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THE THEORY OF DISTRIBUTIONS

by

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0. Introduction

In this report we shall outline the main notions of the theory of distributions. A distribution is defined as a continuous linear functional on a sequentially normed space or a union of such spaces. The elements of the fundamental space Φ are real or complex functions of n real or complex variables in a subset of a real or complex R_n. These functions are called testing functions. As a rule we consider only real testing functions $\varphi(x)$ of a single unrestricted real variable x. However, generalizations will be briefly indicated.

In this report free use will be made of the material of the two preceding reports TW 80 and TW 81 (Topological foundations of the theory of distributions) which will be quoted as FT followed by the number of the section or of the theorem.

The material is mainly derived from Gel'fand and Shilov: Generalized functions Vol.II (Moscow 1958). The many discussions with the members of the applied mathematics department and the pure mathematics department have led to a number of improvements and simplifications of the original text. In this report an advanced theory of the distributions is given. Many more elementary properties are left out of consideration or are mentioned briefly as examples. However, for supplementary material the reader is referred to the first volume of Gel'fand and Shilov in which the elementary theory of distributions is treated in considerable detail.

1. Distributions on K(a)

The space K(a) consists of all infinitely differentiable functions $\varphi(x)$ of the real variable x $(-\infty,\infty)$ which vanish outside the interval (-a,a). In TF 6 it has been proved that K(a) is a complete sequentially normed space with the following sequence of norms

$$\|\varphi\|_p = \sup_{x,q} \left|\varphi^{(q)}(x)\right| \quad , \qquad q \leqq p. \tag{1.1}$$

According to TF 9 this space is also perfect which means that any bounded sequence φ_n (n=1,2,...) contains a subsequence which converges in K(a).

The continuous linear functionals (f,φ) on K(a) are called distributions or generalized functions. The elements of K(a) are also called testing functions.

Example

$$\varphi(x,\varepsilon) = \begin{cases} \exp - \dfrac{\varepsilon^2}{\varepsilon^2 - x^2} & \text{for } -\varepsilon < x < \varepsilon, \\ 0 & \text{elsewhere}, \end{cases} \tag{1.2}$$

represents a testing function for $\varepsilon \leqq a$.

If f(x) is a locally integrable function a distribution is defined by means of

$$(f,\varphi) = \int f(x)\varphi(x)dx. \tag{1.3}$$

A distribution of this type is said to be regular. A distribution which is not regular is said to be singular.

Dirac's delta-function $\delta(x)$ is defined by

$$(\delta(x),\varphi(x)) = \varphi(0). \tag{1.4}$$

Obviously this is a continuous linear functional. It can be easily shown that $\delta(x)$ is a singular distribution.

More generally a distribution is defined by means of

$$(f,\varphi) = \int f_o(x)\ \varphi^{(p)}\ (x)dx, \tag{1.5}$$

where $f_o(x)$ is a locally integrable function. According to TF 7 all distributions on K(a) are necessarily of this form. The lowest integer p for which (1.5) holds is called the order of the distribution f.

Example

$$(\delta,\varphi) = -\int \theta(x)\ \varphi'(x)dx,$$

where $\theta(x)$ denotes the unit-step function

$$\theta(x)=1 \text{ for } x \geqq 0, \qquad \theta(x)=0 \text{ for } x < 0.$$

It is obvious that aequivalent locally integrable functions, i.e. functions differing only on a null-set, determine the same regular distribution. But also the inverse is true.

Theorem 1.1

The values of a regular distribution determine uniquely a class of aequivalent functions f(x).

Proof

It is sufficient to consider the null-distribution

$$(f,\varphi) = \int f(x)\varphi(x)dx \equiv 0.$$

Introducing the integral of f(x)

$$F(x) = \int_{-a}^{x} f(\xi)d\xi$$

which is a continuous function we obtain by partial integration

$$\int_{-a}^{a} F(x)\ \varphi'(x)dx = 0\ .$$

Applying the well-known lemma of Dubois-Reymond from the calculus of variations it follows that F(x) is a constant so that $f(x) \equiv 0$ a.e.

The support of a testing function $\varphi(x)$ is defined as the closure of the set of those x for which $\varphi(x) \neq 0$.

The distribution f(x) is said to be (locally) zero in the open set Ω if $(f,\varphi)=0$ for all φ whose support is contained in Ω.

The distributions f(x) and g(x) are said to be (locally) equal in the open set Ω if $f-g=0$ in Ω.

The support of a distribution f(x) is defined as the closed set of those x for which no open neighbourhood exists where f=0, i.e. the complement of the open set where f=0.

Examples

1. The testing function (1.2) has the support $-\varepsilon \leqq x \leqq \varepsilon$.
2. $\delta(x)=0$ for $x > 0$ and $x < 0$.
3. The support of $\delta(x)$ is the single point x=0.

Next we consider the convergence in the conjugate space K'(a). Since K(a) is a perfect space weak convergence and strong convergence are identical (cf. TF theorem 9.2). According to TF 7 a sequence of distributions f_n converges to the distribution f if

$$(1.6) \qquad (f_n,\varphi)\longrightarrow(f,\varphi) \qquad \text{for all} \quad \varphi\in K(a).$$

According to TF theorem 7.5 K'(a) is complete. Hence if for each $\varphi \in K(a)$ the sequence (f_n,φ) converges to some limit $f(\varphi)$ then also $f(\varphi)$ is a distribution of K'(a).

If f_n is a sequence of regular distributions it may happen that f_n converges in distributional sense to a distribution f and in ordinary sense to a locally integrable function g(x). It is not generally true that f equals the regular distribution g. We have, however, the following criterion which may be derived from a well-known theorem of Lebesque.

Theorem 1.2

If $f_n(x)$ is a sequence of locally integrable functions which converges a.e. to a limit f(x) and if

$$\left| f_n(x) \right| \leqq g(x), \quad \text{for all } n,$$

where g(x) is a locally integrable function, then also

$$f_n(x)\longrightarrow f(x)$$

in distributional sense.

In many important cases a sequence of ordinary functions does not converge in the classical sense but still has a distributional limit.

Examples

1. $$\frac{\varepsilon}{x^2+\varepsilon^2} \longrightarrow \pi\,\delta(x) \qquad \text{for } \varepsilon \longrightarrow +0.$$

2. $$\frac{\sin n x}{x} \longrightarrow \pi\,\delta(x) \qquad \text{for } n \longrightarrow \infty .$$

If $c(x)$ is an infinitely differentiable function then $\varphi(x) \in K(a)$ implies $c(x)\varphi(x) \in K(a)$. Accordingly the product of a distribution $f(x)$ with an infinitely differentiable function $c(x)$ is defined as

$$(c\, f,\varphi) = (f,c\,\varphi). \tag{1.7}$$

This definition is clearly consistent with that of the product of two functions. The simple proof that the operator $c(x)$ is continuous is left to the reader.

The product of two arbitrary distributions is not defined. Expressions like $\delta^2(x)$ are therefore meaningless.

The derivative of a distribution f is defined by

$$(f',\varphi) = (f,-\varphi'). \tag{1.8}$$

The consistence of this definition is obvious. The continuity of the differential operator d/dx follows at once from (1.8). Explicitly this means that

$$f_n \longrightarrow f \quad \text{implies} \quad \frac{df_n}{dx} \longrightarrow \frac{df}{dx} . \tag{1.9}$$

It can easily be shown that the usual rules of the differential calculus also hold for distributions. We have e.g. for the derivative of the product of an infinitely differentiable function $c(x)$ and a distribution $f(x)$

(1.10) $$\frac{d}{dx}(c\, f) = c'f + c\, f'.$$

In many cases by differentiation of regular distributions singular distributions are obtained.

Examples

1. $$\theta'(x) = \delta(x)\ .$$

2. $$\frac{d}{dx} \ln |x| = 1/x\ .$$

3. $$\frac{d}{dx} |x|^{-\frac{1}{2}} = -\tfrac{1}{2}\, |x|^{-\frac{3}{2}} \operatorname{sgn} x.$$

4. $$\frac{d}{dx} \ln |\operatorname{tg} \tfrac{1}{2}x| = 1/\sin x.$$

From (1.5) it follows that any distribution is the derivative of some order p of a regular distribution. The least number p for which this is true is the order of the distribution. Clearly differentiation raises the order of a distribution. The infinitely differentiable functions may be considered as regular distributions of order minus infinity.

Theorem 1.3
A singular distribution the suppost of which is the single point x=0 is of the form

$$f(x) = \sum_{q=0}^{p} a_q\ \delta^{(q)}(x)\ .$$

Proof
Let f(x) be the (p+1)th derivative of the ordinary function F(x). Then for $x > 0$ and for $x < 0$ F(x) equals polynomials of degree p. The required results follows by differentiating p+1 times.

The testing space K(a) can be extended to the wider space K which is the union of all K(a). According to TF 10 convergence in K means convergence in some subspace K(a). Explicitly $\varphi_n \longrightarrow 0$ for $n \longrightarrow \infty$ means

1^o all φ_n are contained in the same interval (-a,a), i.e. the φ_n have uniformly bounded supports.

2^o $$\sup_x \left|\varphi_n^{(q)}(x)\right| \longrightarrow 0 \quad \text{for } q=0,1,2,\ldots \quad .$$

The continuity of a linear functional on K means continuity with respect to each subspace K(a).
The intersection of all K'(a) gives the space of distributions K'. According to TF theorem 10.1 the space K' is complete with respect to (weak) convergence.

2. The testing space $K\{M_p\}$

In this section we consider testing spaces of a very general type which contain K(a) and K as special cases or as subspaces.

We consider a linear topological space Φ consisting of infinitely differentiable functions $\varphi(x)$ which are defined for all (real) x. It is assumed that Φ is a sequentially normed space or a union of such spaces. Further the following natural assumption is made: $\varphi_n \longrightarrow \varphi_o$ in the topology of Φ implies lim $\varphi_n(x) = \varphi_o(x)$ at each individual x.

The continuous linear functionals on Φ are called distributions and can be treated in much the same way as in the previous section. If Φ is also a perfect space a number of simplifications can be obtained as has been shown in the previous section.

Theorem 2.1

If the s.n.s. Φ contains K(a) as a subspace then $\varphi_n \longrightarrow 0$ in the topology of K(a) implies $\varphi_n \longrightarrow 0$ in the topology of Φ .

Proof

It will be shown first that the topologies in K(a) and Φ are concordant in the sense of TF 8. In fact, if $\varphi_n \longrightarrow 0$ in K(a) then lim $\varphi_n(x)=0$ at each x. If $\varphi_n \longrightarrow \varphi_o$ in Φ then

$\varphi_n \longrightarrow \varphi_o$ pointwise, so $\varphi_o(x)$ must be the zero function. The same argument applies in case $\varphi_n \longrightarrow 0$ in Φ and $\varphi_n \longrightarrow \varphi_o$ in K(a). Now TF theorem 8.3 can be applied.

As in the previous section a regular distribution is of the type

$$(f,\varphi) = \int f(x)\varphi(x)dx, \tag{2.1}$$

where f(x) is a locally integrable function. However, this representation imposes some condition on the behaviour of f(x) at infinity which depends on the corresponding behaviour of the testing functions.

It is clear that theorem 1.1 also holds for any testing space containing a K(a) or K.

By the following construction a complete s.n.s. of a very general kind is obtained. This space, called $K\{M_p\}$ depends on a sequence of positive functions $M_o(x), M_1(x), \ldots$ satisfying the inequalities

$$1 \leq M_o(x) \leq M_1(x) \leq \ldots \leq M_p(x) \leq \ldots \, . \tag{2.2}$$

These functions take on a finite or infinite value and it will be assumed that for any fixed x either all $M_p(x)$ are finite or all are infinite. Further we assume that the $M_p(x)$ are continuous where they are finite; thus the set of points where the $M_p(x)$ are finite is open.

The space $K\{M_p\}$ is formed by all infinitely differentiable functions such that for each p and q

$$M_p(x)\,\varphi^{(q)}(x)$$

is continuous and bounded on $-\infty < x < \infty$. This means that $\varphi(x)$ and all its derivatives vanish where ever $M_p(x)$ is infinite.

In $K\{M_p\}$ the following sequence of norms is introduced

$$\|\varphi\|_p = \sup_{x,q} M_p(x)\left|\varphi^{(q)}(x)\right| \, , \; q \leq p, \tag{2.3}$$

It will be shown below that these norms are concordant and that $K\{M_p\}$ is a complete sequentially normed space. Further it will be proved that with some fairly general restriction $K\{M_p\}$ is also a perfect space.

Example

$$M_p(x) \begin{cases} 1 & \text{for } |x| < a, \\ +\infty & \text{for } |x| \geqq a, \end{cases}$$

gives the particular case K(a). Another important case is $M_p(x) = (1+x^2)^p$ which gives the space S.

We shall now prove that $K\{M_p\}$ is a complete s.n.s. The general idea of the proof is as follows. First we consider the set Φ_p of all $\varphi(x)$ which have continuous derivatives up to the p-th order and for which the expressions $M_p(x)\ \varphi^{(q)}(x)$, $(q=0,1,\ldots,p)$ are continuous and bounded on the x-line. Using two lemmas relating pointwise convergence to convergence in norm for sequences $\varphi_n(x)$ it is shown that Φ_p is complete. If further $\overline{\Phi}_p$ is the completion of $K\{M_p\}$ with respect to the p-th norm it turns out that $\Phi_p \subset \overline{\Phi}_p$. Since $\bigcap_{p=1}^{\infty} \overline{\Phi}_p = K\{M_p\}$ it follows that also $\bigcap_{p=1}^{\infty} \Phi_p = K\{M_p\}$. Together with the concordance of the norms which easily follows from the second lemma this shows that $K\{M_p\}$ is a complete s.n.s.

$\overline{\Phi}_p$ is a normed space with the norm (2.3). Let $\varphi_n(x)$ be a f.s. in this space; then the sequence $\varphi_n(x)$ converges uniformly in each bounded interval to a limit function $\varphi_o(x)$. Similarly $\varphi_n^{(q)}(x) \longrightarrow \varphi_o^{(q)}(x)$ for q=1,2,...,p. The following lemma states that $\varphi_o(x) \in \Phi_p$.

Lemma 1

If the sequence $\varphi_n(x) \in \overline{\Phi}_p$ together with its derivatives up to the p-th order converges uniformly in each bounded interval and if $\|\varphi_n\|_p \leqq C$ for all n then the limit function $\varphi_o(x)$ also belongs to Φ_p and $\|\varphi_o\|_p \leqq C$.

Proof

It is obvious that $\varphi_o(x)$ has continuous derivatives up to the p-th order. Let x be a point where $M_p(x) < \infty$.
Since for each q

$$\left|\varphi_n^{(q)}(x)\right| \leqq C/M_p(x) \quad .$$

the same holds for $\varphi_o^{(q)}(x)$ and hence

$$M_p(x) \quad \left|\varphi_o^{(q)}(x)\right| \leqq C \qquad \text{for } q=0,1,\ldots,p.$$

At those x where $M_p(x)=\infty$ all $\varphi_n^{(q)}(x)$ are zero for $q \leqq p$ which implies the same for $\varphi_o^{(q)}(x)$.
Hence $\|\varphi_o\|_p \leqq C$.

Lemma 2

If $\varphi_n(x)$ is a f.s. in $\overline{\Phi}_p$ with $\lim\limits_{\overline{n\to\infty}} \varphi_n(x)=0$ for each x then $\varphi_n \longrightarrow 0$ in the norm of $\overline{\Phi}_p$.

Proof

As in lemma 1 we have $\varphi_n^{(q)}(x) \longrightarrow \varphi_o^{(q)}(x)$ uniformly in each bounded interval for $q \leqq p$.
For any given $\varepsilon > 0$ there is an index n_o such that for $m,n > n_o$ $\|\varphi_n - \varphi_m\| \leqq \varepsilon$. Applying lemma 1 on the sequence $\varphi_n - \varphi_m$ with n fixed and $m \longrightarrow \infty$ it follows that $\|\varphi_n\|_p \leqq \varepsilon$, $n \geqq n_o$.
Finally $\varepsilon \longrightarrow 0$.

Theorem 2.2

The normed space $\overline{\Phi}_p$ is complete.

Proof

Follows easily from lemma 1 and lemma 2.

Corollary

The completion Φ_p of $\Phi = K\{M_p\}$ with respect to the norm $\|\varphi\|_p$ is a subspace of $\overline{\Phi}_p$.

It is obvious that the intersection of all $\overline{\Phi}_p$ coincides with $K(M_p)$. However, in view of the corollary above also

(2.4) $$K\{M_p\} = \bigcap_p^{\infty} \Phi_p.$$

<u>Lemma 3</u>

The following two norms

$$\|\varphi\|_{p_j} = \sup_{x,q} M_{p_j}(x) \left|\varphi^{(q)}(x)\right| , \quad q \leqq p_j , \; j=1 \text{ and } j=2,$$

are concordant.

<u>Proof</u>

Let $\varphi_n(x)$ be a f.s. with respect to both norms which converges to zero in the first norm. Then $\lim_{n\to\infty} \varphi_n(x)=0$ for each x. Now apply lemma 2 on the second norm.

The relation (2.4) and the concordance of the norms $\|\varphi\|_p$ prove that $K\{M_p\}$ is a complete s.n.s.

<u>Theorem 2.3</u>

The space $K\{M_p\}$ is a complete s.n.s.

We shall now show that $K\{M_p\}$ is a perfect space if the following condition, called the P-condition, is satisfied.

<u>Condition P</u>

For each index p there exists an index $p' > p$ with the following property.

For any given $\varepsilon > 0$ we may find an $N(\varepsilon)$ such that

(2.5) $$M_p(x) < \varepsilon \; M_p(x)$$

if

$$|x| > N \quad \text{or} \quad M_p(x) > N.$$

From this condition it follows that the expressions $M_p(x)\,\varphi^{(q)}(x)$ are not only bounded on the x-line but that they tend to zero for $|x| \to \infty$ or $M_p(x) \to \infty$. In fact, if this were not true for some combination p,q there would be a positive constant C and a sequence x_n with either $|x_n| \to \infty$ or $M_p(x) \to \infty$

such that

$$M_p(x_n) \left| \varphi^{(q)}(x_n) \right| > C.$$

But then according to the condition P an index p' and a sequence $\varepsilon_n \longrightarrow 0$ exists with

$$M_p(x_n) \leqq \varepsilon_n M_{p'}(x_n).$$

However, this would imply

$$M_{p'}(x_n) \left| \varphi^q(x_n) \right| > \frac{C}{\varepsilon_n} \longrightarrow \infty$$

which is in contradiction to the fact that $\varphi \in K\{M_p\}$.

In the proof of the perfectness of $K\{M_p\}$ we require the following lemma

Lemma 4

Every sequence $\varphi_n(x)$ which is bounded with respect to each norm $\|\varphi\|_p$ and which converges to zero for each x converges to the zero-function with respect to each norm.

Proof

Starting with a fixed index p we determine an index p' according to the condition P. Suppose that $\|\varphi_n\|_{p'} = C$ for all n. For any given $\varepsilon > 0$ we may find a constant N such that for $|x| > N$ and for $M_p(x) > N$

$$M_p(x) \leqq \frac{\varepsilon}{C} M_{p'}(x).$$

This implies for $q \leqq p$

$$M_p(x) \left| \varphi^{(q)}(x) \right| \leqq \frac{\varepsilon}{C} M_{p'}(x) \left| \varphi^{(q)}(x) \right| \leqq \frac{\varepsilon}{C} \|\varphi\|_{p'} \leqq \varepsilon .$$

For those x where $M_p(x)=M_{p'}(x)=\infty$ we have

$$M_p(x) \left| \varphi^{(q)}(x) \right| =0.$$

For the remaining x, which form a compact set, an index n_o may be found such that for $n \geq n_o$ and all $q \leq p$

$$M_p(x) \quad \left|\varphi_n^{(q)}(x)\right| \leq \varepsilon \ .$$

The latter inequality now holds for all x. Thus for $n > n_o$

$$\|\varphi_n\|_p = \sup_x M_p(x) \quad \left|\varphi^{(q)}(x)\right| \leq \varepsilon \, .$$

Since ε is arbitrary this means that $\|\varphi_n\|_p \longrightarrow 0$ for all p.

Corollary

If $\varphi_n(x)$ is bounded in each norm and if $\varphi_n(x) \longrightarrow \varphi_o(x)$ pointwise then $\varphi_o(x) \in K\{M_p\}$ and $\varphi_n(x) \longrightarrow \varphi_o(x)$ in the topology of $K\{M_p\}$.

Proof

According to lemmas 1 and 2 $\varphi_o(x) \in \Phi_p$ for each p and therefore also $\varphi_o(x) \in K\{M_p\} = \bigcap_p^{\infty} \Phi_p$. The sequence $\varphi_o - \varphi_n$ is bounded in each norm and converges pointwise to zero. Lemma 3 states that $\|\varphi_n - \varphi_o\|_p \longrightarrow 0$ for each p which means that $\varphi_n \longrightarrow \varphi_o$ in the topology of $K\{M_p\}$.

Theorem 2.4

The space $K\{M_p\}$ with the condition P is perfect.

Proof

We shall show that each bounded set is compact. In view of lemma 4 it is sufficient to prove that an arbitrary bounded sequence contains a subsequence that is pointwise convergent.
Let the sequence φ_n be bounded in each norm $\| \ \|_p$; then $\varphi_n^{(1)}(x)$ is uniformly bounded on account of the inequality

$$\left|\varphi_n^{(1)}(x)\right| \leqslant \sup_{q \leqslant p;x} M_p(x) \left|\varphi_n^{(q)}(x)\right| < C_p .$$

Moreover, from the mean value theorem we find

$$\left|\varphi_n(x')-\varphi_n(x'')\right| = \left|\varphi_n'(\xi) \cdot x'-x''\right| < C\left|x'-x''\right| ,$$

so that φ_n is equicontinuous.
The theorem of Arzela-Ascoli yields that φ_n contains a subsequence that is pointwise convergent.

An important particular case of the space $K\{M_p\}$ is obtained by taking

$$M_p(x) = (1+x^2)^p . \tag{2.7}$$

This space is called S and contains these infinitely differentiable functions $\varphi(x)$ for which all expressions

$$x^q \quad \varphi^{(p)}(x) \tag{2.8}$$

are bounded. This means that $\varphi(x)$ and each derivative vanishes at infinity faster than any power of x. This space satisfies the condition N which means that it is a perfect space. Obviously we have $S \supset K$. It is easy to show that K is a subspace of S which is dense in S. In fact, let h(x) be some infinitely differentiable function which equals 1 for $|x| \leqq 1$ and zero for $|x| \geqq 0$. Putting

$$m_p = \max_{q \leqq p} \left|h^{(q)}(x)\right| ,$$

we have for n=1,2,3,...

$$\max_{q \leqq p} \left|h^{(q)}\left(\frac{x}{n}\right)\right| \leqq m_p .$$

Then for any element $\varphi(x)$ of S a sequence of finite testing functions $\varphi_n(x) = \varphi(x)h(\frac{x}{n})$ can be constructed which converges to $\varphi(x)$ in the ordinary sense. According to lemma 4 there remains to prove the boundedness of the norms. In fact

$$M_p(x)\left|\varphi_n^{(q)}(x)\right| = M_p(x)\left|\sum_k \binom{q}{k} h^{(k)}\left(\frac{x}{n}\right)\varphi^{(q-k)}(x)\right| \leqq$$

$$\leqq \sum_k \binom{q}{k} m_p\, M_p(x)\left|\varphi^{(q-k)}(x)\right| \leqq C_p\, m_p \|\varphi\|_p.$$

Of course the same property holds for any $K\{M_p\}$ space where the M_p-functions are everywhere finite.

3. Distributions on $K\{M_p\}$

We consider distributions with respect to a perfect space $K\{M_p\}$ satisfying the condition P of the preceding section.
If Φ_p denotes the completion of $\Phi = K\{M_p\}$ with respect to the p-th norm then for any distribution f on Φ there exists an index p such that it is continuous with respect to Φ_p. We know that then f is continuous on all completions with a larger index. The least possible index p is called the order of the distribution. The space Φ_p is an isometric subspace of the space $\overline{\Phi}_p$ of all functions $\varphi(x)$ for which the expressions $M_p(x)|\varphi(x)|$ are continuous and bounded on the x-line. The latter space may be considered as a closed subspace of the direct sum Ψ of p+1 spaces of functions which are integrable with the fixed weight-function $M_p(x)$. According to the theorem of Hahn and Banach the functional f on Φ_p may be extended to Ψ with preservation of its norm. The general form of a continuous linear functional on Ψ is of the form

$$\sum_{q \leqq p} \int M_p(x)\,\varphi^{(q)}(x)\, d\sigma_q(x),$$

where the $\sigma_q(x)$ are measures concentrated at those points where the $M_p(x)$ are finite. The norm of f equals $\sum_q \operatorname{Var} \sigma_q(x)$

Theorem 3.1

Any continuous linear functional f on $K\{M_p\}$ which satisfies the condition P is of the form

(3.1) $$(f,\varphi) = \sum_{q \leqq p} \int M_p(x)\, \varphi^{(q)}(x)\, d\,\epsilon_q(x),$$

where the $\epsilon_q(x)$ are measures. Further

(3.2) $$\|f\|_p = \sum_{q \leqq p} \text{Var}\ \epsilon_q(x).$$

Specialization of this result for the space K(a) yields the representation (1.5) which is easily obtained from (3.1) by partial integration.

In a similar way any continuous linear functional on S may be represented by

(3.3) $$(f,\varphi) = \int f_o(x)\, \varphi^{(p)}(x)dx\ ,$$

where

$$f_o(x) = O(|x|^p) \quad \text{for} \quad |x| \longrightarrow \infty .$$

This may be interpreted also on the following way. Any distribution of S' is the derivative of some order of a continuous function with a finite algebraical growth at infinity.

If f is a distribution of S' with a bounded support, say within (-a,a), the representation (3.3) holds with a finite function $f_o(x)$ vanishing for $|x| > a+\varepsilon$ where ε is an arbitrarily small positive number. This result may be easily derived in the following way. Let us take an arbitrary infinitely differentiable function h(x) which equals 1 for $|x| \leqq a$ and which vanishes outside $|x| \leqq a+\varepsilon$. Then obviously $(f,\varphi)=(f,\varphi h)$ so that the desired representation can be obtained from (3.3) by replacing φ by φh followed by a simple reduction.

In the following we shall use mostly the testing spaces K(a), K and S. We note that $S \supset K$ so that S' contains less distributions than K'. E.g. the function exp $|x|$ is a regular functional of K' but it does not belong to S.

4. Generalizations

In this section we shall consider testing spaces of complex functions with complex distributions § and testing spaces with functions of several variables.

The natural element of a complex testing space Ψ is a holomorphic analytic function $\psi(z)$ which satisfies some condition at infinity.

Example

The space Z(a) consisting of all holomorphic functions $\psi(z)$ which satisfy the following inequalities

$$(4.1) \qquad |z^k \ \psi(z)| \leq C_k(\psi) \exp a|y| \ , \ z=x+iy.$$

In this space the topology is given by the following sequence of norms

$$(4.2) \qquad \|\psi\|_p = \sup_{k\leq p} \left|z^k \psi(x+iy)\right| e^{-a|y|} \ , \ p=0,1,2,\dots \ .$$

In a similar way as for K(a) it can be shown that with these norms Z(a) is a perfect complete s.n.s. Also we may consider the space $Z=\bigcup_a Z(a)$ and more generally $Z\{M_p\}$ for which all expressions $\sup_z M_p(z)|\psi(z)|$ are finite.

The complex distributions with respect to some complex testing space Ψ are defined as the continuous linear functionals on Ψ.

A distribution (g,ψ) is said to be of regular type if it is of the form

$$(4.3) \qquad (g,\psi) = \int_{-\infty}^{\infty} \overline{g(x)}\,\psi(x)dx,$$

where g(x) is a complex-valued locally integrable function on the real axis. In this case the function g(x) and the functional g may be identified.

A distribution is said to be of the analytic type if more generally

$$(4.4) \qquad (g,\psi) = \int_C \overline{g(z)}\,\psi(z)dz$$

on some contour C in the complex z-plane.
A distribution which is neither regular nor analytic is called singular.

Example

Defining the complex delta-function $\delta(z-c)$ by the functional $\psi(z) \longrightarrow \psi(c)$ a distribution is obtained which is not regular but which is analytic since

$$\psi(c) = \oint \frac{1}{2\pi i} \frac{\psi(z)}{z-c} dz , \tag{4.5}$$

where C is a closed circular contour around c.

We note that here the linear operations on distributions are given by

$$(g_1+g_2,\psi) = (g_1,\psi) + (g_2,\psi) \tag{4.6}$$

and

$$(\alpha\ g,\psi) = \bar{\alpha}\,(g,\psi). \tag{4.7}$$

If c(z) is a holomorphic function such that $\psi \in \Psi$ implies $\bar{c}\,\psi \in \psi$ then the product of c(z) with a distribution g(z) is defined as

$$(c\ g,\psi) = (g,\bar{c}\,\psi) \tag{4.8}$$

As a second generalization we shall consider distributions of n variables. In order not to complicate matters we shall consider real-valued distributions on real testing functions of n real variables $x_1,x_2,\dots,x_n$. Then the following simplifying notations due to L. Schwartz may be used.

x denotes the vector $(x_1,x_2,\dots,x_n)$.

q denotes the set of integers $(q_1,q_2,\dots,q_n)$.

$$|q| \overset{\text{def}}{=} q_1 + q_2 + \dots + q_n .$$

$$D^q \overset{\text{def}}{=} \frac{\partial^{|q|}}{\partial x_1^{q_1} \dots \partial x_n^{q_n}} .$$

$$x^k \overset{\text{def}}{=} x_1^{k_1} \dots x_n^{k_n} .$$

The space K(a) is defined by the set of all infinitely differentiable testing functions $\psi(x)$ vanishing outside the block $-a_i < x_i < a_i$ $(i=1,2,\dots,n)$. The norms are given by

$$\|\varphi\|_p = \sup_{\substack{x \\ |q|\leq p}} \left| D^q \varphi(x)\right| \qquad (p=0,1,2,\dots) . \tag{4.9}$$

The space S is defined by the set of all infinitely differentiable testing functions $\psi(x)$ vanishing at infinity with all derivatives faster than any power of $1/|x|$. The norms are given by

$$\|\varphi\|_p = \sup_{\substack{x \\ |k|,|q|\leq p}} \left| x^k D^q \varphi(x)\right| \qquad (p=0,1,2,\dots). \tag{4.10}$$

The n-dimensional distributions can be treated in almost the same way as the one-dimensional ones. The distributions on $K\{M_p\}$ are of the following form

$$(f,\varphi) = \sum_{|q|\leq p} \int M_p(x)\, D^q \varphi(x)\, f_q(x)dx \tag{4.11}$$

which is only a slight formal generalization of (3.2).

5. Fourier transforms in S

We consider the complex testing space S consisting of all complex-valued infinitely differentiable testing functions $\varphi(x)$ of a single real variable with the property that

$$x^k \varphi^{(q)}(x) \longrightarrow 0 \quad \text{as} \quad |x| \longrightarrow \infty \tag{5.1}$$

for all $k,q = 0,1,2,\dots$.

A simple argument shows that the Fourier transform $\psi(\sigma)$ of $\varphi(x)$ as defined by

$$F(\varphi) \equiv \psi(\sigma) \equiv \widetilde{\varphi(x)} \stackrel{\text{def}}{=} \int e^{i x \sigma} \varphi(x)dx \tag{5.2}$$

also belongs to S.

This means that Fourier transformation is a 1-1 mapping of S onto itself. The inverse transformation is given by

$$F^{-1}(\psi) \equiv \varphi(x) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int e^{-i x \sigma} \psi(\sigma) d\sigma . \tag{5.3}$$

We mention the well-known rules

(5.4) $$\frac{d}{dx} F(\varphi) = F(i\, x\, \varphi),$$

and

(5.5) $$F(\frac{d}{dx}\varphi) = -i\sigma\, F(\varphi).$$

The operator F is continuous on S. This important fact follows easily from the following estimate. We have

$$(-i\sigma)^k \psi^{(q)}(\sigma) = \int (\frac{d}{dx})^k \{(ix)^q \varphi(x)\}\, e^{i\,x\sigma}\, dx =$$

$$= \sum_{j=0}^{k} C_j\, i^q \int x^{q-j}\, \varphi^{(p-j)}(x) e^{i\,x\sigma} dx,$$

where the C_j are certain positive constants. Hence for $k+q \leqq p$

$$\left|\sigma^k\ \psi^{(q)}(\sigma)\right| = \sum_{j=0}^{k} C_j \int \frac{\left|x^{q-j+2}\ \varphi^{(k-j)}(x)\right|}{1+x^2}\, dx \leqq$$

$$= \sum_{j=0}^{k} C'_j\ \|\varphi\|_{p+2} \qquad \text{so that}$$

$$\|\psi\|_p \leqq C_p\ \|\varphi\|_{p+2}.$$

The same argument shows the continuity of the inverse operator F^{-1}. However, this would also follow from Banach's theorem on the inverse operator (cf. TF theorem 8.2). Hence we have proved

<u>Theorem 5.1</u>

The Fourier operator F and F^{-1} are continuous linear operators on S mapping S onto itself.

The Fourier transform F(f) or $g(\sigma)$ of a distribution $f(x) \in S'$ is defined by the Parseval relation

(5.6) $$(F(f), F(\varphi)) = 2\pi (f,\varphi).$$

The consistency of this definition with the ordinary definition of the Fourier transform of an absolutely integrable function can be shown as follows.

$$(f,\psi)=\int \overline{f(x)}\varphi(x)dx=\frac{1}{2\pi}\int \overline{f(x)}\left\{\int \psi(\sigma)e^{-ix\sigma}d\sigma\right\}dx =$$

$$=\frac{1}{2\pi}\int (\sigma)\left\{\int \overline{f(x)\ \exp ix\sigma}\ dx\right\}d\sigma = \frac{1}{2\pi}\int \overline{g(\sigma)}\,\psi(\sigma)d\sigma =$$

$$=\frac{1}{2\pi}(g,\psi).$$

From the definition (5.6) it follows easily that the rules (5.4) and (5.5) also hold for distributions.

(5.7) $$\frac{d}{dx}F(f) = F(ixf)\,,\quad F\left(\frac{df}{dx}\right) = -i\sigma F(f).$$

Example

(5.8) $$F(\delta(x))=1 \qquad F(x^k)=\left(-i\frac{d}{dx}\right)^k \delta(\sigma).$$

Let $f(x)$ be a regular distribution which is aequivalent to a locally integrable function of finite algebraical order at infinity, i.e.

(5.9) $$|f(x)| < C(1+|x|)^p$$

for some constants C and p, then denoting the truncated function at $(-a,a)$ by $f_a(x)$, i.e.

(5.10) $$f_a(x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{for } |x| \leqq a, \\ 0 & \text{for } |x| > a, \end{cases}$$

we obviously have

(5.11) $$f_a(x) \longrightarrow f(x) \quad \text{for } a \longrightarrow +\infty$$

in the topology of S'. From the continuity of the Fourier operator it now follows that

(5.12) $$F(f) = \lim_{a\longrightarrow\infty} \int_{-a}^{a} e^{ix\sigma} f(x)dx.$$

Theorem 5.2

If $f(x)$ is a distribution with a bounded support then

(5.13) $$F(f) = (\overline{f},\ \exp i\sigma x),$$

where $\exp i\sigma$ stands for any testing function which equals $\exp i\sigma$ in an open interval which contains the support of f.

Proof

If f is a finite regular distribution the relation (5.13) holds of course. Otherwise f is the derivative of some ordinary function vanishing outside an open covering of the support of f and we may apply (5.7).

6. Translation and convolutions in S

The translation operator T_h in the one-dimensional testing space S defined by

$$(6.1) \qquad T_h \varphi(x) = \varphi(x+h)$$

is a continuous operator. In fact, we have for $k,q \leqq p$

$$\sup \left| x^k \varphi^{(q)}(x+h) \right| = \sup \left| (x-h)^k \varphi^{(q)}(x) \right| \leqq$$
$$\leqq \sum_{j=0}^{k} \binom{k}{j} h^{k-j} \|\varphi\|_p \leqq (1+h)^p \|\varphi\|_p .$$

The continuity of T_h can be shown to be even uniform in each bounded interval of the parameter h. This means that if A is a bounded set of S also the union of all $T_h A$, $|h| < h_o$ is bounded.

The translation operator T_h is also continuous with respect to the parameter h. This means that for $h \to h_o$ for each $\varphi \in S$ we have $T_h \varphi \longrightarrow T_{h_o} \varphi$ in the topology of S. The proof is straightforward.

However, we may also use the following more general arguments. Let h_n be a sequence converging to h_o then the sequence $\varphi(x+h_n)$ is bounded in S. Since S is perfect the sequence forms a (sequentially) compact set which has a limit. But since the topological limit implies the ordinary pointwise limit there is a unique limit which must be $\varphi(x+h_o)$.

Let φ be a testing function; then for any distribution f the expression

$$f(\xi), \varphi(x+\xi))$$

is a continuous function of x. It may happen that this function is itself a testing function.

Definition

The distribution f_o is said to be a convolutor if

(6.2) $$f_o * \varphi \equiv (f_o(\xi), \varphi(x+\xi)) = \psi(x) \in S$$

for each testing function φ and if $\varphi_n \longrightarrow 0$ implies $f_o * \varphi_n \longrightarrow 0$ in the topology of S.

The Dirac-function $\delta(x-a)$ is a simple example of a convolutor of S since

$$\delta(x-a) * \varphi(x) = \varphi(x+a).$$

The convolution of a distribution f(x) with a convolutor $f_o(x)$ is defined by

(6.3) $$(f_o * f, \varphi) = (f, f_o * \varphi).$$

If f_o and f are ordinary functions which are integrable in $(-\infty, \infty)$ we have

$$(f_o * f, \varphi) = \int \overline{f(x)}dx \int \overline{f_o(\xi)}\varphi(x+\xi)d\xi = \int \overline{f(x)}dx \int \overline{f_o(\eta - x)}\varphi(\eta)d\eta =$$

$$= \int \varphi(\eta)d\eta \int \overline{f(x)f_o(\eta - x)dx},$$

so that

(6.4) $$f_o(x) * f(x) = \int f(\xi)f_o(x-\xi)d\xi \quad ,$$

which shows that the definition (6.3) is consistent with the usual definition of the convolution of functions.

Example

(6.5) $$\delta(x) * f(x) = f(x).$$

(6.6) $$\delta'(x) * f(x) = f'(x).$$

It follows from the definition of the convolutor that the operator $f_o *$ is continuous (in weak sense).

Theorem 6.1

If $f_o(x)$ is a convolutor then the same is true for $f_o'(x)$ and

(6.7) $$\frac{d}{dx}(f_o * f) = f_o' * f = f_o * f' \quad .$$

Proof

From (6.2) it follows at once that $f'_o(x)$ is a convolutor. In view of the continuity of the operator $f_o *$ we have

$$\frac{d}{dx}(f_o * \varphi) = f_o * \frac{d\varphi}{dx} .$$

Then the relation (6.7) follows easily from the definition (6.3).

Theorem 6.2

Every functional of S' with a bounded support is a convolutor of S'.

Proof

Since every functional of S' is the derivative of some order of an ordinary function it is sufficient in view of the preceding theorem to prove the statement for a finite function i.e. a function which vanishes outside a certain interval. But then we have

$$f * \varphi = (f(\xi), \varphi(x+\xi)) = \int_{-a}^{a} f(\xi)\varphi(x+\xi)d\xi$$

and the theorem follows easily.

Theorem 6.3

For a sequence of distributions f_n with a uniformly bounded support converging to the limit f we have

(6.8) $$f_n * g \longrightarrow f * g,$$

where g is an arbitrary functional.

Proof

The only difficult point in the proof of this theorem is to show that for any testing function $\varphi(x)$ $f_n \longrightarrow f$ implies $f_n * \varphi \longrightarrow f * \varphi$. Without loss of generality we may assume that $f \equiv 0$. The set f_n is clearly weakly bounded. According to TF theorem 7.4 this implies boundedness in the strong sense. Then applying TF theorem 7.3 we see that there exists an index p such that all f_n belong to the same normed completion Φ'_p of S.

According to the results of section 3 there exists a uniform representation

(6.9) $$(f_n,\varphi) = \int F_n(x)\ \varphi^{(p)}(x)dx$$

where the $F_n(x)$ are bounded measurable functions vanishing outside some interval, say $F_n(x) \equiv 0$ for $|x| > a$. Since the norms $\|f_n\|_p$ with respect to Φ_p are uniformly bounded $\sup_{x,n} |F_n(x)|$ is finite.

We have further that

$$f_n * \varphi = \int_{-a}^{a} F_n(\xi)\ \varphi^{(p)}(\xi + x)d\xi \quad ,$$

so

$$\| f_n * \varphi\|_m \leq \int_{-a}^{a} |F_n(\xi)|\ \|\varphi^{(p)}(\xi + x)\|_m\ d\xi \ \leq$$

$$\leq 2a \sup_{x,n} |F_n(x)| \sup_{|\xi|\leq a} \|\varphi^{(p)}(\xi + x)\|_m ,$$

or $\|f_n * \varphi\|_m \leq C_m$. Further

$$(f_n * \varphi)^{(p)}(x) = (f_n(\xi), \varphi^{(p)}(\xi + x)) \longrightarrow 0 \quad \text{for each } x.$$

Hence by applying lemma 4 of section 2 it follows that $\|f_n * \varphi\|_m \longrightarrow 0$ for each m.

Corollary

Each distribution may be obtained as the limit of a sequence of testing functions.

Proof

Starting from any sequence of testing functions $f_n(x)$ with support in (-a,a) converging to $\delta(x)$ we observe that

$$f_n(x) * \dot{g}(x) \longrightarrow \delta(x) * g(x) = g(x).$$

We may take e.g.

$$\begin{cases} f_n(x) = \sqrt{\frac{n}{\pi}}\ \dfrac{\exp - n\ tg^2 x}{\cos^2 x} & \text{for } |x| < \frac{1}{2}\pi \ , \\ f_n(x) = 0 & \text{for } |x| \geq \frac{1}{2}\pi \ . \end{cases}$$

7. Fourier transforms and convolutions in S

For ordinary functions $f(x)$ and $g(x)$ of $L(-\infty,\infty)$ the convolution $f * g$ also belongs to the same class and

$$(7.1) \qquad F(f * g) = F(f).F(g).$$

It will be shown that a similar result also holds for distributions.

Theorem 7.1

If f is a regular distribution which is a multiplicator in S then F(f) is a convolutor.

Proof

With a slight change in the notation we write $g(\sigma)=F(f(x))$, $\psi(\sigma)=F(\varphi(x))$ and we assume that $g(\sigma)$ is a multiplicator of S. Then we show that $f(x)$ is a convolutor. In fact

$$f * \varphi = (f(\xi), \varphi(\xi+x)) = \frac{1}{2\pi}(g(\sigma), e^{-i\sigma x}\psi(\sigma)) =$$

$$= \frac{1}{2\pi}\int e^{-i\sigma x}\,\overline{g(\sigma)}\,\psi(\sigma)d\sigma = F^{-1}(\overline{g}\,\psi) \in S,$$

and if $\varphi_n \longrightarrow 0$ then $\psi_n = F(\varphi_n) \longrightarrow 0$ and also $\overline{g}\;\psi_n \longrightarrow 0$ so that $F^{-1}(\overline{g}\;\psi_n) = f * \varphi_n \longrightarrow 0$.

Theorem 7.2

If $g=F(f)$ is a multiplicator of S then

$$(7.2) \qquad F(f * f_1) = F(f).\,F(f_1)\,.$$

Proof

We know already that f is a convolutor. Then

$$(F(f * f_1), \psi) = \frac{1}{2\pi}(f * f_1, \varphi) = \frac{1}{2\pi}(f_1, f * \varphi) =$$

$$= (F(f_1), F(f).F(\varphi)) = (F(f).F(f), \psi).$$

Theorem 7.3

If f is a distribution with a bounded support in S then F(f) is a multiplicator.

Proof

From theorem 5.2 stating that

$$F(f) = (\overline{f}, \exp i\sigma x)$$

it follows that F(f) is infinitely differentiable and that F(f) is of finite algebraical growth at infinity.

8. Fourier transforms in other spaces

We consider a testing space Φ which is contained in S. Simple examples of Φ are K(a) and K. The set of all Fourier transforms of the elements of $\varphi \in \Phi$ will be denoted by $\tilde{\Phi} = F(\Phi) = \Psi$, and its elements by $\tilde{\varphi} = F(\varphi) = \psi$. Obviously Ψ is linearly isomorphic with Φ. The topology of Ψ is defined as follows. The sequence $\psi_n(\sigma) = F(\varphi_n(x))$ converges to zero if $\varphi_n(x) \longrightarrow 0$ in Φ.

The Fourier operator is a continuous linear operator $F(\Phi \longrightarrow \Psi)$ with continuous inverse.

The relations (5.4) and (5.5) hold also in the general case.

The Fourier transform of a distribution is defined as in section 5. We shall now consider the particular space K(a) is more detail. Since we have

$$(8.1) \qquad \psi(s) = \int_{-a}^{a} e^{i x s} \varphi(x) dx,$$

the testing functions of the transformed space F(K(a)) which is called Z(a) are holomorphic functions of the complex variable $s = \sigma + i\tau$ in the entire s-plane. Since each derivative of $\varphi(x)$ is uniformly bounded a simple calculation shows that

$$(8.2) \qquad \left| s^k \psi(s) \right| \leqq C_k \exp a|\tau| .$$

Therefore the topology of Z(a) is that of a s.n.s. with the following set of norms

$$(8.3) \qquad \|\psi\|_p = \sup_{k \leqq p} \left| s^k \psi(\sigma + i\tau) \right| e^{-a|\tau|} .$$

Since K(a) is complete and perfect the same holds for Z(a).

The general representation of a distribution of Z'(a) can easily be obtained from that of K'(a). From the definition (5.6) it follows that

$$(g,\psi)= \int_{-a}^{a} f(x)\, \varphi^{(m)}(x)dx \;.$$

Substitution of

$$\varphi^{(m)}(x) = \int_{-\infty}^{\infty} (-i\sigma)^m \psi(\sigma) e^{-i\,x\sigma}\, d\sigma$$

gives

(8.4) $$(g,\psi) = \int_{-\infty}^{\infty} G(\sigma)\, \psi(\sigma)d\sigma \;,$$

where

$$G(\sigma) = (-i\sigma)^m \int_{-a}^{a} e^{-i\,x\sigma}\, f(x)dx \;,$$

so that G(σ) is a holomorphic function of the order $\leqq 1$ and of type $\leqq$ a which on the real axis has a finite algebraical order of growth at infinity.

9. Generalizations

In this section a number of generalizations will be indicated briefly.

Extension of the Fourier transformation to n-dimensional spaces does not give rise to difficulties. For the S-space of n variables $x_1, x_2, \ldots, x_n$ we define

(9.1) $$\psi = F(\varphi) = \int e^{i(x,\sigma)}\, \varphi(x)dx \;,$$

where

(9.2) $$(x,\sigma) \overset{def}{=} x_1\sigma_1 + x_2\sigma_2 + \ldots + x_n\sigma_n .$$

The inversion of (9.1) is

(9.3) $$\varphi = F^{-1}(\psi) = \frac{1}{(2\pi)^n} \int e^{-i(x,\sigma)}\, \psi(\sigma)d\sigma \;.$$

The extension of the rules (5.4) and (5.5) to partial differentiation is obvious. The continuity of the Fourier operator can be proved as in section 5.
We note that the definition of the Fourier transform of a distribution becomes

(9.4) $$(F(f),F(\varphi)) = (2\pi)^n(f,\varphi).$$

Fourier transformation of the n-dimensional space K(a) gives rise to the space Z(a) of holomorphic functions ψ of n complex variables satisfying

$$|\varphi^k\psi(s)| \leq C_k(\psi)\ \exp\left\{a_1\ |\tau_1| + a_2|\tau_2| + \ldots + a_n|\tau_n|\right\}.$$

F-transformation of K, the union of all K(a) gives Z, the union of all Z(a).
Without proof we mention the following important result.

<u>Theorem 9.1</u>
The equation

(9.5) $$\overline{P}(z_1,z_2,\ldots,z_n)\ f = 1\ ,$$

where P is a polynomial has a solution f in Z'.

<u>Proof</u>
Cf. Gel'fand and Shilov II ch.II § 3.3.

We now consider the translation in an arbitrary testing space Φ . For simplicitly we take the case of a single real variable x. It will be assumed that for all real h the translation operator T_h is determined with

(9.6) $$T_h\ \varphi(x) = \varphi(x+h).$$

Further it will be assumed that the operator T_h is uniformly bounded in any bounded interval $|h| \leq h_o$. This implies that for each fixed h the operator T_h is continuous. Further as in section 6 it can be shown that if Φ is perfect or a union of perfect spaces the operator T_h is continuous with respect to h. As a particular case we note that in Z(a) the translation exists and has the above-mentioned properties.

If Φ is a space with a uniformly bounded translation as explained above the definities of a convolutor can be given

in the usual way.
The results of section 7 can easily be extended to perfect spaces Φ of the above kind but where the translation is not only continuous but also differentiable, i.e. that for each testing function

$$\frac{\varphi(x+h)-\varphi(x)}{h} \longrightarrow \varphi'(x) \quad \text{as } h \longrightarrow 0 \tag{9.7}$$

in the topology of Φ .
For details the reader is referred to Gel'fand and Shilov II ch.III § 3.