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Note on the Computation of the Zeros of Functions Satisfying
a second Order Differential Equation.

by

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TECHNICAL NOTES AND SHORT PAPERS

Note on the Computation of the Zeros of Functions Satisfying a Second Order Differential Equation

By D. J. Hofsommer

It has been pointed out by P. Wynn [1] that, if a function satisfies a second order differential equation, this fact may be used with advantage in the computation of its zeros. In his note he only pays attention to Richmonds formula which, incidentally, was already known to Schröder [2]. We will elaborate his idea to construct another iteration formula.

Let $f(x)$ be the function, the roots of which are to be computed. Let α be such a root and let x be a first approximation. If the approximation is sufficiently close,

$$(1) \quad \alpha = x - f/f' - \frac{1}{2}(f''/f')(f/f')^2 - \frac{1}{6}[3(f''/f')^2 - f'''/f'](f/f')^3 + 0[(f/f')^4].$$

This series may be used either for direct computation in taking enough terms, or for obtaining an iterative process if only few terms are retained. If $f(x)$ satisfies the homogeneous differential equation

$$(2) \quad f'' = 2Pf' + Qf + 2S,$$

substitution in the series (1) yields

$$(3) \quad \alpha = x - f/f' + (P + S/f')(f/f')^2 \\ - \frac{1}{6}(4P^2 - P' + Q + 10PS/f' + S'/f' + 6S^2/f'^2)(f/f')^3 + 0[(f/f')^4]$$

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or,

$$(4) \quad \alpha = x - f/f' - (P + S/f')(f/f')^2 + O[(f/f')^3]$$

$$(5) \quad = x - \frac{1}{f'/f - P - S/f'} + O[(f/f')^3]$$

From (4) and (5) it follows that the iterative processes

$$(6) \quad x_{k+1} = x_k - f(x_k)/f'(x_k) - [P(x_k) + S(x_k)/f'(x_k)][f(x_k)/f'(x_k)]^2$$

or

$$(7) \quad x_{k+1} = x_k - \frac{1}{f'(x_k)/f(x_k) - P(x_k) - S(x_k)/f'(x_k)}$$

are cubically convergent.

Wynn's process gives

$$x_{k+1} = x_k - \frac{1}{f'(x_k)/f(x_k) - P(x_k) - S(x_k)/f'(x_k) - \frac{1}{2}Q(x_k)f(x_k)/f'(x_k)}$$

if $f(x)$ satisfies (2).

From equality (7) it is seen that the last term in the denominator may be left out without affecting the order of the iteration process.

For comparison we have calculated by aid of (7) the first three zeros of J_0 with one iteration step. The results are

x_0	x_1	x_1 (Wynn)
2.405	2.404 825 557 694	2.404 825 557 697
5.520	5.520 078 110 286	5.520 078 110 286
8.654	8.653 727 912 904	8.653 727 912 914

Of course one iteration step amounts to using the development (4) or (5). If in such a case the required accuracy is almost, but not quite, obtained, application of (3) is preferable to taking a second iteration step.

In such a way we have calculated the zeros of $P_{37}(\cos \varphi)$. Using Smith's table [3] the zeros φ_m were found in hundredths of a degree. Then new values for φ_m were calculated by aid of (3) or (4) using the NBS tables [4]. In this way interpolation was avoided. We found

m	φ_m (radians)	m	φ_m (radians)
0	1.570 796 326 8	9	0.816 897 487 6
1	1.487 027 983 4	10	0.733 136 902 8
2	1.403 259 745 5	11	0.649 379 438 3
3	1.319 491 725 5	12	0.565 626 516 9
4	1.235 724 047 9	13	0.481 880 536 8
5	1.151 956 859 3	14	0.398 145 883 1
6	1.068 190 338 7	15	0.314 431 540 1
7	0.984 424 715 0	16	0.230 759 304 7
8	0.900 660 291 7	17	0.147 197 715 6
		18	0.064 126 781 0

The approximation in hundredths of a degree could have been obtained also by

starting with the approximation for the zeros of $P_n(\cos \varphi)$

$$\begin{aligned}\varphi_m &\doteq \left(\frac{1}{2} - \frac{2m}{2n+1}\right)\pi, & n \text{ odd} \\ \varphi_m &\doteq \left(\frac{1}{2} - \frac{2m+1}{2n+1}\right)\pi, & n \text{ even}\end{aligned}$$

and applying one Newton step $x_{k+1} = x_k - f(x_k)/f'(x_k)$.

Finally we remark that if $f(x)$ is a member of a set $f_n(x)$, it often happens that $f'_n(x)$ satisfies a recurrence relation

$$f'_n(x) = A(x)f_{n-1}(x) + B(x)f_n(x).$$

It then may be useful to transform

$$f'_n(x_k)/f_n(x_k) = A(x_k)f_{n-1}(x_k)/f_n(x_k) + B(x_k),$$

especially if $f_n(x)$ is tabulated and $f'_n(x)$ not or if the tables of $f_n(x)$ are more extensive than those of $f'_n(x)$.

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1. P. WYNN, "On a cubically convergent process for determining the zeros of certain functions," *MTAC*, v. 10, 1956, p. 97-100.
2. E. SCHRÖDER, "Ueber unendlich viel Algorithmen zur Auflösung der Gleichungen," *Math. Ann.*, v. 2, 1870, p. 317-363.
3. E. R. SMITH & ARCHIE HIGDON, "Zeros of the Legendre polynomials," *Jn. Science*, Iowa State College, v. 12, 1938, p. 263-274. (*MTAC*, Rev. 132, v. 1, 1944, p. 153-514.)
4. NBSCL, *Table of Sines and Cosines to Fifteen Decimal Places at Hundredths of a Degree* (*NBS Applied Math. Series*, no. 5), Govt. Printing Office, Washington, 1949. (*MTAC*, Rev. 662, v. 3, 1949, p. 515-516.)