STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

AFDELING TOEGEPASTE WISKUNDE

Technical Note TN 33

On the optimum programming of the heat flow equation with special reference to soil consolidation problems

by

M.B. Abbott

August 1963

SUMMARY:

It is observed that in many consolidation problems with equal time steps the isochrones become closer and closer together in time, so that the amount of additional information provided per time step, in the form of additional settlement, becomes progressively less. This behaviour is in fact common to all solutions of the heat flow equation. The present note describes a simple procedure for choosing the time steps such that the settlement is approximately the same at each step.

For the investigation of decreasing rate of settlement with time, a phase space is used in which each point represents a state of the pore-pressure system. The metric of this space is chosen such that the distance between any two representative points in the space is identical to the settlement occurring between the two corresponding pore-pressure states.

The rate of consolidation is equal to the net rate of loss of water, which in the case of a homogeneous soil is itself proportional to the difference between the boundary pore-pressure gradients. By using these gradient it is proved that the distance between successive (equal time-step) image points decreases with time in all common cases, with some variation with choice of difference scheme, thus demonstrating the general nature of the observed behaviour of the isochrones.

From this investigation a simple procedure for obtaining approximately uniform steps in phase space, and thus in consolidation, is determined for linear and non-linear, single and multilayered soils. Using this procedure the amount of computingmachine time used in a settlement problem may be reduced by a factor of between 3 and 10. The procedure may be applied to the numerical solution of all equations of the heat flow type.

THE PHASE SPACE OF A CONSOLIDATION PROBLEM

The properties of decreasing rate of settlement experienced in consolidation problemscan be especially easily visualised, and subsequently precisely formulated, in terms of a phase space, D^J. This is a space in which each coordinate is associated with a product of three terms: the excess pore pressure at each grid point, the local value of the coefficient of volume decrease, and the area of the region to which each grid value is supposed to apply. When equidistant grid points are employed, the last of these terms is associated with the distance between grid points and is, of course, the same for all j and k. Thus the elements of the space D^J are the set of all possible values of

$$p_j = u_j m_j \, \mu(z : z \in Z, \, u(z) = u_j) \qquad (1)$$

This reads: "the product of u and m at the j^{th} grid point ($= u_j m_j$) multiplied by the measure of z upon which u takes the value u_j, z always being an element of the region Z upon which the excess pore pressure is defined". This formulation is illustrated in fig. 1.

It is seen that this representation of the instantaneous state of a pore pressure system corresponds to a transformation of that state into a single point in D^J. This single representative point is accordingly called the image point of the instantanous state. Since the image point corresponding to any particular instant comprehends completely the state of the physical system at that instant, and since that state uniquely determines all future states, then the initial location of the image point in itself uniquely determines the location of all succeeding image points. These succeeding points define a trajectory, so that each and every point in D^J defines a unique trajectory while, conversely, through each point in D^J passes one and only one trajectory. These concepts are schematized in fig. 2.

The complete definition of any space involves not only the set of elements of the space: it is also necessary to define some measure of distance or metric, in the space. The distance between two points $p^{(1)}(=p_1^{(1)}, p_2^{(1)}, \ldots, p_j^{(1)}, \ldots, p_J^{(1)})$ and $p^{(2)}(=p_1^{(2)}, p_2^{(2)}, \ldots, p_j^{(2)}, \ldots, p_J^{(2)})$, written as $\rho(p^{(1)}, p^{(2)})$, will be defined in the present work by

$$\rho(p^{(1)}, p^{(2)}) = \sum_{j=1}^{J} (p_j^{(1)} - p_j^{(2)}) \qquad (2)$$

where $p^{(1)}$ necessarily represents a less consolidated state then $p^{(2)}$. This space is an example of a <u>metric space</u> which is clearly not a Euclidean space and in the sequel it is this ensemble of elements and metric that will be represented by D^J. The metric (2) is introduced because of the physical meaning which is thereby given to distance in D^J, for substituting (1) in (2) shows that

$$\rho(p^{(1)}, p^{(2)}) = \sum_{j=1}^{J} \{ u_j^{(1)} m_j^{(1)} \mu(z : z \in Z, u(z) = u_j^{(1)}) - u_j^{(2)} m_j^{(2)} \mu(z : z \in Z, u(z) = u_j^{(2)}) \}$$

This corresponds, with $\mu_j^{(1)} = \mu_j^{(2)}$, to a Lebesgue integral:

$$\int_Z \{ (um)^{(1)} - (um)^{(2)} \} d\mu = \sum_{j=1}^{J} \{ u_j^{(1)} m_j^{(1)} - u_j^{(2)} m_j^{(2)} \} \mu(z : z \in Z, u(z) = u_j^{(1)}, u(z) = u_j^{(2)} \}$$

The corresponding Riemann integral is given, with $\mu_j^{(1)} = \mu_j^{(2)} = \delta z$ for all j, by

$$\rho(p^{(1)}, p^{(2)}) = \int_Z \{ (um)^{(1)} - (um)^{(2)} \} dz \qquad (3)$$

which is identical to the consolidation occurring between isochrones $(u)^{(1)}$ and $(u)^{(2)}$ with m varying linearly from $(m)^{(1)}$ to $(m)^{(2)}$.

THE RATE OF SETTLEMENT IN TERMS OF THE BOUNDARY PORE-PRESSURE GRADIENTS

For this investigation only the one-dimensional form of the consolidation equation will be considered in detail. The two and three dimensional forms may, however, be treated in an analogous manner if due account is taken of induced shear stresses (Josselin de Jong, 1963). The consolidation equation is then:

$$\frac{\partial u}{\partial t} = \frac{k}{\gamma\omega^{m}} \frac{\partial^2 u}{\partial z^2} = c \frac{\partial^2 u}{\partial z^2} \qquad (4)$$

where $k = k(u)$ and $m = m(u)$. For the purpose of deriving the present results k and m may be treated as constants: it will later be shown that their variation cannot in fact influence these results.

Now, in practice (4) may be approximated by one of several finite-difference relations which, although they generate a solution that deviates slightly from that generated by (4), are ultimately consistent with (4) in that this deviation becomes progressively less with time. Putting aside for the moment all questions of numerical stability, two common types of difference operator may be represented as in fig. 3, and in the following difference equation:

$$\theta(u_{j+1}^{k+1} - 2u_j^{k+1} + u_{j-1}^{k+1}) + (1-\theta)(u_{j+1}^{k} - 2u_j^{k} + u_{j-1}^{k})$$

$$= \frac{(\Delta z)^2}{c(\Delta t)} (u_j^{k+1} - u_j^{k}) \qquad (5)$$

The explicit operator illustrated in fig. 3a then corresponds to equation (5) with $\theta = 0$ while the implicit operator of fig. 3b corresponds to equation (5) with $\theta = 1$. Representing $\frac{c(\Delta t)}{(\Delta z)^2}$ as r the computation with $\theta = 0$ will proceed, at each step, as follows:

$$
\begin{aligned}
u_o^{k+1} &= u_o^k \\
u_1^{k+1} &= ru_o^k + (1-2r)u_1^k + ru_2^k \\
u_2^{k+1} &= \qquad ru_1^k + (1-2r)u_2^k + ru_3^k \\
u_3^{k+1} &= \qquad\qquad ru_2^k + (1-2r)u_3^k + ru_4^k \\
&\vdots \qquad\qquad\qquad \ddots \\
u_{J-2}^{k+1} &= \qquad\qquad\qquad ru_{J-3}^k + (1-2r)u_{J-2}^k + ru_{J-1}^k \\
u_{J-1}^{k+1} &= \qquad\qquad\qquad\qquad ru_{J-2}^k + (1-2r)u_{J-1}^k + ru_J^k \\
u_J^{k+1} &= \qquad\qquad\qquad\qquad\qquad u_J^k
\end{aligned}
$$

so that

$$\sum_{j=0}^{J} u_j^{k+1} = (r+1)u_o^k + (1-r)u_1^k \qquad + (1-r)u_{J-1}^k + (r+1)u_J^k$$

Therefore, from (2) ,

$$\underline{\rho_o(u^k, u^{k+1}) = mr\ \{(u_1^k - u_o^k) - (u_J^k - u_{J-1}^k)\}\Delta z} \qquad (6)$$

The physical interpretation of this is quite simple, for (6) can be written as:

$$\rho_o(u^k, u^{k+1}) = \frac{k}{\gamma\omega} \quad \Delta t \ \{(\frac{u_1^k - u_o^k}{\Delta z}) - (\frac{u_J^k - u_{J-1}^k}{\Delta z})\}$$

and the term

$$k \ \{\frac{u_1^k - u_o^k}{\Delta z} - \frac{u_J^k - u_{J-1}^k}{\Delta z}\}$$

is evidently the net velocity of water leaving the soil. This, under the assumptions of the theory, is indeed identical to the rate of consolidation.

If the above procedure is repeated for the case where $\theta = 1$ it is readily seen that

$$\rho_1(u^k,u^{k+1}) = mr \{(u_1^{k+1} - u_o^{k+1})-(u_J^{k+1} - u_{J-1}^{k+1})\} \Delta \qquad (7)$$

or $$\rho_1(u^k,u^{k+1}) = \rho_o(u^{k+1},u^{k+2}). \qquad (8)$$

Other values of θ, $0 \leq \theta \leq 1$ will evidently give results intermediate between those given in (6) and (7) so that the influence of different values of θ, apart from their effect on stability, is seen to be only to slightly advance or retard the generation of the solution.

The most common pore-pressure distributions are illustrated in fig. 4 where it is readily seen that the quantity

$$\{(u_1^k-u_o^k) - (u_J^k-u_{J-1}^k)\}$$

decreases with time for all these distributions.

Accordingly

$$\underline{\rho(u^k,u^{k+1}) < \rho(u^{k-1},u^k)} \qquad (9)$$

for all k. It is obvious from physical considerations alone that (6), (7) and (9) will hold in non-homogeneous soils if the appropriate values of k are inserted with the gradients in (6) and (7), while with the additional insertion of m values they will also hold for non-linear soil behaviour. In the case of two-dimensional consolidation, neglecing shear release, all the boundary gradients will be involved, but otherwise the procedure will be the same, with the inequality (9) remaining unchanged.

The relation expressed in (9) can be stated as follows:

<u>The sequence $\{\rho(u^k,u^{k+1})\}$ is fundamental in k</u> (10)

Thus if the computation were continued with $k \to \infty$, and if the computer used possessed an infinite "bit", the sequence of points $\{u^k\}$ in D^J would converge towards a limit point $\bar{u}$. This point, which would correspond to the final state of equilibrium of the system, would not however be an element of $\{u^k\}$, as $\rho(u^k,\bar{u})$ could never become identically zero. The set $\{u^k\}$ is thus an open set having as its closure the set $(\bigcup_k u^k) \cup \bar{u}$. This situation is again schematized in fig. 2

Relation (9), or (10), is seen to apply for all cases in which the time step Δt remains constant. The object of the present scheme, on the other hand, is, as far possible, to change the inequality of (9) into an equality. Now if the implicit scheme corresponding to $\theta=1$ were used, so that

$$\rho(u^k,u^{k+1}) = r^k \{(u_1^{k+1}-u_o^{k+1}) - (u_J^{k+1}-u_{J-1}^{k+1})\}$$

then setting

$$r^k = \sigma^k(\Delta t)^k$$

with

$$(\Delta t)^k = \{\frac{(u_1^k-u_o^k)-(u_J^k-u_{J-1}^k)}{(u_1^{k+1}-u_o^{k+1})-(u_J^{k+1}-u_{J-1}^{k+1})}\}\ (\Delta t)^{k-1} \qquad (11)$$

would indeed give

$$\rho(u^k,u^{k+1}) = \sigma^k \{(u_1^k-u_o^k)-(u_J^k-u_{J-1}^k)\} \quad (\Delta t)^{k-1}$$

so that if $\sigma^k = \sigma^{k-1}$,

$$\rho(u^k,u^{k+1}) = \rho(u^{k-1},u^k) = \rho(u^{k-2},u^{k-1})$$

$$\ldots\ldots = \rho(u^o,u^1).$$

A similar result follows for $\theta = 0$ when

$$(\Delta t)^k = \{ \frac{u_1^{k-1}-u_o^{k-1})-(u_J^{k-1}-u_{J-1}^{k-1}}{(u_1^k-u_o^k)-(u_J^k-u_{J-1}^k)} \} \quad (\Delta t)^{k-1} \qquad (12)$$

Other values of θ would then lead to relations intermediate between (11) and (12). Of all these relations, however, only (12) can be computed at the k^{th} time step, i.e. only (12) is explicit.

It is therefore proposed that (12) be used for all values of θ. This would not then provide an exactly equal consolidation step for all θ, but the maximum deviation from this equal step would not be too large. This deviation would in fact become a maximum when $\theta = 1$ when the amplification in time step actually employed would be that properly appropriate to the preceding step in the computation.

In the event of k varying during consolidation the appropriate values of k **must** be entered in (12), while in the case of multi-layered soils the elementary differences in (12) will take different k values.

TEST APPLICATION OF THE OPTIMISING PROCEDURE

It is proposed to apply this procedure to a consolidation computation described previously (Abbott, 1960). In this case a value of Δt = 15 days, approximately, was employed so as to obtain reasonable accuracy at the beginning of the computation. The computation was continued to 360 days, so that approximately 24 time steps were employed. For the purposes of computing settlements, however, some 6 to 8 sets of pore-pressures would have sufficed, so that using the optimising procedure should reduce the machine-time used by a factor of 3 or 4. Generally, however, much longer consolidation times than these are required in practice, so that a greater saving should then be expected, and indeed a reduction in machine time of up to 10 times might be expected.

===========

References.

1) Abbott, M.B. — The one-dimensional consolidation of multi-layered soils. Géotechnique, Dec. 1960.

2) Abbott, M.B. et al. — On the non-linear consolidation of multi-layered soils. To be published.

3) de Josselin de Jong, G. — Consolidatie in drie dimensies. L.G.M. Mededelingen. Laboratorium voor Grondmechanica, Delft, 1963.