stichting
mathematisch
centrum

AFDELING ZUIVERE WISKUNDE ZN 43/72 JUNE

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SOME SIMPLIFIED PROOFS IN INFINITE-DIMENSIONAL TOPOLOGY

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Printed at the Mathematical Centre, 2e Boerhaavestraat 49, Amsterdam.

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§0 Introduction

In §1 a relatively easy way is described to map the pseudoboundary BQ of the Hilbert cube Q in its pseudo interior s, by an autohomeomorphism of the Hilbert cube (theorem 1.4). With similar methods one can obtain somewhat stronger versions of this result, although the final purpose: characterizing all subsets of the Hilbert cube which are embedded as the pseudoboundary, is not reached along these lines. For this, see [4].

In §2 more Hilbert cube topology is given, mainly without proofs, in order to be able to prove the theorems of §3.

In §3 a new proof is given of the theorem that Z-sets in l_2- or Q-manifolds are topologically infinitely deficient. A similar theorem holds for countable unions of Z-sets. Since sets of infinite deficiency are easily seen to be Z-sets, it follows that in l_2- and Q-manifolds infinite deficiency is characterized by property Z [1], and σ-infinite deficiency characterizes σ-Z-sets. From this characterization one can derive a homeomorphism extension theorem for Z-sets in l_2-manifolds (in Q-manifolds the situation is more complicated), and it can be proved that countable unions of Z-sets in l_2-manifolds are strongly negligible.

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1) In [14] Chapman extended this result to F-manifolds, where F is a, not necessarily separable, Fréchet space such that $F \approx F^\infty$.

§1 The Hilbert cube and its pseudoboundary

1.1 Definitions

Let J denote the interval $[-1,1]$ and $Q = J^\infty$ the Hilbert cube. Points of Q can be written as $x = (x_i)_i$, with x_i the i^{th} coordinate of x. A metric for Q is given by $d(x,y) = \sum_i 2^{-i}|x_i - y_i|$. This metric induces the product topology.

Certain subsets of the Hilbert cube are:

The pseudo interior $s = (-1,1)^\infty$. This is a dense G_δ-subset, homeomorphic to l_2(the latter statement is by no means trivial. For a proof, see [5]).

The pseudoboundary $BQ = Q\backslash s = \{x \mid \exists i : |x_i| = 1\}$. This is a dense F_σ-subset. If a set A is homeomorphic to Q and if we have a canonical factorization of A into a product of closed intervals, then $B(A)$ denotes the pseudoboundary of A, relative to the given coordinate structure.

The endslices $W_i^+ = \{x \mid x_i = 1\}$ and $W_i^- = \{x \mid x_i = -1\}$. We write $W_i = W_i^+ \cup W_i^-$.

The pseudoboundary of Q equals $\bigcup_i W_i$. Notice that W_i^+ and W_i^- are homeomorphic to Q.

Further $\pi_i : Q \to J$ and $\pi_\alpha : Q \to J^\alpha (\alpha \subset \mathbb{N})$ denote the projection-maps defined by $\pi_i(x) = x_i$ and $\pi_\alpha(x) = (x_i)_{i\in\alpha}$. Sometimes we write x_α instead of $(x_i)_{i\in\alpha}$ and if α is infinite, Q_α instead of $\prod_{i\in\alpha} J$ or J^α and s_α instead of $\prod_{i\in\alpha}(-1,1)$

By $\overline{n}$ we mean the set $\{1,2,\ldots,n\}$.

Interior, closure and boundary of a set A are denoted by $A°$, $Cl(A)$ and $Bd(A)$ resp.

1.2 Homeomorphisms

If $(f_i)_i$ is a sequence of autohomeomorphisms of a space X, then $\overline{f}_i$ denotes $f_i \circ f_{i-1} \circ \ldots \circ f_1$. If the sequence $(\overline{f}_i)_i$ converges to a homeomorphism $f : X \to X$ then we call f the infinite left product of $(f_i)_i$ and we write $f = L\prod_{i=1}^\infty f_i$. Hence if we speak of an (infinite) left product f, it is implied that f is a homeomorphism. If X is compact metric and the sequence $(\overline{f}_i)_i$ converges uniformly

to f and all f_i are onto, then f is also onto.
For homeomorphisms of the Hilbert cube the following two notions are useful:

$f : Q \to Q$ is simple if for at most finitely many i, π_i of $\neq \pi_i$.
$f : Q \to Q$ is β^* if $f(B(Q)) = B(Q)$. Simple homeomorphisms are β^*.
Further two homeomorphisms are called isotopic if they are homotopic by a homotopy such that each level is a homeomorphism. The infinite left product of a sequence of homeomorphisms isotopic to the identity is again isotopic to the identity.
A homeomorphism $f : X \to X$ is said to have support on $A \subset X$ if $f|(X\setminus A) = id_{X\setminus A}$. For two homeomorphisms f and g from X to Y the distance $d(f,g)$ is defined as $\sup\{d(f(x),g(x)) | x \in X\}$. We have the following criterion for establishing convergence of a sequence of homeomorphisms:

1.3 CONVERGENCE CRITERION (M.K. Fort Jr. [15])

If X and Y are compact metric and $(f_i)_i$ is a sequence of homeomorphisms from X onto Y such that for all i, $d(f_i,f_{i+1}) < 2^{-i} \cdot \inf\{d(f_i(x),f_i(y)) | x, y \in X \text{ and } d(x,y) \geq i^{-1}\}$ then the sequence converges uniformly to a homeomorphism from X onto Y.

Proof: First we prove the existence of a limit map f, by giving an upperbound for $d(f_j,f_i)$: for $j>i>1$ $d(f_j,f_i) \leq d(f_i,f_{i+1})+\ldots+d(f_{j-1},f_j)$. Define $\eta_{i,j} = \inf\{d(f_j(x),f_j(y)) | x,y \in X \text{ and } d(x,y) \geq 1/i\}$.
Then we can write: $d(f_j,f_i) \leq 2^{-i}\eta_{i,i} + 2^{-i-1}\eta_{i+1,i+1}+\ldots$
$\ldots+2^{-j+1}\eta_{j-1,j-1}$.
But $\eta_{i+1,i+1} \leq \eta_{i,i+1} \leq \eta_{i,i} + 2d(f_i,f_{i+1}) \leq \eta_{i,i}(1+2^{-i+1}) \leq \frac{3}{2}\eta_{i,i}$;
and inductively for $k \geq i$ $\eta_{k,k} \leq (\frac{3}{2})^{k-i}\eta_{i,i}$. (*)
Thus $d(f_j,f_i) \leq 2^{-i}\eta_{i,i}(1+\frac{3}{4}+(\frac{3}{4})^2+\ldots+(\frac{3}{4})^{j-1-i}) \leq 2^{-i+2}\eta_{i,i}$,
independent of j. (**)
From (*) and (**) it follows that for $j>i>1$, $d(f_i,f_j) \leq 2^{-i+2}\eta_{i,i} \leq$
$\leq 2^{-i+2}(\frac{3}{2})^{i-2}\eta_{2,2} = (\frac{3}{4})^{i-2}\eta_{2,2}$. Hence there exists a continuous limit map f.

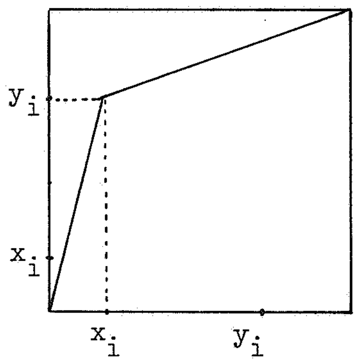
Moreover $\eta_{i,j} \geq \eta_{i,i} - 2d(f_j,f_i) \geq \eta_{i,i}(1-2^{-i+3})$, independent of j. Then also $\inf\{f(x),f(y)|x,y\in X \text{ and } d(x,y) \geq 1/i\} \geq \eta_{i,i}(1-2^{-i+3})$. Since i was arbitrary, it follows that f is one-to-one, hence a homeomorphism.

It is obvious how the statement should be rephrased for infinite left-products. The convergence criterion is of use when we want to construct inductively a sequence of autohomeomorphisms. When at each stage the next homeomorphism can be chosen arbitrarily close to the identity, then the assumptions of the convergence criterion can be fulfilled and we shall not bother how to do it in detail.

1.4 THEOREM The Hilbert cube is homogeneous.

Proof: Although this is an easy corollary to theorem 1.5, it will be proved separately below.

1° For any two points x, $y\in s$, x can be mapped onto y by a Q-autohomeomorphism. For let $\chi_i : J_i \to J_i$ be the piecewise linear homeomorphism that maps x_i onto y_i (see fig. 1.1). Then $h : Q \to Q$ defined by $h(z) = (\chi_i(z_i))_i$ is an autohomeomorphism of Q that maps x onto y.



graph of χ_i

fig.1.1

2° The point $x = (1,1,1,\ldots)$ can be mapped in s by an autohomeomorphism of Q: Let $h_1' : J^{\{1,2\}} \to J^{\{1,2\}}$ be a small rotation of the x_1-x_2 plane as shown in fig. 1.2. We define $h_1 = h_1' \times id_{J^{\mathbb{N}\setminus\{1,2\}}}$. Clearly $\pi_1 h_1(x)<1$. Now $\pi_{\{2,3\}}h_1(x) = (1,1)$ so we get exactly the situation of fig. 1.2

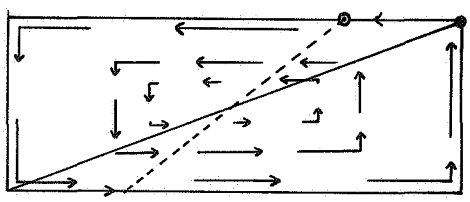


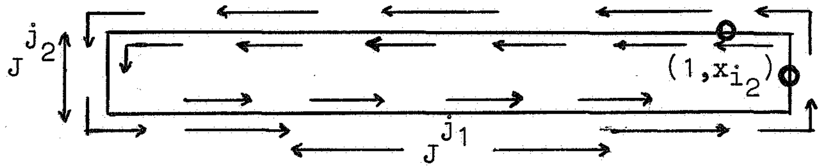
fig. 1.2

when we replace the x_1-x_2 plane by the x_2-x_3 plane and $\pi_{\{1,2\}}(x)$ by $\pi_{\{2,3\}}h_1(x)$. Thus $\pi_{\{2,3\}}h_1(x)$ can be pushed off the second endface by an arbitrarily small rotation h_2' of the x_2-x_3 plane. Hence, if h_2 is defined as $h_2' \times id_{J^{\mathbb{N}\setminus\{2,3\}}}$ then it can be supposed that the

assumptions of the convergence criterion are fulfilled.
Continuing this process we get a sequence of homeomorphism $(h_i)_i$ with left product h, such that for every i, $\pi_i h(x) = \pi_i \bar{h}_i(x)$; i.e. each coordinate of h(x) is determined after finitely many steps. This implies that for every i, $|\pi_i h(x)| < 1$.

<u>3°</u> Suppose x is such that $\alpha = \{i \mid |x_i| = 1\}$ is infinite. We can assume that for these i, $x_i = +1$. Then $h': Q_\alpha \to Q_\alpha$ can be constructed as in <u>1°</u> such that $h'(x_\alpha) \in s_\alpha$. Let $h = h' \times id_{J^{\mathbb{N}\setminus\alpha}}$, then $h(x) \in s$.

<u>4°</u> Suppose $x \in BQ$ is such that $\alpha = \{i \mid |x_i| = 1\} \subset \bar{j}_1$. Let $h_1': J^{j_1} \to J^{j_1}$ be a homeomorphism that maps $x_{\bar{j}_1}$ onto $(0,0,\dots,0,1)$ and define $h_1 = h_1' \times id_{J^{\mathbb{N}\setminus\bar{j}_1}}$. Construct h_2 such that $|\pi_{j_1} h_2 h_1 x| < 1$ in the following way: let $h_2': J^{\{j_1,j_2\}} \to J^{\{j_1,j_2\}}$ be a rotation in the x_{j_1}-x_{j_2} plane (with j_2 to be specified later) as suggested in fig. 1.3.



<u>fig. 1.3</u>

To move $(1,x_{j_2})$ off the small endface, the rotation does not need to move any point over a distance greater than 2^{-j_2+2}. Hence h_2' can be chosen arbitrarily small by choosing j_2 sufficiently large. Define again $h_2 = h_2' \times id_{J^{\mathbb{N}\setminus\{j_1,j_2\}}}$. By repeating this construction infinitely often, we get a sequence $(h_i)_{i\geq 1}$ with h_i (induced by) a rotation in the $x_{j_{i-1}}$-x_{j_i} plane, such that $h = L\Pi_i h_i$ maps x homeomorphically in s.

Now <u>1°</u> - <u>4°</u> imply homogeneity of the Hilbert cube.

1.5 THEOREM (preliminary version)

<u>There exists a sequence $(f_i)_i$ of autohomeomorphisms of Q such that</u>

(a) <u>each f_i is simple, hence β*</u>

(b) <u>the left product $f = L\Pi_i f_i$ exists</u>

(c) <u>$f(BQ) \subset s$</u>

Proof: Consider the i+1-dimensional cube J^{i+1}. Denote its endsliches by $V_j^{\pm}$, $j \leq i+1$. There exists a homeomorphism $h_i : J^{i+1} \to J^{i+1}$ such that $\bigcup_{j\leq i} V_j$ is contracted in V_{i+1}^{+} and h_i is isotopic to the identity. The homeomorphisms h_1 and h_2 are pictured below. (By interpreting J^{i+1} as the suspension over J^i, with top

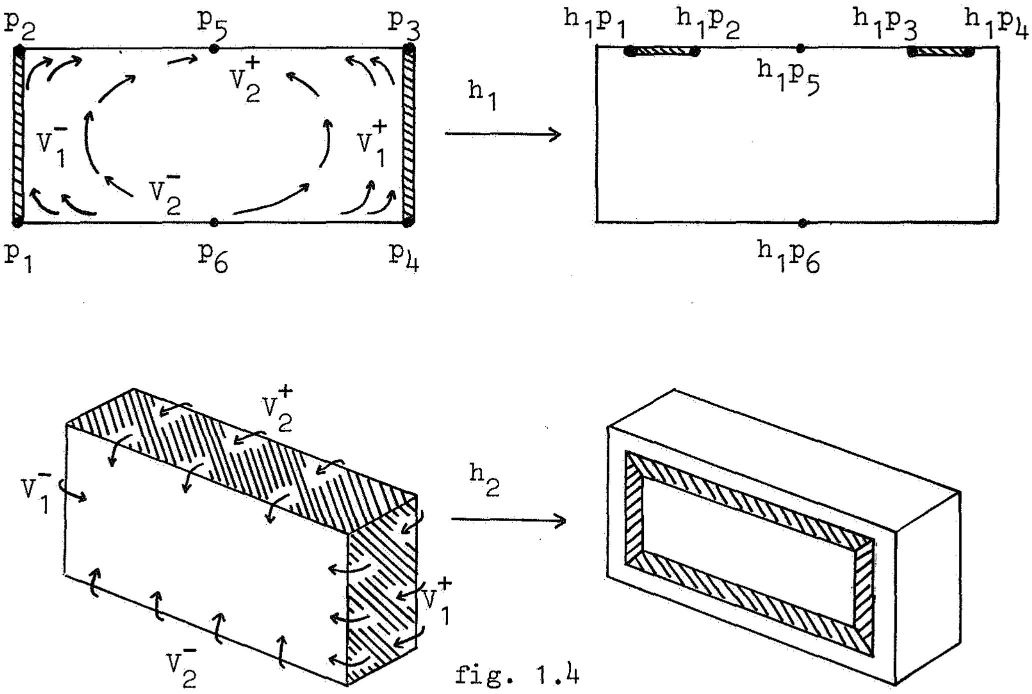


fig. 1.4

and bottom in the middle of V_{i+1}^{+} and V_{i+1}^{-} resp., one could give a formal definition h_i as a contraction to the top along the fibres which leaves the bottom fixed). Let $d : J^{i+1} \times J^{i+1} \to \mathbb{R}$ be the metric defined by $d(x,y) = \sum_{j=1}^{i+1} 2^{-j}|x_j - y_j|$. Pushing $Bd(V_{i+1}^{-})$ to $Bd(V_{i+1}^{+})$ requires a motion over a distance of exactly 2^{-i}. Next, the set $Bd(V_{i+1}^{+})$ can be contracted in the interior of V_{i+1}^{+} by a homeomorphism arbitrarily close to the identity. Let $\bar{h}_i : Bd(V_{i+1}^{-}) \to V_{i+1}^{+}$ be the composition of two such homeomorphisms. It is geometrically obvious that $\bar{h}_i$ can be extended to a homeomorphism h_i from J^{i+1} onto itself which maps $\bigcup_{j\leq i} V_j$ in the interior of V_{i+1}^{+} and has a distance of not much more than 2^{-i} to the identity, e.g. less than 2^{-i+1}. Define $h_i^* : Q \to Q$ as $h_i \times id_{Q\setminus\overline{i+1}}$.

By judicious choice of the sequence $(n_i)_i$, the sequence $(f_i)_i = (h^*_{n_i})_i$ will fulfil (a), (b) and (c). The sequence $(n_i)_i$ will be defined inductively:

First, let $n_{i+1} > n_i$ and let n_{i+1} be so large that $f_{i+1} = h^*_{n_{i+1}}$ meets the conditions of the convergence criterion.

Next, we must ensure that endfaces that are pulled off themselves do not come back.

More precisely: the distance between $\bigcup_{k\leq n_i} W_k$ and $h^*_{n_i}(\bigcup_{k\leq n_i} W_k)$ is positive, say ε. If n_{i+1} is such that $2^{-n_{i+1}+2} < \varepsilon$ then $d(L\Pi_{j>i} f_j, id) < \varepsilon$, irrespective of the choice of subsequent n_j's.

Hence $(L\Pi_{j\geq i} f_j)(\bigcup_{k\leq n_i} W_k)$ will be disjoint from $\bigcup_{k\leq n_i} W_k$ too. Now it follows that $(L\Pi_{j\geq 1} f_j)(BQ) \cap BQ = \emptyset$: for fixed k and $n_j+1\leq k$ $\bar{f}_j = h^*_{n_j} \circ \ldots \circ h^*_{n_1}$ is already defined on the first k coordinates, hence $\bar{f}_j(\bigcup_{k'\leq k} W_{k'}) = \bigcup_{k'\leq k} W_{k'}$ by invariance of domain for $\mathbb{R}^k$. Thus, if i is the first integer such that $n_i+1>k$ then by construction $\bar{f}_i(\bigcup_{k'\leq k} W_{k'}) = h^*_{n_i}(\bigcup_{k'\leq k} W_{k'})$ is disjoint from $\bigcup_{k'\leq k} W_{k'}$. This proves theorem 1.5.

1.6 Apparent boundaries

A set $A \subset Q$ is an <u>apparent boundary</u> if there exists a homeomorphism $h : Q \overset{onto}{\to} Q$ such that $h(A) = B(Q)$. A characterization of apparent boundaries in Q is given in [4] , with help of the notion of property Z. An important non-trivial example of an apparent boundary is a <u>basic core set</u> (bcs) : let $([a_i,b_i])_i$ be a sequence of closed subintervals of $(-1,1)$, then $M = \{x\in s \mid$ for all but finitely many i, $x_i\in[a_i,b_i]\}$ is the basic core set <u>structured on the core</u> $\Pi_i[a_i,b_i]$. For every two basic core sets M and M' obviously

(Q,M) is homeomorphic to (Q,M').

By a slight modification of the homeomorphisms h_i in 1.5, one can ensure that f(BQ) is a bcs : let M be a bcs structured on $\Pi_i[a_i,b_i]$. Suppose for every i the sequences $(a_{i,n})_n$ and $(b_{i,n})_n$ approach to −1 and +1 resp. as n tends to infinity, and $a_{i,1} = a_i$ and $b_{i,1} = b_i$. Then M equals $\cup_n M_n$, with $M_n = \Pi_{i\leq n}[a_{i,n},b_{i,n}] \times$ $\times \Pi_{i>n}[a_i,b_i]$. We shall exhibit a sequence of homeomorphisms $(f_i)_i$ and a sequence $(A_i)_i$ of subsets of BQ such that $(L\Pi_{j\geq i}f_j)(A_i) = M_i$ for some bcs $\cup_i M_i$, and such that $\cup_i \overline{f_i}^{-1}(A_{i+1}) = BQ$. As in 1.5, the f_i's will form a suitable subsequence of a sequence $(h_i^*)_i$ such that $h_i^* = h_i \times id_{J^{\mathbb{N}\setminus\overline{i+1}}}$ for some $h_i : J^{i+1} \to J^{i+1}$.

The homeomorphisms h_1 and h_2 are pictured below. Referring to the notations of 1.5, the extra conditions on h_i are:

(1) $h_i(V_i^+)$ is of the form $[a_1,b_1]\times\ldots\times[a_i,b_i]\times\{1\}$

(2) for j<i, h_i does not affect the j^{th} coordinate of points of $h_{i-1}(V_{i-1})$.

As in 1.5, h_i is chosen smaller than 2^{-i+1}, i.e. $d(h_i,id)<2^{-i+1}$.

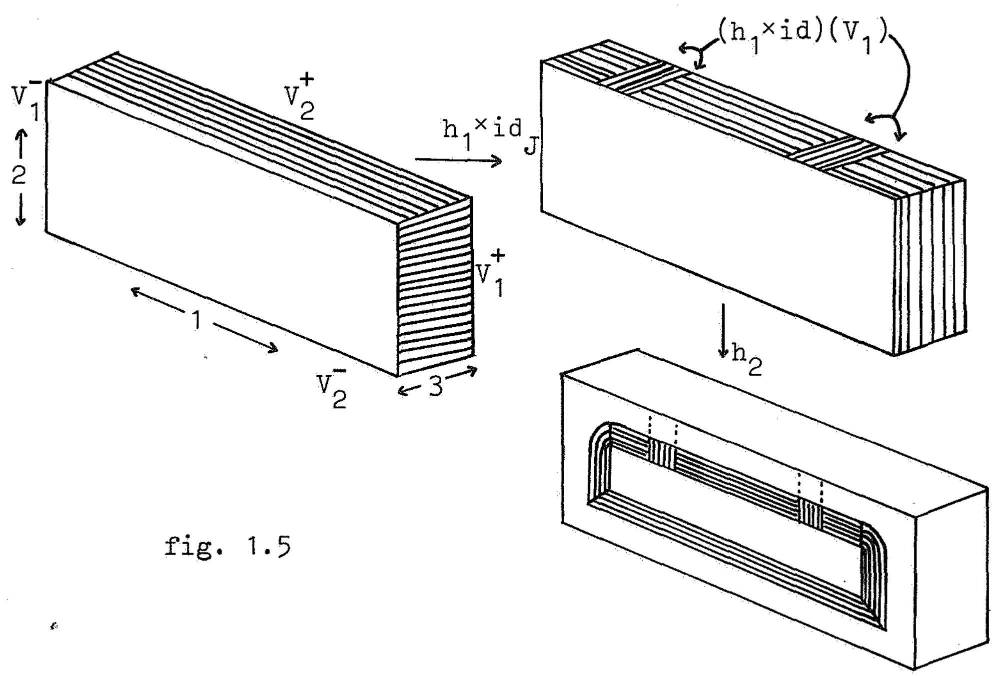


fig. 1.5

Again we choose a subsequence $(h^*_{n_i})_i$ of $(h^*_i)_i$ such that $h = L\Pi_i h^*_{n_i}$ maps BQ inside s by a space homeomorphism. Now (1) and (2) imply that $L\Pi_{j\geq i}\, h^*_{n_j}$ maps $W^+_{n_i}$ onto a product of closed intervals, or, what is equivalent, that h maps $\overline{h}^{*-1}_{n_{i-1}}(W^+_{n_i})$ onto a product of closed intervals. In particular, $h(W^+_{n_1})$ can be written as $\Pi_i[a_i,b_i]$. If we compare, for arbitrary i, $L\Pi_{j\geq i}\, h^*_{n_j}(W^+_{n_i})$ and $h(W^+_{n_1})$, then we see that these sets are, with the exception of the first n_i factors, a product of the same sequence of intervals.
Therefore h(BQ) can be written as $\cup_k M_k$, with $M_k = \Pi_{j\leq n_k}[a_{j,k},b_{j,k}] \times \Pi_{j>n_k}[a_j,b_j]$ (and $[a_j,b_j] =_{def} \pi_j(h(W^+_{n_1})))$. Because $\cup_k M_k$ is dense, being the homeomorphic image of BQ, it follows that $\lim_{k\to\infty} a_{j,k} = -1$ and $\lim_{k\to\infty} b_{j,k} = 1$ (it is easily seen that $(a_{j,k})_k$ and $(b_{j,k})_k$ are monotonously decreasing and increasing resp.)
Hence h maps BQ onto a bcs.

If one could ensure that h keeps some basic core set N (setwise) fixed, which is disjoint from the basic core set M = h(BQ), and if M and N can be interchanged by an order 2 homeomorphism of Q, then $h^{-1}\phi h$ would be an order 2 homeomorphism of Q that interchanges BQ and N. First we prove the second.

1.7 LEMMA <u>Every pair of disjoint basic core sets can be interchanged by an order 2 homeomorphism of Q.</u>

Proof: Suppose M and N are disjoint basic core sets, structured on the cores $\Pi_i[a_i,b_i]$ and $\Pi_i[c_i,d_i]$ resp. Disjointness of M and N is equivalent to disjointness of infinitely many pairs $[a_i,b_i]$ and $[c_i,d_i]$. A permutation of the indices achieves that for all odd i,

$[a_i,b_i]\cap[c_i,d_i] = \emptyset$. Write $A_i = [a_{2i-1},b_{2i-1}] \times [a_{2i},b_{2i}]$ and $C_i = [c_{2i-1},d_{2i-1}] \times [c_{2i},d_{2i}]$, then A_i and C_i are disjoint and $M = \{x|$ for all but finitely many i, $(x_{2i-1},x_{2i})\in A_i\}$ and $N = \{x|$ for all but finitely many i, $(x_{2i-1},x_{2i})\in C_i\}$. For all i, let $\phi_i : J^2 \to J^2$ be an order 2 homeomorphism that interchanges A_i and C_i, then $\phi : Q \to Q$ defined by $\phi(x) = (\phi_i(x_{2i-1},x_{2i}))_i$ is an order 2 homeomorphism that interchanges M and N.

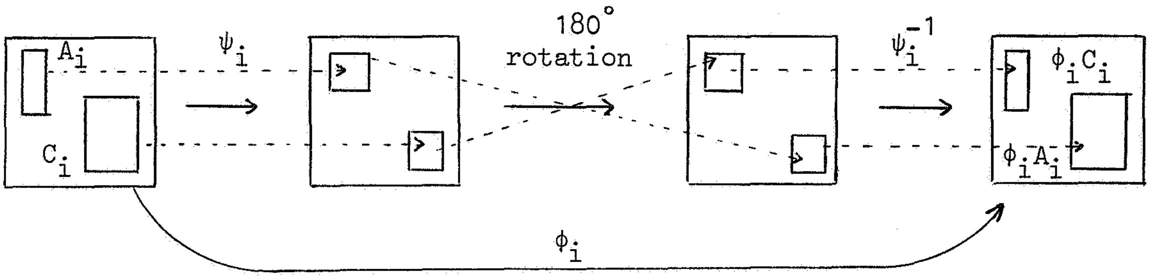


fig.1.6

1.8 THEOREM <u>The pseudoboundary of the Hilbert cube can be mapped onto a basic core set by an order 2 space homeomorphism</u>.

Proof. Choose a bcs N⊂s. Decompose N as $\cup_i N_i$ as indicated in 1.6. It is geometrically obvious that $h_i : J^{i+1} \to J^{i+1}$ (see 1.6) can be chosen to be the identity on a given subcube in the interior. For h_{n_1} we let this subcube be $N_1' = \pi_{\overline{n_1+1}}(N_1)$, the projection of N_1 onto the first n_1+1 coordinates. For h_{n_2} it will be a cube which contains both N_2' and $\pi_{\overline{n_2+1}}\, h^*_{n_1}(N_2)$, for h_{n_3} a cube which contains N_3' and $\pi_{\overline{n_3+1}}\, h^*_{n_2} h^*_{n_1}(N_3)$ etc. The composition $h^*_{n_i} \circ \dots \circ h^*_{n_1}$ is simple, hence β^* and maps every bcs onto itself. Therefore for every i, $\overline{h^*_{n_i}}(N_{i+1}) \subset N$ and $N_i \subset \overline{h^*_{n_{i-1}}}(N)$. Thus $h(N) = \cup_i h(N_i) = \cup_i \overline{h^*_{n_{i-1}}}(N_i) \subset N$ and $N = \cup_i N_i = \cup_i L\prod_{j\geq i} h^*_{n_j}(N_i) \subset h(N)$. As argued above 1.7, this proves the theorem.

§2 Topology of the Hilbert cube

In this paragraph more topology of the Hilbert cube will be given. The concepts of infinite deficiency and property Z will be introduced and of some of the theorems a proof will be given, in order to give some idea how the coordinate structure of Q can be used. For further proofs, see [2] and [3].

2.1 THEOREM For $p = (0,0,0,\ldots)$ there exists an isotopy $\Phi : Q \times I \to Q$ such that $\Phi_0 = \mathrm{id}$, Φ_t is β^* for $t<1$, $\Phi_1(BQ \cup \{p\}) = BQ$ and $\Phi_1(p) = (1,0,0,\ldots)$. (no proof)[1]

2.2 A closed set $A \subset Q$ is deficient in the i^{th} direction if $\pi_i(A)$ is a single point in $(-1,1)$. A closed set $A \subset Q$ is called infinitely deficient if A is deficient in infinitely many directions.

THEOREM If $A \subset Q$ is infinitely deficient, then there exists a homeomorphism $h : Q \to Q$ such that $h(BQ \cup A) = BQ$. Moreover h can be chosen arbitrarily close to the identity.

Proof: Suppose for $\alpha \subset \mathbb{N}$, α infinite, $\pi_\alpha(A)$ is a single point $p = (p_i)_{i\in\alpha} \in s_\alpha$. Let $\Phi : Q_\alpha \times I \to Q_\alpha$ be an isotopy on Q_α such that $\Phi_0 = \mathrm{id}_{Q_\alpha}$, Φ_t is β^* for $t<1$ and $\Phi_1(B(Q_\alpha) \cup \{p\} = B(Q_\alpha)$. Let $\tau : J^{\mathbb{N}\setminus\alpha} \to [0,1]$ be continuous, 1 on $\pi_{\mathbb{N}\setminus\alpha}(A)$ and <1 elsewhere.

Define $h : Q \to Q$ by

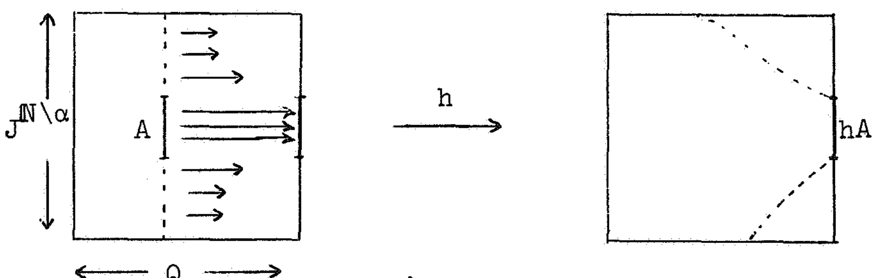


fig.2.1

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1) A modification of the construction in 1.4 sub 2^o gives an isotopy Φ' with $\Phi'_0 = \mathrm{id}$, Φ'_t is β^* for $t<1$ and $\Phi'_1(p) = q$ for some unknown $q \in BQ$. However, the isotopy Φ of theorem 2.1, is constructed along different lines.

$h(x) = (\Phi_{\tau(\pi_{\mathbb{N}\setminus\alpha}(x))}(\pi_\alpha(x)), \pi_{\mathbb{N}\setminus\alpha}(x))$. Using the fact that sets of the form $\pi^{-1}_{\mathbb{N}\setminus\alpha}(y)$ are mapped homeomorphically onto themselves, one readily verifies that h is an onto-homeomorphism such that $h(BQ\cup A) = BQ$. Moreover $d(h,id)<2^{2-n_0}$ if n_0 is the smallest element of α, because if for some x $\pi_i h(x) \neq x_i$ then $i\in\alpha$; i.e. h changes only α-coordinates. Hence h can be chosen arbitrarily close to id_Q by deleting all elements of α which are smaller than N, with N a sufficiently large integer.

2.3 THEOREM <u>If $(A_n)_n$ is a sequence of subsets of Q of infinite deficiency, then there exists a homeomorphism $h : Q \to Q$ such that $h(BQ\cup\bigcup_n A_n) = BQ$. Moreover h can be chosen arbitrarily close to id_Q.</u>

Proof. Select from $\mathbb{N}$ infinitely many infinite disjoint subsets α_n, such that A_n is deficient in the α_n-directions. Choose e.g. inductively a monotonously increasing sequence $(k_i)_i$ such that A_1 is deficient in the k_1^{th} direction, A_2 in the k_2^{nd} direction, A_1 in the k_3^{rd} direction, A_2 in the k_4^{th} direction, A_3 in the k_5^{th} direction, A_1 in the k_6^{th} direction etc.

α_1	α_2	α_3	α_4	α_5	α_6
k_1	k_2				
k_3	k_4	k_5			
k_6	k_7	k_8	k_9		
k_{10}	k_{11}	k_{12}	k_{13}	k_{14}	
k_{15}	k_{16}	k_{17}	k_{18}	k_{19}	k_{20}

. . . .

. . . .

Choose $h_1 : Q \to Q$ such that $h_1(BQ\cup A_1) = BQ$ and $\pi_i h_1(x) = x_i$ for $i\notin\alpha_1$. Now for n>1 $h_1(A_n)$ is still infinitely deficient in the α_n-directions. Choose $h_2 : Q \to Q$ such that $h_2(BQ\cup h_1 A_2) = BQ$ and $\pi_i h_2(x) =$

$= x_i$ for $i \notin \alpha_2$ and h_2 sufficiently small with regard to the convergence criterion. In general choose $h_n : Q \to Q$ such that $h_n(BQ \cup \bar{h}_{n-1}(A_n)) = BQ$ and $\pi_i h_n(x) = x_i$ for $i \notin \alpha_n$, and h_n sufficiently small in regard of the convergence criterion.

Then $h = L\Pi_n h_n$ exists and $h(BQ \cup \bigcup_n A_n) = BQ$: it follows from the definition of h that $\pi_{\alpha_n} h(x) = \pi_{\alpha_n} \bar{h}_n(x)$. We distinguish three cases.

1° If $x \in BQ$ then either for some $i \in \mathbb{N} \setminus \bigcup_n \alpha_n$, $x_i = \pm 1$, or for some n, $x_{\alpha_n} \in B(Q_{\alpha_n})$. In the first case, $\pi_i hx = x_i = \pm 1$. In the latter case $\pi_{\alpha_n} \bar{h}_{n-1} x = x_{\alpha_n}$. Now h_n is defined in such a way that $\pi_{\alpha_n}(\bar{h}_{n-1}(x)) \in B(Q_{\alpha_n})$ implies $\pi_{\alpha_n} \bar{h}_n(x) \in B(Q_{\alpha_n})$. Then also $\pi_{\alpha_n} h(x) = \pi_{\alpha_n}(\bar{h}_n(x) \in B(Q_{\alpha_n})$, thus $h(x) \in BQ$.

2° If $x \in A_n$ then $\pi_{\alpha_n} h(x) = \pi_{\alpha_n}(\bar{h}_n(x)) \in B(Q_{\alpha_n})$, hence $h(x) \in BQ$.

3° If $x \in s \setminus \bigcup_n A_n$ then for all $i \in \mathbb{N} \setminus \bigcup_n \alpha_n, \pi_i h(x) = x_i \in (-1,1)$. For all n, $\pi_{\alpha_n} h(x) = \pi_{\alpha_n} \bar{h}_n(x) \in s_{\alpha_n}$; hence $h(x) \in s$.

It is clear that here again h can be chosen arbitrarily small by deleting initial segments of the sets α_n. In the same way it is seen that $\mathbb{N} \setminus \bigcup_n \alpha_n$ can be made infinite. Hence one can always suppose that $\pi_\alpha o h = \pi_\alpha$ for some infinite set α or for a predetermined finite set α.

2.4 REMARK In theorem 2.2 any finite number of endfaces can be left pointwise fixed if only they are disjoint from A: choose α disjoint from the coordinates that define the endfaces under consideration and let τ be a Urysohnfunction that is 0 on these endfaces. This argument also applies to theorem 2.3, even if $\bigcup_n A_n$ has limit points in the endfaces.

2.5 THEOREM <u>Suppose K_1 and K_2 are closed subsets of Q or s such that $K_1 \cup K_2$ is infinitely deficient and $h : K_1 \to K_2$ is an onto homeomorphism. Then there exists an onto space homeomorphism $h' : Q \to Q$ such that $h'|K_1 = h$. Moreover, if $d(h,id)<\varepsilon$ then we may suppose that $d(h',id)<\varepsilon$ and, in the case of Q, if $K_2 \cap BQ = h(K_1 \cap BQ)$, then h' may be chosen β^*.</u> (no proof).

2.6 A closed set $K \subset X$ is called a <u>Z-set</u> in X (is said to have <u>property Z</u>) if for each non-empty homotopically trivial *) set $O \subset X$, $O \setminus K$ is non-empty and homotopically trivial. For each open set $O \subset X$, $O \cap K$ is a Z-set in O if K is a Z-set in X. A sufficient, and for X = Q or s also necessary condition for property Z is : for all ε there exists an ε-small homotopy of X off K. From this it follows that endslices and sets of infinite deficiency and compact subsets of s are Z-sets. The class of Z-sets is closed under taking closed subsets, finite unions and closed countable unions. Furthermore a set K is a Z-set if K can be covered by open sets O_i such that $O_i \cap K$ is a Z-set in O_i.

2.7 THEOREM <u>Suppose K is a Z-set in X, X = s or Q, then for all ε there exists an ε-small autohomeomorphism h of X such that h(K) is infinitely deficient. Moreover, if X = Q then h can be chosen β^*.</u>
(no proof)

COROLLARY <u>Suppose K_1 and K_2 are Z-sets in X, X = s or Q, and $h : K_1 \to K_2$ is an onto homeomorphism with $d(h,id)<\varepsilon$. Then there exists an onto homeomorphism $h' : X \to X$ such that $h'|K_1 = h$ and $d(h',id_X)<\varepsilon$. If, in the case of Q, $h(K_1 \cap BQ) = K_2 \cap BQ$ then h' can be chosen β^*.</u>

2.8 THEOREM <u>A closed set K in Q is a Z-set in Q iff $K \cap s$ is a Z-set in s</u> (no proof).

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*) in this report a set O is called homotopically trivial if $\forall n$ each map from S^{n-1} in O can be extended to a map from I^n in O.

§3 Infinite deficiency in Q- and l_2-manifolds

3.1 Let F be a topological space. A space M is called an F-manifold and F is a modelspace for M if for each $m \in M$ there exists an open neighborhood U of m such that U is homeomorphic to an open subset of F. In this section F is the Hilbert cube or a separable metric infinite-dimensional Fréchetspace. It will be proved that for F-manifolds infinite deficiency (see 3.7) coincides with property Z (theorems 3.10 and 3.14). This was proved first by Chapman in [13]. Using this result, one can prove a homeomorphism extension theorem for Z-sets in F-manifolds, F an infinite-dimensional separable Fréchetspace (theorem 3.16). See Anderson-McCharen [7]. In [14] Chapman generalizes theorem 3.14 for non-separable Fréchetspaces F with $F \cong F^{\infty}$. The proof given in this section does not generalize to the non-separable case, due to the absence of suitable coverings for the manifold, i.e. such as given in lemmas 3.9 and 3.13.

We suppose throughout that M is connected.

Some theorems we shall need:

3.2 THEOREM (Anderson [1], Anderson-Bing [5]; Kadec [17], [18]) All infinite-dimensional separable Fréchetspaces are homeomorphic to l_2. In particular $l_2 \cong s$.

3.3 THEOREM (Schori [19]) If $F \cong Q$ or if F is a metric topological vector space such that $F \cong F^{\infty}$, and if M is an F-manifold, then $M \cong M \times F$. *)

3.4 THEOREM (Henderson [16]) If F is a metric topological vector space such that $F \cong F^{\infty}$ and if M is an F-manifold then M can be embedded in F as an open subset.

3.5 A homeomorphism $h : X \to X$ is said to have support on $A \subset X$ if $\forall x \notin A$, $h(x) = x$.

*) It is conjectured that $F \cong F^{\infty}$ for all infinite-dimensional Banachspaces F. Bessaga and Pelczynski proved that $F \cong F^{\infty}$ for all infinite-dimensional Hilbert spaces F.

3.6 CONVERGENCE PROCEDURE If $\mathcal{U}$ is a countable star-finite open cover of M, then there exists an ordering (U_i) of $\mathcal{U}$ such that, given a sequence of homeomorphisms $(h_i)_i$ with $h_i : \bar{h}_{i-1}(M) \to \bar{h}_i(M)$, and $h_i|\bar{h}_{i-1}(M)$ has support on U_i then the left product $L\Pi_i h_i$ exists and is a homeomorphism onto $\cap_i \bar{h}_i(M)$.

Proof: Suppose $\mathcal{U}$ is a countable star-finite open cover of M. Let $U_1 \in \mathcal{U}$ be arbitrary. Let $\mathcal{U}_1 = \{U \in \mathcal{U} | U \neq U_1 \wedge U \cap U_1 \neq \emptyset\}$ and inductively $\mathcal{U}_{n+1} = \{U \in \mathcal{U} | U \notin \bigcup_{k \leq n} \mathcal{U}_k \wedge U \cap \mathcal{U}_n \neq \emptyset\}$. Because of star-finiteness of $\mathcal{U}$ all sets $\mathcal{U}_n$ are finite, and because of connectedness of M the $\mathcal{U}_n$ exhaust $\mathcal{U} \setminus \{U_1\}$. Thus let $(U_i)_{i>1}$ be an enumeration of successively $\mathcal{U}_2, \mathcal{U}_1, \mathcal{U}_4, \mathcal{U}_3, \mathcal{U}_6, \mathcal{U}_5, \mathcal{U}_8, \ldots$ This is an enumeration as desired: suppose $(h_i)_i$ is a sequence of homeomorphisms as stated. Now it is easily seen that for every point x of M there exists an i_0 and a neighborhood O_x of x such that $L\Pi h_i | O_x = \bar{h}_{i_0} | O_x$, from which it follows that $L\Pi_i h_i$ is a homeomorphism.

3.7 In the following F = s or F = Q. Suppose M is an F-manifold. A closed set K in M × F is called infinitely deficient if $\pi_F(K)$(with π_F denoting projection onto F) is infinitely deficient. In proving topological infinite deficiency for Z-sets K in M × F (i.e. proving that K can be mapped onto a set of infinite deficiency by a space homeomorphism), we look for an open covering $\{U_i\}_i$ as in 3.6, such that $Bd(U_i)$ is a Z-set in $Cl(U_i)$, and such that $Cl(U_i) \approx F$. Paracompactness gives us a closed refinement $\{V_i\}_i$ with $V_i \subset U_i$. The homeomorphism extension theorem for Z-sets in F implies that we can extend a homeomorphism h_i from $K \cap (V_i \times F)$ onto a suitable infinitely deficient subset of $V_i \times F$, to an autohomeomorphism of $Cl(U_i) \times F$ which is the identity on $Bd(U_i \times F) = Bd(U_i) \times F$. Such a homeomorphism can be extended identically to all of M × F. The part of h_i that constitutes our starting point will be of the form

$h_i(x,y) = (x,\Phi_{\phi(x)}(y))$ for (x,y) a point of some specified Z-set in $Cl(U_i)\times F$, where $\phi : Cl(U_i) \to [0,1]$ is a continuous function and Φ an isotopy defined on all of Q, of which only a small part will be used. Φ is such that for every $t > 0$, $\Phi_t(Q)$ is infinitely deficient.

Description of Φ

We shall write $Q \times Q$ instead of Q. For $t = 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ $\Phi_t(x;y)$ is defined as follows:

$$\Phi_0(x;y) = (x;y) = (x_1,x_2,x_3,x_4,\ldots ;y_1,y_2,y_3,y_4,\ldots\ldots)$$

$$\ldots\ldots$$

$$\ldots\ldots$$

$$\Phi_{\frac{1}{6}}(x;y) = (x_1,x_2,x_3, 0, 0, 0,x_4,\ldots; y_1,y_2,y_3, 0, 0,\ldots)$$

$$\Phi_{\frac{1}{5}}(x;y) = (x_1,x_2,x_3, 0, 0,y_3,x_4,\ldots; y_1,y_2, 0, 0, 0,\ldots)$$

$$\Phi_{\frac{1}{4}}(x;y) = (x_1,x_2, 0, 0,x_3,y_3,x_4,\ldots; y_1,y_2, 0, 0, 0,\ldots)$$

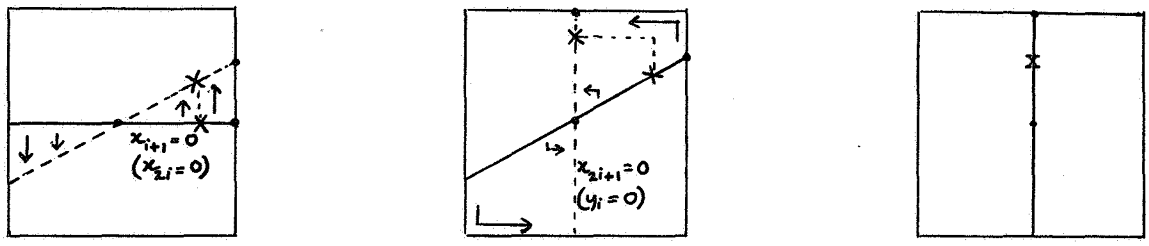
$$\Phi_{\frac{1}{3}}(x;y) = (x_1,x_2, 0,y_2,x_3,y_3,x_4,\ldots; y_1, 0, 0, 0, \ldots\ldots)$$

$$\Phi_{\frac{1}{2}}(x;y) = (x_1, 0,x_2,y_2,x_3,y_3,x_4,\ldots; y_1, 0, 0, 0, \ldots\ldots)$$

$$\Phi_1(x;y) = (x_1,y_1,x_2,y_2,x_3,y_3,x_4,\ldots; 0, 0, 0, 0, \ldots\ldots)$$

The transition from $\Phi_{\frac{1}{2i}}$ to $\Phi_{\frac{1}{2i+1}}$ is performed by a suitable rotation in the x_{i+1}-x_{2i+1}-plane, the transition from $\Phi_{\frac{1}{2i-1}}$ to $\Phi_{\frac{1}{2i}}$ by a rotation in the x_{2i}-y_i-plane. Note that the projection of $\Phi_{\frac{1}{2i}}(Q\times Q)$ onto the x_{i+1}-x_{2i+1}-plane consists of the coordinate axis $x_{i+1} = 0$ and the projection of $\Phi_{\frac{1}{2i-1}}(Q\times Q)$ onto the x_{2i}-y_i-plane of the coordinate-axis $y_i = 0$.

fig. 3.1



3.8 REMARK Observe that $y_i = 0 \Rightarrow$ for all $t\in[0,1]$ the y_i-coordinate of $\Phi_t(x,y)$ is zero.

<u>The case F = Q</u>

3.9 LEMMA <u>Suppose M is a Q-manifold. Then there exists a countable open star-finite covering $\mathcal{U}$ of M such that for all $U\in\mathcal{U}$, $(Cl(U), Bd(U))$ is homeomorphic to Q together with a finite union of endfaces. Moreover, $\mathcal{U}$ can be constructed as a refinement of any given open cover.</u>

Proof: Let $\{O_i\}_i$ be an open cover of M with sets homeomorphic to open subsets of Q; if desired $\{O_i\}_i$ is a refinement of some given open cover. Let $\{O_i'\}_i$ be a star-finite refinement and $\{F_i\}_i$ be a cover of M with compact sets, such that for all i, $F_i\subset O_i'$. Now each F_i can be covered by a finite number of basic open sets $U_{i,1},\ldots, U_{i,k_i}$ with closures in O_i', such that $(Cl(U_{i,j}),Bd(U_{i,j}))$ is homeomorphic to Q together with a finite union of endfaces. 'Basic' means: in the coordinatization of O_i', derived from its open embedding in Q, $U_{i,j}$ is a product of open subintervals Y_n of $[-1,1]$ such that for at most finitely many n, $Y_n \neq [-1,1]$. Then $\mathcal{U} = \{U_{i,j}\}_{i,j}$ is the desired cover.

3.10 THEOREM <u>Suppose M is a Q-manifold and K is a Z-set in $M \times Q$. Then there exists an onto homeomorphism $h : M \times Q \to M \times Q$ such that $h(K)$ is infinitely deficient.</u>

Proof: We shall write $Q \times Q$ instead of Q and also $K\subset M\times Q\times Q$ (which is, by (3.3), homeomorphic to M). Let the covering $\mathcal{U} = \{U_i\}_i$ of M be as in lemma 3.9 and suppose the ordering is as in convergence procedure 3.6. Because of paracompactness of M there exists a closed refinement $\{V_i\}_i$ with $V_i \subset U_i$. Define $U_i' = U_i \times Q \times Q$ and $V_i' = V_i \times Q \times Q$. Then the cover $\{U_i'\}_i$ also possesses the properties listed in lemma 3.9 and in convergence procedure 3.6. Choose $\phi_1 : Cl(U_1) \to [0,1]$ such that $\phi_1(Bd(U_1)) = \{0\}$ and $\phi_1(V_1) = \{1\}$. We construct $h_1 : M \times Q \times Q \to M \times Q \times Q$ as follows : for $(m,(x,y)) \in Bd(U_1') \cup (K \cap U_1')$, define $h_1(m,(x,y)) = (m, \Phi_{\phi_1(m)}(x,y))$. Thus $h_1(K \cap V')$ projects onto a single point in the second Q-factor and $h_1(K \cap U_1')$ onto a countable union of sets of infinite deficiency. Hence $h_1(Bd(U_1') \cup (K \cap U_1')) = Bd(U_1') \cup h_1(K \cap U_1')$ is a closed countable union of Z-sets in $Cl(U_1')$ and therefore a Z-set (2.6). As in 2.2 it is seen that $h_1|Bd(U_1') \cup (K \cap U_1')$ is one-to-one. Bicontinuity is obvious. Since $h_1|Bd(U_1') \cup (K \cap U_1')$ is a homeomorphism between Z-sets in $Cl(U_1')$, we can extend h_1 to an autohomeomorphism of $Cl(U_1')$, and because $h_1|Bd(U_1') = id$, we may extend h_1 identically to the remainder of $M \times Q \times Q$.

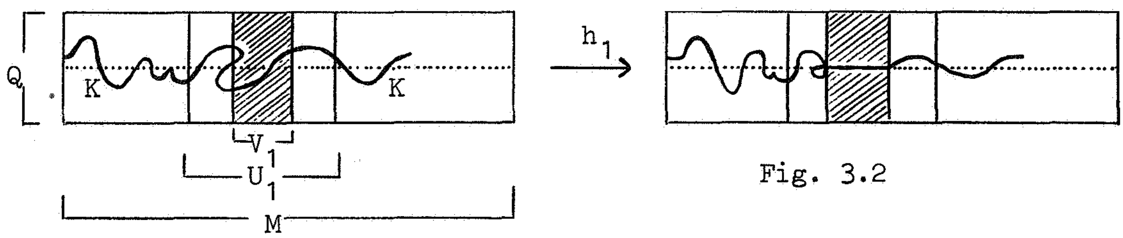


Fig. 3.2

The next step is essentially the inductive step: choose ϕ_2 : $Cl(U_2) \to [0,1]$ such that $\phi_2(Bd(U_2)) = \{0\}$ and $\phi_2(V_2) = \{1\}$. For $(m,(x,y)) \in Bd(U_2') \cup (h_1(K) \cap U_2')$, let $h_2(m,(x,y)) = (m, \Phi_{\phi_2(m)}(x,y))$, and extend h_2 as above to an autohomeomorphism of $M \times Q \times Q$.
If we construct $h_3, h_4, \ldots$ in a similar fashion, then we get, according to 3.6, a sequence of homeomorphisms whose infinite left product $h = L\Pi_i h_i$ is an autohomeomorphism of $M \times Q \times Q$ that maps K into the set $M \times Q \times \{0\}$. To become convinced of this, observe that for every i and every point (m,x,y) of $\bar{h}_{i-1}(K)$, $h_i(m,x,y) =$

$(m,\Phi_{\phi_i(m)}(x,y))$. Hence the M-coordinate of a point of K remains the same all the time. So suppose $(m,x,y)\in K$ and write $\bar{h}_i(m,x,y) = (m,x^{(i)},y^{(i)})$. For some i, $m\in V_i$. Then also $\bar{h}_{i-1}(m,x,y) = (m,x^{(i-1)},y^{(i-1)})\in V_i \times Q \times Q$, thus $h_i(m,x^{(i-1)},y^{(i-1)}) = (m,\Phi_1(x^{(i-1)},y^{(i-1)}))$, i.e. $y^{(i)} = 0$. Because of remark 3.8, for all $j\geq i$ $y^{(j)} = 0$ and hence $h(m,x,y)\in M \times Q \times \{0\}$.

3.11 THEOREM <u>Let M be a Q-manifold and $\bigcup_i K_i$ a countable union of Z-sets in $M \times Q \cong M$. Then there exists an autohomeomorphism h of $M \times Q$ such that $h(\bigcup_i K_i)$ is the union of countably many infinitely deficient sets</u>.

Proof: Let $\{U_i'\}_i = \{U_i \times Q \times Q\}_i$ be as in the proof of theorem 3.10. We construct h as the left product of a sequence of autohomeomorphisms $(h_i)_i$ with support on U_i' that make $\bigcup_j K_j \cap U_i'$ σ-infinitely deficient. In its turn h_i is the left product of a sequence of homeomorphisms $(\chi_{i,j})_j$. We can suppose that $K_1\subset K_2\subset K_3\subset ..$

Choose $\phi_{1,1} : Cl(U_1) \to [0,1]$ zero on $Bd(U_1)$ and positive on U_1 itself. Define $\chi_{1,1}$ by $\chi_{1,1}(m,x,y) = (m,\Phi_{\phi_{1,1}(m)}(x,y))$ for $(m,x,y)\in(K_1\cap U_1')\cup Bd(U_1')$ and extend $\chi_{1,1}$ in the usual way to an autohomeomorphism of $Cl(U_1')$. Then $\chi_{1,1}(K_1\cap U_1')$ is σ-infinitely deficient, even in the following sense: for each point $(m,x,y)\in K_1\cap U_1'$, almost all y-coordinates of $\chi_{1,1}(m,x,y)$ are zero.

Because we are working in a copy of the Hilbert cube, the convergence criterion is the obvious device to ensure convergence of the product of $(\chi_{1,j})_j$. The homeomorphism $\chi_{1,1}$ could be made arbitrarily small by taking $\phi_{1,1}$ sufficiently close to zero, and by restricting the size of the extension of $\chi_{1,1}$ with help of the corollary to theorem 2.7. Thus the inductive construction of $\chi_{1,j}$ is simply as follows:

choose ε such that Φ_ε is as small as necessary in view of the convergence criterion. Choose $\phi_{1,j} : Cl(U_1) \to [0,\varepsilon]$ such that $\phi_{1,j}(m) = 0 \Leftrightarrow m\in Bd(U_1)$. Define, for $(m,x,y)\in\overline{\chi}_{1,j-1}((U_1'\cap K_j)\cup Bd(U_1'))$,

$\chi_{1,j}(m,x,y) = (m,\Phi_{\phi_{1,j}(m)}(x,y))$. Extend $\chi_{1,j}$ in the usual way to a sufficiently small autohomeomorphism of $Cl(U_1')$. Now $h_1 = \amalg_j \chi_{1,j}$ exists and, by remark 3.8, maps $\cup_j K_j \cap U_1'$ into $\{(m,x,y) \mid$ for almost every $n, y_n = 0\}$; hence $h_1(\cup_j K_j \cap U_1')$ is σ-infinitely deficient. Finally, extend h_1 identically to the remainder of $M \times Q \times Q$. For the inductive step, we construct in exactly the same way the homeomorphism h_i such that h_i maps $\bar{h}_{i-1}(\cup_j K_j) \cap U_i'$ into $\{(m,x,y) \mid$ for almost every n, $y_n = 0\}$. In doing this, it may happen that for some point $p \in U_i' \cap \cup_j K_j$ and some $n \in \mathbb{N}$, the y_n-coordinate of $\bar{h}_i(p)$ is zero whereas the y_n-coordinate of $\bar{h}_{i+1}(p)$ is not zero any more. But there exists an i_0 such that $\forall i \geq i_0$ $\bar{h}_i(p) = \bar{h}_{i_0}(p)$ and for this point almost all y_n-coordinates are zero. Hence $\amalg_i h_i$ maps $\cup_j K_j$ onto a countable union of sets of infinite deficiency.

3.12 The case $F = s \cong l_2$

The general ideas of the proofs of theorems 3.10 and 3.11 are the same, although the proof of topological σ-infinite deficiency of σ-Z sets is much more complicated. It is mainly included for completeness. The proof of Chapman [13] is much better for this case. Lemma 3.13, which corresponds to lemma 3.9, is proved with a rather different argument. Throughout Q is used in the compactification of s and s-manifolds.

3.13 LEMMA <u>Suppose M is an s-manifold. Then there exists an embedding i of M in Q and an open subset $\tilde{M}$ of Q with $iM \subset \tilde{M} \subset Cl(M)$ and a star-finite open cover $\mathcal{U}$ of $\tilde{M}$ such that for all $U \in \mathcal{U}$, $(Cl(U), Bd(U))$ is homeomorphic to Q together with a finite union of endslices.</u>

Proof: Let $i : M \hookrightarrow s \subset Q$ be an arbitrary open embedding of M in s (cf. theorem 3.4). Define $\tilde{M} = Q \setminus (Cl_Q(s \setminus iM))$, the largest open subset of Q whose intersection with s is iM. Write $\tilde{M} = \cup_i A_i$, $A_i = Cl(A_i) \subset A_{i+1}^{o}$ for all i. (In the sequel closures and interiors are taken relative to Q). Cover A_1 with finitely many basic open sets (for the precise definition of basic, see 3.9) whose closures are contained in $\overset{\circ}{A}_2$, and cover each set $A_{i+1}\setminus\overset{\circ}{A}_i$ by a finite number of basic open sets whose closures are contained in $\overset{\circ}{A}_{i+2}\setminus A_{i-1}$ (for $i = 1$ in $\overset{\circ}{A}_3$). Then the collection of all these basic open sets constitutes the desired cover.

3.14 THEOREM <u>For M an s-manifold and K a Z-set in $M \times s$ there exists a homeomorphism $h : M\times s \xrightarrow{\text{onto}} M\times s$ such that h(K) is infinitely deficient.</u>

Proof: The same techniques as in 3.10 can be used, as follows from the following observations:

1° For every $U\in\mathcal{U}$, with $\mathcal{U}$ as given by 3.13, instead of 3.9, the set $Bd(U)\cap s$ is a Z-set in $Cl(U)\cap s$, and the same holds for $(Bd(U)\cap s)\times s$ relative to $(Cl(U)\cap s)\times s$. Furthermore $Cl(U)\cap s$ is homeomorphic to s, which is not proved in this note. See e.g. [2] for a proof.

2° For every $t\in[0,1]$ and every $x\in Q \times Q$, $\Phi(x,t)\in B(Q\times Q) \Leftrightarrow x\in B(Q\times Q)$. Now $\Phi_t : Q \times Q \to Q \times Q$ is a closed map because of compactness of $Q \times Q$. Then, for K a closed subset of s, also $\Phi_t(K) = \Phi_t(Cl_Q(K)\cap s) = \Phi(Cl(K))\cap s$ is a closed subset of s. Hence $\Phi_t|s\times s$ is closed and from this bicontinuity of $h_1|Bd(U_1')\cup(K\cap U_1')$ (U_1' as defined in 3.10) is easily derived, so that the homeomorphism extension theorem for s (the corollary to theorem 2.7) can be applied.

3.15 THEOREM <u>If M is an s-(or l_2-)manifold and $\cup_j K_j$ a countable union of Z-sets in $M \times s$ (or $M \times l_2$) then there exists an autohomeomorphism h of $M \times s$ (or $M \times l_2$) such that $h(\cup_j K_j)$ is the union of countably many sets of infinite deficiency.</u>*)

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*) From the characterization of infinite deficiency by property Z for sand l_2 it follows that topological infinite deficiency in s coincides with topological infinite deficiency in l_2.

Proof: We write $s \cong s \times s$ and consider the open subset $\tilde{M} \times Q \times Q$ of $Q \times Q \times Q$ with $\tilde{M}$ as in 3.13. We shall perform similar constructions as in 3.11, except that care has to be taken that we get a β^*-homeomorphism, for only in that case the restriction to $M \times s \times s$ becomes an autohomeomorphism of $M \times s \times s$.

We use a cover $\mathcal{U} = \{U_i\}_i$ as in 3.13, with an ordering as in 3.6. We suppose that $K_1 \subset K_2 \subset \ldots$ The closures $K_1', K_2', \ldots$ are Z-sets in Q (theorem 2.8). Define $U_1' = U_i \times Q \times Q$. Let the union of endfaces $G_j \subset Q \times Q$ be defined by $\{(x,y) | \exists j' \leq j : |x_{j'}| = 1 \text{ or } |y_{j'}| = 1\}$. Let $(t_j)_j$ be a monotonously decreasing sequence of positive real numbers such that for $t' \leq t_j$, $\Phi_{t'}$ does not affect the coordinates which define G_j. Let $(F_j^*)_j$ be an enumeration of the endfaces of $Cl(U_i)$, let $F_j = F_1^* \cup \ldots \cup F_j^*$ and define $G_j' = Cl(U_i) \times G_j$ and $F_j' = F_j \times Q \times Q$. (In these notations the index i is suppressed because we need them for only one i at a time.) Previously we constructed homeomorphisms h_i and $\chi_{i,j}$ by starting with a homeomorphism $(m,x,y) \mapsto (m, \Phi_{\phi(m)}(x,y))$, defined on some subset of the manifold. The function ϕ was chosen dependent on m only, in order to ensure one-to-one-ness of the resulting homeomorphism. Obviously this procedure also works if ϕ depends on $m, x_1, \ldots, x_j, y_1, \ldots, y_j$ and assumes only values smaller than t_j. So in the sequel we shall assume that real-valued functions $\phi_{i,j}(m,x,y)$ are into $[0,t_j)$ and depend on $m, x_1, \ldots, x_j, y_1, \ldots, y_j$ only.

Choose $\phi_{1,1} : Cl(U_1') \to [0,1]$ with $\phi_{1,1}(Bd(U_1') \cup F_1' \cup G_1') = \{0\}$ and $\phi_{1,1}(U_1' \setminus (F_1' \cup G_1')) > 0$ and define $\chi_{1,1}(m,x,y) = (m, \Phi_{\phi_{1,1}(m,x,y)}(x,y))$ for $(m,x,y) \in (K_1' \cap U_1') \cup Bd(U_1') \cup F_1' \cup G_1'$. According to corollary 2.7, $\chi_{1,1}$ can be extended to a β^*-autohomeomorphism of $Cl(U_1')$. Thus we get an autohomeomorphism of $Cl(U_1')$ which leaves $F_1' \cup G_1'$ pointwise fixed and maps $(K_1' \cap U_1') \setminus (F_1' \cup G_1')$ onto a countable union of sets of infinite deficiency, just as in 3.11.

The next step will be essentially the same as the inductive step. Choose $\phi_{1,2} : Cl(U_1') \to [0,1]$ with $\phi_{1,2}(Bd(U_1')\cup F_2'\cup G_2') = \{0\}$ and $\phi_{1,2}(U_1'\setminus(F_2'\cup G_2'))\subset(0,1)$ sufficiently small in view of the convergence criterion and define $\chi_{1,2}(m,x,y) = (m,\Phi_{\phi_{1,2}(m,x,y)}(x,y))$ for $(m,x,y)\in\chi_{1,1}(K_2'\cap U_1')\cup Bd(U_1')\cup F_2'\cup G_2'\cup\chi_{1,1}(F_2'\cup G_2')$. Extend $\chi_{1,2}$ to a sufficiently small β^* onto autohomeomorphism of $Cl(U_1')$. Let $g_1 : Cl(U_1') \to Cl(U_1')$ be the left product of the homeomorphisms $\chi_{1,j}$ thus constructed. Then g_1 maps $\cup_j K_j\cap U_1'$ into a countable union of sets of infinite deficiency.

Since $\phi_{1,n}(m,x,y) = 0$ if $(m,x,y)\in F_n'\cup G_n'$, the set $K_n'\cap\overline{\chi}_{1,n-1}^{-1}(F_n'\cup G_n')$ is not mapped onto a (σ-) infinitely deficient set by $\overline{\chi}_{1,n}$. However, for each n $g_1(K_n'\setminus\overline{\chi}_{1,n-1}^{-1}(F_n'\cup G_n')$ is σ-infinitely deficient. But $\overline{\chi}_{1,n}$ maps the set $K_n'\cap\overline{\chi}_{1,n-1}^{-1}(F_n'\cup G_n')$ on a compact subset of the pseudo-boundary, so that $g_1(K_n'\setminus\overline{\chi}_{1,n-1}^{-1}(F_n'\cup G_n'))$ is indeed an F_σ-set, as the definition of σ-infinite deficiency implies.

Furthermore, $g_1(B(Cl(U_1')))\supseteq B(Cl(U_1'))$ because $g_1(\overline{\chi}_{1,n-1}^{-1}(F_n'\cup G_n')) = F_n'\cup G_n'$ and $B(Cl(U_1')) = \bigcup_n(F_n'\cup G_n')$; and $g_1(B(Cl(U_1')))\setminus B(Cl(U_1')) = \bigcup_n g_1(F_n'\cup G_n')\setminus B(Cl(U_1'))$ is σ-infinitely deficient. Call this set C. Then, by theorem 2.3 and remark 2.4, there exists an autohomeomorphism f_1 of $Cl(U_1')$ that maps $B(Cl(U_1'))\cup C$ onto $B(Cl(U_1'))$ in such a way that $Bd(U_1')$ stays pointwise fixed and $f_1g_1(\cup_j K_j\cap U_1')$ remains σ-infinitely deficient. Now let h_1 be the restriction of f_1g_1 to $(U_1\cap s)\times s\times s$, then $h_1(\cup_j K_j\cap U_1\cap s)$ is σ-infinitely deficient. Next extend h_1 identically to the remainder of $M\times s\times s$ and finally the proof is concluded as in 3.11.

3.16 Theorem 3.14 is the clue to the following result:

<u>THEOREM If K_1 and K_2 are Z-sets in an l_2-manifold M and $h : K_1 \xrightarrow{onto} K_2$ is a homeomorphism homotopic to the identity then h can be extended to an (onto) autohomeomorphism of M isotopic to the identity.</u>

The homotopy condition cannot be omitted, even if isotopy is dropped in the conclusion; let e.g. $M \approxeq S^1 \times l_2$($S^1$ is the 1-sphere) then a homeomorphism from $S^1 \times \{0\}$ onto a contractible 1-sphere in M cannot be extended to a space homeomorphism.

This theorem is proved by first replacing the homotopy from the identity to h by an isotopy $H : K_1 \times I \to M$ which is an embedding of $K_1 \times I$ as a Z-set. Here topological infinite deficiency is used. Thereafter M is embedded as an open subset in s, such that $H(K_1 \times I)$ is mapped onto a closed set (hence a Z-set). Next, using the homeomorphism extension theorem for Z-sets in s, $H(K_1 \times I)$ is brought in a nice position, such that it is easy to extend H to an isotopy with support on the embedding of M. For more details, see Anderson-McCharen [7].

A corresponding theorem for Q-manifolds holds only with certain qualifications. See also Anderson-Chapman [6].

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