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Numerical treatment of the North Sea Problem without friction

by

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Introduction

The North Sea Problem may be approximately solved by the following set of equations [4]

$$(1) \qquad \begin{cases} u_t = -\lambda u + \Omega v - gh\,\zeta_x + U \\ v_t = -\Omega u - \lambda v - gh\,\zeta_y + V\,, \\ \zeta_t = -u_x - u_y \end{cases}$$

with the notation

- u,v the components of the total stream in the x and y direction respectively,
- ζ the elevation of the water surface,
- U,V the surface stress due to the windfield,
- λ a coefficient of friction,
- Ω the coefficient of Coriolis,
- g constant of gravity,
- h depth function.

In [3] we constructed a class of difference schemes by introducing a viscosity term into the scheme used by H.A. Lauwerier and B.R. Damsté [4]. A special case, which is suitable for numerical calculations runs as follows

$$(2) \qquad \begin{cases} u_{k+1} = (1-\lambda\tau)u_k + \Omega\tau v_k - gh\,\tau\,D_x\,\zeta_k + \tau U_k \\ v_{k+1} = -\Omega\tau u_k + (1-\lambda\tau)v_k - gh\,\tau\,D_y\,\zeta_k + \tau V_k\,, \\ \zeta_{k+1} = -\tau D_x\,u_{k+1} - \tau D_y\,v_{k+1} + \zeta_k \end{cases}$$

where τ is the time step and D_x, D_y are certain discrete approximations to the operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ on a rectangular net of elementary meshes $(\Delta x, \Delta y)$. The difference operators depend on two real parameters a and b, which may be chosen arbitrary (cf. [2] and [3]).

If a = 0 and b = 1, difference scheme (2) is exactly the scheme used by G. Fisher [1].

In [3] it was proved that (2) is stable in the sense of Rjabenki and Filippov if the following condition is satisfied

$$\tau \leq \frac{2}{\beta_m \sqrt{gh}} , \tag{3}$$

where

$$\beta_m^2 = \max_{\substack{0 \leq \xi \leq 1 \\ 0 \leq \eta \leq 1}} \frac{\Delta^2 y \, \xi (b + 2a\sqrt{1-\eta})^2 + \Delta^2 x \eta (b + 2a\sqrt{1-\xi})^2}{(b + 2a)^2 \, \Delta^2 x \Delta^2 y} . \tag{4}$$

From this we conclude that the criterium for stability in the sense of Rjabenki and Filippov (R-F stability) does not depend on the friction at the bottom of the North Sea. Hence we may choose $\lambda = 0$ without destroying the R-F stability of scheme (2).

In practice, however, one has to perform calculations over very long time intervals, so that one requires stability in the sense of O'Brien, Hyman and Kaplan (O'B-H-K stability). In [3] it was proved that scheme (2) is O'B-H-K stable if

$$\tau < \frac{\lambda}{\lambda^2 + \Omega^2} , \tag{5}$$

$$gh \, \beta_m^2 \, \tau^2 + 2\lambda\tau - 4 < 0. \tag{6}$$

For small values of λ we have to choose τ so small that scheme (2) is of no practical value. Scheme (2) becomes unstable in the sense of O'Brien, Hyman and Kaplan for $\lambda = 0$, hence North Sea models without friction may not be treated by this scheme.

2. Introduction of an additional friction term

Let us consider the following difference scheme

$$(7)\qquad \begin{cases} u_{k+1} = (1-\lambda\tau-c\tau^2)u_k + \Omega\tau v_k - gh\tau D_x\zeta_k + \tau U_k \\ v_{k+1} = -\Omega\tau u_k + (1-\lambda\tau-c\tau^2)v_k - gh\tau D_y\zeta_k + \tau V_k , \\ \zeta_{k+1} = -\tau D_x u_{k+1} - \tau D_y v_{k+1} \end{cases}$$

where c is uniformly bounded if $\tau \to 0$. This scheme arises from (2) by replacing λ with $\lambda+c\tau$. The addition of such a friction term was proposed by the author in one of his lectures at the Colloquium held at the Mathematical Centre Amsterdam (1965/'66) [5]. We now investigate the effect of such terms more closely.

It is clear that scheme (7) is a consistent approximation to the equations (1).

The conditions (5) and (6) transform to

$$(8)\qquad \tau^3c^2 + (2\tau\lambda - 1)\tau c + (\lambda^2 + \Omega^2)\tau - \lambda < 0$$

$$(9)\qquad 2\tau^2c^2 + gh\,\beta_m^2\,\tau^2 + 2\lambda\tau - 4 < 0.$$

The points (τ,c) which satisfy these inequalities are enclosed by the curves

$$(10)\qquad \Gamma_1^{\pm} : c = \frac{1 - 2\lambda\tau \pm \sqrt{1 - 4\Omega^2\tau^2}}{2\tau^2}$$

$$(11)\qquad \Gamma_2 : c = \frac{2}{\tau^2} - \frac{\lambda}{\tau} - \frac{1}{2}gh\,\beta_m^2 .$$

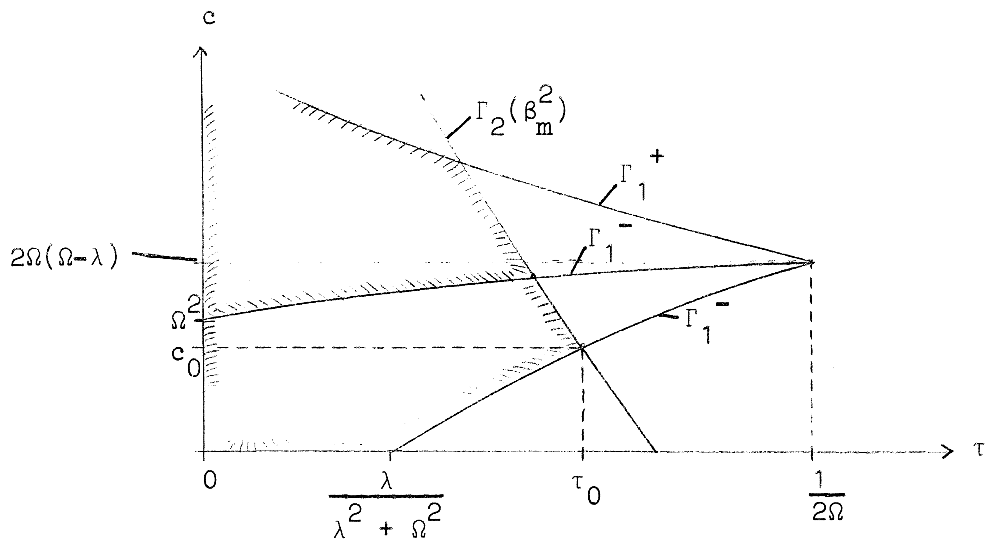

fig. 1

The curve Γ_1^- intersects the τ axis at the point $\tau = \frac{\lambda}{\lambda^2+\Omega^2}$ for $\lambda \neq 0$, and the c axis at the point $c = \Omega^2$ for $\lambda = 0$. The point $(\tau,c) = (\frac{1}{2\Omega}, 2\Omega(\Omega-\lambda))$ has the maximal τ coordinate of all points on the curves Γ_1^+ and Γ_1^-.

In figure 1 the curves $\Gamma_1^\pm$ and Γ_2 are illustrated for the cases $\lambda = 0$ and $\lambda \neq 0$, with a fixed value of β_m^2. If β_m^2 increases, the point of intersection (τ_0,c_0) of the curves Γ_1^- and Γ_2 moves along Γ_1^- to the point $(\frac{\lambda}{\lambda^2+\Omega^2}, 0)$ for $\lambda \neq 0$ and to the point $(0,\Omega^2)$ for $\lambda = 0$.

The points satisfying (8) and (9) lie in the shaded region of figure 1.

We replace (8) and (9) by the following conditions on τ and c:

$$(12)\quad \begin{cases} c = 0 \text{ for } \tau_0 < \dfrac{\lambda}{\lambda^2+\Omega^2} \\[2ex] c=c_0=\dfrac{1-2\lambda\tau_0-\sqrt{1-4\Omega^2\tau_0^2}}{2\tau_0^2} = \dfrac{2}{\tau_0^2} - \dfrac{\lambda}{\tau_0} - \dfrac{1}{2}gh\beta_m^2 \text{ for } \tau_0 \geq \dfrac{\lambda}{\lambda^2+\Omega^2} \end{cases}$$

and

$$(13)\qquad \tau < \tau_0.$$

From figure 1 it is clear that this choice for c is optimal in the sense that τ may be chosen as large as possible. Further, we see that

(14) $$0 \leq c \leq 2\Omega(\Omega - \lambda),$$

so that c is uniformly bounded. In fact, the North Sea value of Ω is approximately 1,25 10^{-4} $\sec^{-1}$, so that

(14') $$0 \leq c \leq 3 \cdot 2\ 10^{-8}.$$

This gives an impression of the change of the difference scheme by introducing the terms $c\tau^2 u_k$ and $c\tau^2 v_k$.

There remains the evaluation of the value τ_0 as a function of β_m^2. Using (12), a straightforward calculation gives

(15) $$\tau_0^2 = \frac{3gh\beta_m^2 - 2\Omega^2 + \sqrt{g^2h^2\beta_m^4 - 12gh\beta_m^2\Omega^2 + 4\Omega^2}}{g^2h^2\beta_m^4} .$$

For large values of β_m^2 this reduces to

(15') $$\tau_0 \sim \frac{2}{\beta_m\sqrt{gh}} .$$

Hence

(13') $$\tau < \frac{2}{\beta_m\sqrt{gh}} ,$$

which is the same condition as obtained for R-F stability except for the equality sign.

It may be remarked that we shall always find $\tau_0 \leq \frac{1}{2\Omega}$ (cf. fig. 1), hence $\frac{1}{2\Omega}$ is an upperbound for the time step τ. The North Sea value of this upperbound is approximately 4000 sec, which is no restriction to the actual time step used in practice.

From these considerations we conclude that difference scheme (7), where c and τ satisfy (12) and (13), may be used to treat North Sea models without friction, and is stable in the sense of O'Brien, Hyman and Kaplan.

References

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