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UNIFORM ASYMPTOTIC EXPANSIONS OF THE INCOMPLETE GAMMA FUNCTIONS AND THE INCOMPLETE BETA FUNCTION

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Uniform asymptotic expansions of the incomplete gamma functions and the incomplete beta function *)

N.M. Temme

ABSTRACT

New asymptotic expansions are derived for the incomplete gamma functions and the incomplete beta function. In each case the expansion contains the complementary error function and an asymptotic series. The expansions are uniformly valid with respect to certain domains of the parameters.

*) This paper is not for review; it is meant for publication in a journal.

1. INTRODUCTION

In this paper we give some new results on the asymptotic expansions of the incomplete beta and gamma functions. The expansions are uniformly valid with respect to certain domains of the parameters, and, in each case, the dominant term is given by the complementary error function.

The incomplete gamma functions may be considered as special cases of the confluent hypergeometric functions or as Whittaker functions. In the last two decades many new results on uniform asymptotic expansions of these functions have been established, and, in a way, it is not necessary to consider special cases of this large class of functions. However, for special cases special methods can be applied, which may lead to deeper and more elegant results.

Formerly, asymptotic formulas were expressed in elementary transcendental functions, such as the exponential, logarithmic, circular and gamma functions. If extra parameters were present in the formulas, as frequently happens in physical problems, the asymptotic expressions often lacked uniformity with respect to these parameters. Nowadays, these inconveniences are faced by using other functions, such as error functions, Airy functions, Bessel functions, incomplete gamma functions, parabolic cylinder functions. It is very important to have a good knowledge of these functions, and, with the results in this paper, we hope to fill up some gaps, as far as it concerns the incomplete beta and gamma functions.

The functions considered in this paper are very important in mathematical statistics and in probability theory. We expect that with our results new formulas can be derived for the approximation of probability functions and, which is of great importance in this area, of the inverses of these functions.

2. DEFINITIONS AND PRELIMINARIES

The *incomplete gamma functions* are defined by the following equations

$$(2.1)\qquad \gamma(a,x) = \int_0^x e^{-t}t^{a-1}dt \quad , \quad \Gamma(a,x) = \int_x^\infty e^{-t}t^{a-1}dt \; .$$

The parameters a and x may be complex (with certain restrictions), but, for the time being, we suppose that a and x are real and positive. At the end of section 3 some remarks will be made about complex parameters.

From the definition it follows immediately that

$$\gamma(a,x) + \Gamma(a,x) = \Gamma(a). \tag{2.2}$$

The functions may be normalized with respect to the gamma functions. In that case we use the notation

$$P(a,x) = \gamma(a,x)/\Gamma(a) \quad , \quad Q(a,x) = \Gamma(a,x)/\Gamma(a). \tag{2.3}$$

The functions P and Q satisfy the relation

$$P(a,x) + Q(a,x) = 1. \tag{2.4}$$

For large values of x we have the well-known asymptotic expansion

$$\Gamma(a,x) \sim x^{a-1}e^{-x}\{1 + \frac{a-1}{x} + \frac{(a-1)(a-2)}{x^2} + \ldots\} , \tag{2.5}$$

which may be derived by partial integration of the integral in (2.1). The asymptotic expansion (2.5) also follows from the integralrepresentation

$$\Gamma(a,x) = e^{-x}x^a \int_0^\infty e^{-xt}(t+1)^{a-1}dt \tag{2.6}$$

by substitution of the binomial expansion of $(t+1)^{a-1}$ and by interchanging the order of summation and integration. By using this method it can easily be proved that the result in (2.5) can be written in the following way

$$\Gamma(a,x) = x^{a-1}e^{-x}\{\sum_{n=0}^{m} \frac{n!}{x^n}\binom{a-1}{n} + R_m(a,x)\} , \qquad m = 0,1,\ldots, \tag{2.7}$$

with

$$R_m(a,x) = O(x^{-m-1}) , \qquad x \to \infty. \tag{2.8}$$

Moreover, if $m > a-2$, we have

$$(2.9) \qquad R_m(a,x) = \theta \frac{(m+1)!}{x^{m+1}} \binom{a-1}{m+1}, \qquad 0 \le \theta < 1.$$

The right-hand side of (2.9) is small for $x \to \infty$. However, if both x and a are large, the expansion is not useful, unless $a = o(x)$.

For large values of a we can better use the function $\gamma(a,x)$. From (2.1) it follows that

$$\gamma(a,x) = e^{-x} x^a \Gamma(a) \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(a+n+1)} ;$$

this series converges for every finite x. It is useful for $a \to \infty$ and $x = o(a)$, since under this condition the series has an asymptotic character.

Expansions with a more uniform character are given by TRICOMI [1], who found among others

$$(2.10) \qquad \frac{\gamma(a+1, a+y\sqrt{2a})}{\Gamma(a+1)} = \tfrac{1}{2} \operatorname{erfc}(-y) - \frac{1}{3} \sqrt{\frac{2}{a\pi}} (1+y^2) e^{-y^2} + O(a^{-1}),$$

y,a real , $a \to \infty$. The function erfc is the *complementary error function* defined by

$$(2.11) \qquad \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt .$$

It is a special case of $\Gamma(a,x)$, viz.

$$\operatorname{erfc}(x) = \frac{1}{\sqrt{\pi}} \Gamma(\tfrac{1}{2}, x^2) .$$

Also a recent result of WONG [4] may be connected with our results. WONG discussed a more general class of integrals, and he applied his methods to the function $S_n(x)$ defined by

$$e^{nx} = \sum_{r=0}^{n} \frac{(nx)^r}{r!} + \frac{(nx)^n}{n!} S_n(x) .$$

The function S_n is a special case of the incomplete gamma functions, and the asymptotic expansion of $S_n(x)$ for $n \to \infty$, $x \sim 1$ can also be expressed in terms of the error function. (WONG interpreted his results only for $0 \le x \le 1$, but not across the "transition point" at $x = 1$).

It will be the purpose of this paper to give some new representations of $\gamma(a,x)$ and $\Gamma(a,x)$ from which asymptotic expansions can be derived, which hold uniformly in $0 \le \lambda < \infty$ where $\lambda = x/a$ and $a \to \infty$ and/or $x \to \infty$.

3. UNIFORM ASYMPTOTIC EXPANSIONS

Considering the integrand $e^{-t}t^{a-1}$ of (2.1) for $a > 1$, we can remark that this function has its maximum at $t_0 = a-1$. For large values of the parameter a the value of $\gamma(a,x)$, or $\Gamma(a,x)$, changes rapidly when x changes from values with $x < a-1$ to values with $x > a-1$. If the maximum of $e^{-t}t^{a-1}$ lies inside the interval of integration (i.e. $x > t_0$), the asymptotic expansion of $\gamma(a,x)$ can be found by the method of LAPLACE. In the other case ($x < t_0$) the integral is of the following type

$$\int_0^T e^{-\omega t} f(t)\,dt \quad , \qquad T > 0 \quad , \quad \omega \text{ large},$$

which can be expanded by using WATSON's lemma. In both cases asymptotic expansions may be derived which are not valid anymore if x and a are almost equal.

The integrals (2.1) are not attractive as starting points for deriving uniform expansions. Therefore we transform the integral for $\gamma(a,x)$ into

$$(3.1) \qquad \frac{\gamma(a,x)}{\Gamma(a)} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{xs} \frac{ds}{s(s+1)^a} , \qquad c > 0,$$

in which $(s+1)^{-a}$ will have its principal value which is real for $s > -1$. Formula (3.1) can be derived by using some elementary properties of Laplace transforms. Namely, the Laplace transform of $\frac{d\gamma(a,x)}{dx}$ is $\Gamma(a)(s+1)^{-a}$, from which it follows that

$$\frac{\Gamma(a)}{s(s+1)^a} = \int_0^\infty e^{-sx}\gamma(a,x)\,dx.$$

On inverting this expression we obtain (3.1).

The contour in (3.1) can be shifted to the left of $s = 0$, but then we have to take into account the residu at this point. It follows that

$$P(a,x) = 1 + \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} e^{sx} \frac{ds}{s(s+1)^a}, \qquad -1 < d < 0,$$

from which we conclude that

$$Q(a,x) = \frac{-1}{2\pi i} \int_{d-i\infty}^{d+i\infty} e^{sx} \frac{ds}{s(s+1)^a}, \qquad -1 < d < 0. \tag{3.2}$$

The contour in (3.2) will be deformed into a path in the s-plane which crosses the saddle point of the integrand. First, we change (3.2) into

$$Q(a,x) = \frac{e^{-\phi(\lambda)}}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{a\phi(t)} \frac{dt}{\lambda - t}, \qquad 0 < c < \lambda, \tag{3.3}$$

where

$$\phi(t) = t - 1 - \ln t, \qquad \lambda = x/a. \tag{3.4}$$

The saddle point t_0 follows from the equation $\phi'(t_0) = 0$. Hence $t_0 = 1$, $\phi(t_0) = \phi'(t_0) = 0$ and $\phi''(t_0) = 1$. The steepest descent path follows from

$$\operatorname{Im} \phi(t) = \operatorname{Im} \phi(t_0) = 0, \tag{3.5}$$

and, by writing $t = \sigma + i\tau$ ($\sigma, \tau \in \mathbb{R}$) we obtain

$$\sigma = \tau \operatorname{ctg} \tau, \qquad -\pi < \tau < \pi. \tag{3.6}$$

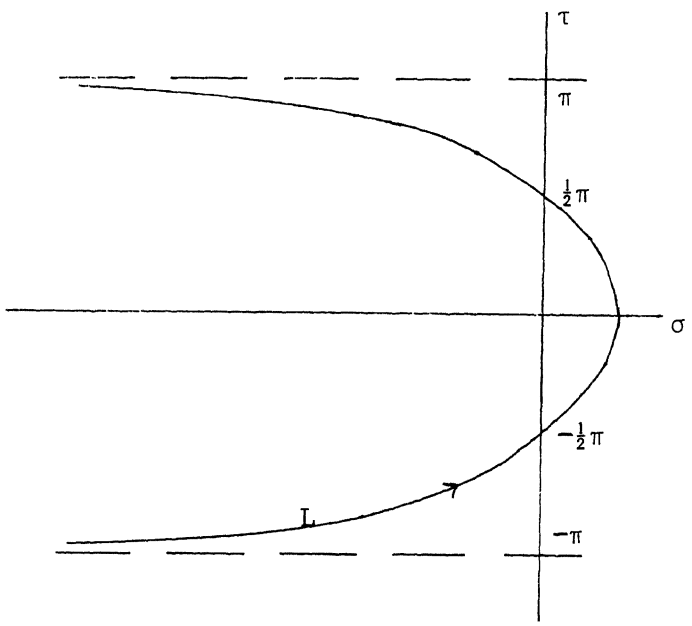

Figure 1

Let temporarily $\lambda > 1$, that is $x > a$. Then the contour in (3.3) may be shifted into the contour L in the t-plane defined by (3.6). According to Cauchy's theorem the integral in (3.3) remains unaltered, and on L $\phi(t)$ is real and negative. Next we define the mapping of the t-plane into the u-plane by the equation

$$(3.7)\qquad -\tfrac{1}{2}u^2 = \phi(t)$$

with the condition that $t \in L$ corresponds with $u \in \mathbb{R}$, and $u < 0$ if $\tau < 0$, $u > 0$ if $\tau > 0$.

The result is

$$(3.8)\qquad Q(a,x) = \frac{e^{-a\phi(\lambda)}}{2\pi i}\int_{-\infty}^{\infty} e^{-\frac{1}{2}au^2}\,\frac{dt}{du}\,\frac{du}{\lambda - t}\,, \qquad \lambda > 1.$$

The presence of the pole at $t = \lambda$ in the integrand of (3.8) is somewhat disturbing, but we will get rid of it. Let u_1 be the point in the u-plane which corresponds to the point $t = \lambda$ in the t-plane, that is

$$-\tfrac{1}{2}u_1^2 = \phi(\lambda) = \lambda - 1 - \ln\lambda,$$

hence $u_1 = \pm\, i\{2(\lambda-1-\ln\lambda)\}^{\frac{1}{2}}$. There is still an ambiguity in the sign. However, the correct sign follows from the conditions imposed on the mapping defined in (3.7). In fact we have

$$u_1 = -i(\lambda-1)\,\{2(\lambda-1-\ln\lambda)/(\lambda-1)^2\}^{\frac{1}{2}}\,, \tag{3.9}$$

where the square root is positive for positive values of the argument.

We can get rid of the pole in (3.8) by writing

$$\frac{dt}{du}\,\frac{1}{\lambda-t} = \frac{dt}{du}\,\frac{1}{\lambda-t} - \frac{A}{u-u_1} + \frac{A}{u-u_1}\,, \tag{3.10}$$

and we choose A such that the first two terms at the right-hand side constitute a regular function at $t = \lambda(u=u_1)$. It turns out that A should be taken to be -1. (According to l'Hôpital's rule: $\lim\limits_{u\to u_1} \frac{dt}{du}\,\frac{u-u_1}{\lambda-t} = -1$). With the partition in (3.10) we obtain

$$Q(a,x) = -\frac{e^{-a\phi(\lambda)}}{2\pi i}\int_{-\infty}^{\infty} e^{-\frac{1}{2}au^2}\,\frac{du}{u-u_1} + R(a,x), \tag{3.11}$$

$$R(a,x) = \frac{e^{-a\phi(\lambda)}}{2\pi i}\int_{-\infty}^{\infty} e^{-\frac{1}{2}au^2}\,\left\{\frac{dt}{du}\,\frac{1}{\lambda-t} + \frac{1}{u-u_1}\right\}du\,, \tag{3.12}$$

and the integral in (3.11) can be expressed in terms of the complementary error function defined in (2.10), so that

$$Q(a,x) = \tfrac{1}{2}\mathrm{erfc}(a^{\frac{1}{2}}\zeta) + R(a,x)\,, \tag{3.13}$$

with

$$\zeta = iu_1/\sqrt{2}. \tag{3.14}$$

From erfc(x) + erfc(-x) = 2 it follows that

$$P(a,x) = \tfrac{1}{2}\mathrm{erfc}(-a^{\frac{1}{2}}\zeta) - R(a,x) \ . \tag{3.15}$$

The treatment of saddle point problems in which the saddle point and a pole of the first order are close together is due to B.L. van der WAERDEN [2].

So far, the results in (3.13) and (3.15) are exact, since no approximations were used. In order to obtain asymptotic expansions for P(a,x) and Q(a,x) the function R(a,x) will be expanded in an asymptotic series. The integrand of R(a,x) is a holomorphic function in the finite u-plane for every $\lambda \geq 0$. If we put the expansion

$$\frac{dt}{du}\frac{1}{\lambda-t} + \frac{1}{u-u_1} = \sum_{k=0}^{\infty} c_k(\lambda)u^k \tag{3.16}$$

in (3.12), and, if we reverse the order of summation and integration, by WATSON's lemma [3], we obtain the expansion

$$R(a,x) \sim \frac{e^{-a\phi(\lambda)}}{2\pi i} \sum_{k=0}^{\infty} c_{2k}(\lambda) \frac{\Gamma(k+\frac{1}{2})}{(\frac{1}{2}a)^{k+\frac{1}{2}}} \ . \tag{3.17}$$

(The conditions for WATSON's lemma are certainly satisfied since the function in (3.16) is bounded for large real values of u).

Each coefficient $c_k(\lambda)$ is an analytic function of λ near $\lambda = 1$, and the expansion (3.17) is not only valid near $\lambda = 1$ but for all $\lambda \geq 0$. That is to say, we can fix x and let a tend to infinity, or conversely . Also x and a may grow dependently or indepently of each other.

The first few coefficients are

$$c_0(\lambda) = \frac{i}{\lambda-1} - \frac{1}{u_1} \ , \qquad c_0(1) = i/3 \ ,$$

$$c_2(\lambda) = \frac{-i(\lambda^2+10\lambda+1)}{12(\lambda-1)^3} - \frac{1}{u_1^3} \ , \qquad c_2(1) = -i/180 \ .$$

Our expansion is more powerful than that of TRICOMI. TRICOMI's formula (2.9) follows from our expansion by replacing ζ in (3.15) by its approximation $(\lambda-1)/\sqrt{2}$ and by using the first term in (3.17). Moreover, for the complete expansion, TRICOMI obtained an infinite series, of which each term contains functions related to the error function. In our expansion just one error function appears and moreover, we obtain expansions for both P and Q. Of course, the coefficients $c_{2k}(\lambda)$ in (3.17) are much more complicated than the coefficients in TRICOMI's expansion.

As remarked before, the expansion (3.17) is also valid for fixed a and $x\to\infty$, in spite of the nature of the series containing terms with negative powers of a. The coefficients however, depend on x and a, and in fact we can say that the sequence $\{d_k\}$, $d_k = c_{2k}(\lambda)a^{-k}$, is an asymptotic sequence. That is,

$$d_{k+1} = o(d_k)$$

if one (or both) of the parameters a and x is (are) large uniformly in $x/a \geq 0$.

Finally, we consider complex values of a and x. The integral (3.1) can be transformed into

$$Q(a,x) = \frac{e^{-x}x^a}{2\pi i}\int_L e^{p-a\ln p}\,\frac{dp}{x-p}\,, \tag{3.18}$$

where we take the branch cut for ln p along the negative p-axis. L is a contour on which at infinity $\frac{1}{2}\pi < |\arg p| < \pi$, L crosses the real p-axis at a positive p-value, and L divides the p-plane in two parts of which one part contains the origin and the other part contains the point x. This is always possible if $|\arg x| < \pi$. A typical situation is sketched below.

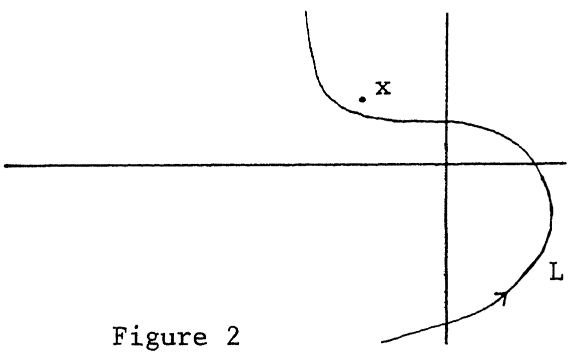

Figure 2

The saddle point in (3.18) is situated at $p = a$. Hence, if $|\arg a| < \pi$, we can deform L such that it crosses the saddle point locally along a path of steepest descent. Therefore, we may conclude that the results of this section are uniformly valid for complex values of the parameters in the ranges $|\arg x| \le \pi-\varepsilon$, $|\arg a| \le \pi \; \delta$. The quantity $\ln \lambda$ has to be interpreted as $\ln x - \ln a$, where the logarithmic function has its principal value. By choosing other directions for the branch cut of the logarithmic function in (3.18), larger domains for arg x can be obtained.

4. THE INCOMPLETE BETA FUNCTION

The *incomplete beta function* is defined by

$$(4.1) \qquad I_x(p,q) = \frac{1}{B(p,q)} \int_0^x t^{p-1}(1-t)^{q-1} dt$$

with $\operatorname{Re} p > 0$, $\operatorname{Re} q > 0$, $0 \le x \le 1$, and

$$(4.2) \qquad B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} .$$

The function in (4.2) is called the *beta function*. Again, we consider real variables x,p and q, and we will derive an asymptotic expansion of $I_x(p,q)$ for large p and q uniformly valid in $0 < \delta \le x \le 1$.

The integrand of (4.1) has its maximum value at the point $t_0 = (p-1)/(p+q-2)$, and for large values of p and q the asymptotic behaviour of I_x depends strongly on the value of x. If $x < t_0$, I_x is small, and if $x > t_0$ we have $I_x \sim 1$.

We first give an integral representation of I_x which resembles those for the incomplete gamma function.

Formula (4.1) is equivalent with

$$(4.3) \qquad I_x(p,q) = \frac{1}{B(p,q)} \int_{\ln \frac{1}{x}}^{\infty} e^{-pt}(1-e^{-t})^{q-1} dt ,$$

and also we have

$$(4.4) \qquad B(p,q) = \int_0^\infty e^{-pt}(1-e^{-t})^{q-1}dt\ ,$$

from which it follows by Laplace inversion

$$(4.5) \qquad (1-e^{-t})^{q-1} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\, B(s,q)ds\ , \qquad c > 0.$$

Using the same technique as in the foregoing section, we obtain

$$I_x(p,q) = \frac{1}{2\pi i\ B(p,q)}\int_{c-i\infty}^{c+i\infty} e^{-(p-s)\ln\frac{1}{x}}\,\frac{B(s,q)}{p-s}ds\ , \qquad 0 < c < p,$$

and this expression can be written as

$$(4.7) \qquad I_x(p,q) = \frac{x^p(1-x)^q}{2\pi i}\,\frac{\Gamma(q)e^q q^{-q}}{B(p,q)}\int_{c-i\infty}^{c+i\infty} e^{q\psi(t)}F_q(t)\,\frac{dt}{t_1-t}\ ,$$

with

$$\psi(t) = t\ln\frac{t}{x} - (1+t)\ln(1+t) - \ln(1-x),$$

$$(4.8) \qquad F_q(t) = \frac{\Gamma(qt)e^{qt}(qt)^{-qt}}{\Gamma(q+qt)e^{q(1+t)}(q+qt)^{-q(1+t)}}\ ,$$

$$t_1 = p/q\ , \quad \text{and} \quad 0 < c < t_1 \qquad .$$

If $c > t_1$, the integral in (4.7) represents the function $-I_{1-x}(q,p)$, as follows from the relation

$$(4.9) \qquad I_x(p,q) = 1 - I_{1-x}(q,p).$$

Of course, the construction of the function $F_q(t)$ in (4.8) is based on the Stirling approximation of the gamma function. For large q we have

$$(4.10) \qquad F_q(t) = \{(1+t)/t\}^{\frac{1}{2}}\,(1+O(q^{-1}))\ , \qquad q \to \infty.$$

The saddle point t_0 of the exponential part in the integrand of (4.7) follows from $\psi'(t_0) = 0$, which gives

$$(4.11)\qquad t_0 = \frac{x}{1-x}\ .$$

Hence, $\psi(t_0) = \psi'(t_0) = 0$, and $\psi''(t_0) = \frac{(1-x)^2}{x}$. The steepest descent line through t_0 follows from $\text{Im}\ \psi(t) = 0$ and has a similar shape as that of the incomplete gamma function. If $x = x_0$, where

$$(4.12)\qquad x_0 = \frac{p}{p+q}\ ,$$

the saddle point at t_0 concides with the pole at t_1.

The calculation of the saddle point t_0 is based on the assumption that the gamma functions in $F_q(t)$ in (4.8) have large arguments. Hence, for small values of x, which correponds to small values of t_0, the calculation is based on false assumptions. Therefore we only consider positive values of x. It is not necessary to have a uniform bound from zero of x. We are even allowing those values of x with $qx \to \infty$.

From now on, details will be omitted, since the method is exactly the same as the one used in the foregoing section. We put

$$(4.13)\qquad -\tfrac{1}{2}u^2 = \psi(t)$$

and the results are

$$(4.14)\qquad I_x(p,q) = \tfrac{1}{2}\text{erfc}(-(q/2)^{\frac{1}{2}}\eta) + S_x(p,q)$$

$$(4.15)\qquad \eta = (x-x_0)\left[2\{\frac{p}{q}\ \ln\frac{x_0}{x} + \ln\frac{1-x_0}{1-x}\}/(x-x_0)^2\right]^{\frac{1}{2}}\ .$$

The square root is positive for positive values of its argument. The function S_x is defined by

$$(4.16)\qquad S_x(p,q) = \frac{x^p(1-x)^q\Gamma(q)e^{-q}q^q}{2\pi i\ B(p,q)}\int_{-\infty}^{\infty} e^{-\frac{1}{2}qu^2}G(u)du\ ,$$

$$(4.17)\qquad G(u) = \frac{F_q(t)}{t_1-t}\frac{dt}{du} + \frac{F_q(t_1)}{u-u_1}\ ,$$

the relation between u and t being defined by (4.13). For u_1 we have

$$(4.18) \qquad -\tfrac{1}{2}u_1^2 = \psi(t_1) \quad , \quad u_1 = i\eta .$$

The role of the parameter λ of the foregoing section is now played by $(x-x_0)$. We have

$$(4.19) \qquad u_1 = i\,\frac{p+q}{q}\left(\frac{p+q}{p}\right)^{\frac{1}{2}}(x-x_0)\ \{1 + \frac{1}{3}\,\frac{p^2-q^2}{pq}(x-x_0) + O(x-x_0)^2\}$$

for $x \to x_0$. An asymptotic expansion of S_x is obtained by expanding $G(u)$ of (4.17) in powers of u, which gives

$$G(u) = \sum_{k=0}^{\infty} d_k u^k \ ,$$

$$(4.20) \qquad S_x(p,q) \sim \frac{x^p(1-x)^q}{2\pi i}\,\frac{\Gamma(p+q)}{\Gamma(p)}\,e^q q^{-q} \sum_{k=0}^{\infty} \frac{d_{2k}\,\Gamma(k+\frac{1}{2})}{(\frac{1}{2}q)^{k+\frac{1}{2}}} \ ,$$

$$(4.21) \qquad d_0 = i\,\frac{F_q(t_0)}{t_1-t_0}\,\frac{\sqrt{x}}{1-x} - \frac{F_q(t_1)}{u_1} \ .$$

The expansion holds for $p \to \infty$ and/or $q \to \infty$, uniformly in $\delta \le x \le 1$, where δ may depend on q, such that $q\delta \to \infty$.

A first approximation for S_x is obtained by replacing the functions F_q in (4.21) by the approximation (4.10). The result is

$$S_x(p,q) = \sqrt{\frac{p}{2\pi q(p+q)}}\left(\frac{x}{x_0}\right)^p\left(\frac{1-x}{1-x_0}\right)^q\left(\frac{1-x_0}{x_0-x} + \frac{i}{u_1\sqrt{x_0}}\right)(1+O(q^{-1})),$$

where u_1 is defined in (4.18) and (4.15). For small values of $(x-x_0)$ this becomes

$$S_x(p,q) = \frac{q-p}{3\{2\pi pq(p+q)\}^{\frac{1}{2}}}(1+O(x-x_0))\left(\frac{x}{x_0}\right)^p\left(\frac{1-x}{1-x_0}\right)^q(1+O(q^{-1})) \ ,$$

$x \to x_0$, $q \to \infty$. In the symmetric case $(p=q)$ higher order coefficients d_2, d_4, ... must be considered in order to get an approximation for $x \sim x_0$. If $x = x_0$ and $p = q$, then the function S_x must vanish. This is a consequence of (4.9) and of $x_0 = \frac{1}{2}$.

5. SOME REMARKS ON THE CONFLUENT HYPERGEOMETRIC FUNCTIONS

As mentioned in the introduction, the incomplete gamma functions are special cases of the confluent hypergeometric functions. Explicitly we have in terms of KUMMER's function

$$\gamma(a,x) = a^{-1}x^{a}{}_1F_1(a;a+1;-x) = a^{-1}x^{a}e^{-x}{}_1F_1(1;a+1;x).$$

For the function ${}_1F_1(a;c;x)$ we have the integral representation

$${}_1F_1(a;c;t) = \frac{t^{1-\lambda}\Gamma(\lambda)}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{st}s^{-\lambda}{}_2F_1(a,\lambda;c;s^{-1})ds\ ,$$

where $d > 1$ and $\mathrm{Re}\ \lambda > 0$. This formula can be proved by expanding the two hypergeometric functions as power series. The parameter λ is a free parameter and can be taken equal to c. In that case the ${}_2F_1$ function reduces to $(1-s^{-1})^{-a}$ and we obtain

$$(5.1)\qquad {}_1F_1(a;c;b) = \frac{t^{1-c}\Gamma(c)}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{st}s^{a-c}(s-1)^{-a}ds\ ,$$

$d > 1$ and $\mathrm{Re}\ c > 0$. The branches of the multi-valued functions s^{a-c} and $(s-1)^{-a}$ emanate from 0 and 1, respectively, to $-\infty$, and the functions will have their principal values. If we take in the integral in (5.1) values of d between 0 and 1, and if we take the branch cut of $(s-1)^{-a}$ from 1 to $+\infty$, the integral represents a multiple of the confluent hypergeometric function, which is usually denoted by $U(a;c;t)$.

If we fix a and take c and/or x large we can use the saddle point method. If $a = n$, a positive integer, we have a pole of order n at $s = 1$, and if $x \sim c$ this pole is close to the saddle point. Again, the uniform asymptotic expansion will contain error functions. For general values of a we have a branch point at $s = 1$, and now the expansion contains parabolic cylinder functions. Finally, if we also take a large, two saddle points have to be considered, which for some complex values of a, c and x may be close together. In that case the expansion can be given in terms of Airy functions. In a future publication these expansions for the confluent hypergeometric functions will be worked out in more detail.

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