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GENERALIZED POWER SERIES EXPANSIONS FOR A CLASS OF ORTHOGONAL POLYNOMIALS IN TWO VARIABLES

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Generalized power series expansions for a class of orthogonal polynomials in two variables*)

by

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ABSTRACT

This paper continues the analysis of a class of orthogonal polynomials in two variables on a region bounded by two straight lines and a parabola touching these lines, which was introduced by the first author. An explicit series expansion for these polynomials is obtained, which generalizes Constantine's expansion of hypergeometric functions of (2×2) matrix argument in terms of James' zonal polynomials. In two special cases the orthogonal polynomials turn out to be Appell's hypergeometric F_4-functions and certain hypergeometric functions in two variables of order three, respectively.

KEY WORDS & PHRASES: *orthogonal polynomials in two variables; series expansions in terms of James type zonal polynomials; hypergeometric functions of matrix argument; Appell's hypergeometric function* F_4; *hypergeometric functions in two variables of order three.*

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*) This paper is not for review; it is meant for publication elsewhere.

CONTENTS

1. INTRODUCTION

This paper continues the analysis of a class of orthogonal polynomials in two variables over a region bounded by two straight lines and a parabola touching these lines. The basic results on this class of polynomials are given in a paper by the first author [21], where the polynomials are introduced, and in another paper by the second author [28]. See also the survey papers [23] and [25] by the first author.

These orthogonal polynomials in two variables, which in this paper will be denoted by $R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)$, can be considered as highly nontrivial generalizations of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. The main purpose of this paper is the derivation of an explicit series expansion for $R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)$ which generalizes the hypergeometric power series expansion for Jacobi polynomials. Such an expansion should have the form

$$(1.1) \qquad R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta) = \sum_{m,\ell} c_{n,k;m,\ell}^{\alpha,\beta,\gamma}\, f_{m,\ell}^{\alpha,\beta,\gamma}(\xi,\eta),$$

where the coefficients $c_{n,k;m,\ell}^{\alpha,\beta,\gamma}$ and the functions $f_{m,\ell}^{\alpha,\beta,\gamma}(\xi,\eta)$ have to be more elementary special functions with well-known explicit expressions. It turns out that if either $k = 0$ or $k = n$ the functions $f_{m,\ell}$ can be chosen as monomials and the coefficients then become quotients of products of gamma functions. For $k = n$ the polynomial can be identified with a terminating Appell's hypergeometric F_4-function in two variables, and for $k = 0$ we obtain a certain hypergeometric function in two variables of order three.

However, if $k \neq 0$ or n then a certain choice of monomials for $f_{m,\ell}$ leads to rather awkward expressions for the coefficients in (1.1). In this general case the best choice for $f_{m,\ell}$ seems to be the so-called James-type zonal polynomial $Z_{m,\ell}^{\gamma}(\xi,\eta)$, which can be expressed in terms of Gegenbauer polynomials. Then the coefficients in (1.1) can be expressed in terms of a hypergeometric ${}_4F_3$-function of unit argument. In doing this choice we were motivated by the fact that $R_{n,n}^{\alpha,\beta,0}(\xi,\eta)$ can be identified with a hypergeometric function of (2×2) matrix argument. CONSTANTINE [10] proved that hypergeometric functions of matrix argument have a nice explicit expansion in terms of the zonal polynomials introduced by JAMES [18]. In the (2×2) case these zonal polynomials can be identified with our polynomials

$Z^{0}_{m,\ell}(\xi,\eta)$.

We already pointed out that the polynomials $R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$ become more simple on the boundary lines $n = 0$ and $n = k$ of the region $\{(n,k) \in \mathbb{Z}^2 \mid n \geq k \geq 0\}$ for which $R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$ is defined. Similarly, the analysis of the polynomials on the boundary of the orthogonality region in the (ξ,η) plane is easier than in the interior of this region. In particular, $R^{\alpha,\beta,\gamma}_{n,n}(\xi,0)$ and $R^{\alpha,\beta,\gamma}_{n,0}(\xi,\frac{1}{4}\xi^2)$ turn out to be Jacobi polynomials of argument $1-2\xi$ and $1-\xi$, respectively. Our proofs exploit these degeneracies in the (n,k) and (ξ,η) planes. The two pairs of differential recurrence relations derived in [21] and [28] will also be used as essential tools.

The results in this paper are not only an interesting part of the analysis of the polynomials $R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$, but they may be important for a wider class of readers. First, there is a close relationship with the theory of Jacobi polynomials. Many results for Jacobi polynomials will be used in this paper, and, on the other hand, some known results for Jacobi polynomials can be better understood from our two-variable point of view. Second, we bring some unity in the bewildering variety of special functions in more than one variable by identifying hypergeometric functions in two variables (in particular F_4) and hypergeometric functions of (2×2) matrix argument with special cases of our polynomials.

In sections 2 and 3 of this paper we summarize the results on Jacobi polynomials and on the polynomials $R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$ which will be needed. In section 4 the James type zonal polynomials are introduced. The boundary values of the polynomials $R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$ are considered in section 5. Section 6 contains the expansion of the polynomials $R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$ in terms of James type zonal polynomials. In section 7 we consider expansions of the form (1.1) with another natural choice for the functions $f^{\alpha,\beta,\gamma}_{m,\ell}(\xi,\eta)$. This leads to the identification of $R^{\alpha,\beta,\gamma}_{n,n}(\xi,\eta)$ with Appell's F_4-function. Finally, in section 8 we derive expansions of $R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$, $k = 0$ or n, as double Jacobi series with positive coefficients. Sections 7 and 8 can be read independently of each other and of sections 4 and 6. However, section 5 is needed for all the subsequent sections.

In a forthcoming paper we will extend the correspondence between $R^{\alpha,\beta,\gamma}_{n,n}(\xi,\eta)$ and Appell's function F_4 to the non-polynomial case. An expansion in terms of James type zonal polynomials will be given for those solutions of the system of partial differential equations for F_4 which are regular in the singular point (1,0). For special values of the parameters these second solutions are precisely the hypergeometric functions of (2×2) matrix argument.

2. PROPERTIES OF JACOBI POLYNOMIALS

In this section we collect all results on Jacobi polynomials which will be needed in this paper. The standard formulas for Jacobi polynomials have been taken from SZEGÖ [29, Chap. 4] and ERDÉLYI [13, Chap. 10]. A useful survey of many recent results on Jacobi polynomials is given by ASKEY [2].

Let $\alpha,\beta > -1$. Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are orthogonal polynomials on the interval $(-1,1)$ with respect to the weight function $(1-x)^\alpha\,(1+x)^\beta$ and with the normalization $P_n^{(\alpha,\beta)}(1) := (\alpha+1)_n/n!$ We will use the renormalized Jacobi polynomials $R_n^{(\alpha,\beta)}(x) := P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$.

Differentiation formulas:

$$(2.1)\qquad (1-x^2)\frac{d^2}{dx^2}R_n^{(\alpha,\beta)}(x) + (\beta-\alpha-(\alpha+\beta+2)x)\frac{d}{dx}R_n^{(\alpha,\beta)}(x) + n(n+\alpha+\beta+1)R_n^{(\alpha,\beta)}(x) = 0,$$

$$(2.2)\qquad \frac{d}{dx}R_n^{(\alpha,\beta)}(x) = \begin{cases} \dfrac{n(n+\alpha+\beta+1)}{2(\alpha+1)}\,R_{n-1}^{(\alpha+1,\beta+1)}(x) & \text{if } n > 0, \\ 0 & \text{if } n = 0, \end{cases}$$

$$(2.3)\qquad (1-x)^{-\alpha}(1+x)^{-\beta}\frac{d}{dx}\left[(1-x)^{\alpha+1}(1+x)^{\beta+1}R_{n-1}^{(\alpha+1,\beta+1)}(x)\right] = -2(\alpha+1)R_n^{(\alpha,\beta)}(x).$$

Series expansions:

$$(2.4)\qquad R_n^{(\alpha,\beta)}(x) = {}_2F_1(-n,n+\alpha+\beta+1;\alpha+1;\tfrac{1}{2}(1-x)) = \sum_{k=0}^{n}\frac{(-n)_k(n+\alpha+\beta+1)_k}{(\alpha+1)_k\;k!}\left(\frac{1-x}{2}\right)^k,$$

$$(2.5)\qquad R_n^{(\alpha,\beta)}(x) = \left(\frac{1+x}{2}\right)^n {}_2F_1\left(-n,-n-\beta;\alpha+1;\frac{x-1}{x+1}\right) =$$

$$= 2^{-n} \sum_{k=0}^{n} \frac{(-n)_k(-n-\beta)_k}{(\alpha+1)_k\ k!} (x+1)^{n-k}(x-1)^k,$$

$$(2.6)\qquad \left(\frac{1-x}{2}\right)^n = \frac{(\alpha+1)_n}{(\alpha+\beta+2)_n} \sum_{k=0}^{n} \frac{(2k+\alpha+\beta+1)(-n)_k(\alpha+\beta+2)_k}{(k+\alpha+\beta+1)(n+\alpha+\beta+2)_k\ k!} R_k^{(\alpha,\beta)}(x).$$

Value for $x = -1$:

$$(2.7)\qquad R_n^{(\alpha,\beta)}(-1) = (-1)^n \frac{(\beta+1)_n}{(\alpha+1)_n}.$$

Linear and quadratic transformations:

$$(2.8)\qquad \frac{R_n^{(\alpha,\beta)}(-x)}{R_n^{(\alpha,\beta)}(-1)} = R_n^{(\beta,\alpha)}(x),$$

$$(2.9)\qquad R_{2n}^{(\alpha,\alpha)}(x) = R_n^{(\alpha,-\frac{1}{2})}(2x^2-1),$$

$$(2.10)\qquad R_{2n+1}^{(\alpha,\alpha)}(x) = x\, R_n^{(\alpha,\frac{1}{2})}(2x^2-1).$$

Gegenbauer and Chebyshev polynomials:

$$(2.11)\qquad R_n^{(\gamma,\gamma)}(x) = \frac{(\gamma+\frac{1}{2})_n}{(2\gamma+1)_n} \sum_{k=0}^{[n/2]} \frac{(-n)_{2k}}{(-n-\gamma+\frac{1}{2})_k\ k!} (2x)^{n-2k},$$

$$(2.12)\qquad R_n^{(-\frac{1}{2},-\frac{1}{2})}(\cos\theta) = \cos n\theta,$$

$$(2.13)\qquad R_n^{(\frac{1}{2},\frac{1}{2})}(\cos\theta) = \frac{\sin(n+1)\theta}{(n+1)\sin\theta}.$$

It follows from these last two formulas that

(2.14) $$R_n^{(-\frac{1}{2},-\frac{1}{2})}(\tfrac{1}{2}(t+t^{-1})) = \tfrac{1}{2}(t^n+t^{-n}),$$

(2.15) $$R_n^{(\frac{1}{2},\frac{1}{2})}(\tfrac{1}{2}(t+t^{-1})) = \frac{t^{n+1}-t^{-n-1}}{(n+1)(t-t^{-1})}.$$

Formula (2.12) is a special case of

(2.16) $$R_n^{(\gamma,\gamma)}(\cos\theta) = \frac{n!}{(2\gamma+1)_n}\sum_{k=0}^{n}\frac{(\gamma+\frac{1}{2})_k(\gamma+\frac{1}{2})_{n-k}}{k!\ (n-k)!}\cos(n-2k)\theta,$$

which formula results in

(2.17) $$R_n^{(\gamma,\gamma)}(\tfrac{1}{2}(t+t^{-1})) = \frac{n!}{(2\gamma+1)_n}\sum_{k=0}^{n}\frac{(\gamma+\frac{1}{2})_k(\gamma+\frac{1}{2})_{n-k}}{k!\ (n-k)!}\ t^{n-2k}.$$

Quadratic norm: Let

(2.18) $$\omega_n^{(\alpha,\beta)} := \frac{\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta dx}{\int_{-1}^{1}(R_n^{(\alpha,\beta)}(x))^2\ (1-x)^\alpha(1+x)^\beta dx}.$$

Then

(2.19) $$\omega_n^{(\alpha,\beta)} = \frac{(2n+\alpha+\beta+1)(\alpha+1)_n(\alpha+\beta+2)_n}{(n+\alpha+\beta+1)(\beta+1)_n\quad n!}.$$

Christoffel-Darboux formula:

(2.20) $$\sum_{k=0}^{n}\omega_k^{(\alpha,\beta)}R_k^{(\alpha,\beta)}(x)R_k^{(\alpha,\beta)}(y) = \frac{2(\alpha+1)(\alpha+2)_n(\alpha+\beta+2)_n}{(2n+\alpha+\beta+2)(\beta+1)_n\quad n!}\cdot$$

$$\cdot\frac{R_{n+1}^{(\alpha,\beta)}(x)R_n^{(\alpha,\beta)}(y) - R_n^{(\alpha,\beta)}(x)R_{n+1}^{(\alpha,\beta)}(y)}{x-y};$$

Limit formulas:

$$\lim_{\beta\to\infty} \frac{R_n^{(\alpha,\beta)}(x)}{R_n^{(\alpha,\beta)}(-1)} = \left(\frac{1-x}{2}\right)^n, \tag{2.21}$$

$$\lim_{\alpha\to\infty} R_n^{(\alpha,\beta)}(x) = \left(\frac{1+x}{2}\right)^n, \tag{2.22}$$

$$\lim_{\gamma\to\infty} R_n^{(\gamma,\gamma)}(x) = x^n. \tag{2.23}$$

These three results follow from (2.4), (2.7), (2.8) and (2.11).

Another pair of differential recurrence relations:

$$(1-x)^{1-\alpha} \frac{d}{dx}\left[(1-x)^{\alpha} R_n^{(\alpha,\beta)}(x)\right] = -\alpha\, R_n^{(\alpha-1,\beta+1)}(x), \tag{2.24}$$

$$(1+x)^{-\beta} \frac{d}{dx}\left[(1+x)^{\beta+1} R_n^{(\alpha-1,\beta+1)}(x)\right] = \alpha^{-1}(n+\alpha)(n+\beta+1) R_n^{(\alpha,\beta)}(x). \tag{2.25}$$

It follows from SLATER [27, (2.5.31)] that

$$R_n^{(\alpha,\beta)}(x)\, R_n^{(\beta,\alpha)}(x) = {}_4F_3\left(\begin{matrix}-n, \frac{1}{2}(\alpha+\beta+1), \frac{1}{2}(\alpha+\beta+2), n+\alpha+\beta+1; \\ \alpha+1, \beta+1, \alpha+\beta+1;\end{matrix}\; 1-x^2\right). \tag{2.26}$$

In particular:

$$\left(R_n^{(\alpha,\alpha)}(x)\right)^2 = {}_3F_2(-n, \alpha+\tfrac{1}{2}, n+2\alpha+1; \alpha+1, 2\alpha+1; 1-x^2). \tag{2.27}$$

Appell's hypergeometric function F_4 is defined by

$$F_4(a,b;c,c';x,y) := \sum_{m=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m (c')_n\, m!\, n!} x^m y^n, \quad |x|^{\frac{1}{2}}+|y|^{\frac{1}{2}} < 1. \tag{2.28}$$

A result of Watson (cf. SLATER [27,(8.4.4)] gives

$$(2.29)\qquad R_n^{(\alpha,\beta)}(x)\, R_n^{(\beta,\alpha)}(y) =$$

$$= F_4(-n,n+\alpha+\beta+1;\alpha+1,\beta+1;\tfrac14(1-x)(1+y),\tfrac14(1+x)(1-y)).$$

THEOREM 2.1. (cf. BATEMAN [4, pp.392,393] and KOORNWINDER [22, §2]).
If

$$(2.30)\qquad R_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} c_k\left(\frac{1+x}{2}\right)^k$$

then

$$(2.31)\qquad R_n^{(\alpha,\beta)}(x)\, R_n^{(\alpha,\beta)}(y) = \sum_{k=0}^{n} c_k\left(\frac{x+y}{2}\right)^k R_k^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right)$$

and

$$(2.32)\qquad c_k = \frac{(-1)^n(\beta+1)_n(-n)_k(n+\alpha+\beta+1)_k}{(\alpha+1)_n(\beta+1)_k\;\; k!}\,.$$

The product formula and addition formula for Gegenbauer polynomials (cf. ERDÉLYI [12, 3.15(19) and 3.15(20)]):

$$(2.33)\qquad R_n^{(\gamma,\gamma)}(x)\, R_n^{(\gamma,\gamma)}(y) = \frac{\Gamma(\gamma+1)}{\pi^{\frac12}\Gamma(\gamma+\frac12)}\,\cdot$$

$$\cdot\int_{-1}^{1} R_n^{(\gamma,\gamma)}(xy+(1-x^2)^{\frac12}(1-y^2)^{\frac12}t)(1-t^2)^{\gamma-\frac12}\,dt,\ \gamma > -\tfrac12,$$

$$(2.34)\qquad R_n^{(\gamma,\gamma)}(xy+(1-x^2)^{\frac12}(1-y^2)^{\frac12}t) = \sum_{k=0}^{n}\frac{(-1)^k(-n)_k(n+2\gamma+1)_k}{2^{2k}(\gamma+1)_k(\gamma+1)_k}\,\cdot$$

$$\cdot(1-x^2)^{\frac12 k}\, R_{n-k}^{(\gamma+k,\gamma+k)}(x)(1-y^2)^{\frac12 k}\, R_{n-k}^{(\gamma+k,\gamma+k)}(y)\,\cdot$$

$$\cdot\,\omega_k^{(\gamma-\frac12,\gamma-\frac12)}R_k^{(\gamma-\frac12,\gamma-\frac12)}(t).$$

The product formula for Jacobi polynomials (cf. KOORNWINDER [22, (3.7)]):

$$(2.35)\qquad R_n^{(\alpha,\beta)}(x)\,R_n^{(\alpha,\beta)}(y) = \frac{2\Gamma(\alpha+1)}{\pi^{\frac{1}{2}}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})}\cdot$$

$$\cdot\int_0^1\int_0^\pi R_n^{(\alpha,\beta)}(\tfrac{1}{2}(1+x)(1+y)+\tfrac{1}{2}(1-x)(1-y)r^2+(1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}\,r\cos\phi-1)\cdot$$

$$\cdot(1-r^2)^{\alpha-\beta-1}r^{2\beta+1}(\sin\phi)^{2\beta}\,dr\,d\phi,\quad \alpha>\beta>-\tfrac{1}{2}$$

THEOREM 2.2. (cf. SZEGÖ [29, Theorem 7.32.1])

(a) *If* $\alpha\ \geq\beta$ *and* $\alpha\ \geq-\frac{1}{2}$ *then*

$$|\,R_n^{(\alpha,\beta)}(x)\,| \leq 1 \textit{ for } -1\leq x\leq 1.$$

(b) *If* $\alpha\leq\beta$ *and* $\beta\ \geq-\frac{1}{2}$ *then*

$$|\,R_n^{(\alpha,\beta)}(x)\,| \leq |\,R_n^{(\alpha,\beta)}(-1)\,| \quad \textit{for } -1\leq x\leq 1.$$

The coefficients in

$$(2.36)\qquad R_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} g_{n;k}^{\alpha,\beta;a,b}\,R_k^{(a,b)}(x)$$

are called connection coefficients. We have

$$(2.37)\qquad g_{n;k}^{\alpha,\beta;a,\beta} = \frac{n!\ (\beta+1)_n(\alpha-a)_{n-k}(n+\alpha+\beta+1)_k\ \ \omega_k^{(a,\beta)}}{(\alpha+1)_n(a+\beta+2)_n(n-k)!\ (n+a+\beta+2)_k},$$

cf. SZEGÖ [29, (9.41)]. Hence

$$(2.38)\qquad g_{n;k}^{\alpha,\beta;a,\beta} > 0 \text{ if } \alpha > a.$$

In the general case the connection coefficients are given by

$$(2.39)\qquad g_{n;k}^{\alpha,\beta;a,b} = \frac{(n+\alpha+\beta+1)_k (a+1)_k \; n!}{(k+a+b+1)_k (\alpha+1)_k (n-k)!k!} \,\cdot$$

$$\cdot {}_3F_2\left(\begin{matrix}-n+k, n+k+\alpha+\beta+1, k+a+1; \\ 2k+a+b+2, k+\alpha+1;\end{matrix}\; 1\right),$$

cf. FELDHEIM [14] or ASKEY & GASPER [3,(2.5),(2.6)].

THEOREM 2.3. *If* $a \le b$, $\alpha + \beta \ge a + b$, $\beta - \alpha \le b - a$ *then* $g_{n;k}^{\alpha,\beta;a,b} \ge 0$ *and the inequality is strict except if* $a = b$, $\alpha = \beta$ *and* $n - k$ *is odd.*

This theorem is essentially a part of Theorem 1 in *Askey & Gasper* [3]. The last statement in Theorem 2.3 is a slight refinement of their result. It follows immediately from the recurrence relation (2.2) in [3].

The coefficients in

$$(2.40)\qquad R_m^{(\alpha,\beta)}(x)\; R_n^{(\alpha,\beta)}(x) = \sum_{k=|m-n|}^{m+n} A_{m,n,k}^{(\alpha,\beta)}\; \omega_k^{(\alpha,\beta)}\; R_k^{(\alpha,\beta)}(x)$$

are called linearization coefficients. We have $A_{m,n,k}^{(\alpha,\alpha)} = 0$ if $m + n + k$ is odd and

$$(2.41)\qquad A_{m,n,k}^{(\alpha,\alpha)} = \frac{(2\alpha+1)_{\frac{1}{2}(m+n+k)}}{(\alpha+3/2)_{\frac{1}{2}(m+n+k)}} \,\cdot$$

$$\cdot \frac{(\alpha+\frac{1}{2})_{\frac{1}{2}(m+n-k)} (\alpha+\frac{1}{2})_{\frac{1}{2}(n+k-m)} (\alpha+\frac{1}{2})_{\frac{1}{2}(k+m-n)} \; m!\; n!\; k!}{(\frac{1}{2}(m+n-k))!(\frac{1}{2}(n+k-m))!(\frac{1}{2}(k+m-n))!(2\alpha+1)_m (2\alpha+1)_n (2\alpha+1)_k}$$

if $m + n + k$ is even. Formula (2.41) was first stated by DOUGALL [11] without proof. See ASKEY [2, lecture 5] for a survey of several proofs of (2.41) which were published afterwards.

THEOREM 2.4. (cf. GASPER [15]).
If $\alpha \ge \beta$ *and* $\alpha + \beta \ge -1$ *then* $A_{m,n,k}^{(\alpha,\beta)} \ge 0$.

In [16] GASPER extended this nonnegativity result for the linearization coefficients to a slightly larger region of the (α,β) plane.

LEMMA 2.5. *Let the polynomials* $p_n(x)$, $n = 0,1,2,\ldots$, *be orthogonal on the interval* (a,b) *with respect to the strictly positive weight function* $w(x)$. *Then any polynomial of the form*

$$f(x) := \sum_{m=k}^{n} c_m p_m(x), \text{ with } c_n \neq 0,$$

has at least k *zeros of odd multiplicity on* (a,b).

PROOF. Suppose that $f(x)$ has only ℓ zeros $x_1,\ldots,x_\ell$ of odd multiplicity on (a,b) with $\ell < k$. Then $f(x)(x-x_1)\ldots(x-x_\ell)$ is either nonnegative or nonpositive on (a,b) and not identically zero. But

$$\int_a^b f(x)(x-x_1)\ldots(x-x_\ell)\, w(x)\, dx = 0.$$

This is a contradiction. □

Finally we mention Saalschütz's formula (cf. SLATER [27, §2.3.1]):

$$(2.42)\qquad {}_3F_2\,(a,b,-n;\; c,1+a+b-c-n;1) = \frac{(c-a)_n\,(c-b)_n}{(c)_n(c-a-b)_n}\,,\quad n = 0,1,2,\ldots\,.$$

3. SOME EARLIER RESULTS FOR THE ORTHOGONAL POLYNOMIALS $R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)$

The purpose of this section is to summarize some of the results on the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ obtained in KOORNWINDER [21] and SPRINKHUIZEN [28]. We will change the notation used in these two papers by introducing new coordinates $\xi := 1 - \frac{1}{2}u$, $\eta := \frac{1}{4}(1-u+v)$ and by renormalizing the polynomials such that they are equal to 1 in the vertex $(\xi,\eta) = (0,0)$. A motivation of this new notation will be given in section 4.

Let Ω be the region

$$(3.1)\qquad \Omega := \{(\xi,\eta) \mid \eta > 0,\; 1-\xi+\eta > 0,\; \xi^2 - 4\eta > 0,\; 0 < \xi < 2\},$$

which is bounded by two straight lines and a parabola touching these lines (cf. Figure 1).

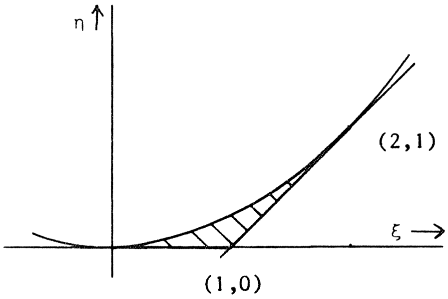

Figure 1

Let

$$w_{\alpha,\beta,\gamma}(\xi,\eta) := \eta^{\alpha}(1-\xi+\eta)^{\beta}(\xi^2-4\eta)^{\gamma}, \quad (\xi,\eta) \in \Omega. \tag{3.2}$$

DEFINITION 3.1. Let $\alpha,\beta,\gamma > -1$, $\alpha + \gamma + \frac{3}{2} > 0$, $\beta + \gamma + \frac{3}{2} > 0$. Let n, k be integers, $n \geq k \geq 0$. Then $R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)$ is a linear combination of monomials $1, \xi, \eta, \xi^2, \xi\,\eta, \eta^2, \xi^3, \xi^2\eta, \ldots, \xi^n, \xi^{n-1}\,\eta, \ldots, \xi^{n-k}\,\eta^k$ such that

(i) $$\iint_{\Omega} R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)\, \xi^{m-\ell}\, \eta^{\ell}\, w_{\alpha,\beta,\gamma}(\xi,\eta)\, d\xi\, d\eta = 0$$

if $m \geq \ell \geq 0$ and if either $m < n$ or $m = n$, $\ell < k$;

(ii) $$R_{n,k}^{\alpha,\beta,\gamma}(0,0) = 1.$$

If $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ is defined as in [21] then

$$R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta) = \frac{p_{n,k}^{\alpha,\beta,\gamma}(2-2\xi,\ 1-2\xi+4\eta)}{p_{n,k}^{\alpha,\beta,\gamma}(2,1)}, \tag{3.3}$$

where the value of $p_{n,k}^{\alpha,\beta,\gamma}(2,1)$ is given in [28, (7.3)].

For $\gamma = \pm\frac{1}{2}$ the polynomials $R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)$ can be expressed in terms of Jacobi polynomials by

$$R_{n,k}^{\alpha,\beta,-\frac{1}{2}}(x+y,xy) = \tfrac{1}{2}\{R_n^{(\alpha,\beta)}(1-2x)\, R_k^{(\alpha,\beta)}(1-2y) + R_k^{(\alpha,\beta)}(1-2x)\cdot R_n^{(\alpha,\beta)}(1-2y)\}, \tag{3.4}$$

$$(3.5)\qquad R_{n,k}^{\alpha,\beta,\frac{1}{2}}(x+y,xy) = \frac{-(\alpha+1)}{(n-k+1)(n+k+\alpha+\beta+2)(x-y)}\cdot$$

$$\cdot\{R_{n+1}^{(\alpha,\beta)}(1-2x)\ R_k^{(\alpha,\beta)}(1-2y) - R_k^{(\alpha,\beta)}(1-2x)\ R_{n+1}^{(\alpha,\beta)}(1-2y)\}.$$

By comparing (2.20) with (3.5) it follows that

$$(3.6)\qquad R_{n,n}^{\alpha,\beta,\frac{1}{2}}\ (1-\tfrac{1}{2}(x+y),\ \tfrac{1}{4}(1-x)(1-y)) =$$

$$= \frac{(\beta+1)_n\ n!}{(\alpha+2)_n(\alpha+\beta+2)_n}\sum_{k=0}^{n}\omega_k^{(\alpha,\beta)}R_k^{(\alpha,\beta)}(x)R_k^{(\alpha,\beta)}(y).$$

Let $\partial_{x_1x_2\ldots x_k}$ denote the partial derivative $\dfrac{\partial^k}{\partial x_1\partial x_2\ldots\partial x_k}$. Consider the second order differential operators

$$(3.7)\qquad D_-^{\gamma} := \tfrac{1}{4}\{\partial_{\xi\xi}+\xi\partial_{\xi\eta}+\eta\partial_{\eta\eta}+(\gamma+\frac{3}{2})\partial_{\eta}\},$$

$$(3.8)\qquad D_+^{\alpha,\beta,\gamma} := 16(w_{\alpha,\beta,\gamma}(\xi,\eta))^{-1}\ D_-^{-\gamma}\circ w_{\alpha+1,\beta+1,\gamma}\ (\xi,\eta),$$

$$(3.9)\qquad E_-^{\alpha,\beta} := \tfrac{1}{2}\{(1-\xi)\partial_{\xi\xi}-2\eta\partial_{\xi\eta}-\eta\partial_{\eta\eta}-(\alpha+\beta+2)\partial_{\xi}-(\alpha+1)\partial_{\eta}\},$$

$$(3.10)\qquad E_+^{\alpha,\beta,\gamma} := 4(w_{\alpha,\beta,\gamma}(\xi,\eta))^{-1}\ E_-^{-\alpha,-\beta}\circ w_{\alpha,\beta,\gamma+1}(\xi,\eta).$$

The operators $D_+^{\alpha,\beta,\gamma}$ and $E_+^{\alpha,\beta,\gamma}$ can be written more explicitly as

$$(3.11)\qquad D_+^{\alpha,\beta,\gamma} = 16\ \eta(1-\xi+\eta)D_-^{\gamma} +$$

$$+ 4\{(\alpha+1)\xi(1-\xi) + (\alpha+\beta+2)\xi\eta-2(\beta+1)\eta\}\ \partial_{\xi} +$$

$$+ 4\ \eta\{-(2\alpha+\beta+3)\xi + 2(\alpha+\beta+2)\eta + 2(\alpha+1)\}\ \partial_{\eta} +$$

$$-\ 4(\alpha+1)(\alpha+\beta+\gamma+\frac{5}{2})\ \xi + 4\ (\alpha+\beta+2)(\alpha+\beta+\gamma+\frac{5}{2})\ \eta + 4\ (\alpha+1)(\alpha+\gamma+\frac{3}{2}),$$

$$(3.12)\qquad E_{+}^{\alpha,\beta,\gamma} = 4(\xi^2-4\eta)\ \{E_{-}^{\alpha,\beta}-(\gamma+1)(2\partial_\xi+\partial_\eta)\} + \\ + 4(\gamma+1)(\xi-2\eta)(2\partial_\xi+\xi\partial_\eta) + \\ - 4(\alpha+\beta+2\gamma+3)(\gamma+1)\xi + 8\ (\gamma+1)(\alpha+\gamma+\tfrac{3}{2})\ .$$

It was proved in [21,§5] and [28,§4] that these differential operators act on $R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)$ as follows:

$$(3.13)\qquad \begin{cases} D_{-}^{\gamma}\ R_{n,0}^{\alpha,\beta,\gamma}(\xi,\eta) = 0, \\ D_{-}^{\gamma}\ R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta) = \\ = \dfrac{k(k+\alpha+\beta+1)(n+\gamma+\frac{1}{2})(n+\alpha+\beta+\gamma+3/2)}{4(\alpha+1)(\alpha+\gamma+3/2)}\ R_{n-1,k-1}^{\alpha+1,\beta+1,\gamma}\ (\xi,\eta) \text{ if } k>0, \end{cases}$$

$$(3.14)\qquad D_{+}^{\alpha,\beta,\gamma}\ R_{n-k,k-1}^{\alpha+1,\beta+1,\gamma}(\xi,\eta) = 4(\alpha+1)(\alpha+\gamma+\tfrac{3}{2})\ R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta),\ k>0,$$

$$(3.15)\qquad \begin{cases} E_{-}^{\alpha,\beta}\ R_{n,n}^{\alpha,\beta,\gamma}(\xi,\eta) = 0, \\ E_{-}^{\alpha,\beta}\ R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta) = \\ = \dfrac{(n-k)(n-k+2\gamma+1)(n+k+\alpha+\beta+1)(n+k+\alpha+\beta+2\gamma+2)}{8(\gamma+1)(\alpha+\gamma+3/2)}\ R_{n-1,k}^{\alpha,\beta,\gamma+1}(\xi,\eta) \text{ if } n>k, \end{cases}$$

$$(3.16)\qquad E_{+}^{\alpha,\beta,\gamma}\ R_{n-1,k}^{\alpha,\beta,\gamma+1}(\xi,\eta) = 8(\gamma+1)(\alpha+\gamma+\tfrac{3}{2})\ R_{n,k}^{\alpha,\beta,\gamma+1}(\xi,\eta),\ n>k.$$

For the calculation of the coefficients in these four differentiation formulas we used the explicit value of $p_{n,k}^{\alpha,\beta,\gamma}\ (2,1)$, cf. [28,(7.3)]. Note that these coefficients are nonzero if α,β,γ satisfy the inequalities of

Definition 3.1.

THEOREM 3.2. (cf. [28, Theorem 8.1])
In the power series

$$R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta) = \sum_{m,\ell} c_{m,\ell}\, \xi^{m-\ell}\eta^{\ell}$$

the coefficient $c_{m,\ell}$ *is nonzero only if* $m \le n$ *and* $m + \ell \le n + k$ *(cf. Figure 2).*

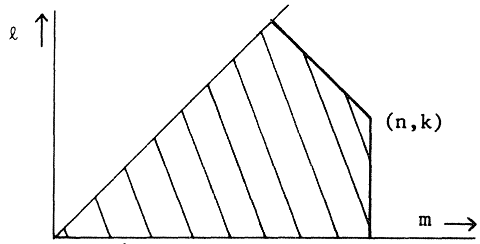

Figure 2

This theorem also follows from the results of section 4, cf. Remark 4.4.

Analogous to (2.7) and (2.8) we have

$$(3.17) \qquad R_{n,k}^{\alpha,\beta,\gamma}(2,1) = \frac{(-1)^{n-k}(\beta+1)_k(\beta+\gamma+3/2)_n}{(\alpha+1)_k(\alpha+\gamma+3/2)_n},$$

$$(3.18) \qquad \frac{R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)}{R_{n,k}^{\alpha,\beta,\gamma}(2,1)} = R_{n,k}^{\beta,\alpha,\gamma}(2-\xi,1-\xi+\eta).$$

Note that the mapping $(\xi,\eta) \to (2-\xi,1-\xi+\eta)$ is a nonorthogonal reflection which maps Ω onto itself, $(0,0)$ to $(2,1)$ and which leaves the points of the line $\xi = 1$ invariant.

Finally we mention the quadratic transformation formulas

$$(3.19) \qquad R_{n+k,n-k}^{\alpha,\alpha,\gamma}(\xi,\eta) = R_{n,k}^{\gamma,-\frac{1}{2},\alpha}(2\xi-4\eta,\xi^2-4\eta),$$

$$R^{\alpha,\alpha,\gamma}_{n+k+1,n-k}(\xi,\eta) = (1-\xi)\, R^{\gamma,\frac{1}{2},\alpha}_{n,k}(2\xi-4\eta,\xi^2-4\eta). \tag{3.20}$$

The quadratic transformation $(\xi,\eta) \to (2\xi-4\eta,\xi^2-4\eta)$ maps both connected components of $\{(\xi,\eta) \in \Omega \mid \xi \neq 1\}$ onto Ω. In fact, $(0,0)$ and $(2,1)$ are both mapped to $(0,0)$, $(1,0)$ is mapped to $(2,1)$, and $(1,\frac{1}{4})$ is mapped to $(1,0)$.

Note that formulas (3.19), (3.20) and (3.4) together imply that

$$R^{-\frac{1}{2},-\frac{1}{2},\gamma}_{n,k}(1-xy,\tfrac{1}{4}(x-y)^2) = \tfrac{1}{2}\,\{R^{(\gamma,\gamma)}_{n+k}(x)\, R^{(\gamma,\gamma)}_{n-k}(y) + R^{(\gamma,\gamma)}_{n-k}(x)\, R^{(\gamma,\gamma)}_{n+k}(y)\}, \tag{3.21}$$

$$R^{-\frac{1}{2},\frac{1}{2},\gamma}_{n,k}(1-xy,\tfrac{1}{4}(x-y)^2) = (x+y)^{-1}\{R^{(\gamma,\gamma)}_{n+k+1}(x)\, R^{(\gamma,\gamma)}_{n-k}(y) + R^{(\gamma,\gamma)}_{n-k}(x)\, R^{(\gamma,\gamma)}_{n+k+1}(y)\}. \tag{3.22}$$

4. JAMES TYPE ZONAL POLYNOMIALS

As was pointed out in section 1, the main problem to be solved in this paper is the derivation of an explicit expression of $R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$ in terms of certain polynomials $Z^{\gamma}_{m,\ell}(\xi,\eta)$ called James type zonal polynomials. In this section we introduce these polynomials $Z^{\gamma}_{m,\ell}(\xi,\eta)$, we give some motivation for the choice of these polynomials and we derive some simple properties of the expansion coefficients. To a large extent, the contents of this section coincide with KOORNWINDER [25, §4.4].

<u>DEFINITION 4.1</u>. Let $\gamma > -1$. Let n, k be integers such that $n \geq k \geq 0$. Then the James type zonal polynomial $Z^{\gamma}_{n,k}(\xi,\eta)$ is defined by

$$Z^{\gamma}_{n,k}(\xi,\eta) := \frac{(2\gamma+1)_{n-k}}{(\gamma+\frac{1}{2})_{n-k}}\, \eta^{\frac{1}{2}(n+k)}\, R^{(\gamma,\gamma)}_{n-k}(\tfrac{1}{2}\eta^{-\frac{1}{2}}\xi). \tag{4.1}$$

It follows from (2.11) that

$$Z^{\gamma}_{n,k}(\xi,\eta) = \sum_{i=0}^{[\frac{1}{2}(n-k)]} \frac{(-n+k)_{2i}}{(-n+k-\gamma+\frac{1}{2})_i \; i!} \xi^{n-k-2i} \eta^{k+i}. \tag{4.2}$$

Note that

$$Z^{\gamma}_{n,k}(\xi,\eta) = \xi^{n-k}\eta^{k} + \text{polynomial of degree less than } n, \tag{4.3}$$

$$\lim_{\gamma\to\infty} Z^{\gamma}_{n,k}(\xi,\eta) = \xi^{n-k}\,\eta^{k}. \tag{4.4}$$

From (2.14) and (2.15) we get the special cases

$$Z^{-\frac{1}{2}}_{n,k}(x+y,xy) = (1+\delta_{n,k})^{-1}(x^n y^k + x^k y^n), \tag{4.5}$$

$$Z^{\frac{1}{2}}_{n,k}\,(x+y,xy) = (x-y)^{-1}(x^{n+1}y^k - x^k y^{n+1}). \tag{4.6}$$

From (2.17) we derive

$$Z^{\gamma}_{n,k}(x+y,xy) = \frac{(n-k)!}{(\gamma+\frac{1}{2})_{n-k}} \sum_{i=0}^{n-k} \frac{(\gamma+\frac{1}{2})_i\,(\gamma+\frac{1}{2})_{n-k-i}}{i!\;(n-k-i)\;!}\, x^{n-i}\;y^{k+i}. \tag{4.7}$$

Note also the boundary values

$$Z^{\gamma}_{n,k}(\xi,0) = \begin{cases} \xi^n & \text{if } k = 0, \\ 0 & \text{if } k > 0, \end{cases} \tag{4.8}$$

$$Z^{\gamma}_{n,k}(\xi,\tfrac{1}{4}\xi^2) = \frac{(2\gamma+1)_{n-k}}{(\gamma+\frac{1}{2})_{n-k}}\,(\tfrac{1}{2}\xi)^{n+k}. \tag{4.9}$$

In view of (4.3) any polynomial

$$P(\xi,\eta) := \sum_{\ell=0}^{n} \sum_{m=\ell}^{n} c_{m,\ell}\, \xi^{m-\ell}\, \eta^{\ell}$$

has a unique expansion

$$P(\xi,\eta) = \sum_{\ell=0}^{n} \sum_{m=\ell}^{n} c^{\gamma}_{m,\ell}\, Z^{\gamma}_{m,\ell}\,(\xi,\eta)$$

for each $\gamma > -1$. This can be considered as a generalized power series expansion.

Note that $c^{\gamma}_{m,0} = c_{m,0}$ by (4.8).

In particular, let us define the coefficients $c^{\alpha,\beta,\gamma}_{n,k;m,\ell}$ by

<u>DEFINITION 4.2.</u>

$$R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta) = \sum_{\ell=0}^{n} \sum_{m=\ell}^{n} c^{\alpha,\beta,\gamma}_{n,k;m,\ell}\; Z^{\gamma}_{m,\ell}(\xi,\eta). \tag{4.10}$$

We claim that the generalized power series expansion (4.10) is a suitable analogue of the ordinary power series expansion (2.4) for Jacobi polynomials. This also justifies the introduction of the new coordinates ξ,η in section 3. Below we give a number of arguments for considering the expansion (4.10).

(a) It follows from (3.4), (3.5) and (2.4) that $R^{\alpha,\beta,-\frac{1}{2}}_{n,k}(x+y,xy)$ has a natural expansion in terms of $x^m y^\ell + x^\ell y^m$, $m \geq \ell$, and, similarly, $R^{\alpha,\beta,\frac{1}{2}}_{n,k}(x+y,xy)$ in terms of $(x-y)^{-1}(x^{m+1}y^\ell - x^\ell y^{m+1})$, $m \geq \ell$. In both cases the expansion coefficients can be given explicitly. By (4.5) and (4.6) this leads to the expansion (4.10) in the case $\gamma = \pm\frac{1}{2}$ and we obtain

$$c^{\alpha,\beta,-\frac{1}{2}}_{n,k;m,\ell} = \{(-n)_m(-k)_\ell(n+\alpha+\beta+1)_m(k+\alpha+\beta+1)_\ell + \tag{4.11}$$

$$+ (-k)_m(-n)_\ell(k+\alpha+\beta+1)_m(n+\alpha+\beta+1)_\ell\}\{2(\alpha+1)_m(\alpha+1)_\ell\; m!\; \ell!\}^{-1},$$

$$(4.12)\qquad c^{\alpha,\beta,\frac{1}{2}}_{n,k;m,\ell} = -\ \{(-n-1)_{m+1}(-k)_\ell(n+\alpha+\beta+2)_{m+1}(k+\alpha+\beta+1)_\ell\ +$$

$$-\ (-k)_{m+1}(-n-1)_\ell(k+\alpha+\beta+1)_{m+1}(n+\alpha+\beta+2)_\ell\}.$$

$$.\ \{(n-k+1)(n+k+\alpha+\beta+2)(\alpha+2)_m(\alpha+1)_\ell(m+1)!\ \ell!\}^{-1}.$$

(b) It was pointed out in KOORNWINDER [25, §4.4] that

$$(4.13)\qquad R^{\alpha,\beta,0}_{n,n}(x+y,xy) = {}_2F_1\left(-n,n+\alpha+\beta+\frac{3}{2};\alpha+\frac{3}{2};\begin{bmatrix} x & 0\\ 0 & y\end{bmatrix}\right),$$

where ${}_2F_1(a,b;c;X)$ is the hypergeometric function of matrix argument X which was introduced by HERZ [17]. CONSTANTINE [10] proved that there are natural power series expansions of such hypergeometric functions in terms of so-called zonal polynomials which (in the (2×2) case) are spherical functions on GL(2,ℝ)/O(2) belonging to finite dimensional irreducible representations of GL(2,ℝ). These zonal polynomials were introduced by JAMES [18]. Furthermore, JAMES [19,(7.9)] showed that in the (2×2) case these zonal polynomials coincide up to a constant factor with our polynomials $Z^0_{m,\ell}(x+y,xy)$. By using formula (25) in CONSTANTINE [10] it follows that

$$(4.14)\qquad {}_2F_1\left(a,b;c;\begin{bmatrix} x & 0\\ 0 & y\end{bmatrix}\right) =$$

$$= \sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{(a)_m(a-\frac{1}{2})_\ell(b)_m(b-\frac{1}{2})_\ell(3/2)_{m-\ell}}{(c)_m(c-\frac{1}{2})_\ell(3/2)_m\ell!(m-\ell)!}\,Z^0_{m,\ell}(x+y,xy).$$

Now (4.13) and (4.14) together give

$$(4.15)\qquad c^{\alpha,\beta,0}_{n,n;m,\ell} = \frac{(-n)_m(-n-\frac{1}{2})_\ell(n+\alpha+\beta+3/2)_m(n+\alpha+\beta+1)_\ell(3/2)_{m-\ell}}{(\alpha+3/2)_m(\alpha+1)_\ell(3/2)_m\ell!(m-\ell)!}.$$

(c) It can be proved that there is an interpretation of the polynomials $R_{n,k}^{\frac{1}{2}(q-3),\frac{1}{2}(d-q-3),0}(\xi,\eta)$ as so-called intertwining functions on the group $O(d)$, which are right invariant with respect to $O(2) \times O(d-2)$, left invariant with respect to $O(q) \times O(d-q)$, and which belong to some irreducible representation of $O(d)$. In particular, for $q = 2$ we obtain the spherical functions on the Grassmann manifold $O(d) / O(2) \times O(d-2)$. According to JAMES & CONSTANTINE [20] group theoretic considerations give a motivation for expanding these intertwining functions in terms of zonal polynomials. In particular, it follows from JAMES & CONSTANTINE [20, (15.4)] that

$$c_{n,0;m,\ell}^{\alpha,\beta,0} = \begin{cases} \dfrac{(-n)_m (n+\alpha+\beta+2)_m (\frac{1}{2})_m}{(\alpha+3/2)_m \; m! \; m!} & \text{if } \ell = 0, \\[2ex] 0 & \text{if } \ell \neq 0, \end{cases} \tag{4.16}$$

for integer or halfinteger α and β.

(d) It will be proved in section 7 that

$$\lim_{\beta\to\infty} \frac{R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)}{R_{n,k}^{\alpha,\beta,\gamma}(2,1)} = \frac{Z_{n,k}^{\gamma}(\xi,\eta)}{Z_{n,k}^{\gamma}(2,1)} . \tag{4.17}$$

Note that the pair of formulas (4.10) and (4.17) is analogous with the formulas (2.4) and (2.21) for Jacobi polynomials.

A final motivation for considering the expansion (4.10) is given by the following differentiation formula, which is easily verified.

$$\begin{cases} D_-^{\gamma} \, Z_{n,0}^{\gamma}(\xi,\eta) = 0, \\[2ex] D_-^{\gamma} \, Z_{n,k}^{\gamma}(\xi,\eta) = \frac{1}{4}k(n+\gamma+\frac{1}{2})Z_{n-1,k-1}^{\gamma}(\xi,\eta) \text{ if } k > 0. \end{cases} \tag{4.18}$$

On comparing this result with (3.13) we obtain the recurrence relation

$$(4.19)\qquad c_{n,k;m,\ell}^{\alpha,\beta,\gamma} = \frac{k(k+\alpha+\beta+1)(n+\gamma+\frac{1}{2})(n+\alpha+\beta+\gamma+3/2)}{\ell(\alpha+1)(m+\gamma+\frac{1}{2})(\alpha+\gamma+3/2)}\,\cdot$$

$$\cdot c_{n-1,k-1;m-1,\ell-1}^{\alpha+1,\beta+1,\gamma}\,,\quad \ell > 0,\ k > 0.$$

Since $D_{-}^{\gamma}\, R_{n,0}^{\alpha,\beta,\gamma}(\xi,\eta) = 0$, it also follows that

$$(4.20)\qquad R_{n,0}^{\alpha,\beta,\gamma}(\xi,\eta) = \sum_{m=0}^{n} c_{n,0;m,0}^{\alpha,\beta,\gamma}\; Z_{m,0}^{\gamma}(\xi,\eta),$$

i.e. $c_{n,0;m,\ell}^{\alpha,\beta,\gamma} = 0$ if $\ell > 0$. Formulas (4.19) and (4.20) together imply:

THEOREM 4.3.

$c_{n,k;m,\ell}^{\alpha,\beta,\gamma} \neq 0$ *only if* $\ell \leq k$ (*cf. Figure* 3).

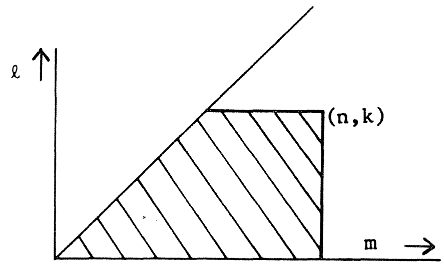

Figure 3

REMARK 4.4. Theorem 4.3 together with formula (4.2) provides a new proof of Theorem 3.2.

REMARK 4.5. It follows from (4.10) and (4.8) that

$$(4.21)\qquad R_{n,k}^{\alpha,\beta,\gamma}(\xi,0) = \sum_{m=0}^{n} c_{n,k;m,0}^{\alpha,\beta,\gamma}\; \xi^{m}.$$

Hence, in view of (4.19), the general coefficients $c_{n,k;m,\ell}^{\alpha,\beta,\gamma}$ are known as soon as we know the explicit power series expansion (4.21) of the boundary value $R_{n,k}^{\alpha,\beta,\gamma}(\xi,0)$ for all values of α,β,γ,n,k. In particular, we know the coefficients $c_{n,n;m,\ell}^{\alpha,\beta,\gamma}$ as soon as we know the explicit power series expansion of $R^{\alpha,\beta,\gamma}_{n,n}(\xi,0)$ for all values of α,β,γ,n. This power series expansion will be obtained in section 5.

REMARK 4.6. It follows from (4.20) and (4.9) that

$$R^{\alpha,\beta,\gamma}_{n,0}(\xi,\tfrac{1}{4}\xi^2) = \sum_{m=0}^{n} c^{\alpha,\beta,\gamma}_{n,0;m,0} \frac{(2\gamma+1)_m}{(\gamma+\frac{1}{2})_m} (\tfrac{1}{2}\xi)^m. \tag{4.22}$$

Hence, the general expansion (4.20) of $R^{\alpha,\beta,\gamma}_{n,0}(\xi,\eta)$ is known as soon as we know the explicit power series expansion (4.22) of the boundary value $R^{\alpha,\beta,\gamma}_{n,0}(\xi,\frac{1}{4}\xi^2)$. This power series expansion will also be obtained in section 5.

5. SOME BOUNDARY VALUES

In this section it will be shown that the polynomials $R^{\alpha,\beta,\gamma}_{n,n}(\xi,\eta)$ become Jacobi polynomials on the boundary lines $\eta = 0$ and $1 - \xi + \eta = 0$, and that $R^{\alpha,\beta,\gamma}_{n,0}(\xi,\eta)$ can be expressed as a Jacobi polynomial on the parabola $\xi^2 - 4\eta = 0$. In the case of general degree (n,k) certain Jacobi expansions of the boundary values of $R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$ will be considered, for which the Fourrier-Jacobi coefficients can be expressed in terms of the corresponding coefficients for $R^{\alpha+k,\beta+k,\gamma}_{n-k,0}(\xi,0)$ and $R^{\alpha,\beta,\gamma+n-k}_{k,k}(\xi,\frac{1}{4}\xi^2)$, respectively.

The key for deriving these results is the following lemma.

LEMMA 5.1.

(a) *On the line* $\eta = 0$ *the second order partial differential operator* $D^{\alpha,\beta,\gamma}_{+}$ *reduces to a first order ordinary differential operator involving only derivatives* $\partial_\xi = d/d\xi$, *which is given by*

$$D^{\alpha,\beta,\gamma}_{+}\Big|_{\eta=0} = 4(\alpha+1)\ \xi^{-(\alpha+\gamma+\frac{1}{2})}(1-\xi)^{-\beta}\frac{d}{d\xi} \circ \xi^{\alpha+\gamma+3/2}(1-\xi)^{\beta+1}. \tag{5.1}$$

(b) *On the parabola* $\xi^2-4\eta = 0$ *the operator* $E^{\alpha,\beta,\gamma}_{+}$ *reduces to a first order differential operator involving only derivatives* $\partial_\xi + \frac{1}{2}\xi\,\partial_\eta = d/d\xi$, *which is given by*

$$(5.2)\qquad E_+^{\alpha,\beta,\gamma}\Big|_{\xi^2-4\eta\,=\,0} = 4(\gamma+1)\ \xi^{-(\alpha+\gamma+\frac{1}{2})}(2-\xi)^{-(\beta+\gamma+\frac{1}{2})}\frac{d}{d\xi}\circ\xi^{\alpha+\gamma+3/2}\cdot$$

$$\cdot(2-\xi)^{\beta+\gamma+3/2}.$$

PROOF. The proof follows immediately by substitution of $\eta = 0$ in (3.11) and $\xi^2 - 4\eta = 0$ in (3.12), respectively. □

Formulas (5.1), (5.2) and (2.3) imply that

$$(5.3)\qquad D_+^{\alpha,\beta,\gamma}\Big|_{\eta\,=\,0} R_{n-1}^{(\alpha+\gamma+3/2,\beta+1)}(1-2\xi)$$

$$= 4(\alpha+1)(\alpha+\gamma+\tfrac{3}{2})\ R_n^{(\alpha+\gamma+\frac{1}{2},\beta)}(1-2\xi),$$

$$(5.4)\qquad E_+^{\alpha,\beta,\gamma}\Big|_{\xi^2-4\eta\,=\,0} R_{n-1}^{(\alpha+\gamma+3/2,\beta+\gamma+3/2)}(1-\xi) =$$

$$= 8(\gamma+1)(\alpha+\gamma+\tfrac{3}{2})\ R_n^{(\alpha+\gamma+\frac{1}{2},\beta+\gamma+\frac{1}{2})}(1-\xi).$$

Now we can prove the important

THEOREM 5.2.

$$(5.5)\qquad R_{n,n}^{\alpha,\beta,\gamma}(\xi,0) = R_n^{(\alpha+\gamma+\frac{1}{2},\beta)}(1-2\xi),$$

$$(5.6)\qquad R_{n,0}^{\alpha,\beta,\gamma}(\xi,\tfrac{1}{4}\xi^2) = R_n^{(\alpha+\gamma+\frac{1}{2},\beta+\gamma+\frac{1}{2})}(1-\xi),$$

$$(5.7)\qquad \frac{R_{n,n}^{\alpha,\beta,\gamma}(\xi,\xi-1)}{R_{n,n}^{\alpha,\beta,\gamma}(2,1)} = R_n^{(\beta+\gamma+\frac{1}{2},\alpha)}(2\xi-3).$$

PROOF. Comparison of (3.14) with (5.3) and of (3.16) with (5.4) and complete induction with respect to n results in (5.5) and (5.6). The boundary value of $R_{n,n}^{\alpha,\beta,\gamma}(\xi,\eta)$ for $1 - \xi + \eta = 0$ follows from (5.5) and (3.18). □

Next we will consider Jacobi expansions for the polynomials $R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)$ on the boundary curves $\eta = 0$ and $\xi^2 - 4\eta = 0$.
Let us define the coefficients $a_{n,k;m}^{\alpha,\beta,\gamma}$ and $b_{n,k;m}^{\alpha,\beta,\gamma}$ by

DEFINITION 5.3.

$$R_{n,k}^{\alpha,\beta,\gamma}(\xi,0) = \sum_m a_{n,k;m}^{\alpha,\beta,\gamma} R_m^{(\alpha+\gamma+\frac{1}{2},\beta)}(1-2\xi), \tag{5.8}$$

$$R_{n,k}^{\alpha,\beta,\gamma}(\xi,\tfrac{1}{4}\xi^2) = \sum_m b_{n,k;m}^{\alpha,\beta,\gamma} R_m^{(\alpha+\gamma+\frac{1}{2},\beta+\gamma+\frac{1}{2})}(1-\xi). \tag{5.9}$$

Formulas (5.3) and (3.14), (5.4) and (3.16) yield the following equalities for the coefficients $a_{n,k;m}^{\alpha,\beta,\gamma}$ $(k>0)$ and $b_{n,k;m}^{\alpha,\beta,\gamma}$ $(n>k)$, respectively:

$$a_{n,k;m}^{\alpha,\beta,\gamma} = \begin{cases} a_{n-1,k-1;m-1}^{\alpha+1,\beta+1,\gamma} & \text{if } m > 0, \\ 0 & \text{if } m = 0, \end{cases} \tag{5.10}$$

$$b_{n,k;m}^{\alpha,\beta,\gamma} = \begin{cases} b_{n-1,k;m-1}^{\alpha,\beta,\gamma+1} & \text{if } m > 0, \\ 0 & \text{if } m = 0. \end{cases} \tag{5.11}$$

Use of (5.10), (5.11) and complete induction with respect to k and n-k, respectively, results in

$$a_{n,k;m}^{\alpha,\beta,\gamma} \neq 0 \quad \text{only if } k \le m \le n, \tag{5.12}$$

$$b_{n,k;m}^{\alpha,\beta,\gamma} \neq 0 \quad \text{only if } n - k \le m \le n + k, \tag{5.13}$$

$$a_{n,k;m}^{\alpha,\beta,\gamma} = a_{n-k,0;m-k}^{\alpha+k,\beta+k,\gamma}, \tag{5.14}$$

$$b_{n,k;m}^{\alpha,\beta,\gamma} = b_{k,k;m-n+k}^{\alpha,\beta,\gamma+n-k}. \tag{5.15}$$

Thus we obtain

LEMMA 5.4. *We have*

$$R_{n,k}^{\alpha,\beta,\gamma}(\xi,0) = \sum_{m=k}^{n} a_{n-k,0;m-k}^{\alpha+k,\beta+k,\gamma} R_m^{(\alpha+\gamma+\frac{1}{2},\beta)}(1-2\xi), \tag{5.16}$$

$$R_{n,k}^{\alpha,\beta,\gamma}(\xi,\tfrac{1}{4}\xi^2) = \sum_{m=n-k}^{n+k} b_{k,k;m-n+k}^{\alpha,\beta,\gamma+n-k} R_m^{(\alpha+\gamma+\frac{1}{2},\beta+\gamma+\frac{1}{2})}(1-\xi), \tag{5.17}$$

$$\frac{R_{n,k}^{\alpha,\beta,\gamma}(\xi,\xi-1)}{R_{n,k}^{\alpha,\beta,\gamma}(2,1)} = \sum_{m=k}^{n} a_{n-k,0;m-k}^{\beta+k,\alpha+k,\gamma} R_m^{(\beta+\gamma+\frac{1}{2},\alpha)}(2\xi-3). \tag{5.18}$$

The last part of the lemma follows from (5.16) and (3.18).

Next we give some formulas for $a_{n,k;m}^{\alpha,\beta,\gamma}$ and $b_{n,k;m}^{\alpha,\beta,\gamma}$ in the case of special values of the parameters.

From (3.4) and (2.40) it follows that $b_{n,k;m}^{\alpha,\beta,-\frac{1}{2}}$ can be expressed in terms of the linearization coefficients $A_{n,k,m}^{\alpha,\beta,\gamma}$ of the Jacobi polynomials:

$$b_{n,k;m}^{\alpha,\beta,-\frac{1}{2}} = A_{n,k,m}^{(\alpha,\beta)} \omega_m^{(\alpha,\beta)}. \tag{5.19}$$

If $\alpha = \beta$ then application of the quadratic transformation formulas (3.19) and (2.9), (3.20) and (2.10), respectively, results in

$$b_{n+k,n-k;2m}^{\alpha,\alpha,-\frac{1}{2}} = a_{n,k;m}^{-\frac{1}{2},-\frac{1}{2},\alpha}, \tag{5.20}$$

$$b_{n+k+1,n-k;2m+1}^{\alpha,\alpha,-\frac{1}{2}} = a_{n,k;m}^{-\frac{1}{2},\frac{1}{2},\alpha}. \tag{5.21}$$

From 3.18) and (5.9) with $\alpha = \beta$ it follows that

$$b_{n,n;2m+1}^{\alpha,\alpha,\gamma} = b_{n+1,n;2m}^{\alpha,\alpha,\gamma} = 0 \tag{5.22}$$

Combination of (5.19), (5.20), (5.21) and (5.22) gives an expression for the linearization coefficients of order (α,α) in terms of $a^{-\frac{1}{2},\pm\frac{1}{2},\alpha}_{n,k;m}$. In section 6 we will derive the explicit values of the coefficients $a^{\alpha,\beta,\gamma}_{n,k;m}$ and thus we will find a new derivation of the linearization coefficients for the Gegenbauer polynomials.

It follows from the quadratic transformation formulas (3.19), (3.20), (2.9) and (2.10) that

$$(5.23)\qquad b^{\alpha,\alpha,\gamma}_{n,n;2m} = a^{\gamma,-\frac{1}{2},\alpha}_{n,0;m},$$

$$(5.24)\qquad b^{\alpha,-\frac{1}{2},\gamma}_{n,n;m} = a^{\gamma,\gamma,\alpha}_{2n,0;m},$$

$$(5.25)\qquad b^{\alpha,\alpha,\gamma+1}_{n,n;2m} = a^{\gamma,\frac{1}{2},\alpha}_{n,0;m}.$$

For the proof of (5.25) we used (5.11) once.

From Lemma 5.4 and Lemma 2.5 we can derive a corollary about the number of zeros of $R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$ on the boundary.

COROLLARY 5.5.

(a) $R^{\alpha,\beta,\gamma}_{n,k}(\xi,0)$ *has at least* k *zeros of odd multiplicity for* $\xi\in(0,1)$.

(b) $R^{\alpha,\beta,\gamma}_{n,k}(\xi,\xi-1)$ *has at least* k *zeros of odd multiplicity for* $\xi\in(1,2)$.

(c) $R^{\alpha,\beta,\gamma}_{n,k}(\xi,\frac{1}{4}\xi^2)$ *has at least* n-k *zeros of odd multiplicity for* $\xi\in(0,2)$.

6. EXPANSION OF THE POLYNOMIALS $R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$ IN TERMS OF JAMES TYPE ZONAL POLYNOMIALS

In this section we will derive the explicit value of the coefficients $c^{\alpha,\beta,\gamma}_{n,k;m,\ell}$ in formula (4.10) giving the expansion of $R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$ in terms of the James type zonal polynomials $Z^{\gamma}_{m,\ell}(\xi,\eta)$. We will proceed in the following way. From the boundary value $R^{\alpha,\beta,\gamma}_{n,0}(\xi,\frac{1}{4}\xi^2)$ we obtain the coefficients $c^{\alpha,\beta,\gamma}_{n,0;m,0}$ and hence the boundary value $R^{\alpha,\beta,\gamma}_{n,0}(\xi,0)$. By rewriting $R^{\alpha,\beta,\gamma}_{n,0}(\xi,0)$ as a Jacobi series we derive the coefficients $a^{\alpha,\beta,\gamma}_{n,0;m}$ defined by (5.8). This also gives the coefficients $a^{\alpha,\beta,\gamma}_{n,k;m}$. Next the Jacobi series of $R^{\alpha,\beta,\gamma}_{n,k}(\xi,0)$ can be rewritten as a power series and we obtain the coefficients $c^{\alpha,\beta,\gamma}_{n,k;m,0}$. Finally $c^{\alpha,\beta,\gamma}_{n,k;m,\ell}$ can be expressed in terms of $c^{\alpha+\ell,\beta+\ell,\gamma}_{n-\ell,k-\ell;m-\ell,0}$.

At the end of this section several interesting corollaries will be discussed. We mention the expression of $R^{\alpha,\beta,\gamma}_{n,0}(x+y,xy)$ as a generalized hypergeometric function in the two variables x and y, the expression of $R^{\alpha,\beta,\gamma}_{n,k}(1,0)$ in terms of a ${}_3F_2$-function of argument 1, and a new derivation of the linearization coefficients for Gegenbauer polynomials.

Let us consider $R^{\alpha,\beta,\gamma}_{n,0}(\xi,\eta)$. Combination of (5.6), (4.22) and (2.4) results in

$$(6.1)\qquad c^{\alpha,\beta,\gamma}_{n,0;m,0} = \frac{(-n)_m(n+\alpha+\beta+2\gamma+2)_m(\gamma+\frac{1}{2})_m}{(\alpha+\gamma+3/2)_m(2\gamma+1)_m\ m!}.$$

So we have the first explicit expansion (cf. (4.20)):

$$(6.2)\qquad R^{\alpha,\beta,\gamma}_{n,0}(\xi,\eta) = \sum_{m=0}^{n} \frac{(-n)_m(n+\alpha+\beta+2\gamma+2)_m(\gamma+\frac{1}{2})_m}{(\alpha+\gamma+3/2)_m(2\gamma+1)_m\ m!}\, Z^{\gamma}_{m,0}(\xi,\eta).$$

REMARK 6.1. If $\gamma = -\frac{1}{2}$ then the right hand side of formula (6.2) has to be interpreted as the limit case for $\gamma \to -\frac{1}{2}$. A similar interpretation has to be used on many other places.

Similar to (6.1), it follows from (5.5), (4.21) and (2.4) that

$$(6.3)\qquad c^{\alpha,\beta,\gamma}_{n,n;m,0} = \frac{(-n)_m(n+\alpha+\beta+\gamma+3/2)_m}{(\alpha+\gamma+3/2)_m\ m!}.$$

Hence, by (4.19) we have

$$(6.4)\qquad c^{\alpha,\beta,\gamma}_{n,n;m,\ell} = \frac{(-n)_m(-n-\gamma-\frac{1}{2})_\ell(n+\alpha+\beta+\gamma+3/2)_m(n+\alpha+\beta+1)_\ell(\gamma+3/2)_{m-\ell}}{(\alpha+\gamma+3/2)_m(\alpha+1)_\ell(\gamma+3/2)_m\ \ell!\ (m-\ell)!}$$

and thus the expansion (4.10) for the polynomials $R^{\alpha,\beta,\gamma}_{n,n}(\xi,\eta)$.

LEMMA 6.2. *The power series expansion of* $R^{\alpha,\beta,\gamma}_{n,0}(\xi,\eta)$ *for* $\eta = 0$ *is given by*

$$(6.5)\qquad R^{\alpha,\beta,\gamma}_{n,0}(\xi,0) = {}_3F_2\left(\begin{matrix}-n, n+\alpha+\beta+2\gamma+2, \gamma+\frac{1}{2};\\ \alpha+\gamma+3/2, 2\gamma+1;\end{matrix}\ \xi\right).$$

PROOF. The proof follows immediately from (4.21) and (6.2). □

LEMMA 6.3. *The coefficients* $a^{\alpha,\beta,\gamma}_{n,0;m}$ *in the Jacobi expansion* (5.8) *of* $R^{\alpha,\beta,\gamma}_{n,0}(\xi,0)$ *are given by*

$$(6.6)\qquad a^{\alpha,\beta,\gamma}_{n,0;m} = \frac{n!\,(n+\alpha+\beta+2\gamma+2)_m(m+\alpha+\beta+2)_{n-m}(\gamma+\frac{1}{2})_m(\gamma+\frac{1}{2})_{n-m}}{(2\gamma+1)_n(m+\alpha+\beta+\gamma+3/2)_m(2m+\alpha+\beta+\gamma+5/2)_{n-m}\ m!\,(n-m)!}.$$

PROOF. It follows from (6.3), (2.6) and (5.8) that

$$a^{\alpha,\beta,\gamma}_{n,0;m} = \frac{(-1)^m(-n)_m(n+\alpha+\beta+2\gamma+2)_m(\gamma+\frac{1}{2})_m}{(m+\alpha+\beta+\gamma+3/2)_m(2\gamma+1)_m\ m!}\ .$$

$$\cdot\ {}_3F_2\left(\begin{matrix}-n+m, n+m+\alpha+\beta+2\gamma+2, m+\gamma+\frac{1}{2};\\ 2m+\alpha+\beta+\gamma+5/2, m+2\gamma+1;\end{matrix}\ 1\right),$$

which can be evaluated by using (2.42). □

THEOREM 6.4. *The explicit form of the Jacobi expansion* (5.8) *is*

$$(6.7)\qquad R^{\alpha,\beta,\gamma}_{n,k}(\xi,0) = \frac{(n-k)!}{(2\gamma+1)_{n-k}} \sum_{m=k}^{n} \frac{(n+k+\alpha+\beta+2\gamma+2)_{m-k}}{(m+k+\alpha+\beta+\gamma+3/2)_{m-k}}\ .$$

$$\cdot\ \frac{(m+k+\alpha+\beta+2)_{n-m}(\gamma+\frac{1}{2})_{m-k}(\gamma+\frac{1}{2})_{n-m}}{(2m+\alpha+\beta+\gamma+5/2)_{n-m}(m-k)!\ (n-m)!}\ R_m^{(\alpha+\gamma+\frac{1}{2},\beta)}(1-2\xi).$$

Note that

$$(6.8)\qquad a^{\alpha,\beta,\gamma}_{n,k;m} > 0 \quad \text{if} \quad \gamma > -\tfrac{1}{2}.$$

PROOF. Use formulas (5.16) and (6.6). □

THEOREM 6.5. *The coefficients* $c^{\alpha,\beta,\gamma}_{n,k;m,0}$ *in the power series expansion* (4.10) *of* $R^{\alpha,\beta,\gamma}_{n,k}(\xi,0)$ *are given by*

$$(6.9)\qquad c^{\alpha,\beta,\gamma}_{n,k;m,0} = \frac{(n+\alpha+\beta+\gamma+3/2)_m(-n)_m}{(\alpha+\gamma+3/2)_m\; m!}\,.$$

$$\cdot\ {}_4F_3\left(\begin{matrix}-m,-n+k,-n-k-\alpha-\beta-1,\gamma+\frac{1}{2};\\ -n,-n-m-\alpha-\beta-\gamma-\frac{1}{2},2\gamma+1;\end{matrix}\ 1\right).$$

PROOF. We will give two different proofs.

(a) It follows from (6.7) and (2.4) that

$$c^{\alpha,\beta,\gamma}_{n,k;m,0} = \frac{(\gamma+\frac{1}{2})_{n-k}(n+k+\alpha+\beta+2\gamma+2)_{n-k}(n+\alpha+\beta+\gamma+3/2)_m(-n)_m}{(2\gamma+1)_{n-k}(n+k+\alpha+\beta+\gamma+3/2)_{n-k}(\alpha+\gamma+3/2)_m\; m!}\,.$$

$$\cdot\ {}_7F_6(-n+k,-n+m,\gamma+\tfrac{1}{2},-n-k-\alpha-\beta-1,-n-\alpha-\beta-\gamma-\tfrac{1}{2},$$

$$,-2n-\alpha-\beta-\gamma-\tfrac{3}{2}, -n-\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta-\tfrac{1}{2}\gamma+\tfrac{1}{4};\ -n,-n+k-\gamma+\tfrac{1}{2},$$

$$,-n-k-\alpha-\beta-\gamma-\tfrac{1}{2},-n-m-\alpha-\beta-\gamma-\tfrac{1}{2},-2n-\alpha-\beta-2\gamma-1,$$

$$,-n-\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta-\tfrac{1}{2}\gamma-\tfrac{3}{4};\ 1).$$

By using a result of Whipple (cf. SLATER [27, (2.4.1.1)] this well-poised terminating ${}_7F_6$ can be rewritten as a Saalschützian terminating ${}_4F_3$ and the theorem follows.

(b) Combination of (3.14), (5.1) and (4.21) gives the recurrence relation

$$(6.10)\qquad c^{\alpha,\beta,\gamma}_{n,k;m,0} = \frac{m+\alpha+\gamma+3/2}{\alpha+\gamma+3/2}\, c^{\alpha+1,\beta+1,\gamma}_{n-1,k-1;m,0} - \frac{m+\alpha+\beta+\gamma+3/2}{\alpha+\gamma+3/2}\, c^{\alpha+1,\beta+1,\gamma}_{n-1,k-1;m-1,0}$$

for $n \geq k > 0$, $n \geq m > 0$. For $k = 0$ the ${}_4F_3$ in (6.9) becomes a terminating Saalschützian ${}_3F_2$ which can be evaluated by using (2.42). In view of (6.5) the theorem turns out to be true for $k = 0$. Clearly, the theorem is true for $m = 0$. Using (6.10) we can now prove the general case of (6.9) by complete induction with respect to k. □

COROLLARY 6.6. *We have the expansion*

$$(6.11)\qquad R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta) = \sum_{\ell=0}^{k}\sum_{m=\ell}^{n} c^{\alpha,\beta,\gamma}_{n,k;m,\ell}\; Z^{\gamma}_{m,\ell}(\xi,\eta),$$

where

$$(6.12)\qquad c^{\alpha,\beta,\gamma}_{n,k;m,\ell} = \frac{(-k)_\ell(-n)_m(-n-\gamma-\frac{1}{2})_\ell(n+\alpha+\beta+\gamma+3/2)_m}{(-n)_\ell(\alpha+\gamma+3/2)_m(\alpha+1)_\ell(\gamma+3/2)_m}\;\cdot$$

$$\cdot\frac{(k+\alpha+\beta+1)_\ell}{\ell!}\frac{(\gamma+3/2)_{m-\ell}}{(m-\ell)!}\;{}_4F_3\left(\begin{matrix}-m+\ell,-n+k,-n-k-\alpha-\beta-1,\gamma+\frac{1}{2};\\ -n+\ell,-n-m-\alpha-\beta-\gamma-\frac{1}{2},2\gamma+1;\end{matrix}\,1\right).$$

In this expansion $Z^{\gamma}_{m,\ell}(\xi,\eta)$ is defined by (4.1).

PROOF. By using complete induction with respect to k the result follows from (6.9) and (4.19). □

REMARK 6.7. In a number of special cases of $m,\ell,n,k,\alpha,\beta,\gamma$ the expression (6.12) can be simplified. If one of the equalities $m = n$, $\ell = k$, $m = \ell$, $k = 0$ or $n = k$ holds, then the coefficient $c^{\alpha,\beta,\gamma}_{n,k;m,\ell}$ can be written as a quotient of products of gamma functions depending linearly on $m,\ell,n,k,\alpha,\beta,\gamma$. If $\gamma = \pm\frac{1}{2}$ then we get back (4.11) and (4.12).

We would like to write $R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$ as a linear combination of elementary expressions $(a/b)\,\xi^{m-\ell}\eta^{\ell}$, where a and b are products of gamma functions depending linearly on $m,\ell,n,k,\alpha,\beta,\gamma$. The best possible result would be a double sum, which indeed can be obtained for $R^{\alpha,\beta,\gamma}_{n,0}(\xi,\eta)$ and for $R^{\alpha,\beta,\gamma}_{n,n}(\xi,\eta)$ (see section 7). However, (6.11) expresses $R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$ as a quadruple sum of elementary terms. It is not clear to the authors how this can be simplified.

We conclude this section with a number of corrollaries to the results earlier obtained in this section.

Combination of (6.2) and (4.7) gives:

COROLLARY 6.8. *We have*

$$(6.13) \qquad R_{n,0}^{\alpha,\beta,\gamma}(x+y,xy) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-n)_{i+j}(n+\alpha+\beta+2\gamma+2)_{i+j}(\gamma+\frac{1}{2})_i(\gamma+\frac{1}{2})_j}{(\alpha+\gamma+3/2)_{i+j}(2\gamma+1)_{i+j}\, i!\, j!} x^i y^j .$$

By using the notation of BURCHNALL & CHAUNDY [7,§ 1] this becomes

$$R_{n,0}^{\alpha,\beta,\gamma}(x+y,xy) = F\left(\begin{matrix}-n,n+\alpha+\beta+2\gamma+2:\gamma+\frac{1}{2};\gamma+\frac{1}{2}; \\ \alpha+\gamma+3/2,2\gamma+1 \ : \quad ; \quad ;\end{matrix} \ x,y\right),$$

a hypergeometric series in two variables of order three (cf. ERDÉLYI [12, § 5.7]). According to CARLSON [9,(1.8)] it follows from (6.13) that

$$R_{n,0}^{\alpha,\beta,\gamma}(x+y,xy) = \mathcal{R}_n\left(n+\alpha+\beta+2\gamma+2,-n-\beta-\gamma-\tfrac{1}{2}; \begin{bmatrix}1-x & 1-y\\ 1 & 1\end{bmatrix}; \gamma+\tfrac{1}{2},\gamma+\tfrac{1}{2}\right),$$

where the function $\mathcal{R}_t(\mu,\mu';Y;\nu,\nu')$ is defined by CARLSON [8, § 2].

COROLLARY 6.9. *The value of* $R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)$ *in the vertex* (1,0) *is given by*

$$(6.14) \qquad R_{n,k}^{\alpha,\beta,\gamma}(1,0) = \frac{(-1)^k(\beta+1)_k}{(\alpha+\gamma+3/2)_k}\ {}_3F_2\left(\begin{matrix}-n+k,n+k+\alpha+\beta+2\gamma+2,\gamma+\frac{1}{2}; \\ k+\alpha+\gamma+3/2,2\gamma+1;\end{matrix}\ 1\right).$$

PROOF. By restricting (3.14) to $(\xi,\eta) = (1,0)$ and by using (3.11) with $(\xi,\eta) = (1,0)$ it follows that

$$(6.15) \qquad R_{n,k}^{\alpha,\beta,\gamma}(1,0) = -\frac{\beta+1}{\alpha+\gamma+3/2} R_{n-1,k-1}^{\alpha+1,\beta+1,\gamma}(1,0).$$

The corollary follows by iteration of this result and by using (6.5). □

Two special cases of (6.14) are $n = k$ and $\alpha = \beta$. In these cases we have, respectively,

$$(6.16) \qquad R_{n,n}^{\alpha,\beta,\gamma}(1,0) = \frac{(-1)^n(\beta+1)_n}{(\alpha+\gamma+3/2)_n},$$

$$(6.17) \qquad \begin{cases} R_{n+k,n-k}^{\alpha,\alpha,\gamma}(1,0) = (-1)^{n-k}\dfrac{(\alpha+1)_n(\frac{1}{2})_k}{(\alpha+\gamma+3/2)_n(\gamma+1)_k}, \\[2ex] R_{n+k+1,n-k}^{\alpha,\alpha,\gamma}(1,0) = 0. \end{cases}$$

Formula (6.17) can be proved by application of Watson's formula (cf. SLATER [27,(2.3.3.13)] or directly from (3.19), (3.20) and (3.17).

COROLLARY 6.10. *If* $\alpha \geq \beta$, $\gamma \geq -\frac{1}{2}$ *and* $\max(\alpha,\beta+\gamma+\frac{1}{2}) \geq -\frac{1}{2}$ *then*

$$(6.18) \qquad |R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)| \leq 1 \textit{ for } \eta = 0,\ 0 \leq \xi \leq 1,$$

$$\textit{or } 1-\xi+\eta = 0,\ 1 \leq \xi \leq 2.$$

PROOF. We use (6.7) together with the nonnegativity of the coefficients for $\gamma \geq -\frac{1}{2}$ and the inequalities for Jacobi polynomials (cf. Theorem 2.2). The inequalities for α,β,γ imply that $\alpha+\gamma+\frac{1}{2} \geq \beta$ and $\alpha+\gamma+\frac{1}{2} \geq -\frac{1}{2}$. Hence

$$|R_{n,k}^{\alpha,\beta,\gamma}(\xi,0)| \leq 1 \quad \text{for} \quad 0 \leq \xi \leq 1.$$

In particular, $|R_{n,k}^{\alpha,\beta,\gamma}(1,0)| \leq 1$. Since $\alpha \geq \beta$ we also have $|R_{n,k}^{\alpha,\beta,\gamma}(2,1)| \leq 1$ by (3.18).
Again by (3.18) we have

$$\left|\frac{R_{n,k}^{\alpha,\beta,\gamma}(2-\xi,1-\xi+\eta)}{R_{n,k}^{\alpha,\beta,\gamma}(2,1)}\right| = |R_{n,k}^{\beta,\alpha,\gamma}(\xi,0)| \leq$$

$$\leq \max\left\{|R_{n,k}^{\beta,\alpha,\gamma}(0,0)|,|R_{n,k}^{\beta,\alpha,\gamma}(1,0)|\right\},$$

since $\max(\alpha,\beta+\gamma+\frac{1}{2}) \geq -\frac{1}{2}$. Hence

$$|R_{n,k}^{\alpha,\beta,\gamma}(2-\xi,1-\xi+\eta)| \leq \max\left\{|R_{n,k}^{\alpha,\beta,\gamma}(2,1)|,|R_{n,k}^{\alpha,\beta,\gamma}(1,0)|\right\} \leq 1. \qquad \square$$

COROLLARY 6.11. *If* $\alpha \geq -\frac{1}{2}$, $\alpha \geq \beta$, $\gamma \geq -\frac{1}{2}$ *and if one of the equalities* $\alpha = \beta$, $\beta = -\frac{1}{2}$ *or* $\gamma = -\frac{1}{2}$ *holds then*

$$(6.19) \qquad |R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)| \leq 1 \quad , \quad (\xi,\eta) \in \partial\Omega.$$

PROOF. Use corollary (6.10), formula (5.9) or (5.17), the nonnegativity of $b^{\alpha,\beta,\gamma}_{n,k;m}$ in the cases given in the corollary (cf. (5.22), (5.23), (5.24), (5.19) and Theorem 2.4), and the inequalities for the Jacobi polynomials (Theorem 2.2). □

REMARK 6.12. We need the restricting equalities $\alpha = \beta$, $\beta = -\frac{1}{2}$ or $\gamma = -\frac{1}{2}$ in Corollary 6.11 because in other cases the nonnegativity of $b^{\alpha,\beta,\gamma}_{n,k;m}$ is not yet proved. It is the authors' hypothesis that the coefficients $b^{\alpha,\beta,\gamma}_{n,k;m}$ are positive for $\alpha \geq -\frac{1}{2}$, $\alpha \geq \beta$ and $\gamma \geq -\frac{1}{2}$. If this is true then formula (6.19) would hold for all α, β, γ such that $\alpha \geq -\frac{1}{2}$, $\alpha \geq \beta$, $\gamma \geq -\frac{1}{2}$.

REMARK 6.13. Combination of formulas (6.7), (5.19), (5.20), (5.21) and (5.22) results in a new proof for the linearization coefficients $A^{(\alpha,\alpha)}_{n,k,m}$ for the Gegenbauer polynomials (cf. (2.40) and (2.41)).

We can use (6.11) with coefficients given by (6.12) in order to derive the following pair of differential recurrence relations, which are the analogues of (2.24), (2.25) for Jacobi polynomials.

COROLLARY 6.14. *We have*

$$(6.20) \qquad \eta^{1-\alpha}\, D_-^{\gamma} \circ \eta^{\alpha}\, R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta) = \tfrac{1}{4}\,\alpha(\alpha+\gamma+\tfrac{1}{2})\, R^{\alpha-1,\beta+1,\gamma}_{n,k}(\xi,\eta),$$

$$(6.21) \qquad (1-\xi+\eta)^{-\beta}\, D^{\gamma} \circ (1-\xi+\eta)^{\beta+1}\, \frac{R^{\alpha-1,\beta+1,\gamma}_{n,k}(\xi,\eta)}{R^{\alpha-1,\beta+1,\gamma}_{n,k}(2,1)} =$$

$$= \tfrac{1}{4}\,(\beta+1)(\beta+\gamma+3/2)\,\frac{R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)}{R^{\alpha,\beta,\gamma}_{n,k}(2,1)}\,.$$

PROOF. Formula (4.18) can be extended to

$$(6.22) \qquad \eta^{1-\alpha}\, D_-^{\gamma} \circ \eta^{\alpha}\, Z^{\gamma}_{n,k}(\xi,\eta) = \tfrac{1}{4}(k+\alpha)(n+\alpha+\gamma+\tfrac{1}{2}) Z^{\gamma}_{n,k}(\xi,\eta).$$

Substitution of (6.22) in (6.11) gives (6.20). Combination of (6.20) and (3.18) gives (6.21).

7. ANOTHER EXPANSION OF THE POLYNOMIALS $R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$ AND THEIR RELATION WITH APPELL'S FUNCTION F_4

In this section we will consider an expansion of $R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$ in terms of the polynomials

$$(7.1)\qquad (1-\xi)^m\, R^{(\alpha,\beta)}_{\ell}\,(1+2(1-\xi)^{-1}\eta),\ m \geq \ell.$$

These polynomials play a similar role with respect to the operator $E^{\alpha,\beta}_{-}$ as the James type zonal polynomials do with respect to D^{γ}_{-}. In particular, it will be proved that the expansion of $R^{\alpha,\beta,\gamma}_{n,n}(\xi,\eta)$ only contains polynomials for which $m = \ell$. It will follow from this expansion that $R^{\alpha,\beta,\gamma}_{n,n}(\xi,\eta)$ can be expressed as an Appell function F_4, which seems to be a quite important result. This section will be concluded with an interpretation of $Z^{\gamma}_{m,\ell}(\xi,\eta)$ and the polynomials (7.1) as limit cases of $R^{\alpha,\beta,\gamma}_{m,\ell}(\xi,\eta)$ for $\beta \to\infty$, $\gamma \to\infty$, respectively.

Let us consider the polynomial (7.1). By (2.4) its power series expansion in $1 - \xi$ and η equals

$$(7.2)\qquad (1-\xi)^n\, R^{(\alpha,\beta)}_{k}(1+2(1-\xi)^{-1}\eta) = \sum_{i=0}^{k} \frac{(-k)_i(k+\alpha+\beta+1)_i}{(\alpha+1)_i\ i!}\,(1-\xi)^{n-i}(-\eta)^i.$$

On the boundary line $\eta = 0$ the polynomial reduces to

$$(7.3)\qquad (1-\xi)^n\, R^{(\alpha,\beta)}_{k}(1+2(1-\xi)^{-1}\eta)\Big|_{\eta=0} = (1-\xi)^n.$$

By using (7.2) the polynomial restricted to the axis of reflection $\xi = 1$ becomes

$$(7.4)\qquad (1-\xi)^n\, R^{(\alpha,\beta)}_{k}(1+2(1-\xi)^{-1}\eta)\Big|_{\xi=1} = \begin{cases} \dfrac{(n+\alpha+\beta+1)_n}{(\alpha+1)_n}\,\eta^n & \text{if } n = k,\\ 0 & \text{if } n > k.\end{cases}$$

In view of (7.2) any polynomial in ξ and η has a unique expansion in terms of the polynomials (7.1) (α,β fixed). In particular, we will consider the expansion of $R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$ in terms of these polynomials.

DEFINITION 7.1. The coefficients $d^{\alpha,\beta,\gamma}_{n,k;m,\ell}$ are given by

$$(7.5)\qquad R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta) = \sum_{\ell=0}^{n}\sum_{m=\ell}^{n} d^{\alpha,\beta,\gamma}_{n,k;m,\ell}(1-\xi)^m R^{(\alpha,\beta)}_{\ell}(1+2(1-\xi)^{-1}\eta).$$

REMARK 7.2. If $\xi = 1$ then substitution of (7.4) in (7.5) results in

$$(7.6)\qquad R^{\alpha,\beta,\gamma}_{n,k}(1-\eta) = \sum_{m=0}^{n} d^{\alpha,\beta,\gamma}_{n,k;m,m}\frac{(m+\alpha+\beta+1)_m}{(\alpha+1)_m}\eta^m.$$

The following theorem gives a motivation for considering the expansion (7.5).

THEOREM 7.3. *We have*

$$(7.7)\qquad E^{\alpha,\beta}_{-}(1-\xi)^n R^{(\alpha,\beta)}_{k}(1+2(1-\xi)^{-1}\eta) =$$

$$= \begin{cases} \tfrac{1}{2}(n-k)(n+k+\alpha+\beta+1)(1-\xi)^{n-1}R^{(\alpha,\beta)}_{k}(1+2(1-\xi)^{-1}\eta) & \textit{if } n > k,\\ 0 & \textit{if } n = k.\end{cases}$$

PROOF. Use (3.9) and (2.1). □

On comparing (7.7) with (7.5) and (3.15) we obtain the recurrence relation

$$(7.8)\qquad d^{\alpha,\beta,\gamma}_{n,k;m,\ell} = \frac{(n-k)(n-k+2\gamma+1)(n+k+\alpha+\beta+1)(n+k+\alpha+\beta+2\gamma+2)}{4(\gamma+1)(\alpha+\gamma+3/2)(m-\ell)(m+\ell+\alpha+\beta+1)} d^{\alpha,\beta,\gamma+1}_{n-1,k;m-1,\ell}$$

if $m > \ell$ and $n > k$.

Since $E^{\alpha,\beta}_{-} R^{\alpha,\beta,\gamma}_{n,n}(\xi,\eta) = 0$, it also follows that

$$(7.9)\qquad R^{\alpha,\beta,\gamma}_{n,n}(\xi,\eta) = \sum_{m=0}^{n} d^{\alpha,\beta,\gamma}_{n,n;m,m}(1-\xi)^m R^{(\alpha,\beta)}_{m}(1+2(1-\xi)^{-1}\eta).$$

THEOREM 7.4. *The coefficients* $d^{\alpha,\beta,\gamma}_{n,k;m,\ell}$ *in* (7.5) *are nonzero only if* $m - \ell \le n - k$ *and* $m + \ell \le n + k$ (*cf. Figure* 4).

PROOF. The inequality $m + \ell \le n + k$ follows from (7.1) and Theorem 3.2. The inequality $m - \ell \le n - k$ is a consequence of (7.8) and (7.9). □

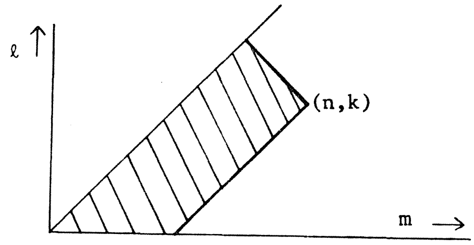

Figure 4

In view of (7.6), (7.8) and Theorem 7.4, we obtain the coefficients $d^{\alpha,\beta,\gamma}_{n,k;m,\ell}$ as soon as we know the expansion of $R^{\alpha,\beta,\gamma+m-\ell}_{n-m+\ell,k}(1,\eta)$ as a power series in η. Here we restrict ourselves to the case $n = k$.
It follows from (7.9) that

$$(7.10)\qquad R^{\alpha,\beta,\gamma}_{n,n}(\xi,0) = \sum_{m=0}^{n} d^{\alpha,\beta,\gamma}_{n,n;m,m}\,(1-\xi)^m.$$

From (5.5), (2.7), (2.8) and (2.4) we know

$$(7.11)\qquad \frac{R^{\alpha,\beta,\gamma}_{n,n}(\xi,0)}{R^{\alpha,\beta,\gamma}_{n,n}(1,0)} = \sum_{m=0}^{n} \frac{(-n)_m(n+\alpha+\beta+\gamma+3/2)_m}{(\beta+1)_m\, m!}(1-\xi)^m.$$

Comparison of (7.10) and (7.11) yields (by using (6.16))

$$(7.12)\qquad d^{\alpha,\beta,\gamma}_{n,n;m,m} = \frac{(-1)^n(\beta+1)_n(-n)_m(n+\alpha+\beta+\gamma+3/2)_m}{(\alpha+\gamma+3/2)_n(\beta+1)_m\, m!}\,.$$

So we have the expansion

$$(7.13)\qquad \frac{R^{\alpha,\beta,\gamma}_{n,n}(\xi,\eta)}{R^{\alpha,\beta,\gamma}_{n,n}(1,0)} = \sum_{m=0}^{n} \frac{(-n)_m(n+\alpha+\beta+\gamma+3/2)_m}{(\beta+1)_m\, m!}\,(1-\xi)^m\, R^{(\alpha,\beta)}_m(1+2(1-\xi)^{-1}\eta).$$

In the case $\gamma = -\frac{1}{2}$ formulas (7.11) and (7.13) together are equivalent with Theorem 2.1. For $\xi = 1$ we obtain

$$(7.14)\qquad \frac{R^{\alpha,\beta,\gamma}_{n,n}(1,\eta)}{R^{\alpha,\beta,\gamma}_{n,n}(1,0)} = \sum_{m=0}^{n} \frac{(-n)_m (n+\alpha+\beta+\gamma+3/2)_m (m+\alpha+\beta+1)_m}{(\beta+1)_m (\alpha+1)_m\, m!}\, \eta^m =$$

$$= {}_4F_3\left(\begin{matrix}-n, n+\alpha+\beta+\gamma+3/2, \tfrac{1}{2}(\alpha+\beta+1), \tfrac{1}{2}(\alpha+\beta+2);\\ \alpha+1, \beta+1, \alpha+\beta+1;\end{matrix}\ 4\eta\right).$$

This formula generalizes (2.26).

Now we can prove the following interesting theorem which connects $R^{\alpha,\beta,\gamma}_{n,n}(\xi,\eta)$ with Appell's function F_4 (cf. (2.28)).

THEOREM 7.5. *We have*

$$(7.15)\qquad R^{\alpha,\beta,\gamma}_{n,n}(\xi,\eta) \ / \ R^{\alpha,\beta,\gamma}_{n,n}(1,0) =$$

$$= F_4\ (-n, n+\alpha+\beta+\gamma+3/2;\ \alpha+1, \beta+1; \eta, 1-\xi+\eta) =$$

$$= \sum_{i+j\le n} \frac{(-n)_{i+j} (n+\alpha+\beta+\gamma+3/2)_{i+j}}{(\alpha+1)_i (\beta+1)_j\ i!\ j!}\, \eta^i (1-\xi+\eta)^j .$$

PROOF. It follows from (2.5) that

$$(1-\xi)^m\ R^{(\alpha,\beta)}_m\ (1+2(1-\xi)^{-1}\eta) =$$

$$= (\beta+1)_m\ m!\ \sum_{i+j=m} \frac{\eta^i (1-\xi+\eta)^j}{(\alpha+1)_i (\beta+1)_j\ i!\ j!} .$$

Substitution of this formula in (7.13) proves the theorem. $\square$

This theorem generalizes (2.29).

Next we will prove that the polynomials

$$Z^{\gamma}_{n,k}(\xi,\eta) \quad \text{and} \quad (1-\xi)^n\ R^{(\alpha,\beta)}_k (1+2(1-\xi)^{-1}\ \eta)$$

can be obtained as limit cases of $R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)$ for $\beta \to \infty$, $\gamma \to \infty$, respectively. Thus, in view of (2.21), the expansions (4.10) and (7.5) are quite similar to the expansion (2.4) for the Jacobi polynomials. First we note

LEMMA 7.6. *We have*

$$(7.16)\qquad \lim_{\beta\to\infty} \frac{R_{n,0}^{\alpha,\beta,\gamma}(\xi,\eta)}{R_{n,0}^{\alpha,\beta,\gamma}(2,1)} = \frac{(\gamma+\frac{1}{2})_n}{(2\gamma+1)_n} Z_{n,0}^{\gamma}(\xi,\eta),$$

$$(7.17)\qquad \lim_{\gamma\to\infty} R_{n,n}^{\alpha,\beta,\gamma}(\xi,\eta) = (1-\xi)^n R_n^{(\alpha,\beta)}(1+2(1-\xi)^{-1}\eta).$$

PROOF. Use (6.2), (3.17), (7.13) and (6.16). □

THEOREM 7.7. *We have*

$$(7.18)\qquad \lim_{\beta\to\infty} \frac{R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)}{R_{n,k}^{\alpha,\beta,\gamma}(2,1)} = \frac{(\gamma+\frac{1}{2})_{n-k}}{(2\gamma+1)_{n-k}} Z_{n,k}^{\gamma}(\xi,\eta),$$

$$(7.19)\qquad \lim_{\gamma\to\infty} R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta) = (1-\xi)^n R_k^{(\alpha,\beta)}(1+2(1-\xi)^{-1}\eta).$$

PROOF. In order to prove (7.18) use complete induction with respect to k. For k = 0 (7.18) becomes (7.16). From (3.17) and (3.14) we have

$$\frac{R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)}{R_{n,k}^{\alpha,\beta,\gamma}(2,1)} = \frac{1}{4(\beta+1)(\beta+\gamma+3/2)} D_{+}^{\alpha,\beta,\gamma} \frac{R_{n-1,k-1}^{\alpha+1,\beta+1,\gamma}(\xi,\eta)}{R_{n-1,k-1}^{\alpha+1,\beta+1,\gamma}(2,1)} .$$

For $\beta \to \infty$ (3.11) together with the induction hypothesis gives

$$\lim_{\beta\to\infty} \frac{R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)}{R_{n,k}^{\alpha,\beta,\gamma}(2,1)} = \eta \lim_{\beta\to\infty} \frac{R_{n-1,k-1}^{\alpha+1,\beta+1,\gamma}(\xi,\eta)}{R_{n-1,k-1}^{\alpha+1,\beta+1,\gamma}(2,1)} =$$

$$= \frac{(\gamma+\frac{1}{2})_{n-k}}{(2\gamma+1)_{n-k}} \; Z^{\gamma}_{n,k}(\xi,\eta).$$

In order to prove (7.19) we use complete induction with respect to n-k, the case n - k = 0 being clear from (7.17). It follows from (3.16), (3.12) together with the induction hypothesis that

$$\lim_{\gamma\to\infty} R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta) = (1-\xi) \lim_{\gamma\to\infty} R^{\alpha,\beta,\gamma+1}_{n-1,k}(\xi,\eta) =$$

$$= (1-\xi)^n \; R^{(\alpha,\beta)}_k(1+2(1-\xi)^{-1}\eta). \qquad \square$$

<u>REMARK</u> 7.8. Let us consider the recurrence relations

$$(7.20) \qquad (1-\xi)R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta) = \sum_{m,\ell} b_{m,\ell} \; R^{\alpha,\beta,\gamma}_{m,\ell}(\xi,\eta),$$

$$(7.21) \qquad \eta \; R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta) = \sum_{m,\ell} c_{m,\ell} \; R^{\alpha,\beta,\gamma}_{m,\ell}(\xi,\eta).$$

By the use of the expansion (7.5), Theorem 7.4 and the orthogonality of the polynomials $R^{\alpha,\beta,\gamma}_{n,k}(\xi,\eta)$ it is directly proved that the coefficients $b_{m,\ell}$ in (7.20) are nonzero only if $(m,\ell) \in \{(n+1,k),(n,k+1),(n,k),(n,k-1),(n-1,k)\}$ (cf. Figure 5). Similarly, the coefficients $c_{m,\ell}$ in (7.21) are nonzero only if $(m,\ell) \in \{(n+1,k+1),(n+1,k),(n+1,k-1),(n,k+1),(n,k),(n,k-1),(n-1,k+1),$ $(n-1,k),(n-1,k-1)\}$ (cf. Figure 6). This can be proved by use of the expansion (4.10) and Theorem 4.3. The proofs sketched here are much shorter than those given in SPRINKHUIZEN [28, § 9].

	x				x	x	x
x	⊗	x			x	⊗	x
	x				x	x	x

Figure 5 Figure 6

8. CONNECTION COEFFICIENTS

In this section we shall consider the connection coefficients in the formula

$$R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta) = \sum_{m,\ell} c_{n,k;m,\ell}\, R_{m,\ell}^{a,b,c}(\xi,\eta). \tag{8.1}$$

It will turn out that for $k = 0$ or $k = n$ these coefficients coincide with certain connection coefficients for Jacobi polynomials. If $k = n$ and $(a,b,c) = (\alpha,\beta,-\frac{1}{2})$, or $k = 0$ and $(a,b,c) = (-\frac{1}{2},-\frac{1}{2},\gamma)$ then we obtain explicit expressions of $R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)$, $k = 0$ or n, as double Jacobi series. In these cases there will follow important inequalities. We conclude this section by deriving integral representations for $R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)$, $k = 0$ or n, in terms of Jacobi polynomials.

First note the following corollary of Theorem 3.2.

<u>LEMMA 8.1.</u> *The coefficients* $c_{n,k;m,\ell}$ *in* (8.1) *are nonzero only if* $m \le n$ *and* $m + \ell \le n + k$ *(cf. Figure 2 in section 3).*

<u>THEOREM 8.2.</u>

(a) *The coefficients* $c_{n,k;m,\ell}$ *in the formula*

$$R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta) = \sum_{m,\ell} c_{n,k;m,\ell}\, R_{m,\ell}^{a,b,\gamma}(\xi,\eta)$$

are nonzero only if $m \le n$ *and* $\ell \le k$ *(cf. Figure 3 in section 4).*

(b) *The coefficients* $c_{n,k;m,\ell}$ *in the formula*

$$R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta) = \sum_{m,\ell} c_{n,k;m,\ell}\, R_{m,\ell}^{\alpha,\beta,c}(\xi,\eta)$$

are nonzero only if $m-\ell \le n-k$ *and* $m+\ell \le n+k$ *(cf. Figure 4 in section 7).*

<u>PROOF</u>. In both cases we can first use Lemma 8.1. Part (a) of the theorem follows by $(k+1)$-fold application of the operator D_-^{γ} to both sides of the formula and by using (3.13). Similarly, in view of (3.15), part (b) of the

theorem is proved by (n-k+1)-fold application of $E_{-}^{\alpha,\beta}$ to both sides of the formula. □

Let the coefficients $g_{n;k}^{\alpha,\beta;a,b}$ be defined by (2.36).

THEOREM 8.3. *There are expansions*

$$R_{n,0}^{\alpha,\beta,\gamma}(\xi,\eta) = \sum_{m=0}^{n} g_{n;m}^{\alpha+\gamma+\frac{1}{2},\beta+\gamma+\frac{1}{2};a+\gamma+\frac{1}{2},b+\gamma+\frac{1}{2}} R_{m,0}^{a,b,\gamma}(\xi,\eta), \tag{8.2}$$

$$R_{n,n}^{\alpha,\beta,\gamma}(\xi,\eta) = \sum_{m=0}^{n} g_{n;m}^{\alpha+\gamma+\frac{1}{2},\beta;\alpha+c+\frac{1}{2},\beta} R_{m,m}^{\alpha,\beta,c}(\xi,\eta). \tag{8.3}$$

PROOF. It follows from Theorem 8.2(a) that $R_{n,0}^{\alpha,\beta,\gamma}(\xi,\eta)$ can be expanded in terms of $R_{m,0}^{a,b,\gamma}(\xi,\eta), m = 0,1,\ldots,n$. Now restrict to $\eta = \frac{1}{4}\xi^2$ and apply (5.6) and (2.36). This proves (8.2). Similarly, for the proof of (8.3) use Theorem 8.2(b), restrict to $\eta = 0$ and apply (5.5) and (2.36). □

The coefficients in (8.3) are positive if $\gamma > c$ (cf. (2.38)). See Theorem 2.3 for the cases that the coefficients in (8.2) are positive.

THEOREM 8.4.

(a) *If* $$R_n^{(\alpha+\gamma+\frac{1}{2},\beta+\gamma+\frac{1}{2})}(x) = \sum_{m=0}^{n} c_{n;m} R_m^{(\gamma,\gamma)}(x)$$

then $$R_{n,0}^{\alpha,\beta,\gamma}(1-xy,\tfrac{1}{4}(x-y)^2) =$$

$$= \sum_{m=0}^{n} c_{n;m} R_m^{(\gamma,\gamma)}(x) R_m^{(\gamma,\gamma)}(y)$$

and $$c_{n;m} = g_{n;m}^{\alpha+\gamma+\frac{1}{2},\beta+\gamma+\frac{1}{2};\gamma,\gamma} =$$

$$= \frac{(n+\alpha+\beta+2\gamma+2)_m(\gamma+1)_m\, n!}{(m+2\gamma+1)_m(\alpha+\gamma+3/2)_m(n-m)!\; m!}\; {}_3F_2\left(\begin{matrix}-n+m, n+m+\alpha+\beta+2\gamma+2, m+\gamma+1;\\ 2m+2\gamma+2, m+\alpha+\gamma+3/2;\end{matrix}\; 1\right).$$

If either $\alpha > \beta$ *and* $\alpha + \beta \geq -1$ *or* $\alpha = \beta > -\frac{1}{2}$ *and* $n - m$ *is even then in the above formulas* $c_{n;m} > 0$.

(b) *If* $$R_n^{(\alpha+\gamma+\frac{1}{2},\beta)}(x) = \sum_{m=0}^{n} c_{n;m} R_m^{(\alpha,\beta)}(x)$$

then $$R_{n,n}^{\alpha,\beta,\gamma}(1-\tfrac{1}{2}(x+y),\ \tfrac{1}{4}(1-x)(1-y)) = \sum_{m=0}^{n} c_{n;m} R_m^{(\alpha,\beta)}(x)\ R_m^{(\alpha,\beta)}(y)$$

and $$c_{n;m} = g_{n;m}^{\alpha+\gamma+\frac{1}{2},\beta;\alpha,\beta} =$$

$$= \frac{n!(\beta+1)_n(\gamma+\frac{1}{2})_{n-m}(n+\alpha+\beta+\gamma+3/2)_m}{(\alpha+\gamma+3/2)_n(\alpha+\beta+2)_n(n-m)!(n+\alpha+\beta+2)_m}\,\omega_m^{(\alpha,\beta)}.$$

If $\gamma > -\frac{1}{2}$ *then in the above formulas* $c_{n;m} > 0$.

PROOF. Part (a) of the theorem follows from (8.2) and (3.21). The coefficients are given in (2.39) and Theorem 2.3 implies the positivity result. For part (b) use (8.3), (3.4) and (2.37). □

Theorem 8.4 (a) gives an explicit expression for $R_{n,0}^{\alpha,\beta,\gamma}(\xi,\eta)$ and it shows that $R_{n,0}^{\alpha,\beta,\gamma}(1-xy,\frac{1}{4}(x-y)^2)$ is the generalized translate of the Jacobi polynomial $R_n^{(\alpha+\gamma+\frac{1}{2},\beta+\gamma+\frac{1}{2})}(x)$ expressed as a Gegenbauer series of order (γ,γ) (See ASKEY [2, Lecture 2] for the definition of generalized translates).

Similarly, Theorem 8.4 (b) gives an explicit expression for $R_{n,n}^{\alpha,\beta,\gamma}(\xi,\eta)$ and it shows that $R_{n,n}^{\alpha,\beta,\gamma}(1-\frac{1}{2}(x+y),\frac{1}{4}(1-x)(1-y))$ is the generalized translate of the Jacobi polynomial $R_n^{(\alpha+\gamma+\frac{1}{2},\beta)}(x)$ expressed as a Jacobi series of order (α,β). Hence we also have a new expression for the generalized translate of the Jacobi polynomial kernel (cf. BAVINCK [5, § 5.8], [6]). This kernel gives a summation method for Fourier-Jacobi expansions. If $\gamma \to \infty$ in Theorem 8.4 (b) then, using (7.17), we obtain Bateman's bilinear sum, which can be interpreted as the De la Vallée Poussin kernel (cf. ASKEY [1]).

For $\gamma = \frac{1}{2}$ Theorem 8.4 (b) implies (3.6), and thus the Christoffel-Darboux formula (2.20) for Jacobi polynomials.

COROLLARY 8.5.

(a) *Let* $\alpha \geq \beta$, $\alpha + \beta \geq -1$, $(\alpha,\beta) \neq (-\frac{1}{2},-\frac{1}{2})$. *Then* $R^{\alpha,\beta,\gamma}_{n,0}(\xi,0) > 0$ *for* $\xi \in [0,1]$ *except if* $\alpha = \beta$, n *is odd,* $\xi = 1$.
If $\alpha \geq \beta$, $\alpha + \beta \geq -1$, $\gamma \geq -\frac{1}{2}$ *then*

$$|R^{\alpha,\beta,\gamma}_{n,0}(\xi,\eta)| \leq 1 \text{ on } \bar{\Omega}.$$

(b) *Let* $\alpha \leq \beta$, $\alpha + \beta \geq -1$, $(\alpha,\beta) \neq (-\frac{1}{2},-\frac{1}{2})$. *Then* $(-1)^n R^{\alpha,\beta,\gamma}_{n,0}(\xi,\xi-1) > 0$ *for* $\xi \in [1,2]$ *except if* $\alpha = \beta$, n *is odd,* $\xi = 1$. *If* $\alpha \leq \beta$, $\alpha + \beta \geq -1$, $\gamma \geq -\frac{1}{2}$ *then*

$$|R^{\alpha,\beta,\gamma}_{n,0}(\xi,\eta)| \leq |R^{\alpha,\beta,\gamma}_{n,0}(2,1)| \text{ on } \bar{\Omega}.$$

(c) *If* $\gamma > -\frac{1}{2}$ *then* $R^{\alpha,\beta,\gamma}_{n,n}(\xi,\frac{1}{4}\xi^2) > 0$ *for* $\xi \in [0,2]$. *If* $\gamma \geq \frac{1}{2}$, $\max(\alpha,\beta) \geq -\frac{1}{2}$, $(\xi,\eta) \in \bar{\Omega}$ *then*

$$|R^{\alpha,\beta,\gamma}_{n,n}(\xi,\eta)| \leq \begin{cases} 1 & \text{if } \alpha \geq \beta, \\ |R^{\alpha,\beta,\gamma}_{n,n}(2,1)| & \text{if } \alpha \leq \beta. \end{cases}$$

PROOF.

(a) It follows from Theorem 8.4 (a) that

$$R^{\alpha,\beta,\gamma}_{n,0}(1-x^2,0) = \sum_{m=0}^{n} c_{n;m} \, (R^{(\gamma,\gamma)}_m(x))^2.$$

If $\alpha > \beta$, $\alpha + \beta \geq -1$ then $c_{n;n}$ and $c_{n;n-1}$ are both positive. By SZEGÖ [29, Theorem 3.3.3] $R^{(\gamma,\gamma)}_n(x)$ and $R^{(\gamma,\gamma)}_{n-1}(x)$ cannot have common zeros. Hence $R^{\alpha,\beta,\gamma}_{n,0}(1-x^2,0) > 0$ for $0 \leq x \leq 1$. If $\alpha = \beta > -\frac{1}{2}$ then $c_{n;n}$ and $c_{n;n-2}$ are both positive. The positivity results again from [29, Theorem 3.3.3] together with (2.9) and (2.10). The second statement follows from Theorem 8.4 (a) and the fact that $|R^{(\gamma,\gamma)}_m(x)| \leq 1$ if $\gamma \geq -\frac{1}{2}$ and $-1 \leq x \leq 1$, cf. Theorem 2.2.

(b) Use part (a) of the corollary together with (3.18).

(c) It follows from Theorem 8.4 (b) that

$$R_{n,n}^{\alpha,\beta,\gamma}(1-x,\tfrac{1}{4}(1-x)^2) = \sum_{m=0}^{n} c_{n;m}(R_m^{(\alpha,\beta)}(x))^2.$$

A similar argument as in the proof of (a) gives the positivity result. The second statement follows from Theorem 8.4 (b) together with the inequalities for Jacobi polynomials (cf. Theorem 2.2). □

The above corollary confirms part of the hypothesis that for $\alpha \geq \beta \geq -\frac{1}{2}$, $\gamma \geq -\frac{1}{2}$ the **inequality**

$$|R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)| \leq 1$$

is valid on $\bar{\Omega}$, cf. SPRINKHUIZEN [28, § 7].

Let us conclude this paper by deriving integral representations for $R_{n,0}^{\alpha,\beta,\gamma}(\xi,\eta)$ and $R_{n,n}^{\alpha,\beta,\gamma}(\xi,\eta)$. Combination of Theorem 8.4 (a) and formula (2.33) gives

$$R_{n,0}^{\alpha,\beta,\gamma}(1-xy,\tfrac{1}{4}(x-y)^2) = \frac{\Gamma(\gamma+1)}{\pi^{\frac{1}{2}}\Gamma(\gamma+\frac{1}{2})} \cdot \tag{8.4}$$

$$\cdot\int_{-1}^{1} R_n^{(\alpha+\gamma+\frac{1}{2},\beta+\gamma+\frac{1}{2})}(xy+(1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}t)(1-t^2)^{\gamma-\frac{1}{2}}\,dt, \gamma > -\tfrac{1}{2},$$

which can be considered as a generalization of the product formula (2.33).

Similarly, Theorem 8.4 (b) and formula (2.35) imply the following generalization of the product formula (2.35):

$$R_{n,n}^{\alpha,\beta,\gamma}(1-\tfrac{1}{2}(x+y),\tfrac{1}{4}(1-x)(1-y)) = \frac{2\Gamma(\alpha+1)}{\pi^{\frac{1}{2}}\Gamma(\alpha-\beta)\ \Gamma\ (\beta+\frac{1}{2})} \cdot \tag{8.5}$$

$$\cdot\int_0^1\int_0^{\pi} R_n^{(\alpha+\gamma+\frac{1}{2},\beta)}(\tfrac{1}{2}(1+x)(1+y)+\tfrac{1}{2}(1-x)(1-y)r^2 +$$

$$+ (1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}\, r\cos\phi - 1)(1-r^2)^{\alpha-\beta-1} r^{2\beta+1}(\sin\phi)^{2\beta}\, dr\, d\phi,$$

$$\alpha > \beta > -\tfrac{1}{2}.$$

Both in (8.4) and (8.5) the left hand sides can be considered as the first term of an orthogonal expansion of the integrand. The full orthogonal expansion (a generalized addition formula) can be obtained by means of the techniques described in KOORNWINDER [24], i.e. by using integration by parts and differential recurrence relations for Jacobi polynomials. In particular, from (8.4) we get

$$(8.6)\qquad R_n^{(\alpha+\gamma+\frac{1}{2},\beta+\gamma+\frac{1}{2})}(xy+(1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}t) =$$

$$= \sum_{k=0}^{n} \frac{(-1)^k(-n)_k(n+\alpha+\beta+2\gamma+2)_k}{2^{2k}(\alpha+\gamma+3/2)_k(\gamma+1)_k}(1-x^2)^{\frac{1}{2}k}(1-y^2)^{\frac{1}{2}k}.$$

$$\cdot R_{n-k,0}^{\alpha,\beta,\gamma+k}(1-xy,\tfrac{1}{4}(x-y)^2)\ \omega_k^{(\gamma-\frac{1}{2},\gamma-\frac{1}{2})} R_k^{(\gamma-\frac{1}{2},\gamma-\frac{1}{2})}(t),$$

which is a generalization of the addition formula (2.34) for Gegenbauer polynomials. See MANOCHA [26] and CARLSON [9, § 3] for related generalizations of this addition formula.

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