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Three notes on orthogonal polynomials

by

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An inequality for the classical polynomials

Richard Askey [(1)]

In the last twenty years a number of inequalities for determinants of orthogonal polynomials have been found. The earliest such inequality is much older and follows from the Christoffel-Darboux formula. It is

$$(1) \qquad C_n(x) = \begin{vmatrix} p_n(x) & p_{n+1}(x) \\ p_n'(x) & p_{n+1}'(x) \end{vmatrix} > 0, \ -\infty < x < \infty,$$

where the orthogonal polynomials $p_n(x)$ are standardized by $p_n(x) = k_n x^n + \ldots$, $k_n > 0$. In the 1940's Turán found a new inequality for determinants of Legendre polynomials which is more sensitive than (1). He proved that

$$(2) \qquad D_n(x) = \begin{vmatrix} P_n(x) & P_{n+1}(x) \\ P_{n+1}(x) & P_{n+2}(x) \end{vmatrix} < 0, \ -1 < x < 1.$$

$D_n(x)$ is a more sensitive function since it distinguished between $-1 < x < 1$ and $x^2 > 1$. This is because $D_n(x) > 0$, $x^2 > 1$. There are now many proofs and extensions of (2). The most elaborate treatment is given in the important paper of Karlin and Szegö [2], where even $k \times k$ determinants are treated.

Our aim is much more modest. We would like to find an inequality like (1) which is sensitive to the spectral interval of the orthogonal polynomials. There is no hope of finding such an inequality for general orthogonal polynomials, but it should be possible to find one for the classical polynomials. These inequalities are given in Therems 1 and 2.

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Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial and $L_n^{\alpha}(x)$ the Laguerre polynomial with the usual normalizations. See Szegö [3]. We set $Df(x) = f'(x)$.

Theorem 1. For α, $\beta > -1$ define $\Delta_n(x,\alpha,\beta) = \Delta_n(x)$ by

$$(3) \qquad \Delta_n(x) = \begin{vmatrix} \dfrac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)} & \dfrac{P_{n+1}^{(\alpha,\beta)}(x)}{P_{n+1}^{(\alpha,\beta)}(1)} \\ \\ \dfrac{DP_n^{(\alpha,\beta)}(x)}{DP_n^{(\alpha,\beta)}(1)} & \dfrac{DP_{n+1}^{(\alpha,\beta)}(x)}{DP_{n+1}^{(\alpha,\beta)}(1)} \end{vmatrix} .$$

Then

$$\Delta_n(x) \text{ is } \begin{matrix} > 0 & -1 < x < 1 \\ = 0 & x^2 = 1 \\ < 0 & x^2 > 1. \end{matrix}$$

Theorem 2. Let $\alpha > -1$ and define $\Delta_n(x,\alpha) = \Delta_n(x)$ by

$$\Delta_n(x) = \begin{vmatrix} \dfrac{L_n^{\alpha}(x)}{L_n^{\alpha}(0)} & \dfrac{L_{n+1}^{\alpha}(x)}{L_{n+1}^{\alpha}(0)} \\ \\ \dfrac{DL_n^{\alpha}(x)}{DL_n^{\alpha}(0)} & \dfrac{DL_{n+1}^{\alpha}(x)}{DL_{n+1}^{\alpha}(0)} \end{vmatrix} .$$

Then

$$\Delta_n(x) \text{ is } \begin{matrix} > 0 & x > 0 \\ = 0 & x = 0 \\ < 0 & x < 0. \end{matrix}$$

The proofs of Theorems 1 and 2 are similar so we will only prove Theorem 1. Recall that

$$(4) \qquad (1-x^2)y'' = [\alpha-\beta + (\alpha+\beta+2)x]y' - n(n+\alpha+\beta+1)y, \quad y = P_n^{(\alpha,\beta)}(x),$$

$$(5) \qquad P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)} ,$$

$$(6) \qquad \frac{d}{dx} P_n^{(\alpha,\beta)}(x) = \frac{n+\alpha+\beta+1}{2} P_{n-1}^{(\alpha+1,\beta+1)}(x).$$

See Szego [3, Chapter IV].

If we differentiate $\Delta_n(x)$ we get

$$\Delta_n'(x) = DP_n^{(\alpha,\beta)}(x)\, DP_{n+1}^{(\alpha,\beta)}(x) \left[\frac{1}{P_n^{(\alpha,\beta)}(1) DP_{n+1}^{(\alpha,\beta)}(1)} - \frac{1}{P_{n+1}^{(\alpha,\beta)}(1) DP_n^{(\alpha,\beta)}(1)} \right]$$
$$+ \frac{P_n^{(\alpha,\beta)}(x) D^2 P_{n+1}^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1) DP_{n+1}^{(\alpha,\beta)}(1)} - \frac{P_{n+1}^{(\alpha,\beta)}(x) D^2 P_n^{(\alpha,\beta)}(x)}{P_{n+1}^{(\alpha,\beta)}(1) DP_n^{(\alpha,\beta)}(1)} .$$

Then using (4), (5), and (6) we see that

$$(7) \quad (1-x^2)\Delta_n'(x) = \frac{-[\Gamma(\alpha+1)]^2 \Gamma(n)\Gamma(n+2)}{\Gamma(n+\alpha+1)\Gamma(n+\alpha+2)} (1-x^2) P_{n-1}^{(\alpha+1,\beta+1)}(x)\, P_n^{(\alpha+1,\beta+1)}(x)$$
$$+ [\alpha-\beta+(\alpha+\beta+2)x]\ \Delta_n(x) =$$
$$= -k_n(1-x^2) P_{n-1}^{(\alpha+1,\beta+1)}(x)\, P_n^{(\alpha+1,\beta+1)}(x)$$
$$+ [(\alpha-\beta) + (x+\beta+2)x]\ \Delta_n(x).$$

The solution to (7) is

$$(8) \quad |1-x|^{\alpha+1}\ |1+x|^{\beta+1}\ \Delta_n(x) =$$
$$= k_n \int_x^1 |1-u|^{\alpha+1}\ |1+u|^{\beta+1}\ P_{n-1}^{(\alpha+1,\beta+1)}(u)\, P_n^{(\alpha+1,\beta+1)}(u)du ,$$

For $x > 1$ we have $P_n^{(\alpha,\beta)}(x) > 0$, and so $\Delta_n(x) < 0$ follows immediately from (8). For $x < -1$ we have $(-1)^n P_n^{(\alpha,\beta)}(x) > 0$, so $\Delta_n(x) < 0$ also follows from (8). For $-1 < x < 1$ we either differentiate (8) or partially solve (7) to get

$$[(1-x)^{\alpha+1}(1+x)^{\beta+1}\Delta_n(x)]' =$$

$$= -k_n(1-x)^{\alpha+1}(1+x)^{\beta+1}P_{n-1}^{(\alpha+1,\beta+1)}(x)\,P_n^{(\alpha+1,\beta+1)}(x).$$

Thus the extrema of $(1-x)^{\alpha+1}(1+x)^{\beta+1}\Delta_n(x)$ occur at $x = \pm 1$ and at the zeros of $P_{n-1}^{(\alpha+1,\beta+1)}(x)$ and $P_n^{(\alpha+1,\beta+1)}(x)$. For $x = 1$, $\Delta_n(1) = 0$, and a simple calculation using $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\alpha,\beta)}(x)$ shows that $\Delta_n(-1) = 0$. Also $\Delta_n(x) > 0$ for $x^2 < 1$ if x is close enough to $x = 1$, i.e. either $x_{1,n}^{\alpha+1,\beta+1} < x < 1$ or $-1 < x < x_{n,n}^{\alpha+1,\beta+1}$, where $x_{i,n}^{\alpha+1,\beta+1}$ are the n zeros of $P_n^{(\alpha+1,\beta+1)}(u)$ ordered by $-1 < x_{n,n} < \ldots < x_{1,n} < 1$. This follows from (8). At a zero of either $P_n^{(\alpha+1,\beta+1)}(x)$ or $P_{n-1}^{(\alpha+1,\beta+1)}(x)$ we have

$$\Delta_n(x) = a_n\, C_{n-1}(x)$$

for some positive a_n, and so $\Delta_n(x) > 0$ by (1). This completes the proof of Theorem 1.

It is interesting to observe that these theorems are more sensitive about the spectrum than Turán type inequalities. For Jacobi polynomials with $\alpha \neq \beta$ and for $L_n^{\alpha}(x)$ Turán type inequalities off the spectrum do not take the simple form that these inequalities do. I was surprised to see the inequality for the integral

$$\int_x^1 (1-u)^{\alpha+1}(1+u)^{\beta+1}P_{n-1}^{(\alpha+1,\beta+1)}(u)\,P_n^{(\alpha+1,\beta+1)}(u)\,du > 0,\quad -1 < x < 1.$$

This integral, among others, arose in work of Hirschman [1] on the problem dual to mean convergence. It is easy to see that most of the other integrals Hirschman obtained do not keep the same sign for all x, $-1 < x < 1$.

Once you start considering other normalizations of orthogonal polynomials a further inequality suggests itself. This is

$$\begin{vmatrix} \dfrac{p_n(x)}{p_n(a)} & \dfrac{p_n'(x)}{p_n'(a)} \\[2ex] \dfrac{p_n'(x)}{p_n'(a)} & \dfrac{p_n''(x)}{p_n''(a)} \end{vmatrix} < 0, \quad x \text{ in the spectral interval,} \tag{9}$$

where a = 1 for the ultraspherical (or Jacobi) polynomials and a = 0 for Laguerre polynomials. This inequality fails for Laguerre polynomials for x large and this suggests that a different normalization might lead to an inequality like (9). The appropriate normalization is probably $L_n^\alpha(x) = x^n + \ldots$. For the ultraspherical polynomials I have been unable to prove (9). In fact, I cannot even prove it in the case that has always before been trivial, that of Tchebycheff polynomials of the first kind. Without the normalization, inequalities like (9) are mentioned by Karlin and Szegö [2], but they have not been able to give them the same exhaustive treatment that they gave Turán type inequalities.

[1] I.I. Hirschman, Jr., Projections associated with Jacobi polynomials, Proc. Amer. Math. Soc. 8 (1957), 286-290.

[2] S. Karlin and G. Szegö, On certain determinants whose elements are orthogonal polynomials, Journal d'Analyse Mathematique, 8 (1961), 1-157.

[2] G. Szegö, Orthogonal polynomials, American Mathematical Society Colloquium Publications, vol. 23, New York, 1959.

Linearization of the product of orthogonal polynomials II

Richard Askey [(1)]

In a recent issue of the Australian Journal of Statistics Eagleson [2] has defined the concept of positive definite sequences with respect to Krawtchouk polynomials, the orthogonal polynomials associated with the binomial distribution. The essential fact about these polynomials that allows him to characterize positive definite sequences is that you can write the product of two polynomials as a linear combination with nonnegative coefficients of these polynomials.
Recently there have been a number of theorems of this type proven for other polynomials, [1],[3],[4]. In particular in [1] a general theorem was given which gave this result for two other discrete polynomials, the polynomials of Charlier and Meixner. Krawtchouk polynomials are closely related to Meixner polynomials so a natural question to ask is whether or not we can obtain Eagleson's result from a general theorem. We will show how to modify the proof given in [1] to obtain a new general theorem for orthogonal polynomials on a finite set which implies Eagleson's result for Krawtchouk polynomials.

Krawtchouk polynomials can be defined by

$$k_n(x) = \frac{(-N)_n p^n}{n!} \, {}_2F_1\left(-n,-x;-N;\frac{1}{p}\right) .$$

They have the generating function

$$(1+qz)^x (1-pz)^{N-x} = \sum_{n=0}^{N} k_n(x) z^n ,$$

which Eagleson used. Generating functions are known for very few orthogonal polynomials, so we will use the recurrence formula instead..

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For Krawtchouk polynomials it is given by

$$K_1(x)\, K_n(x) = K_{n+1}(x) + n(q-p)\, K_n(x) + pqn(N+1-n)\, K_{n-1}(x),$$

where we have renormalized these polynomials so that now

$$K_n(x) = x^n + \dots$$

$$K_1(x) = x - pN.$$

Here $0 < p < 1$, $p + q = 1$ and N is a fixed positive integer. $k_n(x)$ are orthogonal on $[0,1,\dots,N]$ with respect to the weight function

$$j(x) = \binom{N}{x} p^x q^{N-x}, \quad x = 0,1,\dots,N.$$

We will consider orthogonal polynomials whose recurrence formula is

$$p_1(x)\, p_n(x) = p_{n+1}(x) + \alpha_n p_n(x) + \beta_n p_{n-1}(x),\ p_1(x) = x + a,\ p_0(x) = 1.$$

To obtain polynomials orthogonal on a finite set we assume that

$$\beta_0 = \beta_{N+1} = 0\ ,\quad \beta_n > 0,\ n = 1,2,\dots,N.$$

For the Krawtchouk polynomials the essential properties of α_n and β_n are

(i) $\alpha_{n+1} \geqq \alpha_n \geqq 0$, $n = 1,\dots,N$, if $q \geqq p$, $\alpha_0 = 0$,

(ii) $\beta_n = \beta_{N+1-n}$, $n = 0,1,\dots,\left[\frac{N+1}{2}\right]$,

(iii) $0 < \beta_1 \leqq \beta_2 \leqq \dots \leqq \beta_{\left[\frac{N+1}{2}\right]}$.

We will show that if we have contions (i), (ii), and (iii) for α_n and β_n then we have

(1) $$p_n(x)\, p_m(x) = \sum_{k=|n-m|}^{n+m} a(k,m,n)\, p_k(x),\quad a(k,m,n) \geqq 0,\ n + m \leqq N.$$

Instead of proving (1) directly we will prove a maximal principle for a hyperbolic difference equation and then show how this implies (1).

I have found that it is easier for most people to follow the ensuing argument if it is stated for difference equations rather than for orthogonal polynomials.

For $k(n)$ a sequence defined on $0,1,\ldots,n,\ldots$ we define

$$\Delta_n k(n) = k(n+1) + \alpha_n k(n) + \beta_n k(n-1), \quad n = 1,2,\ldots .$$

Theorem 1. Let $a(n,m)$ satisfy the difference equation

$$\Delta_n a(n,m) = \Delta_m a(n,m) . \tag{2}$$

Then if $\beta_0 = \beta_{N+1} = 0$,

(α) $0 \leqq \alpha_n \leqq \alpha_{n+1}$, $\quad \alpha_{N+1-n} \geqq \alpha_n$, $\quad n = 1,2,\ldots,\left[\frac{N+1}{2}\right]$

(β) $0 < \beta_n \leqq \beta_{n+1}$, $\quad \beta_n \leqq \beta_{N+1-n}$, $\quad n = 1,2,\ldots,\left[\frac{N+1}{2}\right]$

and if $a(n,0) = a(0,n) \geqq 0$, $a(-1,n) = a(n,-1) = 0$, $n = 0,1,\ldots,N$, then

$$a(n,m) \geqq 0, \quad n,m = 1,2,\ldots, \quad n + m \leqq N. \tag{3}$$

The proof is by induction on m. Assume we have proven (3) for $0,1,\ldots,m$ and consider $a(n,m+1)$. From (2) we have

$$a(n,m+1) + \alpha_m a(n,m) + \beta_m a(n,m-1) = a(n+1,m) + \alpha_n a(n,m) + \beta_n a(n-1,m)$$

so

$$a(n,m+1) = a(n+1,m) + (\alpha_n - \alpha_m)\, a(n,m) + (\beta_n - \beta_m)\, a(n-1,m)$$

$$+ \beta_m\left[a(n-1,m) - a(n,m-1)\right] .$$

Since $a(n,m) = a(m,n)$ we may assume that $m + 1 \leqq n$ or $m < n$. Also we have $m + n \leqq N$ so $m < N + 1 - n$. Thus from (α) we have

$$\alpha_n - \alpha_m \geqq 0 \quad \text{if} \quad m < n \leqq \left[\frac{N}{2}\right]$$

and

$$\alpha_n - \alpha_m \geqq \alpha_{N+1-n} - \alpha_m \geqq 0 \quad \text{if} \quad \left[\frac{N+1}{2}\right] \leqq n \leqq N, \text{ since } m < N+1-n.$$

Similarly $\beta_n - \beta_m \geqq 0$. Also we can estimate $a(n-1,m) - a(n,m-1)$ by recurrence; for

$$\begin{aligned}a(n,m+1) - a(n+1,m) &\geqq \beta_m\left[a(n-1,m) - a(n,m-1)\right]\\ &\geqq \beta_m\beta_{m-1}\left[a(n-2,m-1) - a(n-1,m-2)\right]\\ &\geqq \beta_m\beta_{m-1}\beta_1\left[a(n-m-1,1) - a(n-m,0)\right]\\ &\geqq \beta_m\beta_{m-1} \cdots \beta_1\beta_{n-m-1}a(n-m-2,0) \geqq 0.\end{aligned}$$

Thus $a(n,m) \geqq 0$ for $n,m = 1,2,\ldots,\ n + m \leqq N$.

To obtain the theorem for orthogonal polynomials we observe that if

$$p_n(x)\ p_m(x) = \sum_{k=|n-m|}^{n+m} a(k,m,n)\ p_k(x)$$

then

$$a(k,m,n) = \frac{\int p_n(x)\ p_m(x)\ p_k(x)\ d\alpha(x)}{\int p_k^2(x)\ d\alpha(x)} \tag{4}$$

for a nonnegative measure $d\alpha(x)$. In our case the measure is a finite number of point masses but that is not necessary for this result.

Corollary 1. Let α_n and β_n satisfy the conditions of Theorem 1 and define $a(k,m,n)$ by (4). Then $a(k,m,n) \geqq 0$, $n,m = 0,1,\ldots,\ n+m \leqq N$.

We need only show that

$$\Delta_n a(k,m,n) = \Delta_m a(k,m,n) \tag{5}$$

and that

$$a(k,0,n) = a(k,n,0) \geqq 0, \tag{6}$$

$$a(k,-1,n) = a(k,n,-1) = 0. \tag{7}$$

(5) follows from the recurrence formula for

$$\Delta_n a(k,m,n) = \frac{\int p_n(x)\, p_m(x)\, p_k(x)\, p_1(x)\, d\alpha(x)}{\int p_k^2(x)\, d\alpha(x)} = \Delta_m a(k,m,n).$$

Also

$$a(k,n,0) = a(k,0,n) = \frac{\int p_n(x)\, p_k(x)\, d\alpha(x)}{\int p_k^2(x)\, d\alpha(x)} = \begin{cases} 0 & n \neq k \\ 1 & n = k. \end{cases}$$

If $p_{-1}(x)$ is defined to be zero then the recurrence formula holds so we have $a(k,-1,n) = a(k,n,-1) = 0$.

It is of some interest to compare Theorem 1 with the corresponding Theorem in [1]. There we assumed that $\alpha_n \leqq \alpha_{n+1}$ and $\beta_n \leqq \beta_{n+1}$ for all n. If we let $N \to \infty$ in Theorem 1 we formally recover our previous theorem.

Eagleson's theorem for $q \geqq p$ is an immediate consequence of Corollary 1. For $p \leqq q$ he also proves a similar result, but now his polynomials are normalized to be positive at $x = 0$. This also follows from our work if we make the appropriate changes.

A natural question to ask is whether this theorem contains anything for the other classical discrete polynomials. The answer is no, for the theorem fails for all of the Hahn polynomials, and these are the only other classical discrete polynomials.

[1] R. Askey, Linearization of the product of orthogonal polynomials, to appear in volume dedicated to S. Bochner, Princeton Univ. Press.

[2] G.K. Eagleson, A characterization theorem for positive definite sequences on the Krawtchouk polynomials, Australian Jour. of Statistics, 11 (1969), 29-38.

[3] G. Gasper, Linearization of the product of Jacobi polynomials, I, to appear in Can. Jour. Math.

[4] G. Gasper, Linearization of the product of Jacobi polynomials, II, to appear in Can. Jour. Math.

Orthogonal expansions with positive coefficients, II

Richard Askey[(1)]

In a number of problems that have recently been considered, ranging from differential geometry [1] to numerical analysis [5], the essential problem reduced to when one set of orthogonal polynomials could be written as a linear combination of a different set of orthogonal polynomials with non negative coefficients. For the classical polynomials we have a fairly good understanding of this problem and many of the known results are summarized in [3]. However for other orthogonal polynomials our knowledge is very slight. We will show how to obtain a general theorem using only the recurrence formulas satisfied by the orthogonal polynomials. Then we will show how to use this result to obtain new results for some interesting classes of polynomials.

Theorem I. Let $p_n(x)$ and $g_n(x)$ satisfy

(1) $\quad x\, p_n(x) = p_{n+1}(x) + \alpha_n p_n(x) + \beta_n p_{n-1}(x), \qquad n = 0, 1, \ldots,$

(2) $\quad x\, g_n(x) = g_{n+1}(x) + \gamma_n g_m(x) + \delta_n g_{n-1}(x), \qquad n = 0, 1, \ldots,$

$p_{-1}(x) = g_{-1}(x) = \beta_0 = \delta_0 = 0$, $p_0(x) = g_0(x) = 1$. To insure that $p_n(x)$ and $g_n(x)$ are orthogonal we assume that α_{n-1}, γ_{n-1} are real and $\beta_n > 0$, $\delta_n > 0$, $n = 1, 2, \ldots$.

Let $g_n(x) = \sum_{k=0}^{n} a(k,n)\, p_k(x)$. Then $a(k,n) \geqq 0$ if we assume

(3) $\quad \alpha_k \geqq \gamma_n, \qquad k = 0, 1, \ldots, n, \quad n = 0, 1, \ldots$.

(4) $\quad \beta_k \geqq \delta_n, \qquad k = 1, 2, \ldots, n, \quad n = 1, 2, \ldots$.

Proof. We have $g_{n+1}(x) = \sum_{k=0}^{n+1} a(k, n+1)\, p_k(x)$

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Also

$$g_{n+1}(x) = x\, g_n(x) - \gamma_n g_n(x) - \delta_n g_{n-1}(x)$$

$$= x \sum_{k=0}^{n} a(k,n)p_k(x) - \gamma_n \sum_{k=0}^{n} a(k,n)p_k(x) - \delta_n \sum_{k=0}^{n-1} a(k,n-1)p_k(x)$$

$$= \sum_{k=0}^{n} a(k,n) \left[p_{k+1}(x) + \alpha_k p_k(x) + \beta_k p_{k-1}(x)\right] - \gamma_n \sum_{k=0}^{n} a(k,n)\, p_k(x)$$

$$- \delta_n \sum_{k=0}^{n} a(k,n-1)p_k(x)$$

$$= p_{n+1}(x) + \left[a(n-1,n) + (\alpha_n - \gamma_n)\right] p_n(x)$$

$$+ \sum_{k=1}^{n-1} \left[a(k-1,n) + (\alpha_k - \gamma_n)\, a(k,n) + \beta_{k+1}\, a(k+1,n) - \delta_n\, a(k,n-1)\right] p_k(x)$$

$$+ \left[(\alpha_0 - \gamma_n)\, a(0,n) + \beta_1\, a(1,n) - \delta_n\, a(0,n-1)\right] p_0(x).$$

Thus we have

(5) $\quad a(n+1,\ n+1) = 1$

(6) $\quad a(n,n+1) = a(n-1,n) + (\alpha_n - \gamma_n),$

(7) $\quad a(k,n+1) = a(k-1,n) + (\alpha_k - \gamma_n)\, a(k,n) + (\beta_{k+1} - \delta_n)\, a(k+1,n)$

$$+ \delta_n \left[a(k+1,n) - a(k,n-1)\right],\ k = 1, \ldots, n-1,$$

(8) $\quad a(0,n+1) = (\alpha_0 - \gamma_n)\, a(0,n) + (\beta_1 - \delta_n)\, a(1,n) + \delta_n \left[a(1,n) - a(0,n-1)\right].$

If we adopt the convention that $a(n+1,n) = a(-1,n) = 0$, then (6) and (8) are just (7) for $k = n$ and $k = 0$.

We will show that $a(k,n) \geqq 0$ by an induction on n. Assume that we have shown that $a(k,m) \geqq 0$, $k \leqq m$, $m \leqq n$ and consider $a(k,n+1)$. If $k = n+1$, then $a(n+1,n+1) = 1 > 0$. If $k = n$ then $a(n,n+1) = a(n-1,n) + (\alpha_n - \gamma_n)$ and so $a(n,n+1) = (\alpha_n - \gamma_n) + (\alpha_{n-1} - \gamma_{n-1}) + \ldots + (\alpha_0 - \gamma_0) \geqq 0$ since $\alpha_j \geqq \gamma_j$.

If $k \leqq n-1$ then

$$a(k,n+1) = a(k-1,n) + (\alpha_k-\gamma_n)\, a(k,n) + (\beta_{k+1}-\delta_n)\, a(k+1,n) + \delta_n [a(k+1,n) - a(k,n-1)]$$

Each of the terms on the right hand side is nonnegative except for possibly the last term, $a(k+1,n) - a(k,n-1)$. Using (8) again we see that

$$a(k,n+1) - a(k-1,n) = (\alpha_k-\gamma_n)\, a(k,n) + (\beta_{k+1}-\delta_n)\, a(k+1,n) + \delta_n [a(k+1,n) - a(k,n-1)]$$

$$\geqq \delta_n [a(k+1,n) - a(k,n-1)] \geqq \cdots \geqq \delta_n\, \delta_{n-1} \cdots \delta_{n-j} [a(k+j+1,n-j) - a(k+j,n-j-1)].$$

It is now sufficient to choose j so that $2j > n-k-1$, for then

$k+j+1 > n-j$ and $k+j > n-j-1$ and the last term vanishes. Thus $a(k+1,n) - a(k,n-1) \geqq 0$ and so we have shown that $a(k,n+1) \geqq 0$, $k = 0,1, \ldots, n+1$.

This proof is very similar to the proof in [2] which gives

$$(9) \qquad p_n(x)\, p_m(x) = \sum a(k,m.n)\, p_k(x), \qquad a(k,m.n \geqq 0,$$

under certain conditions on the coefficients in (1).

Both this proof and the proof of (9) have the strange defect that they prove too much. In this case we not only prove that $a(k,n) \geqq 0$ but we have shown that $a(k,n) \geqq a(k-1,n-1)$.

One appliciation concerns Pollaczek polynomials. A special case of these interesting polynomials, which generalize the ultraspherical polynomials, satisfy

$$x\, R_n^\lambda(x,a) = R_{n+1}^\lambda(x,a) + \frac{n(n+2\lambda-1)}{4(n+\lambda+a)(n+\lambda+a-1)}\, R_{n-1}^\lambda(x,a) \quad ,$$

and for $a = 0$ reduce to the ultrasperical polynomials. In [3] we proved that

$$|R_n^\lambda(x,a)| \leqq R_n^\lambda(1,a), \quad -1 \leqq x \leqq 1, \quad \text{for}$$

$a \geqq 0$, $a \geqq (\lambda-\lambda^2)/(1+\lambda)$, $\lambda > 0$. The restruction $a \geqq (\lambda-\lambda^2)/(1+\lambda)$ is artificial and we show how to remove it.

For $a = 0$, $0 < \lambda < 1$ we have that

$$\beta_n = \frac{n(n+2\lambda-1)}{4(n+\lambda)(n+\lambda-1)}$$

is a decreasing sequence. Thus

$$\beta_1 \geqq \ldots \geqq \beta_n > \delta_n = \frac{n(n+2\lambda-1)}{4(n+\lambda+a)(n+\lambda+a-1)}$$

for $a > 0$.

From Theorem 1 we then have

$$R_n^\lambda(x,a) = \sum_{k=0}^{n} a(k,n)\, R_k^\lambda(x,0)$$

with $a(k,n) \geqq 0$ if $0 < \lambda < 1$, $a > 0$. Then

$$(10) \qquad |R_n^\lambda(x,a)| \leqq \sum_{k=0}^{n} a(k,n)\, |R_k^\lambda(x,0)| \leqq R_n^\lambda(1,a)$$

since $|R_k^\lambda(x,0)| \leqq R_k^\lambda(1,0)$ is a well known result.

One other interesting application is to the associated polynomials. If $p_n(x)$ satisfies

$$x\, p_n(x) = p_{n+1}(x) + \alpha_n\, p_n(x) + \beta_n\, p_{n-1}(x)$$

we define $p_n(x,\nu)$ by

$$x\, p_n(x,\nu) = p_{n+1}(x,\nu) + \alpha_{n+\nu}\ p_n(x,\nu) + \beta_{n+\nu}\, p_{n-1}(x,\nu), \tag{11}$$

$p_{-1}(x,\nu) = 0$, $p_0(x,\nu) = 1$. For general orthogonal polynomials we must have $\nu = 1, 2, \ldots$, but for some of the classical polynomials (11) makes sense for $\nu > 0$, or even some $\nu < 0$. While we can prove some results for associated polynomials some general orthogonal polynomials we restrict ourself to ultraspherical polynomials. These polynomials $S_n^\nu(x,\mu)$ satisfy

$$x\, S_n^\nu(x,\mu) = S_{n+1}^{\ \nu}(x,\mu) + \frac{(n+2\lambda+\mu-1)\ (n+\mu)}{4(n+\lambda+\mu)\ (n+\lambda+\mu-1)}\ S_{n+1}^{\ \nu}(x,\mu).$$

$$= S_{n+1}^{\ \nu}(x,\mu) + S_n^\mu\, S_{n-1}^{\ \nu}(x,\mu).$$

We have $S_n^\mu < \frac{1}{4}$ if $\nu > 1$ and $S_n^\mu = \frac{1}{4}$ if $\nu = 1$. Then from Theorem 1 we have

$$S_n^\nu(x,\mu) = \sum_{k=0}^{n} a(k,n) \qquad S_k^1(x,\mu) \tag{12}$$

with $a(k,n) \geqq 0$ for $\nu > \mu > -1$. If we let $\mu \to \infty$ in δ_n^μ we see that $S_n^\nu(x,\infty) = S_n^{(1)}(x,0) = S_n^{(1)}(x,\mu)$ so (12) is

$$S_n^\nu(x,\mu) = \sum_{k=0}^{n} a(k,n)\ S_k^\nu(x,\infty), \tag{13}$$

$a(k,n) \geqq 0$ for $\nu > 1$. Since $|S_k^1(x,\mu)| \leqq S_k^1(1,\mu)$ we have

$$|S_n^\nu(x,\mu)| \leqq S_n^\nu(1,\mu),\ \nu > 1,\ -1 \leqq x \leqq 1, \quad \mu > -1.$$

For $0 < \nu < 1$ we can prove a more general result than (13). It now becomes

$$S_n^\nu(x,\mu) = \sum_{k=0}^{n} a(k,n)\ S_k^\nu(x,\lambda) \tag{14}$$

with $a(k,n) \geqq 0$ for $0 < \nu < 1$, $\mu > \lambda \geqq 0$. For $\lambda = 0$, $\nu = \frac{1}{2}$ the coefficients $a(k,n)$ were calculated by Barrucand and Dickinson [4] and a moments reflection on their expression for $a(k,n)$ (the product of 14 gamma functions) shows that these numbers are nonnegative. It was this result that started me thinking of the possibility of proving a general theorem of the type of Theorem 1. If we can find a stronger theorem, then I suspect that we could show that $a(k,n) \geqq 0$ in (13) for $\nu > 1$ if $0 \leqq \mu < \lambda$.

From (14) we also have $|S_n^\nu(x,\mu)| \leqq S_n^\nu(1,\mu)$, $-1 \leqq x \leqq 1$, $\mu > 0$, $0 < \nu < 1$, since this holds for $\mu = 0$.

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