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Combinatorial Set Theory

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PREREQUISITS

notation and conventions

$A, B, C, D, E, F, O, U, V, S, T, A', A'_\alpha \ldots$ stand for	ordinary sets in naive set theory, or e.q. the Zermelo-Fraenkel set theory with the axiom of choice, but without CH or GCH.
$\mathfrak{A}, \mathfrak{L}, \mathfrak{C}, \ldots \mathfrak{A}, \mathfrak{A}_\alpha \ldots$	families of sets
$\mathcal{P}(A)$	the powerset of A
$\emptyset$	the empty set
$\mathfrak{A}^\cap (\mathfrak{A}^{\hat{f}})$	the family consisting of all intersections of (finite) subfamilies of $\mathfrak{A}$.
$\mathfrak{A}^\cup (\mathfrak{A}^{\check{f}})$	.... unions .....
Or	the class of all ordinals
$\xi, \eta, \zeta, \rho, \mu, \nu, \xi', \xi_\eta, \ldots$	ordinals ($\omega = \omega_0$ is the first infinite ordinal)
Each ordinal is the set of its predecessors:	$\xi = \{\eta \mid \eta < \xi\}$ (Hence $\eta < \xi \leftrightarrow \eta \in \xi$)
(For $A \subset$ Or) sup A	$\bigcup A$ (Hence sup $\xi = \xi$, and sup $\emptyset = \emptyset = 0$)
$\xi \dotplus 1$	$\xi \cup \{\xi\} \in$ Or
$\xi \dotplus \eta$	the ordinal which is the (ordinal) sum of ξ and η (defined as usual)
ξ is cofinal with ρ	$\exists f : \xi \to \rho \ \forall \rho' < \rho \ \exists \xi' < \xi \ f\xi' > \rho'$
cf(ρ), the cofinality of ρ	min $\{\xi \mid \xi$ is cofinal with $\rho\}$
ρ is regular	cf(ρ) = ρ

Card	the class of all cardinals (i.e. initial ordinals)
$\|A\|$, $\|\zeta\|$	the cardinality of the set A, of the ordinal ζ
n,i,k,l,r	finite cardinals (members of ω)
$\alpha, \beta, \gamma, \delta, \ldots \alpha', \alpha_\xi, \ldots$	infinite cardinals, or if explicitly stated, arbitrary (finite or infinite) cardinals

Card is, like Or, well-ordered by $<$ (or $\in$). The infinite members are indexed by ordinals:

$\omega = \omega_0, \omega_1, \omega_2, \ldots, \omega_\omega, \ldots \omega_\xi, \ldots$	$\omega_\xi = \aleph_\xi$
ω_ξ is limit cardinal (successor)	ξ is limit ordinal (successor)
α is regular	$\mathrm{cf}(\alpha) = \alpha$ (like before)
α is singular	$\mathrm{cf}(\alpha) \neq \alpha$ (i.e. $\mathrm{cf}(\alpha) < \alpha$)

examples: for each ordinal ρ $\mathrm{cf}(\rho)$ is a regular cardinal; ρ is cofinal with ρ, hence $\mathrm{cf}(\rho) \leq \rho$; ρ is a successor ordinal $\Rightarrow \mathrm{cf}(\rho) = 1$; α is a successor cardinal $\Rightarrow \mathrm{cf}(\alpha) = \alpha$, i.e. α is regular; $\mathrm{cf}(\omega) = \mathrm{cf}(\omega \dot{+} \omega) = \omega$; $\mathrm{cf}(\omega_1) = \omega_1$; ω_ω is singular; α is regular iff $\alpha = \sum_{\eta<\xi} \beta_\eta$ implies

$|\xi| \geq \alpha$ or $\exists \eta < \xi \quad |\beta_\eta| = \alpha$

(for $A \subset \mathrm{Card} \subset \mathrm{Or}$) sup A	$\bigcup A$ (like before; note that $A \subset \mathrm{Card} \Rightarrow \bigcup A \in \mathrm{Card}$).
α^+	$\omega_{\xi+1}$ if $\alpha = \omega_\xi$

$\sum_{\xi<\sigma} \alpha_\xi$, $\prod_{\xi<\rho} \alpha_\xi$, α^β are cardinals defined as usual (Note that $\alpha + \beta = |\alpha \dot{+} \beta|$, where $\alpha \dot{+} \beta$ stands for the ordinal sum of the (initial) ordinals α and β . Iff $\alpha < \beta$ then $\alpha + \beta = \alpha \dot{+} \beta$)

CH	$\omega_1 = 2^{\omega_0}$
GCH	$\forall \alpha \quad \alpha^+ = 2^\alpha$
$\log \beta$	$\min \{\gamma \mid 2^\gamma \geq \beta\}$
${}^\alpha \log \beta$	$\min \{\gamma \mid \alpha^\gamma \geq \beta\}$

1. Regressive functions

1.1 Definition. Let M be a set of ordinals.
A function $\phi : M \to \mathrm{Or}$ is regressive if

$$\forall \xi \in M \quad \phi(\xi) < \xi$$

and $\phi(0) = 0$ if $0 \in M$.

1.2 THEOREM [ALEXANDROV-URYSOHN].
Let $f : \omega_1 \to \omega_1$ be regressive, then $\exists \xi_0 < \omega_1 \quad |f^{-1}(\xi_0)| = \omega_1$

Proof. Put $A_n = \{\xi \in \omega_1 | \ f^{(n)}\xi = 0\}$. Since $(f^{(n)}\xi)_{n\in\omega}$ is a non-increasing sequence for each ξ, it must stop, i.e. $\exists_n \ \forall_{m>n}$ $f^{(n)}\xi = f^{(m)}\xi$, and hence $\xi \in A_n$ for that n. Thus $\bigcup_{n\in\omega} A_n = \omega_1$, so that some A_{n_0} must have cardinality ω_1. As $|f^{(0)}(A_{n_0})| = \omega_1$ and $|f^{(n_0)}(A_{n_0})| = 1$
we can find $k < n_0$ such that $|f^{(k)}(A_{n_0})| = \omega_1$ but $|f^{(k+1)}(A_{n_0})| \le \omega_0$

Now we can choose $\xi_0 \in f^{(k+1)}(A_{n_0})$ in such a way that

$|f^{-1}(\xi)| = |f^{(k)}(A_{n_0})| = \omega_1$

1.3 Definitions. Let ρ be a limit ordinal, and $M \subset \rho$ an arbitrary subset of ρ.
A function $\phi : M \to \rho$ is definitely diverging if

$$\forall \xi < \rho \ \exists \eta \in M \ \forall \mu \in M \setminus \eta \quad \phi(\mu) > \xi \quad .$$

This is (by definition) equivalent to

$$\lim_{\eta \in M} \phi(\eta) = \rho$$

It means that the functionvalues of ϕ eventually exceed any ordinal $\xi < \rho$.
The set M is cofinal in ρ if $\forall \xi < \rho \ \exists_\eta \in M \quad \xi < \eta$ (i.e. M possesses arbitrarily large members) .

The set M is <u>stationary in ρ</u> if $M \cap C \neq \emptyset$ for each closed cofinal subset of ρ .
Note that, in the case $cf(\rho) > \omega_0$, the intersection of two closed cofinal subsets of ρ is again cofinal (and closed, in ρ). Hence any subset M of ρ containing a closed cofinal subset of ρ is then stationary. The converse does not hold . However we have:

1.4 THEOREM. <u>If $cf(\rho) > \omega$ and $M \subset \rho$, then M is not stationary iff $\exists \phi : M \to \rho$ ϕ is regressive and definitely diverging.</u>

Proof. Necessity: let M not be stationary. Then there is a closed cofinal subset C of ρ, which is disjoint from M.
Define $\phi : M \to \rho$ as follows

$$\phi(\mu) = \sup \{\alpha \in A \mid \alpha < \mu\}.$$

Note that $\sup \emptyset = 0$, and that $\phi(\mu) \in A$ for each $\mu \in M$ since A is closed. It is easily seen that ϕ is both regressive and definitely diverging.

Sufficiency. Assume that M is a stationary subset of ρ and ϕ : $M \to \rho$ and is regressive and definitely diverging. Define $h : \rho \to \rho$ by transfinite induction:

$$h(0) = \min\{\xi \mid \phi^{-1}(0) \subseteq \xi\}$$

if $\nu < \rho$: $h(\nu+1) = \min \{\xi \mid \forall \mu<\nu \quad \phi^{-1}(h(\mu)) \subseteq \xi\}$
if $\eta < \rho$, η is a limitordinal:

$$h(\eta) = \sup\{h(\nu) \mid \nu < \eta\}$$

Notice that h is continuous by definition.
Let $\eta_0 = \min \{\eta \mid h(\eta) = \rho\}$,
$A = \{h(\eta) \mid \eta < \eta_0 \wedge \eta \text{ is a limitordinal}\}$.

Notice that η_0 is a limitordinal $\geq \omega_1$,since $cf(\rho) \geq \omega_1$.
From $cf(\rho) \geq \omega_1$ and $h(\eta) > \eta$ for all $\eta < \rho$, it also follows that A is cofinal in ρ, and it is easily seen that moreover A is a closed subset of ρ. Hence $A \cap M \neq \emptyset$.

So there exists a limitordinal, η_1, such that $h(\eta_1) \in A \cap M$. By definition of h we have that

$$\forall \eta < \eta_0 \; \forall \xi \in M \setminus h(\eta+1) \quad \phi(\xi) > h(\eta)$$

Applying this to $\{\eta \mid \eta < \eta_1\}$ we find

$$\forall \eta < \eta_1 \qquad \phi(h(\eta_1)) > h(\eta)$$

and thus $\qquad \phi(h(\eta_1)) \geq h(\eta_1)$,

contradicting that ϕ is regressive.

1.5 <u>Remarks</u>. Note that if $cf(\rho) = \omega$, then clearly M is stationary iff $\exists \xi < \rho \quad M \cup \xi = \rho$. Moreover for any $M \subset \rho$ which is cofinal with ρ, there is a regressive, definitely diverging $\phi : M \to \rho$. For if $\rho = \sup\{\rho_i | i \in \omega\}$ and $\rho_i < \rho_{i+1}$ for all $i \in \omega$, then we may put

$$\phi(\mu) = \begin{cases} 0 & \text{if } \rho_1 \leq \mu \\ \max \{\rho_i | \rho_i < \mu\} \end{cases}$$

APPLICATIONS TO TOPOLOGY

$D(\xi)$ will denote ξ with the discrete topology.

1.6 THEOREM [MYCIELSKI [6]]
$D(\alpha+)$ can be embedded as a closed subset in $(D(\alpha))^{\alpha+}$

Remark. A topological space T is called <u>α-compact</u> if each open cover of T has a subcover of power $< \alpha$. So ω-compact = compact and ω_1-compact = Lindelöf. The Tychonoff-theorem states: "a product of α-compact spaces is α-compact if $\alpha = \omega$". Notice that a closed subset of an α-compact space is again α-compact. Hence the above theorem of Mycielski shows that a product of α^+ many α^+-compact spaces is not α^+-compact. This is well-known for $\alpha^+ = \omega_1$: the product of even two Lindelöfspaces (e.g. the half-open interval space) need not be Lindelöf.

The existence of cardinals α for which the "α-Tychonoff-theorem" holds (the so-called strongly compact cardinals) does not follow from the ordinary axioms of settheory. In fact, "if they exist" they are measurable, and hence inaccessible.

Proof of the theorem.

Let $R = \prod\{D(\xi) : \alpha \leq \xi < \alpha^+\} \sim (D(\alpha))^{\alpha^+}$, i.e. R is the set of regressive functions from $\alpha^+\setminus\alpha$ to α^+. Now $\alpha^+\setminus\alpha$ is stationary in α^+ and α^+ is regular. Hence, by 1.4 , each $f \in R$ is constant on a cofinal subset of α^+. For each $\zeta < \alpha^+$ we choose one regressive $f_\zeta \in R$ with the following properties: (i) $\forall\ \xi \in \alpha^+\setminus\zeta \quad f(\xi) = \zeta$

(ii) $\quad f|(\zeta\setminus\alpha)\colon \zeta\setminus\alpha \to \alpha$ is bijective.

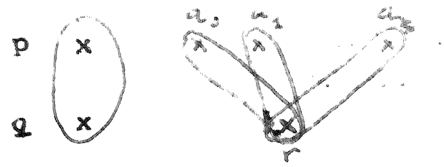
Now we only have to show that $A = \{f(\zeta) \mid \zeta < \alpha^+\}$ has no accumulationpoints in R.

Let $g \in T$, g is regressive, and is constant on a cofinal set. Certainly $\exists\, \xi_1, \xi_2, \zeta \ \ (\alpha \leq \xi_1 < \xi_2 < \alpha^+ \wedge g(\xi_1) = g(\xi_2) = \zeta)$.

Then $\{f \in R \mid f(\xi_1) = f(\xi_2) = \zeta\}$ is a neighbourhood of g which contains at most one element, f_ζ, of A, since f_ζ is the only element of A which assumes the value ζ more then once.

2. Large quasidisjoint subfamilies of large families

2.1 If $\mathfrak{A}$ is a large family of finite sets, then does there exist a big disjoint subfamily of $\mathfrak{A}$? Not necessarily.



$\mathfrak{A} = \{\{p,q\}\} \cup \{\{r,a_\xi\} \mid \xi < n\}$

Note that $\bigcap \mathfrak{A} = \emptyset$.

This situation suggests the following definition:

A family Σ of sets is quasidisjoint if $\forall A,B \in \Sigma \quad A \neq B \Rightarrow A \cap B = \bigcap \Sigma$

2.2 Remarks 1° The following three conditions are equivalent:

(i) Σ is a quasidisjoint family

(ii) $\{B \setminus \bigcap \Sigma \mid B \in \Sigma\}$ is a disjoint family

(iii) each three-element subfamily of Σ is quasidisjoint.

2° It follows easily from the Teichmüller-Tukey lemma (or the equivalent Zorn-lemma) that any family contains maximal quasidisjoint and maximal disjoint subfamilies.

We will show that if $\mathfrak{A}$ is a large family of sets of small cardinality, then there is a large subfamily $\Sigma \subset \mathfrak{A}$ which is quasidisjoint (Theorem 2.3-2.5).

2.3 THEOREM. Let n be a fixed integer. If $\mathfrak{A}$ is a family of n-element sets, and $|\mathfrak{A}| = \alpha$ is regular, then $\exists \Sigma \quad \Sigma \subset \mathfrak{A} \wedge \Sigma$ is quasidisjoint $\wedge \ |\Sigma| = |\mathfrak{A}| = \alpha$. (cf. Theorem 2.5)

Proof. The proof will go by induction on n. For n = 1, $\mathfrak{A}$ is disjoint and we may take $\Sigma = \mathfrak{A}$.

Let the lemma be true for all numbers smaller then n.

Let $\mathfrak{A}_0$ be a maximal disjoint subfamily of $\mathfrak{A}$, and suppose $\beta = |\mathfrak{A}_0| < \alpha$. Since each $A \in \mathfrak{A}$ meets at least one member of A_0 of $\mathfrak{A}_0$, and since α is regular

$$\exists A_0 \in \mathfrak{A}_0 \qquad |\{A \in \mathfrak{A} \mid A \cap A_0 \neq \emptyset\}| = \alpha$$

As A_0 is finite

$$\exists x \in A_0 \qquad |\{A \in \mathfrak{A} \mid x \in A\}| = \alpha.$$

Consider $\{A \setminus \{x\} \mid x \in A \in \mathfrak{A}\}$. By the inductionhypothesis this family has a quasidisjoint subfamily Σ_1 of cardinality α. Then

$$\Sigma = \{B \cup \{x\} \mid B \in \Sigma'\}$$

is a quasidisjoint subfamily of $\mathfrak{A}$ of cardinality α.

2.4 <u>Corollary. Let $\mathfrak{A}$ be an uncountable family of finite sets, and assume that $|\mathfrak{A}| = \alpha$ is regular. Then $\mathfrak{A}$ has a quasidisjoint subfamily Σ of cardinality α.</u>

Proof. By the regularity of α there is an $n < \omega$ such that $\mathfrak{A}$ has a family of α sets consisting of exactly n elements. Apply the above theorem to this family.

2.5 <u>THEOREM [ERDÖS-RADO [2], MICHAEL [5]].</u>

<u>If α is an infinite cardinal, β is any cardinal (either finite or infinite) and the family $\mathfrak{A}$ has the following properties</u>

(i) $\forall A \in \mathfrak{A} \qquad |A| \leq \beta$

(ii) $\forall \Sigma \subseteq \mathfrak{A} \qquad (\Sigma \text{ is quasidisjoint}) \Rightarrow |\Sigma| \leq \alpha$

<u>then $|\mathfrak{A}| \leq \alpha^\beta$</u>

Proof (of E. MICHAEL). For each $\nu < (\alpha^\beta)^+$ we define a subcollection $\mathfrak{A}_\nu$ of $\mathfrak{A}$ such that

(a) $\forall \nu < (\alpha^\beta)^+ \qquad |\mathfrak{A}_\nu| \leq \alpha^\beta$

(b) $\qquad \bigcup\{\mathfrak{A}_\nu \mid \nu < (\alpha^\beta)^+\} = \mathfrak{A}$

From this it follows that $|\mathfrak{A}| \leq 2^{\beta} \cdot \alpha^{\beta} = \alpha^{\beta}$.
Let $\mathfrak{A}_0$ be a maximal disjoint subfamily of $\mathfrak{A}$. If $\mathfrak{A}_\mu$ are defined for $\mu < \nu$, then we put

$$A_\nu = \bigcup \{\bigcup \mathfrak{A}_\mu | \mu < \nu\} = \bigcup (A_\mu | \mu < \nu\}$$

For each $K \subset A_\nu$, such that $|K| \leq \beta$, let

$$\mathfrak{A}_{K,\nu} = \{A \in \mathfrak{A} \setminus \bigcup_{\mu<\nu} \mathfrak{A}_\mu | A \cap A_\nu = K\}$$

If there exists A, $A' \in \mathfrak{A}_{K,\nu}$ such that $A \cap A' = K$, then let $\mathfrak{A}^*_{K,\nu}$ be a maximal quasidisjoint subfamily of

$\mathfrak{A}_{K,\nu}$ containing $\{A,A'\}$. In this case

(c) $$\bigcap \mathfrak{A}^*_{K,\nu} = K$$

If such A, A' do not exist, then let $\mathfrak{A}^*_{K,\nu}$ be an arbitrary maximal quasidisjoint subfamily of $\mathfrak{A}_{K,\nu}$. Now $\mathfrak{A}^*_{K,\nu} = \emptyset$ iff $\mathfrak{A}_{K,\nu} = \emptyset$.
Finally let

$$\mathfrak{A}_\nu = \bigcup \{ \mathfrak{A}^*_{K,\nu} \mid K \subset A_\nu \wedge |K| \leq \beta\}$$

let us verify (a) and (b).

To verify (a), note first that $|\mathfrak{A}_0| < \alpha$ and $|A_\nu| \leq |\nu| \cdot \sup \{|A_\mu| \ |\mu < \nu\} \leq \alpha^\beta \cdot \alpha^\beta = \alpha^\beta$ (by induction on ν). Now A_ν has at most $(\alpha^\beta)^\beta = \alpha^\beta$ β-element subsets K, and by (ii) $|\mathfrak{A}^*_{K,\nu}| \leq \alpha$. It follows that $|\mathfrak{A}_\nu| \leq \alpha^\beta \cdot \alpha = \alpha^\beta$.

To verify (b), suppose that, on the contrary there is an $A \in \mathfrak{A}$ which is not in any $\mathfrak{A}_\nu$ $(\nu < (\alpha^\beta)^+)$. We will show that such an A meets $A_{\nu+1} \setminus A_\nu$ for each $\nu < (\alpha^\beta)^+$, whence $|A| \geq \alpha^\beta > \beta$, in contraction to (i) . Let $K = A \cap A_\nu$. Now $\mathfrak{A}^*_{K,\nu} \neq \emptyset$, because $A \in \mathfrak{A}_{K,\nu}$. There are two possibilities : either there exists an $A' \in \mathfrak{A}^*_{K,\nu}$ such that $A \cap A' \setminus K \neq \emptyset$, and so $A \cap A_{\nu+1} \setminus A_\nu \neq \emptyset$; or for each $A' \in \mathfrak{A}^*_{K,\nu}$ we have $A \cap A' \subset K$, and hence $A \cap A' = K$. In this case however $\mathfrak{A}^*_{K,\nu} \cup \{A\}$ is quasidisjoint by (c), contradicting the maximality of $\mathfrak{A}^*_{K,\nu}$.

Remark. Note that theorem 3 is an immediate consequence of theorem 5.

APPLICATIONS TO TOPOLOGY.

A topological space is said to have the Suslin-property if every disjoint family of open subsets is countable. Whether a product of two (or finitely many) spaces with the Suslin-property again has the Suslin-property depends on the choosen axioms of set theory. The following, however, holds in "general" (i.e. Z.F.+ Choise).

2.6 THEOREM. A topological product $X = \prod_{i \in J} X_i$ has the Suslin property iff every finite subproduct $\prod_{i \in J'} X_i$, $|J'| < \omega$ has the Suslin property.

Proof. Suppose $\{O_\xi' \mid \xi < \omega_1\}$ is a disjoint family of non-empty open subsets of X. For each ξ we choose a non-empty basic open set $O_\xi \subset O_\xi'$. Let $J_\xi = \{i \in J \mid \pi_i O_\xi = X_i\}$. There is a subfamily $A \subset \omega_1$ of cardinality ω_1, such that

$$\{J_\xi \mid \xi \in A\} \text{ is quasidisjoint (corollary 4).}$$

If $\xi \neq \xi'$ then $O_\xi \cap O_{\xi'} = \emptyset$ and hence $J_\xi \cap J_{\xi'} \neq \emptyset$. Thus

$\bar{J} =_{\text{def}} \cap\{J_\xi \mid \xi \in A\} = J_\xi \cap J_{\xi'} \neq \emptyset$, for all $\xi \neq \xi'$, $\xi, \xi' \in A$.

From this it follows that $\{\pi_{\bar{J}} O_\xi \mid \xi \in A\}$ is an uncountable disjoint family of open subsets of the finite subproduct $\prod_{i \in \bar{J}} X_i$. Contradiction.

2.7 Remarks. Let us define the cellurarity number $c(X)$ of a topological space as follows: $c(X) = \sup\{|\mathfrak{A}| \mid \mathfrak{A}$ is a family of disjoint open subsets of X$\}$. Modifying the proof given above a little we obtain the following result:

If $\alpha = \sup\{c(X_{i_1} \times X_{i_2} \times \dots X_{i_n}) \mid i_1, \dots i_n \in J, n < \omega\}$ is regular, then $\alpha = c \prod_{i \in J} X_i$.

Note that "X has the Suslin property" is equivalent to "each uncountable family of open sets has an uncountable subfamily without disjoint members". Now Šanin investigated the following modification of the Suslin property:
Any uncountable family of open sets has an uncountable subfamily with a non-empty intersection.
It is easily seen that if two spaces have this property then so has their product. For infinite products this also holds as can be proved quite similarly to the proof of theorem 6.

3. Partition calculus

3.1 In the preceding paragraph we have mentioned that a product of two spaces with the Suslinproperty does not necessarily have the Suslinproperty. Let us, naively, try to proof this and take a look at the point where we get stuck.

Suppose $R = R_1 \times R_2$ and R_1 and R_2 both satisfy the Suslinproperty. Let $\{G_\xi \mid \xi < \omega_1\}$ be an uncountable family of disjoint basic open sets, i.e. $G_\xi = G_\xi^1 \times G_\xi^2$ for each $\xi < \omega_1$, and G_ξ^k is open in R_k, $k = 1, 2$. We will try (and fail) to deduce a contradiction. Since the G_ξ's are disjoint,

$$\forall \xi \neq \eta \quad G_\xi^1 \cap G_\eta^1 = \emptyset \quad \vee \quad G_\xi^2 \cap G_\eta^2 = \emptyset.$$

Put $I_k = \{\{\xi,\eta\} \mid \xi < \eta < \omega_1 \wedge G_\xi^k \cap G_\eta^k = \emptyset\}$ $k = 1, 2$.

For any set A, we denote the family of all 2-element subsets of A by $[A]^2$. So

$$[\omega_1]^2 = I_1 \cup I_2.$$

In order to derive a contradiction we should like to find an uncountable set $A \subset \omega_1$ such that $\forall \xi \neq \eta\ (\xi,\eta \in A \Longrightarrow G_\xi^k \cap G_\eta^k = \emptyset)$, for some fixed k. Thus we ask: If $[\omega_1]^2 = I_1 \cup I_2$, then does there exist an $A \subset \omega_1$ such that $|A| = \omega_1 \wedge ([A]^2 \subset I_1 \vee [A]^2 \subset I_2)$? The answer is negative, as was shown first by Sierpinsky, who proved the even stronger:

3.2 THEOREM. There exists a partition $[2^\omega]^2 = I_0 \cup I_1$, such that for each $A \subset 2^\omega$ $([A]^2 \subset I_0 \vee [A^2] \subset I_1) \Longrightarrow |A| \leq \omega_0$. (cf. Theorem 35).

Proof. We "represent" 2^ω by $\mathbb{R}$. Let $<$ be the usual ordening on $\mathbb{R}$, and $\prec$ an arbitrary but fixed wellordening. Put

$$\{x,y\} \in I_0 \text{ iff } x < y \Longleftrightarrow x \prec y$$
$$\{x,y\} \in I_1 \text{ iff } y < x \Longleftrightarrow x \prec y.$$

Clearly $[\mathbb{R}]^2 = I_0 \cup I_1$. Suppose $A \subset \mathbb{R}$ and $[A]^2 \subset I_0$. Then the elements of A are wellordered by $<$ (since $<$ and $\prec$ coincide on A).

Suppose A is uncountable. Then put $r = \inf\{ r' \in \mathbb{R} | (-\infty, r'] \cap A$ is uncountable$\}$.
Notice that $r < \infty$, since $A = \bigcup \{(-\infty, n] \cap A | n \in \mathbb{N}\}$. It is easily seen that under these assumptions r has no countable nbd-base. So $|A| \leq \omega_0$. If $[A]^2 \subset I_1$ then the elements of A are wellordered by $>$ and similarly $|A| \leq \omega_0$.

3.3 We can visualise theorem 2 in the following way:
A graph is an ordered pair (V,S) consisting of a set of "vertices" V, and a subset $S \subset [V]^2$ whose elements are called sides. A graph (V,S) is complete if for each $v, w \in V$, $v \neq w$, there exists a side $s \in S$ joining v and w (i.e. $s = \{v,w\}$). Now theorem 2 states:

<u>There exist a partition of a complete graph with 2^ω vertices (i.e. a partition of the set of all sides), in two subsets, such that each complete subgraph of any element of the partition has at most ω vertices</u>.

3.4 We will now investigate some, more general cases of this partition-problem. Let, as in [8],

(1) $\quad \alpha \to (\beta_\xi)^r_{\xi<\nu}$ $\qquad$ "α arrows β_ξ, $\xi < \nu$, r"

stand for the following statement: If

(2) $\quad |S| = \alpha$ and $[S]^r := \{X \subseteq S \mid |X| = r\} = \bigcup_{\xi < \nu} I_\xi$

(i.e. "we have an r-partition of S"), then $\exists A \subseteq S \; \exists \xi <$ $[A]^r \subseteq I_\xi \wedge |A| = \beta_\xi$.
If $\beta_\xi = \beta$ for all $\xi < \nu$, then we may also write

$$\alpha \to (\beta)^r_\nu \quad ,$$

or $\quad \alpha \to (\beta, \ldots \beta)^r$ with ν β's, if ν is infinite.
For the negation of (1) we write

$$\alpha \not\to (\beta_\xi)_{\xi<\nu} \; .$$

Remarks.

1°. Theorem 2 can be written as $2^{\omega} \not\rightarrow (\omega_1,\omega_1)^2$.

2°. Suppose (1) holds. What is the effect if we change one of α, β_ξ, ν, r or commute the β_ξ?

(a) If $\alpha' > \alpha$ then also $\alpha' \rightarrow (\beta_\xi)^r_{\xi<\nu}$.

(b) If $|\nu'| = |\nu|$, and $f:\nu' \to \nu$ is any bijection then also $\alpha \to (\beta_{f(\xi)})^r_{\xi<\nu'}$.

(c) If $\nu' < \nu$ then also $\alpha \rightarrow (\beta_\xi)^r_{\xi<\nu'}$.

We obtain (b) and (c) as a special case of:

(d) If $|\nu'| \leq |\nu|$, and the β'_ξ, $\xi < \nu'$ are such that $\exists f$: $\nu' \to \nu$ satisfying $\beta'_\xi \leq \beta'_{f(\xi)}$ for each $\xi < \nu'$, then also $\alpha \rightarrow (\beta'_\xi)^r_{\xi<\nu'}$

(e) If

3°. $cf(\alpha) = \min\{v \mid \alpha \rightarrow (\alpha)^1_\nu\}$

4°. Note that in (1) r is (as usual) finite and the β_ξ and α are infinite. Though it might seem rasonable to drop these conditions, it turns out that if this is done, the theory is complicated very much, without yielding proportionally nicer results. In fact we might as well put some further restrictions to the use of (1), since (1) is only interesting if

(a) $r \geq 1$, or rather even $r > 1$ (cf 3°).

(b) $2 \leq |\nu| < \alpha$

(c) $\beta_\xi \leq \alpha$ for all $\xi < \nu$.

For suppose (2) holds. In case (a), if $r = 1$, then we just deal with partitions of S itself. If $r = 0$ then $[S]^r = [S]^0 = \{\emptyset\}$. If (b) is not satisfied, and $\nu < 2$, then we are not looking at partitions at all. If $\alpha < |\nu| \leq \nu$ then let I_ξ, $\xi < \nu$ be a partition of $[S]^r$ consisting of singletons and empty sets. It follows that $\alpha \not\rightarrow (\beta_\xi)^r_{\xi<\nu}$. If (c) is not true, and $\beta_{\xi_0} > \alpha$ then let I_ξ, $\xi < \nu$ be the trivial partition: $I_{\xi_0} = [S]^r$ and $I_\xi = \emptyset$ for other $\xi < \nu$. Again it follows that $\alpha \not\rightarrow (\beta_\xi)^r_{\xi<\nu}$.

3.5 Using some well-known lemma's from the theory of ordered sets we will prove the following generalization of theorem 3.2:

<u>THEOREM</u> $2^\alpha \nrightarrow (\alpha^+,\alpha^+)^2$

DEFINITION. An ordered set A is <u>complete</u> or <u>completely ordered</u> if it has one (and hence both) of the following two equivalent properties:

(a) each subset A' of A has an inf which belongs to A (we put $\inf \emptyset = \sup A \in A$).

(b) each subset A' of A has an inf and a sup which belongs to A.

<u>LEMMA A</u>. A_ξ is a completely ordered set for each $\xi < \nu$, then $\prod\{A_\xi \mid \xi < \nu\}$ is complete with respect to the lexicographic order (i.e. $(a_\xi)_{\xi<\nu} < (b_\xi)_{\xi<\nu}$ iff $(\alpha_\xi)_{\xi<\nu} \neq (\beta_\xi)_{\xi<\nu}$ and $\alpha_\xi < \beta_\xi$ for the first ξ for which $\alpha_\xi \neq \beta_\xi$).

Proof. We use induction to ν, and so may assume that $\prod\{A_\xi \mid \xi < \nu'\}$ is complete for all $\nu' < \nu$. Suppose $A' \subset A$. Put $A'_{\nu'} = \{(a_\xi)_{\xi<\nu} \mid (a_\xi)_{\xi<\nu} \in A'\}$ for all $\nu' < \nu$, and $a(\nu') = \inf A'_{\nu'}$. Suppose ν is a successor. If $a(\nu-1) = (a_\xi)_{\xi<\nu-1}$ for some $(a_\xi)_{\xi<\nu} \in A$, then consider $A'' = \{(a_\xi)_{\xi<\nu} \in A' \mid (a_\xi)_{\xi<\nu-1} = a(\nu-1)\}$. The points of this set are ordered according to their last coordinate, $a_{\nu-1}$, since the other coördinates are equal. So this set has an inf in A, and since all other $(\alpha_\xi)_{\xi<\nu} \in A' \setminus A''$ are bigger then all elements of A'', this is also the inf of A'. If $a(\nu-1) = (a_\xi)_{\xi<\nu-1}$ is not a member of $A'_{\nu-1}$, then clearly $\inf A' = (a_\xi)_{\xi<\nu}$ if $a_{\nu-1} = \sup A$.
Let ν be a limit ordinal. Notice that if $\nu'' < \nu' < \nu$ and $a(\nu'') = (a''_\xi)_{\xi<\nu''}$, and $a(\nu') = (a'_\xi)_{\xi<\nu}$ then $a''_\xi = a'_\xi$ for all $\xi < \nu''$. So there exist $a_\xi \in A_\xi$ such that $a(\nu') = (a_\xi)_{\xi<\nu'}$ for all $\nu' < \nu$. It is easy to check that now $\inf A' = (a_\xi)_{\xi<\nu}$.

<u>LEMMA B.</u> If $A = \{f : \alpha \to \{0,1\}\} = \prod\{\{0,1\} \mid \xi < \alpha\}$ with the lexicographic order $<$, and A' is a subset of A which is well ordered by $<$, then $|A'| \leq \alpha$.

Remark. Notice that we used a similar lemma in 3.2 for $\mathbb{R}$ instead of A, but that, if $\alpha = \omega$, $\mathbb{R} = A \setminus \{f : \omega \to \{0,1\} \mid \exists n \in \omega \ \forall m > n \ f(m) = 0\}$. We might as well have put a similar condition on A too, but since this is not necessary we have choosen for the above form of the lemma.

Proof of lemma B. Suppose, on the contrary, that there is a subset $A' = \{f_\xi \mid \xi < \alpha^+\} \subset A$ such that $\xi < \eta < \alpha^+$ implies $f_\xi < f_\eta$. Let $f = \sup A' = \inf \{g \in A \mid \forall g' \in A' \quad g' \leq g\}$. Since α^+ has no largest element (i.e. α^+ is a limit ordinal) $f \notin A'$, and so it cannot be that $\exists \nu < \alpha \ \forall \nu < \mu < \alpha \quad f(\mu) = 0$, (since such an f has, in A, an immediate predecessor, c.f. the remark made above). Now for each $\nu < \mu$ let f_ν be defined by $f_\nu(\mu) = f(\mu)$ if $\mu < \nu$ and $f_\nu(\mu) = 0$ if $\nu \leq \mu < \alpha$. So we just noticed that $f_\nu < f$ for each $\nu < \alpha^+$. Hence $|\{g \in A' \mid g < f_\nu\}| \leq \alpha$ and since $A' = \bigcup_{\nu<\alpha} \{g \in A' \mid g < f_\nu\}$, we find that $|A'| \leq \alpha$, a contradiction.

Proof of the theorem. Take A as in lemma B, and let $\prec$ be an arbitrary, but fixed well-ordening of A. Consider the following partition of $[A]^2$:

$$I_0 = \{\{f,g\} \mid f,g \in A \wedge f \neq g \wedge f < g \Leftrightarrow f \prec g\}$$

$$I_1 = \{\{f,g\} \mid f,g \in A \wedge f \neq g \wedge g < f \Leftrightarrow f \prec g\}$$

lemma B tells us that any $A' \subset A$ for which $[A']^2 \subset I_0$, satisfies $|A'| < \alpha^+$. Since $(A,<)$ and $(A,>)$ are order-isomorphic, we have the same for I_1. This shows that $|A| \not\to (\alpha^+,\alpha^+)^2$.

3.6 THEOREM [RAMSES]

$$\omega \to (\omega)^r_n$$

Remark. $\alpha \to (\alpha,\alpha)^2$ (which immediately implies $\alpha \to (\alpha)^2_n$) holds only for $\alpha = \omega$, and for "very big" cardinals α. So we might ask whether maybe $\alpha \to (\alpha,\omega)^2$ holds more generally. Erdős proved that this is indeed true for all cardinals α, and we will prove it now for regular α, and later (maybe) for singular α.

3.7 THEOREM [ERDÖS]

For regular α: $\qquad \alpha \to (\alpha,\omega)^2$.

The proofs will come forth next week. Impatient readers are referred to [8].