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RELAXATION OSCILLATIONS OF A VAN DER POL EQUATION WITH LARGE CRITICAL FORCING TERM

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Relaxation oscillations of a van der Pol equation with large critical forcing term

by

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ABSTRACT

A van der Pol equation with sinusoidal forcing term is analyzed with singular perturbation methods for large values of the parameter. Asymptotic approximations of (sub)harmonic solutions with period $T = 2\pi(2n-1)$, $n = 1,2,\ldots$ can be constructed when the amplitude of the forcing term is within an interval that depends on n. These intervals overlap so that two periodic solutions with period $T = 2\pi(2n\pm1)$ may coexist.

KEYWORDS & PHRASES: *Van der Pol equation, relaxation oscillation, subharmonic entrainment, singular perturbation.*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In this paper we consider a Van der Pol equation for large parameter values with a periodic forcing term of a same order of magnitude

$$(1.1) \qquad \frac{d^2x}{dt^2} + \nu\,(x^2-1)\,\frac{dx}{dt} + x = b(\nu)\cos t, \qquad \nu \gg 1$$

with $b(\nu) = O(\nu)$. This equation was first investigated with analytical-topological methods by LITTLEWOOD [6], who proved the existence of (sub)-harmonic solutions of period

$$(1.2) \qquad T = 2\pi(2n-1), \qquad n = 1,2,\dots\,.$$

Littlewood stated that for $b = \alpha\nu$, $\alpha > 2/3$ only globally asymptotically stable solutions of period 2π are found, see also [7]. The proof of this statement has been given by LLOYD [8]. For decreasing α there also occur solutions of period 6π. As α further decreases the 2π-periodic solution dissappear; alternately α passes intervals where one subharmonic solution of period $T = 2\pi(2n-1)$ exists and intervals where two subharmonic solutions of period $T = 2\pi(2n\pm1)$ coexist, $n = 1,2,\dots$.

We will analyse the problem (1.1) for a specific choice of $b(\nu)$ and write

$$(1.3) \qquad b = \alpha\,\nu + \beta.$$

Using singular perturbation techniques we will construct asymptotic approximations of (sub)harmonic solutions of (1.1), (1.3) with $\alpha = 2/3$. The periods of these solutions satisfy (1.2) with n independent of ν. In the process of construction of the approximation we will have to impose conditions upon β of the type

$$(1.4) \qquad \underline{\beta}_n < \beta \le \overline{\beta}_n.$$

It turns out that

$$(1.5) \qquad \underline{\beta}_n < \overline{\beta}_{n+1} = \underline{\beta}_{n-1} < \overline{\beta}_n .$$

This overlapping of invervals differs slightly from Littlewoods results and from numerical results by FLAHERTY and HOPPENSTEADT [2], as we only find intervals with two subharmonic solutions of period $T = 2\pi(2n\pm1)$. This difference is explicable for one part from the fact that we study the case $\alpha = 2/3$ instead of $\alpha < 2/3$ in the limit $\nu \to \infty$, while also the dependence upon the initial values may play a role. We found $\overline{\beta}_2 = 11/6\sqrt{3}$ which refines Littlewood's statement about the existence of solely 2π-periodic solutions for b sufficiently large. In [3,4] the case $\alpha = 0$ was also analyzed with asymptotic techniques. There the subharmonics had a period $T = 2\pi m$ with $m = O(\nu)$. The choice $\alpha = 0$ or $\alpha = 2/3$ leads to solutions with a completely different asymptotic behaviour, which makes it necessary to consider them as separate problems. In [3] we met an unusual structure of two-variable expansions matched with boundary layer solutions. We will see here that the case $\alpha = 2/3$ also exhibits an exceptional structure. The global behaviour of the solution depends strongly on local conditions: each time the solutions passes a neighbourhood of the lines $x = \pm 1$ some quantity is increased with a given value. When it reaches a threshold value the solution enters a phase of rapid change characteristic for a relaxation oscillation. This part of the solution is approximated by a boundary layer type of solution. For the regions sketched in figure 1 separate local approximations have been constructed from the differential equation. Integration constants in there local asymptotic solutions are determined by matching pairs of local solutions of adjacent regions.

Thus, in this paper we investigate the equation

$$(1.6) \qquad \frac{d^2x}{dt^2} + \nu(x^2-1)\frac{dx}{dt} + x = (\frac{2}{3}\nu + \beta)\cos t .$$

It is expected that the study of this problem with a critical forcing term may bring us in the position to deal succesfully with the more complicated problem of $0 < \alpha < 2/3$. It is anticipated that periodic solutions of this problem have a behaviour in which elements of both the case $\alpha = 0$ and $\alpha = 2/3$ are present. A formal asymptotic analysis may help us to get a better understanding of the essentials of this problem, which has the

reputation of being hard to investigate by rigourous analytical methods, see [2,5 and 7].

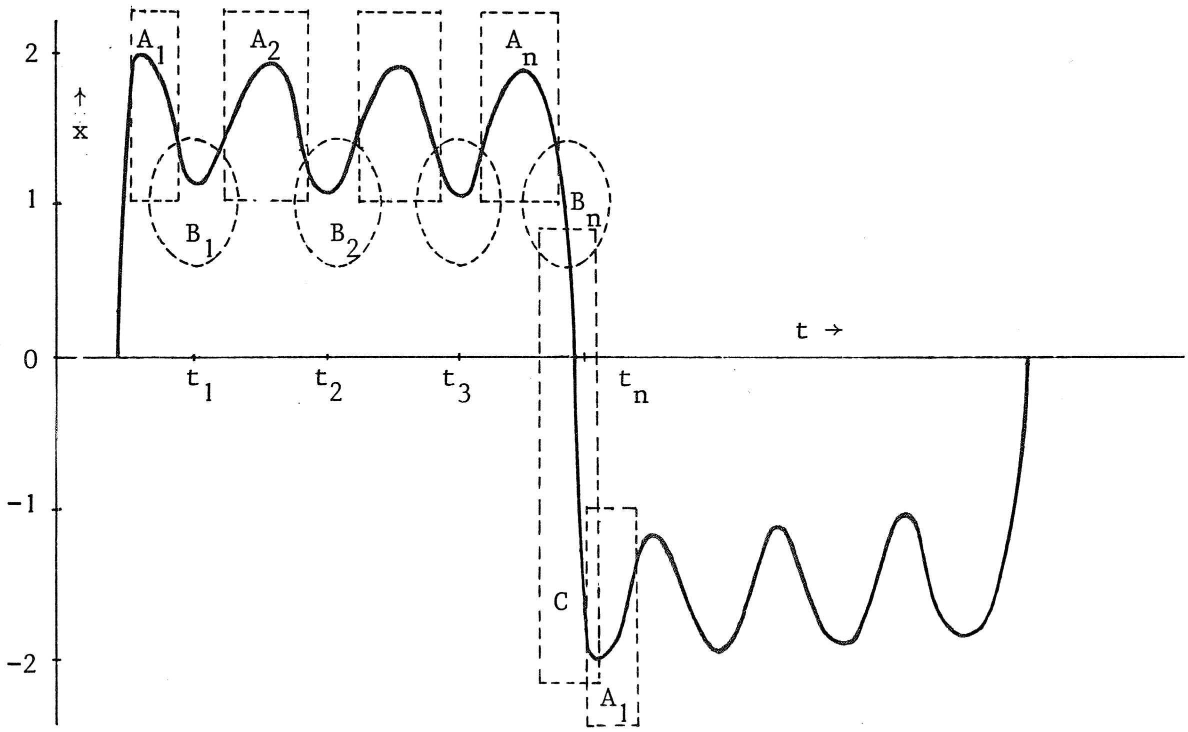

Fig. 1 Characteristic regions for a periodic solution of (1.6)

2. ASYMPTOTIC SOLUTIONS FOR THE REGIONS A_m

It is supposed that in the regions A_m where $1 < x < 2$ the solution can be expanded as

$$x(t;\varepsilon) = x_{m0}(t) + \nu^{-1} x_{m1}(t) + \dots \quad (2.1)$$

Substituting (2.1) into equation (1.6) and equating the terms of order $O(\nu)$ and $O(1)$ we obtain

$$(x_{m0}^2 - 1)\frac{dx_{m0}}{dt} = \frac{2}{3}\cos t, \quad (2.2)$$

(2.3) $$(x_{m0}^2-1)\frac{dx_{m1}}{dt} + 2x_{m0}\,x_{m1}\frac{dx_{m0}}{dt} = -\frac{d^2x_{m0}}{dt^2} - x_{m0} + \beta\,\mathrm{cos}t.$$

Integration of equation (2.2) gives

(2.4) $$\frac{1}{3}x_{m0}^3 - x_{m0} = \frac{2}{3}\,\mathrm{sin}t + C_0^{(m)}.$$

Since in the regions A_m the value of the left-hand side of this equation varies from -2/3 to 2/3, we have to take $C_0^{(m)} = 0$. For this value of $C_0^{(m)}$ the solution of (2.4) reads

(2.5) $$x_{m0}(t) = 2\cos\{\tfrac{1}{3}\arccos(\mathrm{sin}t)\}.$$

Integrating (2.3), while making use of (2.2), we obtain

(2.6) $$(x_{m0}^2-1)\,x_{m1} = -\frac{2\,\mathrm{cos}t}{3(x_{m0}^2-1)} - \int_{t_{m-1}}^{t} x_{m0}(\tau)\,d\tau + \beta\,\mathrm{sin}t + C_1^{(m)},$$

(2.7) $$t_m = 2\pi m - \frac{\pi}{2}.$$

When t approaches t_m from below, x_{m0} and x_{m1} behave as

(2.8ab) $$x_{m0} \approx 1 - (t-t_m)\,/\,\sqrt{3}, \qquad x_{m1} \approx K_m\,/\,(t-t_m),$$

where

(2.9ab) $$K_m = -\frac{1}{2} + \frac{1}{2}\sqrt{3}\,(\,-C_1^{(m)} + I), \quad I = \int_{t_{m-1}}^{t_m} x_{m0}(t)\,dt = 6\sqrt{3}.$$

Thus, for $t \uparrow t_m$ the asymptotic solution (2.1) looses its validity.

3. ASYMPTOTIC SOLUTION FOR THE REGIONS B_m

We analyse the local behaviour of the solution near $(x,t) = (1,t_m)$, $m = 1,2,\ldots$ by introducing a stretching transformation in both the dependent and independent variable

$$\text{(3.1ab)} \qquad x = 1 + V_m(\xi)\nu^{-\gamma}, \qquad t = t_m + \xi\,\nu^{-\alpha}.$$

Substitution into the differential equation yields

$$\text{(3.2)} \qquad \nu^{-\gamma+2\alpha}\frac{d^2V_m}{d\xi^2} + \nu^{1-2\gamma+\alpha}\,(2V_m + \nu^{-\gamma}V_m^{\,2})\,\frac{dV_m}{d\xi} + 1 + V_m\,\nu^{-\gamma} = (\tfrac{2}{3}\,\nu+\beta)\,(\xi\nu^{-\alpha} - \frac{\xi^3\nu^{-3\alpha}}{3!} + \ldots).$$

We see that for $\alpha = \gamma = 1/2$ the second derivative becomes of the same order of magnitude in ν as the leading terms constituting equation (2.2). Multiplying the equation with $\nu^{-1/2}$ and letting ν to infinity, we obtain the limit equation

$$\text{(3.3)} \qquad \frac{d^2V_{m0}}{d\xi^2} + 2\,V_{m0}\,\frac{dV_{m0}}{d\xi} = \frac{2}{3}\,\xi.$$

The function $V_{m0}(\xi)$ expresses the local limit behaviour of the solution for $\nu \to \infty$. In order to match the solution of region A_m it must satisfy

$$\text{(3.4)} \qquad V_{m0}(\xi) \approx \frac{\xi}{\sqrt{\xi}} + \frac{K_m}{\xi}$$

for $\xi \to \infty$, see (2.8). Such a function indeed exists and has the form

$$\text{(3.5)} \qquad V_{m0}(\xi) = a\,\frac{D'_{K_m}(-a\xi)}{D_{K_m}(-a\xi)}, \qquad a = \sqrt[4]{4/3},$$

where $D_\mu(z)$ is the so-called parabolic cylinder function of order μ, see WHITTAKER and WATSON [9,p.347]. For $z \to \infty$ we have that

$$D_\mu(z) = e^{-\frac{1}{4}z^2}\,z^\mu\,\{1 - \frac{\mu(\mu-1)}{2z^2} + \ldots\},$$

while for $z \to -\infty$

$$D_\mu(z) = e^{-\frac{1}{4}z^2}\,z^\mu\,\{1 - \frac{\mu(\mu-1)}{2z^2} + \ldots\} + - \frac{\sqrt{2\pi}}{\Gamma(-\mu)}\,e^{\frac{1}{4}z^2}\,z^{-\mu-1}\,\{1 + \frac{(\mu+1)(\mu+2)}{2z^2} + \ldots\}.$$

Assuming that $K_m \leq 0$ the function $V_{m0}(\xi)$ will be regular for finite ξ, while for $\xi \to \infty$

$$(3.6) \qquad V_{m0}(\xi) \approx \frac{\xi}{\sqrt{3}} - \frac{K_m+1}{\xi} .$$

On the other hand at region A_{m+1} the solution is approximated by

$$x(t) = 1 + \frac{(t-t_m)}{\sqrt{3}} + \frac{-\frac{1}{2}+\frac{1}{2}\sqrt{3}(C_1^{(m+1)}-\beta)}{(t-t_m)} + O((t-t_m)^{-3})$$

as $t \downarrow t_m$. Consequently, (3.6) matches the local solution for region A_{m+1} if

$$K_m = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}\,(\beta - C_1^{(m+1)})$$

or using (2.9a)

$$(3.7) \qquad C_1^{(m+1)} = C_1^{(m)} - I$$

with $I = 6\sqrt{3}$, see (2.). Obviously, we will arrive in the situation that for some m, say m = n,

$$(3.8) \qquad K_{n-1} \leq 0 < K_n \leq \tfrac{1}{2}\sqrt{3}\, I$$

(if n = 1, inequality (3.8) reads $K_1 > 0$). The parabolic cylinder function $D_\mu(z)$ with $\mu > 0$ vanishes for certain value(s) of the argument z. Let z_0 be the largest zero. For $\xi \uparrow \xi_0$ with $z_0 = a\,\xi_0$ we have

$$(3.9) \qquad V_{m0}(\xi) \approx (\xi-\xi_0)^{-1} + \frac{1}{3}a^2 \left(\frac{1}{4}a^2\xi_0^2 - K_n - \frac{1}{2}\right)(\xi-\xi_0),$$

so $V_{n0} \to -\infty$ and the local solution at region B_n becomes singular at $\xi = \xi_0$.

4. ASYMPTOTIC SOLUTION FOR REGION C

At this point the solution enters the boundary layer region C with local coordinate

$$(4.1)\qquad \eta = (t-t_n-\xi_0\nu^{-1/2})\nu.$$

We assume that the solution can be expanded as

$$(4.2)\qquad x = W_0(\eta) + \nu^{-1}\, W_1(\eta) + \nu^{-3/2}\, W_2(\eta) + \dots\ .$$

Substituting (4.1) and (4.2) into (1.6) and equating the terms of order $0(\nu^2)$ and of $0(\nu)$ we obtain, respectively,

$$(4.3)\qquad \frac{d^2W_0}{d\eta^2} + (W_0{}^2-1)\,\frac{dW_0}{d\eta} = 0,$$

$$(4.4)\qquad \frac{d^2W_1}{d\eta^2} + (W_0{}^2-1)\,\frac{dW_1}{d\eta} + 2\,W_0\,W_1\,\frac{dW_0}{d\eta} = 0.$$

The solution of the first equation matches the local solution for region B_n if

$$(4.5)\qquad W_0(\eta) \approx 1 + 1/\eta$$

as $\eta \to -\infty$, see (3.9). This condition is satisfied by the class of solutions

$$(4.6)\qquad \frac{1}{1-W_0} + \frac{1}{3}\log\frac{W_0+2}{1-W_0} = -\eta + H_0,$$

where the integration constant H_0 is found from matching with higher order terms of the asymptotic solution for region B_n. It turns out that

$$(4.7)\qquad H_0 = \frac{1}{6}\log\nu + \frac{1}{3}\log 3.$$

From (3.9) we also deduce that W_1 should behave as

$$(4.8)\qquad W_1(\eta) \approx \frac{1}{3}\,a^2\,(\tfrac{1}{4}\,a^2\xi_0{}^2-K_n-\tfrac{1}{2})\,\eta$$

for $\eta \to -\infty$, so that the integrated equation (4.4) will have the form

(4.9) $$\frac{dW_1}{d\eta} + (W_0^2-1)\, W_1 = a^2(\tfrac{1}{4}\, a^2\xi_0^{\,2}-K_n-\tfrac{1}{2}).$$

On the other hand for $\eta \gg 1/6 \log \nu$ we have

(4.10a) $$W_0(\eta) = -2 + 0(\nu^{1/2}e^{-3\eta})$$

(4.10b) $$W_1(\eta) = \frac{1}{3}\, a^2\, (\tfrac{1}{4}\, a^2\xi_0^{\,2}-K_n-\tfrac{1}{2}) + 0(e^{-3\eta}).$$

The boundary layer solution matches the solution for region $\overline{A}_1$ if

(4.11) $$\overline{x}_{10}(t_n+\xi_0\nu^{-1/2}) + \nu^{-1}\, \overline{x}_{11}(t_n+\xi_0\nu^{-1/2}) =$$
$$-\,2 + \frac{1}{3}\, a^2\, (\tfrac{1}{4}\, a^2\, \xi_0^{\,2}-K_n-\tfrac{1}{2})\;\; \nu^{-1} + o(\nu^{-1}),$$

where $\overline{x}_{1i}(t)$ are the coefficients of an expansion for region $\overline{A}_1$ of the form (2.1).

5. PERIODICITY CONDITIONS

Let us assume that the periodic solutions we are looking for are symmetric in the sense that $x(t) = -x(t-\frac{1}{2}T)$. Then we have completed the local approximations. Transposing (4.11) to the complementary phase $t = t_1 - \pi + \xi_0\, \nu^{-1/2}$ in region A_1 we have

(5.1) $$x_{10}(t_1-\pi+\xi_0\nu^{-1/2}) + \nu^{-1}\, x_{11}(t_1-\pi+\xi_0\nu^{-1/2}) =$$
$$2 - \frac{1}{3}\, a^2\, (\tfrac{1}{4}\, a^2\xi_0^{\,2}-K_n-\tfrac{1}{2})\, \nu^{-1} + o(\nu^{-1})$$

or

(5.2) $$K_n = -\,\frac{1}{2} + \frac{1}{2}\sqrt{3}\, (\beta+C_1^{(1)}-\tfrac{1}{2}\, I).$$

Using (2.9a), (3.7) and (5.2) we find

(5.3) $$\beta = \frac{1}{\sqrt{3}}\, (2K_n+1) - \frac{1}{2}\, (n-\tfrac{1}{2})\, I.$$

From (3.8) we know that K_n ranges from 0 to 9, so β has to satisfy

$$(5.4) \qquad 3\sqrt{3}\,\left(\frac{11}{18}-n\right) < \beta \leq 3\sqrt{3}\,\left(\frac{47}{18}-n\right), \qquad n = 2,3,\ldots .$$

Solutions of period 2π are found for $\beta > -7/6\,\sqrt{3}$.

6. SOME REMARKS

We have found an interval for β, see (5.4), where an asymptotic approximation of a symmetric periodic solution of period $2\pi(2n-1)$ can be constructed. The lower bounds $\underline{\beta}_n$ agree quite well with the numerical results of FLAHERTY and HOPPENSTEADT [2,fig.3], see table I. They took $(\alpha,\beta) = (.475,0)$ and varied the value of ν. In this way intervals for $1/\nu$ were constructed, where numerical solutions of period $T = 2\pi(2n-1)$ exist. Our β relates to their ν as $\beta = -.192\nu$. In the limit $\nu \to \infty$ this problem differs from the one we study with $(\alpha,\beta) = (2/3,\beta)$, but for finite (large) values of ν one may expect some agreement. However, our upperbounds $\overline{\beta}_n$ are systematically above their values. It is possible that there are more conditions to be imposed upon β. They may be of a type we do not meet in our construction procedure. Apart from the different choice of α, we also expect complications because of the strong dependence upon the initial values. It is possible that for β just below $\overline{\beta}_{n+1}$ the set of initial values leading to solutions of period $T = 2\pi(2n+1)$ becomes so small, that only solutions of period $T = 2\pi(2n-1)$ are found in numerical experiments. Our results as given in formula (1.5) support the view that $\alpha = 2/3$ is a critical value (no subharmonics above this value). We see that for $\alpha = 2/3$ the system is at the point to have overlap by three β-intervals.

		Asympt. bounds of intervals		Numeric. bounds of intervals			
n	T	$\underline{\beta}_n$	$\overline{\beta}_n$	$1/\underline{\nu}_n$	$\sim \underline{\beta}_n$	$1/\overline{\nu}_n$	$\sim \overline{\beta}_n$
2	6π	- 7.2	3.2	.028	- 6.8	.148	- 1.3
3	10π	-12.4	- 2.0	.016	-12.0	.041	- 4.7
4	14π	-17.6	- 7.2	.012	-16.0	.018	-10.7
5	18π	-22.8	-12.4	.009	-21.3	.012	-16.0

Table I

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