s)|\cdot\{1 + (1-\Delta H(s)\vee\Delta H'(s))^{-1}\}.$$

So for $t < \tau$

(15) $$\sum_{s\in T\cap[0,t]}|\{\Delta H(s) + \log(1-\Delta H(s))\}-\{\Delta H'(s) + \log(1-\Delta H'(s))\}|$$
$$\le \delta^{-1}H(\tau)\ 2\varepsilon(1 + (1-\sup_{s<\tau}\Delta H(s)-2\varepsilon)^{-1}) = R_3(\delta,\varepsilon).$$

Therefore combining (13), (14) and (15)

$$\sup_{t<\tau}\Big|\sum_{s\le t}\{\Delta H(s) + \log(1-\Delta H(s))\} - \sum_{s\le t}\{\Delta H'(s) + \log(1-\Delta H'(s))\}\Big|$$

$$\le R_1(\delta) + R_2(\delta,\varepsilon) + R_3(\delta,\varepsilon),$$

the right hand member of which can be made arbitrarily small by choice of δ and then ε. In view of (11) this shows continuity of $\log(1-\phi)$ as a function of H uniformly for $t \in [0,\tau)$. So ϕ is also continuous uniformly for $t \in [0,\tau)$, and by (12) uniformly for $t \in [0,\tau]$. □

REFERENCES

[1] AALEN, O.O., (1976), *Statistical inference for a Family of Counting Processes*, Inst. of Math. Stat., Univ. of Copenhagen, Copenhagen.

[2] AALEN, O.O., (1977), *Weak convergence of Stochastic Integrals Related to Counting Processes*, Z. Wahrscheinlichkeitstheorie und verw. Gebiete 38 261-277.

[3] AALEN, O.O., (1978), *Nonparametric Inference for a Family of Counting Processes*, to appear in Ann. Statist. 1978.

[4] BARLOW, R.E. & F. PROSCHAN, (1975), *Statistical Theory of Reliability and Life Testing: Probability Models*, Holt, Rinehart and Winston, New York.

[5] BILLINGSLEY, P., (1968), *Weak Convergence of Probability Measures*, Wiley, New York.

[6] BOEL, R., P. VARAIYA & E. WONG, (1975), *Martingales on Jump Processes I: Representation Results*, SIAM J. Control 13 999-1021.

[7] BRESLOW, N. & J. CROWLEY, (1974), *A Large Sample Study of the Life Table and Product Limit Estimates under Random Censorship*, Ann. Statist. 2 437-453.

[8] CHOU, C.S. & P.A. MEYER, (1975), *Sur la Representation des Martingales comme Integrales Stochastiques dans les Processus Ponctuels*, Lecture Notes in Mathematics 465 226-236. Springer-Verlag, Berlin.

[9] DOOB, J.L., (1953), *Stochastic Processes*, Wiley, New York.

[10] FELLER, W., (1971), *An Introduction to Probability Theory and its Applications*, Volume 2, 2nd edition, Wiley, New York.

[11] JACOD, J., (1975), *Multivariate Point Processes: Predictable Projection, Radon-Nikodym Derivatives, Representation of Martingales*, Z. Wahrscheinlichkeitstheorie und verw. Gebiete 31 235-253.

[12] KAPLAN, E.L. & P. MEIER, (1958), *Nonparametric Estimation from Incomplete Observations*, J. Amer. Statist. Assoc. 53 457-481.

[13] MEYER, P.A., &1976), *Un cours sur les Intégrals Stochastiques*, Lecture Notes in Mathematics 511 245-400 Springer-Verlag, Berlin.

[14] PETERSON, A.V., (1975), *Nonparametric Estimation in the Competing Risks Problem*, Techn. Rep. no. 73, Dept. of Statistics, Stanford University, Stanford.