MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

AFD. TOEGEPASTE WISKUNDE

ON KELVIN AND POINCARÉ-WAVES IN A STRIP

G.W. Veltkamp

Rapport TW 37

Mei
1956

TABLE OF CONTENTS

On Kelvin and Poincaré-waves in a strip.[1)]

1. Introduction.

In this report we shall consider elementary solutions of the set of differential equations [2)]

$$\left.\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\omega\zeta = 0 \\ &(i\omega+\lambda)u - \Omega v + \frac{\partial \zeta}{\partial x} = 0 \\ &\Omega u + (i\omega+\lambda)v + \frac{\partial \zeta}{\partial y} = 0, \end{aligned}\right\} \qquad (1.1)$$

which are defined in the region D: $0 \leq x \leq \pi$, $\pi b_1 < y < \pi b_2$ (we may have $b_1 = -\infty$ and / or $b_2 = +\infty$) and which are such that at x=0 and at $x=\pi$ the following boundary-condition applies [3)]:

u(x,y) is continuous up to x=0 and $x=\pi$ and $u(0,y)=0$, $u(\pi,y)=0$, for $\pi b_1 < y < \pi b_2$. (1.2)

In the equations (1.1) Ω and λ are non-negative real numbers; ω is a complex number which we assume, however, to be limited by the conditions [4)]

$$\operatorname{Re}\omega > 0, \quad 0 \leq \operatorname{Im}\omega < \lambda \, . \qquad (1.3)$$

It will be convenient to introduce the complex number χ, defined by

$$\chi^2 \stackrel{\text{def}}{=} \frac{\omega - i\lambda}{\omega}, \ -\frac{\pi}{2} < \arg\chi \leq 0 \qquad (1.4)$$

Since by (1.3) $0 \leq \arg\omega < \frac{\pi}{2}$, $-\frac{\pi}{2} < \arg(\omega - i\lambda) \leq 0$, we have $-\pi < \arg\frac{\omega - i\lambda}{\omega} \leq 0$, hence the condition in (1.4) can always be satisfied.

By variation of Im ω (between zero and λ) it is easily seen that we shall always have

$$-\text{arc tg}\,\frac{\lambda}{2\operatorname{Re}\omega} \leq \arg\chi \leq 0,$$

the minimum being attained for $\operatorname{Im}\omega = \frac{1}{2}\lambda$. On the other hand, if ω is real then $\arg\chi = -\frac{1}{2}\,\text{arc tg}\,\frac{\lambda}{\omega} > -\frac{\pi}{4}$ for all values of λ.

Obviously, if λ tends to zero, χ tends to one. The factor χ can be used for a formal elimination of the parameter λ from the equations (1.1). Indeed, putting $\omega' = \chi\omega$, $\Omega' = \Omega/\chi$, $u' = \chi u$, $v' = \chi v$, we see that (1.1) reduces to

----

1) Research carried out under the direction of Prof.Dr D. van Dantzig.

2) For the physical meaning of the variables and constants we refer to foregoing reports.

3) In physical terms: the lines x=0 and $x=\pi$ are coasts.

4) It may be shown that the second condition is always satisfied if ω is a free frequency of the system (1.1) in a closed domain at the boundaries of which either $\zeta=0$ or $n_1u+n_2v=0$ (with n_1, n_2 components of the normal).

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + i\,\omega'\zeta = 0$$
$$i\,\omega' u' - \Omega' v' + \frac{\partial \zeta}{\partial x} = 0$$
$$\Omega' u' + i\,\omega' v' + \frac{\partial \zeta}{\partial y} = 0,$$

which is formally equivalent with (1.1) when there λ is taken to be zero. We shall not use this formalism in the sequel. But the above remarks may illuminate the fact that in the following formulae χ always will occur in one of the combinations $\chi\omega$, Ω/χ, χu or χv.

2. Kelvin- and Poincaré-waves.

We first may try to find solutions of (1.1) for which $u(x,y)=0$ throughout D. In this case the boundary-condition (1.2) is automatically satisfied.

Such solutions indeed exist and they are easily seen to be

$$\begin{aligned} \zeta_0(x,y) &\stackrel{\text{def}}{=} C_0^+ \exp\left(\frac{\Omega}{\chi}\left(x-\frac{\pi}{2}\right) - i\,\omega\chi y\right) + \\ &\quad + C_0^- \exp\left(-\frac{\Omega}{\chi}\left(x-\frac{\pi}{2}\right) + i\,\omega\chi y\right), \\ v_0(x,y) &\stackrel{\text{def}}{=} \frac{C_0^+}{\chi} \exp\left(\frac{\Omega}{\chi}\left(x-\frac{\pi}{2}\right) - i\,\omega\chi y\right) + \\ &\quad - \frac{C_0^-}{\chi} \exp\left(-\frac{\Omega}{\chi}\left(x-\frac{\pi}{2}\right) + i\,\omega\chi y\right). \end{aligned} \tag{2.1}$$

The constants C_0^+ and C_0^- are arbitrary complex numbers.

If $C_0^-=0$ then (2.1) clearly represents a progressive wave in the direction of the positive y-axis (when the time-factor $e^{i\omega t}$ is adjoined). If $\lambda=0$ and accordingly ω is positive (and $\chi=1$) then the wave is undamped and moves with unit velocity. The particle-velocity is everywhere parallel to the coasts but the force of Coriolis causes the amplitude of the waves to be increasing towards the right side of the wave. If $\chi \neq 1$ then the behaviour of the wave is somewhat more complicated. If ω is real (so that we have strictly periodic motions) and $\lambda > 0$ then we see that the amplitude decreases exponentially towards the direction of the wave. Some aspects of these damped waves have been discussed recently by PROUDMAN (1954) and SCHONFELD (1955).

Of course similar statements apply to the case $C_0^+=0$, in which case we have a wave in the direction of the negative y-axis.

We shall call these types of solution Kelvin-waves.

Another type of solutions of (1.1) which satisfy (1.2) is found if we suppose $u(x,y)$ to be proportional to $\sin nx$, $n=1,2,\ldots$.

Let for $n=1,2,\ldots$ the functions $f_n^{(1)}(y)$ and $f_n^{(2)}(y)$ be defined by

$$f_n^{(1)}(y) \stackrel{\text{def}}{=} C_n^+ e^{-\mu_n y} + C_n^- e^{\mu_n y}$$
$$f_n^{(2)}(y) \stackrel{\text{def}}{=} C_n^+ e^{-\mu_n y} - C_n^- e^{\mu_n y}, \tag{2.2}$$

where

$$\mu_n^2 \stackrel{\text{def}}{=} n^2 - \omega^2\chi^2 + \Omega^2/\chi^2 , \quad -\tfrac{\pi}{2} < \arg \mu_n \leq \tfrac{\pi}{2} \tag{2.3}$$

and $C_n^{\pm}$ are arbitrary complex numbers.

Then by substitution it is easily verified that we have the following solutions of the system (1.1)

$$\zeta_n(x,y) \stackrel{\text{def}}{=} f_n^{(1)}(y)\cos nx - \frac{i\Omega\mu_n}{\chi^2 n\,\omega} f_n^{(2)}(y)\sin nx$$
$$u_n(x,y) \stackrel{\text{def}}{=} -i\,\frac{n^2+\Omega^2/\chi^2}{\chi^2 n\,\omega} f_n^{(1)}(y)\sin nx \tag{2.4}$$
$$v_n(x,y) \stackrel{\text{def}}{=} -\frac{i\mu_n}{\chi^2\omega}\left\{f_n^{(2)}(y)\cos nx + \frac{i\Omega\omega}{n\mu_n} f_n^{(1)}(y)\sin nx\right\}.$$

The waves which result when to this type of solutions the time-factor $e^{i\omega t}$ is adjoined, are called <u>Poincaré-waves</u>, these are of a more complex nature.

If $\lambda=0$ and accordingly ω is positive and $\chi=1$, μ_n is either positive real (if $n^2-\omega^2+\Omega^2>0$) or a positive imaginary. If $C_n^-=0$ we have in the first case something like a standing wave in the x-direction with an amplitude decreasing exponentially in the direction of the positive y-axis. The particles move in elliptic orbits, except at the boundaries x=0 and $x=\pi$, where u=0. In the second case we have products of progressive waves in the direction of the positive y-axis and standing waves in the x-direction. The velocity of propagation is in this case

$$c_p = \frac{\omega}{|\mu_n|} = \frac{\omega}{\sqrt{\omega^2-n^2-\Omega^2}} ,$$

hence depending on ω and (for fixed n) decreasing to unity if ω tends to infinity [5].

If $\lambda>0$ then the behaviour of the Poincaré-waves is still more complex. Since then Re $\mu_n \geqslant 0$ we have, for all values of n, products of damped progressive waves in the direction of the y-axis and standing waves in the x-direction. But of course, if λ is small then either the exponential decrease or the progressive-wave-character dominates. For more information we again refer to PROUDMAN and SCHONFELD.

It should be remarked that if ω is such that for some integer N $\mu_N=0$ then for n=N the solution (2.4) only contains one arbitrary constant. It may be verified, however, that in this case solutions are

----

5) The group-velocity, defined by $c_g=c_p\left(1-\frac{\omega}{c_p}\frac{dc_p}{d\omega}\right)^{-1}$ proves to be $c_g=c_p^{-1}$ and thus is always less than unity.

$$\zeta_N(x,y) = (C_N - D_N y)\cos Nx - \frac{i\Omega}{\chi^2 N\omega} D_N \sin Nx$$
$$u_N(x,y) = -\frac{i\omega}{N}(C_N - D_N y)\sin Nx \qquad (2.5)$$
$$v_N(x,y) = -\frac{i}{\chi^2\omega} D_N \cos Nx + \frac{\Omega}{\chi^2 N}(C_N - D_N y)$$

which can be obtained formally from (2.3) by taking $f_N^{(1)}(y)=C_N-D_N y$, $\mu_n f_n^{(2)}(y)=D_N$, $\omega^2\chi^2-\Omega^2/\chi^2=N^2$.

3. Completeness of the system of Kelvin and Poincaré-waves.

In a certain way (to be made precise below) the solutions considered in section 2 constitute a complete system.

If we are concerned with flows in the domain $0 \leq x \leq \pi$, $\pi b_1 \leq y \leq \pi b_2$ then it is natural to require that ζ, u and v are regular in the interior of the domain and that the only singularities will occur at the boundaries and more specially near the corners. To be more specific, we shall require that u(x,y) is twice continuously differentiable in the semi-closed domain $0 \leq x \leq \pi$, $\pi b_1 < y < \pi b_2$. We then may prove the following

Theorem

If 1° the functions $\zeta(x,y)$, $u(x,y)$, $v(x,y)$ satisfy the differential equations (1.1) in the domain $0 < x < \pi$, $\pi b_1 < y < \pi b_2$;

2° $u(x,y)$ and $v(x,y)$ are twice continuously differentiable for $0 \leq x \leq \pi$, $\pi b_1 < y < \pi b_2$;

3° $u(0,y)=u(\pi,y)=0$ for $\pi b_1 < y < \pi b_2$,

then uniquely determined constants $C_n^{\pm}$ $(n=0,1,\ldots)$ exist such that

$$\zeta(x,y) = \sum_{n=0}^{\infty} \zeta_n(x,y)$$
$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) \qquad (3.1)$$
$$v(x,y) = \sum_{n=0}^{\infty} v_n(x,y)$$

where the functions ζ_n, u_n and v_n are defined by (2.1), (2.2) and (2.4). The series in (4.1) shall converge uniformly in any closed domain $0 \leq x \leq \pi$, $\pi b_1 + \delta \leq y \leq \pi b_2 - \delta$ with $\delta > 0$, arbitrary.

Proof.

Let ζ, u and v be given functions which satisfy the conditions of the theorem.

From the equations (1.1) we have

$$\left.\begin{aligned} u &= -\chi^{-1}k^{-2}\left\{ i\omega\chi \frac{\partial\zeta}{\partial x} + (\Omega/\chi)\frac{\partial\zeta}{\partial y}\right\} \\ v &= -\chi^{-1}k^{-2}\left\{ -(\Omega/\chi)\frac{\partial\zeta}{\partial x} + i\omega\chi\frac{\partial\zeta}{\partial y}\right\}\end{aligned}\right\} \qquad (3.2)$$

$$\Delta\zeta - k^2\zeta = 0, \tag{3.3}$$

where $k^2 \overset{\text{def}}{=} (\Omega/\chi)^2 - \omega^2\chi^2 = \frac{\omega}{\omega - i\lambda}\left(\Omega^2 - (\omega - i\lambda)^2\right).$ (3.4)

If u and v are twice differentiable, then from (1.1) it appears that ζ is three times differentiable, whence from (3.2) and (3.3) we find

$$\Delta u - k^2 u = 0 \tag{3.5}$$

Now let the functions $c_n(y)$ $(n=1,2,\dots)$ be defined by

$$c_n(y) \overset{\text{def}}{=} \frac{2}{\pi}\int_0^\pi u(x,y)\sin nx\,dx. \tag{3.6}$$

Then we have, since u and $\frac{\partial u}{\partial x}$ are continuous for $0 \leq x \leq \pi$ and $u(0,y) = u(\pi,y) = 0$

$$\frac{2}{\pi}\int_0^\pi \frac{\partial^2 u}{\partial x^2}\sin nx\,dx = -\frac{2n}{\pi}\int_0^\pi \frac{\partial u}{\partial x}\cos nx\,dx =$$

$$= -\frac{2n^2}{\pi}\int_0^\pi u(x,y)\sin nx\,dx = -n^2 c_n(y) \tag{3.7}$$

And, since $\frac{\partial^2 u}{\partial y^2}$ is (two-dimensional) continuous,

$$\frac{2}{\pi}\int_0^\pi \frac{\partial^2 u}{\partial y^2}\sin^2 nx\,dx = \frac{d^2 c_n}{dy^2} \tag{3.8}$$

Combining (3.5), (3.6), (3.7) and (3.8) we find

$$\frac{d^2 c_n}{dy^2} - (k^2+n^2)c_n = 0 \tag{3.9}$$

Hence the functions $c_n(y)$, which are derived from the given function $u(x,y)$ by (3.6) satisfy the ordinary differential equation (3.9). It thus follows, that uniquely defined constants $C_n^{\pm}$ $(n=1,2,\dots)$ exist, such that

$$c_n(y) = -i\,\frac{n^2 + \Omega^2/\chi^2}{\chi^2 n\,\omega}\left(C_n^+ e^{-\mu_n y} + C_n^- e^{\mu_n y}\right), \tag{3.10}$$

when μ_n is defined by (2.3):

$$\mu_n^2 = n^2 - \omega^2\chi^2 + \Omega^2/\chi^2 = n^2 + k^2, \quad -\frac{\pi}{2} < \arg\mu_n \leq \frac{\pi}{2}$$

Let δ be an arbitrarily small positive number. Then since the function $\frac{\partial^2 u}{\partial x^2}$ is bounded in the closed domain $0 \leq x \leq \pi$, $\pi b_1 + \delta \leq y \leq \pi b_2 - \delta$ we have from (3.7) and (3.10)

$$C_n^+ e^{-\mu_n y} + C_n^- e^{\mu_n y} = O(n^{-3}),$$

uniformly in the interval $\pi b_1 + \delta \leq y \leq \pi b_2 - \delta$, whence, since $\operatorname{Re}\mu_n > 0$ for sufficiently great n, it is easily derived that

$$C_n^{+} = O\,(n^{-3}\ \exp\ ((\pi b_1 + \delta)\mathrm{Re}\ \mu_n))$$
$$C_n^{-} = O\,(n^{-3}\ \exp\ ((\pi b_2 - \delta)\mathrm{Re}\ \mu_n)), \tag{3.11}$$

which implies

$$C_n^{+}\, e^{-\mu_n y} \pm C_n^{-}\, e^{\mu_n y} = O(n^{-2}\, e^{-\delta\,\mathrm{Re}\,\mu_n}), \tag{3.12}$$

uniformly in the interval $\pi b_1 + 2\delta \leq y \leq \pi b_2 - 2\delta$.

Now let the constants $C_n^{\pm}$ be determined by the given function $u(x,y)$ in the way indicated above. With these constants we construct the functions $\zeta_n(x,y)$, $u_n(x,y)$ and $v_n(x,y)$, as defined by (2.2) and (2.4). Then, in view of the estimation (3.12) and the fact that $\mathrm{Re}\ \mu_n \sim n$ for $n \to \infty$ the series

$$\sum_{n=1}^{\infty} u_n(x,y) = u(x,y)$$
$$\sum_{n=1}^{\infty} \zeta_n(x,y) \stackrel{\mathrm{def}}{=} \zeta'(x,y)$$
$$\sum_{n=1}^{\infty} v_n(x,y) \stackrel{\mathrm{def}}{=} v'(x,y)$$

will converge uniformly in the closed domain D': $0 \leq x \leq \pi$, $\pi b_1 + 2\delta \leq y \leq \pi b_2 - 2\delta$. By (3.6) and (3.10) the sum of the first series must be the original function $u(x,y)$. Furthermore the series are differentiable term-by-term in D' and since each of the sets (ζ_n, u_n, v_n) satisfies the differential equation (1.1), the set (ζ', u, v') does so as well.

From the linearity of the differential equations (1.1) we now infer that the set ($\zeta'' = \zeta - \zeta'$, $u''=0$, $v''=v-v'$) also satisfies (1.1), which equations reduce to

$$\frac{\partial v''}{\partial y} + i\,\omega\,\zeta'' = 0$$
$$i\,\omega\,\chi^2 v'' + \frac{\partial \zeta''}{\partial y} = 0$$
$$-\Omega v'' + \frac{\partial \zeta''}{\partial x} = 0.$$

From the first two equations we infer

$$\zeta'' = C_o^{+}(x)\, e^{-i\omega\chi y} + C_o^{-}(x) e^{i\omega\chi y}$$
$$v'' = \chi^{-1}\, C_o^{+}(x) e^{-i\omega\chi y} - \chi^{-1}\, C_o^{-}(x) e^{i\omega\chi y},$$

whereas from the third equation we have

$$\frac{dC_o^{\pm}}{dx} = \pm\, (\Omega/\chi)\ C_o^{\pm}(x)$$

hence $C_o^{\pm}(x) = C_o^{\pm}\, e^{\mp(\Omega/\chi)x}$, so ζ'' and v'' must have the form (2.1) with uniquely determined constants $C_o^{\pm}$.

Remarks.

1. From the proof of the theorem it is clear that if we are dealing with flows in the semi-infinite domain $\pi b_1 < y < \infty$ and ζ, u, v remain finite for $y \to \infty$, then $C_n^-=0$ for $n=0,1,\ldots$, unless $\lambda=0$, ω is real. In the latter case we may have $C_o^- \neq 0$ and $C_n^- \neq 0$ for those values of $n \geqslant 1$ for which μ_n is a pure imaginary (i.e. for $n < \sqrt{\omega^2-\Omega^2}$).

2. It is clear that if ω and Ω are such that $\sqrt{\omega^2-\Omega^2}$ is a positive integer, the theorem and the proof must be modified somewhat. We shall not go into these details however.

3. In later work we shall meet the case that on the line $y=\pi b_1$, $0 \leqslant x \leqslant \pi$ $u(x,y)$ is continuous except near the points $x=0$, $x=\pi$, where we shall have $u(x,\pi b_1) \sim x^{-1+\alpha_1}$, $u(x,\pi b_1) \sim (\pi-x)^{-1+\alpha_2}$, with $0 < \alpha_{1,2} < 2$. It then may be shown [6] that we have

$$C_n(\pi b_1) = O(n^{-\alpha}),$$

with $\alpha = \mathrm{Min}(\alpha_1,\alpha_2)$. This implies that the series for $\zeta(x,y)$ will converge uniformly up to $y=\pi b_1$.

4. Some auxiliary estimates.

In the course of following work we shall need some information concerning the complex numbers

$$\gamma_n \stackrel{\text{def}}{=} \frac{i\chi^2 n\omega}{\Omega\mu_n}, \quad n=1,2,\ldots, \tag{4.1}$$

the inverses of which occur in the formula (2.4) for $\zeta(x,y)$ in the case of Poincaré-waves. And especially we want to know whether $|\gamma_n| > 1$ or < 1.

A. We shall first consider the case $\lambda=0$, ω real (and hence $\chi^2=1$). In this case

$$\gamma_n = i\,\frac{\omega}{\Omega}\cdot\frac{1}{\sqrt{1+\frac{\Omega^2-\omega^2}{n^2}}}. \tag{4.2}$$

We shall now prove the following statements:

I. If $0 < \Omega \leqslant 1$, and

α) $0 < \omega < \Omega$, then $|\gamma_n| < 1$, $n=1,2,\ldots$,

β) $\omega=\Omega$, then $\gamma_n=i$, $n=1,2,\ldots$,

γ) $\omega > \Omega$, then $|\gamma_n| > 1$, $n=1,2,\ldots$.

---

6) Compare e.g. BROMWICH (1949), p.494, ex.5.

II. If $\Omega > 1$ and

α) $0 < \omega < \Omega$, then $|\gamma_n| < 1$, $n=1,2,\ldots$,

β) $\omega = \Omega$, then $\gamma_n = i$, $n=1,2,\ldots$,

γ) $\Omega < \omega < \Omega\sqrt{\frac{\Omega^2+1}{\Omega^2-1}}$, then $|\gamma_n| > 1$, $n=1,2,\ldots$,

δ) $\omega \geq \Omega\sqrt{\frac{\Omega^2+1}{\Omega^2-1}}$, then

$$|\gamma_n| \lesseqgtr 1 \text{ for } n \lesseqgtr \Omega\sqrt{\frac{\omega^2-\Omega^2}{\omega^2+\Omega^2}}.$$

The statements I α, I β, II α, II β are obvious from (4.2). If $\omega > \Omega$ and $n^2 > \omega^2 - \Omega^2$ then $\sqrt{1-\frac{\omega^2-\Omega^2}{n^2}}$ is positive and < 1, so γ_n is pure imaginary and $|\gamma_n| > \frac{\omega}{\Omega} > 1$.
We thus are left with the cases $\omega > \Omega$, $1 \leq n < \sqrt{\omega^2-\Omega^2}$ (if $n^2 = \omega^2 - \Omega^2$ then $\gamma_n = \infty$). In these cases γ_n is positive and

$$\gamma_n^2 = \frac{\omega^2 n^2}{\Omega^2(\omega^2-\Omega^2-n^2)} = 1 + \frac{\omega^2+\Omega^2}{\Omega^2(\omega^2-\Omega^2-n^2)} \cdot \left\{ n^2 - \Omega^2 \frac{\omega^2-\Omega^2}{\omega^2+\Omega^2} \right\}. \tag{4.3}$$

If

$$\Omega^2 \frac{\omega^2-\Omega^2}{\omega^2+\Omega^2} < 1, \tag{4.4}$$

clearly $\gamma_n^2 > 1$ for all values of n of the range to be considered ($1 \leq n < \omega^2 - \Omega^2$). Since (4.4) is equivalent with

$$\omega^2(\Omega^2-1) < \Omega^2(\Omega^2+1),$$

this situation occurs if either $0 < \Omega \leq 1$ or $\Omega > 1$ and $\omega < \Omega\sqrt{\frac{\Omega^2+1}{\Omega^2-1}}$.
This completes the proof of I γ and II γ.

If, however, $\Omega^2 > 1$ and $\omega^2 \geq \Omega^2\sqrt{\frac{\Omega^2+1}{\Omega^2-1}}$, we have

$$1 \leq \Omega^2 \frac{\omega^2-\Omega^2}{\omega^2+\Omega^2} < \omega^2 - \Omega^2,$$

and from (4.3) we see that in this case

$$\gamma_n^2 \gtreqless 1 \text{ for } n \gtreqless \Omega\sqrt{\frac{\omega^2-\Omega^2}{\omega^2+\Omega^2}},$$

which completes the proof of II δ.

B. In the case $\lambda > 0$ and ω complex the situation is more difficult. Here we shall only prove the following partial result:

If $0 \leq \lambda \leq \frac{1}{2}\sqrt{2}$ Re ω and $|\omega + i\lambda| < \Omega$ (4.5)
then $|\gamma_n| < 1$ and $0 < \arg \gamma_n \leq \frac{\pi}{2}$ for $n=1,2,\ldots$.

This clearly corresponds with the statements I α and II α made above.

Since $0 \leq \mathrm{Im}\ \omega < \lambda$, we may put for shortness $\mathrm{Re}\ \omega = \omega_o$, $\mathrm{Im}\ \omega = \theta\lambda$, $0 \leq \theta < 1$. Then

$$\omega = \omega_o + i\theta\lambda \quad , \quad \chi^2 = \frac{\omega_o - i(1-\theta)\lambda}{\omega_o + i\theta\lambda}$$

and (4.5) is equivalent with

$$0 \leq \lambda \leq \tfrac{1}{2}\sqrt{2}\ \omega_o \ ; \quad \Omega^2 > \omega_o^2 + (1+\theta)^2\lambda^2 . \tag{4.6}$$

Now

$$\mathrm{Re}(\Omega^2/\chi^2 - \omega^2\chi^2) =$$

$$= \mathrm{Re}\left[\Omega^2 \frac{\omega_o + i\theta\lambda}{\omega_o - i(1-\theta)\lambda} - (\omega_o + i\theta\lambda)(\omega_o - i(1-\theta)\lambda)\right] =$$

$$= \Omega^2 \frac{\omega_o^2 - \theta(1-\theta)\lambda^2}{\omega_o^2 + (1-\theta)^2\lambda^2} - \omega_o^2 - \theta(1-\theta)\lambda^2,$$

$$\mathrm{Im}\ (\Omega^2/\chi^2 - \omega^2\chi^2) = \lambda\omega_o\left[\frac{\Omega^2}{\omega_o^2 + (1-\theta)^2\lambda^2} + 1 - 2\theta\right]$$

Hence, by (4.6)

$$\mathrm{Re}(\Omega^2/\chi^2 - \omega^2\chi^2) >$$

$$> \frac{\{\omega_o^2 + (1+\theta)^2\lambda^2\}\{\omega_o^2 - \theta(1-\theta)\lambda^2\} - \{\omega_o^2 + (1-\theta)^2\lambda^2\}\{\omega_o^2 + \theta(1-\theta)\lambda^2\}}{\omega_o^2 + (1-\theta)^2\lambda^2} =$$

$$= \frac{2\theta\lambda^2\left[(1+\theta)\omega_o^2 - 2(1-\theta)(1+\theta^2)\lambda^2\right]}{\omega_o^2 + (1-\theta)^2\lambda^2} \geq 0,$$

since by the first condition (4.6)

$$2(1-\theta)(1+\theta^2)\lambda^2 \leq (1-\theta)(1+\theta^2)\omega_o^2 \leq (1+\theta)\omega_o^2 .$$

Furthermore, by (4.6)

$$\mathrm{Im}\ (\Omega^2/\chi^2 - \omega^2\chi^2) \geq \lambda\omega_o\left\{\frac{\Omega^2}{\omega_o + (1-\theta)^2\lambda^2} - 1\right\} \geq 0.$$

So, since $\dfrac{\mu_n^2}{n^2} = 1 + \dfrac{\Omega^2/\chi^2 - \omega^2\chi^2}{n^2}$, we have

$$\mathrm{Re}\frac{\mu_n^2}{n^2} > 1, \quad \mathrm{Im}\frac{\mu_n^2}{n^2} \geq 0, \text{ for all } n=1,2,\ldots ,$$

hence

$$\left|\frac{\mu_n}{n}\right| > 1 \quad \text{and } 0 \leq \arg\frac{\mu_n}{n} < \frac{\pi}{4} .$$

Finally, since $\dfrac{\chi^2\omega}{\Omega} = \dfrac{\omega - i\lambda}{\Omega} = \dfrac{\omega_o - i(1-\theta)\lambda}{\Omega}$,

$$\left|\frac{\chi^2\omega}{\Omega}\right| < 1 \text{ and } -\frac{\pi}{4} < \arg\frac{\chi^2\omega}{\Omega} \leq 0.$$

Combining these results, we see that for $\gamma_n = i\,\frac{\chi^2\omega}{\Omega} \cdot \frac{n}{\mu_n}$ the desired properties follow.

5. Propagation of energy by Kelvin-and Poincaré-waves.

We shall now suppose that $\lambda = 0$ and that ω is real. Then by the theorem of section 3 every solution of (1.1) in the domain $0 \leq x \leq \pi$, $\pi b_1 < y < \pi b_2$ which satisfies the boundary-condition (1.2) can be represented by the series

$$\left.\begin{aligned} \zeta(x,y) &= \sum_{n=0}^{\infty} \zeta_n(x,y) \\ u(x,y) &= \sum_{n=1}^{\infty} u_n(x,y) \\ v(x,y) &= \sum_{n=0}^{\infty} v_n(x,y)\ , \end{aligned}\right\} \qquad (5.1)$$

where now

$$\zeta_0(x,y) = C_0^+ \exp(\Omega(x-\tfrac{\pi}{2}) - i\omega y) + C_0^- \exp(-\Omega(x-\tfrac{\pi}{2}) + i\omega y),$$

$$v_0(x,y) = C_0^+ \exp(\Omega(x-\tfrac{\pi}{2}) - i\omega y) - C_0^- \exp(-\Omega(x-\tfrac{\pi}{2}) + i\omega y),$$

$$\zeta_n(x,y) = f_n^{(1)}(y)\cos nx - \frac{i\Omega\mu_n}{n\,\omega} f_n^{(2)}(y)\sin nx\ ,$$

$$u_n(x,y) = -i\,\frac{n^2+\Omega^2}{\omega\,n}\, f_n^{(1)}(y)\sin nx\ .$$

$$v_n(x,y) = -\frac{i\mu_n}{\omega}\left\{ f_n^{(2)}(y)\cos nx + \frac{i\Omega\omega}{n\mu_n} f_n^{(1)}(y)\sin nx \right\}$$

$$f_n^{(1)}(y) = C_n^+ e^{-\mu_n y} + C_n^- e^{\mu_n y}, \quad f_n^{(2)}(y) = C_n^+ e^{-\mu_n y} - C_n^- e^{\mu_n y},$$

$$\mu_n = \sqrt{n^2 - \omega^2 + \Omega^2}\ , \ \arg \mu_n = 0 \text{ or } \tfrac{\pi}{2}\ .$$

We shall now calculate the following integral

$$S \overset{\text{def}}{=} \tfrac{1}{2} \int_0^{\pi} \operatorname{Re}\left[v(x,y) . \bar{\zeta}(x,y) \right] dx, \qquad (5.2)$$

where $\bar{\zeta}$ denotes the complex conjugate of ζ.
From the differential equations (1.1) it may be proved that $\operatorname{Re}\left[\frac{\partial}{\partial x}(u\bar{\zeta}) + \frac{\partial}{\partial y}(v\bar{\zeta})\right] = 0$, whence by Gauss'theorem and the fact that $u=0$ for $x=0$, $x=\pi$ it follows that S is independent of y (for $\pi b_1 < y < \pi b_2$).

In an other report we shall motivate that S may be called the mean flux of energy through the line $y=\text{const}$, $0 \leq x \leq \pi$ in the direction of the positive y-axis.

The calculation of S is most easily effectuated with the aid of the following remarks.

We have

$$\zeta_0(x,y) \pm v_0(x,y) = 2\, C_0^{\pm} \exp\,(\pm\Omega(x-\tfrac{\pi}{2}) \mp i\,\omega y)$$

$$\zeta_n(x,y) \pm v_n(x,y) = \{f_n^{(1)}(y) \mp \frac{i\,\mu_n}{\omega} f_n^{(2)}(y)\}(\cos nx \pm \frac{\Omega}{n} \sin nx).$$

Now by direct calculation it appears that both sets $\{\varphi_n^+(x)\}$ and $\{\varphi_n^-(x)\}$, $n=0,1,\dots$,

where

$$\varphi_0^{\pm}(x) = \exp\,(\pm\,\Omega\,(x-\tfrac{\pi}{2})),$$

$$\varphi_n^{\pm}(x) = \cos nx \pm \frac{\Omega}{n} \sin nx, \quad n=1,2,\dots,$$

are orthogonal over $(0,\pi)$ 7).

Since furthermore

$$\int_0^{\pi} \exp\,(\pm\, 2\,\Omega\,(x-\tfrac{\pi}{2}))dx = \frac{\mathrm{sh}\,\pi\Omega}{\Omega},$$

$$\int_0^{\pi} (\cos nx \pm \frac{\Omega}{n} \sin nx)^2\, dx = \frac{\pi}{2}\,(1+\frac{\Omega^2}{n^2}),$$

we have (in virtue of the uniform convergence of the series (5.1))

$$\int_0^{\pi} |\zeta \pm v|^2\, dx = 4\,|C_0^{\pm}|^2\, \frac{\mathrm{sh}\,\pi\Omega}{\Omega} + \frac{\pi}{2}\sum_{n=1}^{\infty}(1+\frac{\Omega^2}{n^2})\,|f_n^{(1)}(y) \mp \frac{i\,\mu_n}{\omega} f_n^{(2)}(y)|^2.$$

With these results we find for S:

$$S = \frac{1}{8}\int_0^{\pi}\{|\zeta+v|^2 - |\zeta-v|^2\}\,dx = \frac{\mathrm{sh}\,\pi\Omega}{2\,\Omega}\,(|C_0^+|^2 - |C_0^-|^2) + \frac{\pi}{16}\sum_{n=1}^{\infty}(1+\frac{\Omega^2}{n^2})\Big\{|f_n^{(1)}(y) - \frac{i\,\mu_n}{\omega} f_n^{(2)}(y)|^2 - |f_n^{(1)}(y) + \frac{i\mu_n}{\omega} f_n^{(2)}(y)|^2\Big\} =$$

$$= \frac{\mathrm{sh}\,\pi\Omega}{2\,\Omega}\,(|C_0^+|^2 - |C_0^-|^2) - \frac{\pi}{4}\sum_{n=1}^{\infty}(1+\frac{\Omega^2}{n^2})\mathrm{Re}\Big\{\frac{i\,\mu_n}{\omega}\,\overline{f_n^{(1)}(y)}\, f_n^{(2)}(y)\Big\}.$$

Now $\overline{f_n^{(1)}(y)}\, f_n^{(2)}(y) = |C_n^+|^2\, e^{-2y\,\mathrm{Re}\,\mu_n} - |C_n^-|^2\, e^{2y\mathrm{Re}\,\mu_n} +$

$$+\, C_n^+\bar{C}_n^-\, e^{-2iy\,\mathrm{Im}\,\mu_n} - \bar{C}_n^+ C_n^-\, e^{2iy\,\mathrm{Im}\,\mu_n} =$$

$$= |C_n^+|^2\, e^{-2y\,\mathrm{Re}\,\mu_n} - |C_n^-|^2\, e^{2y\mathrm{Re}\,\mu_n} + 2i\,\mathrm{Im}(C_n^+\bar{C}_n^-\, e^{-2iy\,\mathrm{Im}\,\mu_n}).$$

----

7) By application of a criterion of DALZELL (1945) (see also TRICOMI (1955)) it may be proved that these sets are complete over $L_2(0,\pi)$ as well. We shall not use this, however.

Hence

$$\mathrm{Re}\left\{\frac{i\mu_n}{\omega}\overline{f_n^{(1)}(y)}\, f_n^{(2)}(y)\right\} =$$

$$= \begin{cases} -\dfrac{2\mu_n}{\omega}\ \mathrm{Im}\ C_n^{+}\overline{C_n^{-}} & \text{if } \mu_n \text{ is real } (n^2 > \omega^2 - \Omega^2), \\ -\dfrac{|\mu_n|}{\omega}\left(|C_n^{+}|^2 - |C_n^{-}|^2\right) & \text{if } \mu_n \text{ is a pure imaginary } (n^2 < \omega^2 - \Omega^2). \end{cases}$$

So finally [8)]

$$S = \frac{\mathrm{sh}\,\pi\Omega}{2\Omega}\left(|C_0^{+}|^2 - |C_0^{-}|^2\right) +$$

$$+ \frac{\pi}{4\omega}\sum_{1 \le n^2 < \omega^2 - \Omega^2}\left(1 + \frac{\Omega^2}{n^2}\right)|\mu_n|\left(|C_n^{+}|^2 - |C_n^{-}|^2\right) +$$

$$+ \frac{\pi}{2\omega}\sum_{n^2 > \omega^2 - \Omega^2}\left(1 + \frac{\Omega^2}{n^2}\right)\mu_n\ \mathrm{Im}\left(C_n^{+}\overline{C_n^{-}}\right). \qquad (5.3)$$

Formula (5.3) admits a number of statements in physical terms.

a). By Kelvin-waves an amount of energy is propagated in the direction of the waves, which is proportional to the squared modulus of the "amplitude". By interaction of Kelvin-waves of different direction no energy is propagated.

b). By Poincaré-waves of progressive-wave-type (μ_n imaginary, $n^2 < \omega^2 - \Omega^2$) an amount of energy is propagated which is proportional to the squared modulus of the amplitude. By interaction of such waves of different direction no energy is propagated.

c) By Poincaré-waves of damped type (μ_n real, $n^2 > \omega^2 - \Omega^2$) no energy is propagated. By interaction of such waves of different direction energy may be transported, however.

d). No energy is transported by interaction of Kelvin-waves and Poincaré-waves nor by interaction of Poincaré-waves of different index.

If the region under consideration is infinite: $\pi b_1 < y < \infty$, and ζ, u and v remain finite for $y \to \infty$ we have $C_n^{+} = 0$ for those values of $n \geq 1$ for which $n^2 > \omega^2 - \Omega^2$. Hence the third sum of (5.3) vanishes in this case.

---

8) If it happens that $\omega^2 - \Omega^2 = N^2$ (N a positive integer), a term $\frac{\pi}{4N^2}\,\mathrm{Im}(C_N\overline{D}_N)$, where C_N, D_N are the constants of the solution (2.5), should be adjoined.

References.

Proudman, J. (1954), Note on the dynamics of storm surges. Mon.Not. Roy.Astr.Soc., Geop. Suppl.,7, 44.

Schönfeld, J.C. (1955), Tides and storm surges in a shallow sea. Rapport C.S.D.55-3-N, Rijkswaterstaat, The Hague.

Bromwich, J.T.I'A. (1949), Infinite Series, IInd ed. London.

Dalzell, D.P. (1945), On the completeness of a series of normal orthogonal functions, J.Lond.Math.Soc.20,87.

Tricomi, F. (1955), Vorlesungen über Orthogonalreihen. Berlin-Göttingen-Heidelberg.