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A note on a problem of heat transport

by

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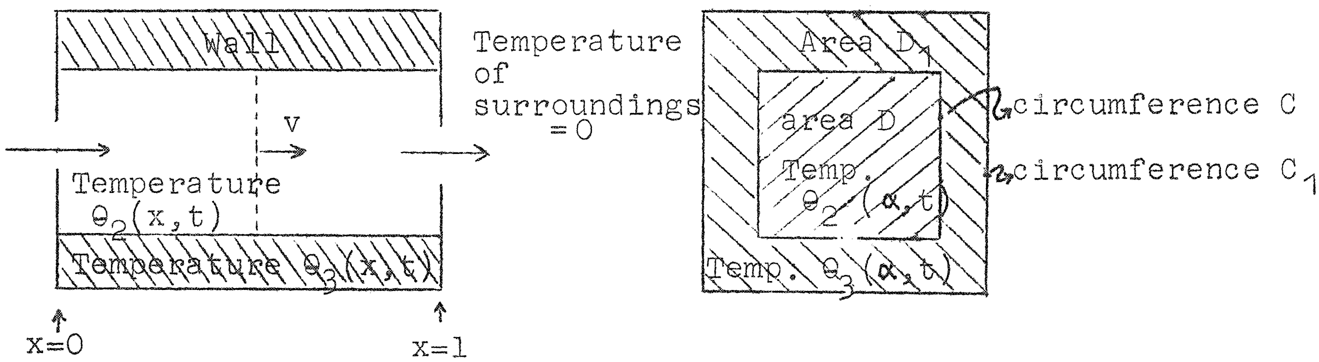
The mathematical model under consideration describes the heating of a railroadcar by hot air, which is blown into it at one end (x=0) and which leaves the car at the opposite end (x=1).**



Longitudinal section Cross section at $x=\alpha$

For the temperature of the air entering the car, $\theta_1(t)$, we have

(1) $\theta_1(t)$ = arbitrary given function of time t.

(For the particular case $\theta_1(t) = A(1-e^{-\lambda t})$ see (15) seqq.)

The air inside the car is thought of as moving in the x direction only, with a constant speed v. It is assumed that inside the car there is no temperature gradient normal to the x direction, so that for the temperature θ_2 inside the car we have $\theta_2=\theta_2(x,t)$.

For x=0 we have the boundary condition

(2) $$\theta_2(0,t) = \theta_1(t)$$

The car loses heat to the wall, which again gives off heat to the environment.

The temperature inside the wall, θ_3, is assumed to be a function of x and t, $\theta_3=\theta_3(x,t)$, so that inside the wall

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**) For a list of the symbols used see p.7.

there is no temperature gradient normal to the x direction. Furthermore we assume that inside the wall no transport of heat takes place in the x direction. The local loss of heat from the car to the wall is taken to be proportional to $\Theta_2(x,t) - \Theta_3(x,t)$, that from the wall to the surroundings proportional to $\Theta_3(x,t)$.
The proportionality factors are q_{23} and q_{30} respectively per unit of area and per unit of time.
The constant temperature of the surroundings is taken as zero.
For the circumferences C and C_1 and the areas D and D_1 see the cross section.
The specific heat of air is q_2, the specific heat of the wall is q_3.
For t=0 we have the initial conditions

(3) $\Theta_2(x,0) = 0$ and $\Theta_3(x,0) = \Theta_{30}(x)$.

We now introduce the constants

(4) $$a = \frac{Cq_{23}}{Dq_2}, \quad b = \frac{Cq_{23}}{D_1q_3}, \quad c = \frac{C_1q_{30}}{D_1q_3}.$$

This gives us the following simultaneous equations:

(5) $$\frac{\partial\Theta_2(x,t)}{\partial t} + v\,\frac{\partial\Theta_2(x,t)}{\partial x} = -a(\Theta_2(x,t)-\Theta_3(x,t))$$

(6) $$\frac{\partial\Theta_3(x,t)}{\partial t} = b(\Theta_2(x,t)-\Theta_3(x,t)) - c\Theta_3(x,t).$$

By applying Laplace transformation to (5) and (6), and using the initial conditions (3), we get the system

(5a) $$p\overline{\Theta}_2(x,p) + v\,\frac{\partial\overline{\Theta}_2(x,p)}{\partial x} = -a(\overline{\Theta}_2(x,p)-\overline{\Theta}_3(x,p))$$

(6a) $$p\overline{\Theta}_3(x,p)-\Theta_{30}(x) = b(\overline{\Theta}_2(x,p)-\overline{\Theta}_3(x,p)) - c\overline{\Theta}_3(x,p),$$

in which p is the variable of Laplace transformation and the bar indicates the Laplace transform.
Elimination of Θ_3 gives

(7) $$v\,\frac{\partial\overline{\Theta}_2}{\partial x} + \frac{p^2+p(a+b+c)+ac}{p+b+c}\,\overline{\Theta}_2 = \frac{a\,\Theta_{30}}{p+b+c}.$$

The solution of this differential equation is

(8) $$\overline{\Theta}_2(x,p)=\overline{K}(p)\ \exp\left\{-\frac{p^2+p(a+b+c)+ac}{v(p+b+c)}\,x\right\}+\frac{a\Theta_{30}}{p^2+p(a+b+c)+ac}$$

in which $\overline{K}(p)$ is a function of p which has to be determined from the boundary condition (2).

From (2) we see that

(9) $$\overline{\Theta}_2(0,p) = \overline{\Theta}_1(p).$$

From (8) we get

(10) $$\overline{\Theta}_2(0,p) = \overline{K}(p) + \frac{a\Theta_{30}}{p^2+p(a+b+c)+ac} .$$

Denoting the inverse of $\frac{a\Theta_{30}}{p^2+p(a+b+c)+ac}$ by T(t) we have

(11) $$T(t)=\begin{cases} H(t)\dfrac{a\Theta_{30}}{\sqrt{(\frac{a+b+c}{2})^2-ac}}\,e^{-\frac{(a+b+c)}{2}t}\,sh\left(t\sqrt{(\frac{a+b+c}{2})^2-ac}\right) & \text{for } (a+b+c)^2>4ac \\ H(t)\dfrac{a\Theta_{30}}{\sqrt{ac-(\frac{a+b+c}{2})^2}}\,e^{-\frac{a+b+c}{2}t}\,\sin\left(t\sqrt{ac-(\frac{a+b+c}{2})^2}\right) & \text{for } (a+b+c)^2<4ac \\ H(t)a\Theta_{30}te^{-\frac{a+b+c}{2}t} & \text{for } (a+b+c)^2=4ac \end{cases}$$

In these formulae H(t) is Heaviside's unit step function

$$H(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$$

We can easily find K(t), the inverse Laplace transform of $\overline{K}(p)$, from (9), (10) and (11) as

(12) $$K(t) = \left\{\Theta_1(t)-T(t)\right\}\ H(t)$$

For the exponential factor in (8) we have

(13) $\overline{U}(p) \overset{\text{def}}{=} \exp\left\{-\frac{p^2+(a+b+c)p+ac}{v(p+b+c)}x\right\} = e^{-\frac{ax}{v}} \exp\left\{\left(-\frac{p}{v}+\frac{ab/v}{p+b+c}\right)x\right\}.$

Erdélyi et al.: Tables of integral transforms I section 5.5 formula (31) gives:

$e^{\frac{a}{p}}-1 \doteqdot \sqrt{\frac{a}{t}}\; I_1(2\sqrt{at})$,

so that the inverse transform of $\overline{U}(p)$ is

(14) $U(t)=e^{-\frac{ax}{v}}\Big\{\delta(t-\frac{x}{v}) +$

$+ e^{-(b+c)(t-\frac{x}{v})} \sqrt{\frac{abx}{vt-x}}\; I_1\left(2\sqrt{\frac{abx}{v}(t-\frac{x}{v})}\right) H(t-\frac{x}{v})\Big\}$,

We thus find the following result for $\theta_2(x,t)$, in which the symbol $*$ denotes convolution:

(15) $\theta_2(x,t) = K(t) * U(t) + T(t)$.

For $\theta_3(x,t)$ we have from (6) together with the boundary condition (3):

(16) $\theta_3(x,t) = \theta_{30}\, e^{-(b+c)t} + \frac{b}{b+c}\,\theta_2(x,t)$.

A particular case

We now use the heating function

(17) $\theta_1(t) = A\,(1-e^{-\lambda t})$.

in which A and λ are positive constants.
Then (12) becomes

(18) $K(t) = \left\{A(1-e^{-\lambda t}) - T(t)\right\} H(t)$.

For $\theta_2(x,t)$ we find from (15), (18) and (14)

(19) $\theta_2(x,t)= T(t)+e^{-\frac{ax}{v}} \int_0^t \left\{A(1-e^{-\lambda(t-\tau)})-T(t-\tau)\right\} H(t-\tau)\,.$

$.\left\{e^{-(b+c)(\tau-\frac{x}{v})} \sqrt{\frac{abx}{v\tau-x}}\; I_1\left(2\sqrt{\frac{abx}{v}(\tau-\frac{x}{v})}\right)H(\tau-\frac{x}{v}) + \delta(\tau-\frac{x}{v})\right\} d\tau.$

For $t < \frac{x}{v}$ we have obviously

(20) $\theta_2(x,t) = T(t)$.

Assuming now $t > \frac{x}{v}$, (19) may be reduced to

(21) $$\theta_2(x,t) = T(t) + e^{-\frac{ax}{v}}\left\{A\left(1-e^{-\lambda(t-\frac{x}{v})}\right) - T(t-\frac{x}{v})\right\} +$$

$$+ Ae^{(-a+b+c)\frac{x}{v}} \int_{x/v}^{t} e^{-\tau(b+c)} \sqrt{\frac{abx}{v\tau - x}}\, I_1\left(\frac{2}{v}\sqrt{abx(v\tau-x)}\right) d\tau +$$

$$- Ae^{(-a+b+c)\frac{x}{v} - \lambda t} \int_{x/v}^{t} e^{-\tau(b+c-\lambda)} \sqrt{\frac{abx}{v\tau - x}}\, I_1\left(\frac{2}{v}\sqrt{abx(v\tau-x)}\right) d\tau +$$

$$- e^{(-a+b+c)\frac{x}{v}} \int_{x/v}^{t} e^{-\tau(b+c)} T(t-\tau) \sqrt{\frac{abx}{v\tau - x}}\, I_1\left(\frac{2}{v}\sqrt{abx(v\tau-x)}\right) d\tau,$$

which may be simplified to

(22) $$\theta_2(x,t) = T(t) + e^{-\frac{ax}{v}}\left\{A\left(1-e^{-\lambda(t-\frac{x}{v})}\right) - T(t-\frac{x}{v})\right\} +$$

$$+ Ae^{-\frac{ax}{v}} \int_0^{\varphi(x,t)} e^{-\frac{v(b+c)}{4abx}w^2} I_1(w)\,dw +$$

$$- Ae^{-\frac{ax}{v} - \lambda(t-\frac{x}{v})} \int_0^{\varphi(x,t)} e^{-\frac{v(b+c-\lambda)}{4abx}w^2} I_1(w)\,dw +$$

$$- e^{-\frac{ax}{v}} \int_0^{\varphi(x,t)} e^{-\frac{v(b+c)}{4abx}w^2} T\left(t - \frac{x}{v} - \frac{vw^2}{4abx}\right) I_1(w)\,dw$$

in which $\varphi(x,t) \stackrel{\text{def}}{=} \frac{2}{v}\sqrt{abx(vt-x)}$.

The temperature $\theta_3(x,t)$ follows from (16).

For $t \to \infty$ the fourth term in the right-hand side of (22) converges even for $\lambda > b+c$ by virtue of the factor $e^{-\lambda t}$ with which the integral is multiplied. The other terms give no difficulties, which means that eventually a steady state is reached.

We are now going to determine the behaviour of the solution $\theta_2(x,t)$ as $t \to \infty$.

Since a steady state is reached, we may take $\frac{\partial \theta_2}{\partial t} = \frac{\partial \theta_3}{\partial t} = 0$ in (5) and (6), which then become

$$(23)\quad \begin{cases} v\,\dfrac{\partial \theta_2}{\partial x} = -a(\theta_2 - \theta_3) \\ \\ b\theta_2 = (b+c)\theta_3. \end{cases}\quad (24)$$

Elimination of θ_3 gives

$$(25)\quad v\,\frac{\partial \theta_2}{\partial x} = -\frac{ac}{b+c}\,\theta_2$$

which, together with the boundary condition (2), leads to

$$(26)\quad \theta_2(x,t) \to A \exp\left\{-\frac{acx}{v(b+c)}\right\} \qquad \text{for } t \to \infty .$$

For the steady-state solution of $\theta_3(x,t)$ we find from (24) and (26)

$$(27)\quad \theta_3(x,t) \to \frac{Ab}{b+c} \exp\left\{-\frac{acx}{v(b+c)}\right\} \qquad \text{for } t \to \infty .$$

List of symbols:

v: speed of the air inside the car.

x: the variable of place.

t: the variable of time.

l: length of the car.

p: the variable of Laplace transformation.

$\Theta_2(x,t)$: the temperature of the air inside the car.

$\Theta_3(x,t)$: the temperature of the wall.

$\Theta_{30}(x)$: the initial temperature of the wall.

$\Theta_1(t)$: the temperature of the air which is blown into the car at $x=0$.

A, λ : constants in the equation of $\Theta_1(t)$.

q_{23}: the loss of heat from the inside of the car to the wall is proportional to q_{23} per unit of area and per unit of time.

q_{30}: the loss of heat from the wall to the surroundings is proportional to q_{30} per unit of area and per unit of time.

q_2: specific heat of air.

q_3: specific heat of wall.

C: inner circumference of cross section of car wall.

C_1: outer " " " " " " "

D: area of cross section of car interior.

D_1: " " " " " wall.

a: $\frac{Cq_{23}}{Dq_2}$, b: $\frac{Cq_{23}}{D_1q_3}$, c: $\frac{C_1q_{30}}{D_1q_3}$.

$H(t)$: Heaviside's unit step function $H(t)=\begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}$

The symbol $\doteqdot$ indicates correspondence in Laplace transformation.

$\left.\begin{matrix}\overline{\Theta}_2(x,p) \\ \overline{\Theta}_3(x,p)\end{matrix}\right\}$: the bar indicates the Laplace transform of the function.