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The asymptotic expansion of the statistical distributions of N. V. Smirnov

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by

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Introduction

Let $F(\lambda)$ be a continuous distribution function of the observable stochastic variable λ and let $F_n(\lambda)$ be the corresponding empirical distribution function which is obtained after n independent observations. Smirnov (1) studied the stochastic variable

$$D_n^+ = \sqrt{n} \sup_x (F_n - F) \tag{1.1}$$

and showed that for $0 < \lambda < \sqrt{n}$

$$\phi_n^+(\lambda) = P\left\{D_n^+ < \lambda\right\} = 1 - \frac{\alpha n!}{n^n} \sum_{k>\alpha}^{n} \frac{(k-\alpha)^k (n-k+\alpha)^{n-k-1}}{k!\,(n-k)!}, \tag{1.2}$$

where $\alpha = \lambda\sqrt{n}$.

He also showed that for $n \to \infty$

$$\phi_n^+(\lambda) = 1 - e^{-2\lambda^2}\left\{1 - \frac{2\lambda}{3\sqrt{n}} + O(\frac{1}{n})\right\} \tag{1.3}$$

with $\lambda = O(n^{1/6})$.

In a recent paper Chan Li Tsian (2) was able to extend the expansion (1.3) by two more terms. The fact that both authors used rather complicated methods prompted us to find the asymptotic expansion of $\phi_n^+(\lambda)$ by some other method. It will be shown here that it is sufficient to use only some elementary notions of the theory of analytic functions. In particular we need some properties of the complex transformation z=w exp-w. Our method enables us to extend Smirnov's expansion by any number of terms. In particular Chan Li Tsian's result is corroborated as well. Before sketching our method we give the following result

$$e^{2\lambda^2}\left\{1 - \phi_n^+(\lambda)\right\} = 1 - \frac{He_1(2\lambda)}{3\sqrt{n}} + \frac{3 - He_4(2\lambda)}{36n} - \tag{1.4}$$

$$\frac{15He_1(2\lambda) - 12He_3(2\lambda) - 5He_4(2\lambda)}{540\, n\sqrt{n}} + \frac{45 - 60He_4(2\lambda) + 32He_6(2\lambda) + 5He_8(2\lambda)}{12960n^2} + \ldots,$$

where He_m denotes the m^{th} polynomial of Hermite.

Let the functions $\varphi(z,\alpha)$ and $\psi(z,\alpha)$ be determined by

$$(1.5) \qquad \varphi(z,\alpha) = \alpha \sum_{k=0}^{\infty} \frac{(k+\alpha)^{k-1}}{k!} z^k , \qquad |z| < e^{-1},$$

and

$$(1.6) \qquad \psi(z,\alpha) = \sum_{k>\alpha}^{\infty} \frac{(k-\alpha)^k}{k!} z^k , \qquad |z| < e^{-1}.$$

Putting

$$(1.7) \qquad \varphi(z,\alpha)\,\psi(z,\alpha) = \sum_{n>\alpha}^{\infty} S_n(\alpha) z^n,$$

it follows from (1.2) that

$$(1.8) \qquad \Phi_n^+(\lambda) = 1 - n!n^{-n} S_n(\alpha).$$

Hence the asymptotic behaviour of $\Phi_n^+(\lambda)$ is essentially that of the n^{th} coefficient of the power series expansion of $\varphi\psi$. Using the well-known formula

$$(1.9) \qquad S_n(\alpha) = \frac{1}{2\pi i} \oint \frac{\varphi(z,\alpha)\,\psi(z,\alpha)}{z^{n+1}} dz ,$$

the required expansion follows easily from the analytic behaviour of φ and ψ in the complex z-plane. It can be shown that φ and ψ have only a branch point of order two at $z=e^{-1}$. The behaviour of $\varphi\psi$ at this point is as follows

$$(1.10) \qquad \varphi\psi \approx (2-2ez)^{-\frac{1}{2}} \exp\left\{-2\alpha\,(2-2ez)^{\frac{1}{2}}\right\}.$$

The contour in (1.9) may be deformed into the upper and lower part of the cut at (e^{-1},∞). Then making the substitution $z=e^{-1}(1-\frac{1}{2}s^2)$ we obtain

$$(1.11) \qquad S_n(\alpha) \approx \frac{e^n}{2\pi i} \int_{-i\infty}^{i\infty} (1-\tfrac{1}{2}s^2)^{-n-1} e^{-2\alpha s} ds.$$

If now s is replaced by $t/\sqrt{n}$ and α by $\lambda\sqrt{n}$ we obtain in view of (1.8)

$$(1.12) \qquad 1-\Phi_n^+(\lambda) \approx \frac{n!}{n^{n+\frac{1}{2}}e^{-n}} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left(1-\frac{t^2}{2n}\right)^{-n-1} e^{-2\lambda t} dt \rightarrow$$

$$\rightarrow \frac{1}{i\sqrt{2\pi}} \int_{-i\infty}^{i\infty} e^{+\frac{1}{2}t^2-2\lambda t} dt = e^{-2\lambda^2},$$

which gives the leading term of (1.3). More detailed information about the behaviour of φ and ψ at $z=e^{-1}$ may be used to determine further terms. It will be seen below that this involves a careful study of the many-valued inverse w(z) of z=w exp-w. This will be done in the subsequent section. The result (1.4) will be derived in the third section along the lines indicated above. However, in order to reduce technical calculations we shall not use the variable $s=(2-2ez)^{\frac{1}{2}}$ but a slightly different equivalent variable.

2. The inverse of $z=we^{-w}$

We shall first consider the case of positive real z and w. Writing z=x+iy and w=u+iv we note that the function u exp-u has two inverses f(x) and g(x) satisfying

(2.1) $$x = fe^{-f} = ge^{-g} ,$$

with

(2.2) $$0 < f(x) \leq 1 \leq g(x) \qquad \text{for } 0 < x < e^{-1} .$$

It will be found convenient to use $p=\frac{1}{2}(g-f)$ as a new independent variable. This implies

(2.3) $$f = pe^{-p}/\text{sh } p \quad , \quad g = pe^{p}/\text{sh } p$$

and

(2.4) $$x = \frac{p}{\text{sh } p} \exp - p \text{ cth } p .$$

We have the following expansion

(2.5) $$- \ln ex = \sum_{k=1}^{\infty} \frac{(2k+1)B_{2k}}{2k\ (2k)!}(2p)^{2k} = \frac{1}{2}p^2 - \frac{1}{36}p^4 + \frac{1}{405}p^6 - \ldots ,$$

where the B_{2k} are the Bernoulli numbers

$$B_2 = \frac{1}{6} , B_4 = -\frac{1}{30} , B_6 = \frac{1}{42} , B_8 = -\frac{1}{30} ,\ldots$$

We shall also need the expansion

$$\frac{d\ \ln g}{dp} = 1 - 2 \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} (2p)^{2k-1} =$$

(2.6)
$$= 1 - \frac{1}{3} p + \frac{1}{45} p^3 - \frac{2}{945} p^5 + \dots$$

Taking now z and w complex we consider the conformal mapping $z = w \exp - w$. The figure (on the last page) shows that the z-plane is mapped upon an infinite number of regions of the w-plane. These regions are separated by the curves

(2.7)
$$u = v \operatorname{cotg} v.$$

If they are numbered by I, II, etc. as shown in the figure we see that the region I corresponds to the branch f(z) and II to the branch g(z). The other branches of the inverse w(z) will be denoted by $g_j(z)$, $j = \pm 1, \pm 2, \dots$.

The same figure shows the curves in the w-plane which correspond to circles $|z| = r$ in the z-plane. These curves are given by the equation

(2.8)
$$u^2 + v^2 = r^2 e^{2u} .$$

For $r < e^{-1}$ the curve consists of two parts, a small closed curve around the origin and a paraboloidal branch. For $r = e^{-1}$ the curve consists of a single track with a node at $w = 1$. For $r > e^{-1}$ we obtain a single simple curve. We note that $\frac{dz}{dw} = (1-w)e^{-w}$ so that $z = e^{-1}$, $w = 1$. gives a branch point of order two.

The branches f(z), g(z) and $g_j(z)$, $j = \pm 1, \pm 2, \dots$, correspond to the various sheets of a Riemannian plane. The sheet of the main branch f(z) has a cut at the positive real axis $e^{-1} < x < \infty$.

The sheet of g(z) has the same cut but also a cut at the negative real axis $-\infty < x < 0$. These sheets are connected at the cut $e^{-1} < x < \infty$. All other sheets have the single cut $-\infty < x < 0$ and are connected to each other and to the second sheet in a screw-like fashion.

The main branch f(z) is holomorphic at the origin and has a branch point of order two at $z = e^{-1}$. It is a classical result *) that

(2.9)
$$f(z) = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} z^n, \qquad |z| < e^{-1} .$$

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*) Due to Eisenstein, Crelle's Journal 28, p.49, (1844). Cf. also Hurwitz-Courant: Funktionentheorie, Kap.7, § 2, Beispiel 1, and E.M. Wright: Solution of the equation $ze^z = a$, Bulletin of the American Mathematical Society, vol.65, pp.89-93 (1959).

A simple calculation shows that at $z=e^{-1}$

$$f(z) = 1 - (2-2ez)^{\frac{1}{2}} + \frac{1}{3}\cdot(2-2ez) + \dots \tag{2.10}$$

and

$$g(z) = 1 + (2-2ez)^{\frac{1}{2}} + \frac{1}{3}(2-2ez) + \dots . \tag{2.11}$$

The branches $g(z)$ and $g_j(z)$, $j=\pm1,\pm2,\dots$, have only an essential singularity at $z=0$.

3. The asymptotic expansion of $\Phi_n^+(\lambda)$

The functions φ and ψ determined by (1.5) and (1.6) may be continued analytically in the following way. We have for $|z| < e^{-1}$

$$\varphi(z,\alpha) = \sum_{k=0}^{\infty} \left(\frac{(k+\alpha)^k}{k!} - \frac{(k+\alpha)^{k-1}}{(k-1)!} \right) z^k =$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{2\pi i} \int_L e^{w(k+\alpha)}\, w^{-k-1}\, (1-w)\, dw =$$

$$= \frac{1}{2\pi i} \int_L e^{\alpha w} \frac{(1-w)e^{-w}}{we^{-w}-z}\, dw, \tag{3.1}$$

where L is a contour such that $|we^{-w}| > |z|$.
In view of the discussion of the preceding section, we may take for L the vertical line $\operatorname{Re} w = 1$. Since $e^{\alpha w}$ tends to zero on the left of L the integral last obtained equals the residue at the only pole of $w=f(z)$, i.e.

$$\varphi(z,\alpha) = \exp \alpha f(z) = z^{-\alpha} f^{\alpha}(z). \tag{3.2}$$

We note in passing that from $\frac{\partial \varphi}{\partial \alpha} \to f(z)$ for $\alpha \to 0$ the result (2.9) follows.
The expression (3.2) determines the complete analytical behaviour of $\varphi(z,\alpha)$. In particular $\varphi(z,\alpha)$ has the branch point $z=e^{-1}$ as the only singularity.

In a similar way we have

$$\psi(z,\alpha) = \sum_{k=0}^{\infty} \frac{z^k}{2\pi i} \int_L e^{w(k-\alpha)} w^{-k-1}\, dw =$$

$$= \frac{1}{2\pi i} \int_L e^{-\alpha w} \frac{e^{-w}}{we^{-w}-z}\, dw,$$

where L is the vertical line Re w=1. We note that the terms with $k \leqq \alpha$ which are included here do not contribute to the sum.
The integral (3.3) now equals the sum of the residues w=g(z) and $g_j(z)$, $j=\pm 1,\pm 2,\ldots$, which are situated on the right of L, i.e.

(3.4) $$\psi(z,\alpha) = \frac{e^{-\alpha g}}{g-1} + \sum_j \frac{e^{-\alpha g_j}}{g_j-1} .$$

Substitution of these results in (1.9) gives

(3.5) $$S_n(\alpha) = \frac{1}{2\pi i} \oint \frac{e^{-\alpha(g-f)}}{g-1} z^{-n-1}\, dz + R,$$

where the remainder contains similar terms with g_j instead of g. However, the behaviour of f,g and g_j at $z=e^{-1}$ shows that R is asymptotically negligible. In fact, a slightly more accurate estimate shows that

(3.6) $$R = n^{-\frac{3}{2}} e^n\, O(e^{-1.6\alpha}).$$

The leading term of (3.5) may be transformed into

(3.7) $$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} x^{-n} e^{-2\alpha p} \frac{d \ln g}{dp}\, dp$$

by using the substitution (2.4) .
From (2.5) we get

(3.8) $$x^{-n} = e^{n(1+\frac{1}{2}p^2)} \exp -n(\frac{1}{36} p^4 - \frac{1}{405} p^6 + \ldots).$$

Hence by performing the substitution $p=t/\sqrt{n}$ the last integral may be written as

(3.9) $$\frac{n^{-\frac{1}{2}} e^n}{2\pi i} \int_{-i\infty}^{i\infty} e^{\frac{1}{2}t^2-2\lambda t} F(t,n)\, dt,$$

where

(3.10) $$F(t,n) = \left(1 - \frac{1}{3}\frac{t}{\sqrt{n}} + \frac{1}{45}\frac{t^3}{n\sqrt{n}} - \ldots\right) \exp\left(\frac{1}{36}\frac{t^4}{n} - \frac{1}{405}\frac{t^6}{n^2} + \ldots\right) .$$

A simple calculation shows that

(3.11) $$F(t) = \sum_{s=0}^{\infty} f_j(t) n^{-\frac{1}{2}j},$$

with

$$f_o = 1, \qquad f_1 = -\frac{1}{3}t, \qquad f_2 = -\frac{1}{36}t^4,$$

$$f_3 = \frac{1}{45}t^3 + \frac{1}{108}t^5, \quad f_4 = \frac{1}{405}t^6 + \frac{1}{2592}t^8, \ldots$$

Noting that

(3.12) $$\int_{-i\infty}^{i\infty} e^{\frac{1}{2}t^2 - 2\lambda t}\, t^m \, dt = i\sqrt{2\pi} \quad e^{-2\lambda^2} He_m(2\lambda),$$

where He_m is the m^{th} polynomial of Hermite we obtain the following asymptotic expansion

(3.13) $$S_n(\alpha) = \frac{e^{n-2\lambda^2}}{\sqrt{2\pi n}} \left\{1 - \frac{1}{3\sqrt{n}} He_1(2\lambda) - \frac{1}{36n} He_4(2\lambda) + \ldots\right\} .$$

This combined with the well-known result

(3.14) $$\frac{n!}{n^n} = \frac{\sqrt{2\pi n}}{e^n} \left\{1 + \frac{1}{12n} + \frac{1}{288n^2} + \ldots\right\}$$

leads at once to the result (1.4).

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(3) Z.W. Birnbaum and F.H. Tingey, Ann.Math.Stat. 22 (1951), 592-596.

(4) B.V. Gnedenko, Tagung in Berlin Wahrscheinlichkeitsrechnung und Math. Stat. Deutscher Verl. d. Wiss. (1954), 97-107.

(5) B.V. Gnedenko, V.S. Koroluk, A.V. Skorokhod, Proc. fourth Berkeley Symp. on mathematical statistics and probability, Berkeley, Univ. of Calif.,(1961), 153-170.

Conformal mapping
$z = we^{-w}$.
$z = re^{i\varphi}$, $w = u + iv$.