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HOMOTOPY CLASSIFICATION OF TYPE $(\Sigma c, 1)$ ANR AND APPLICATION TO PERIODIC ACTIONS ON (I-D) SPACES

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HOMOTOPY CLASSIFICATION OF TYPE (Zq, 1) ANR and APPLICATION to PERIODIC ACTIONS on (I-D) SPACES

Raymond Y.T. Wong *)

1. The purpose of this paper is to prove a homotopy classification theorem (Theorem 1) for ANR and to outline some of its consequences which, using a different lemma, are results already mentioned in [12]. Let Zq denote the integers modulo q, $q \geq 1$ ($Z_1 = \{0\}$). A connected. locally path connected metric space X is said to be an Eilenberg-Maclane space of type ($\mathbb{Z}q$, 1), or simply, of <u>type</u> ($\mathbb{Z}q$, 1), provided the fundamental group $\pi_1(X)$ is isomorphic to $\mathbb{Z}q$ and $\pi_n(X) = \{0\}$ for all $n > 1$. Let E denote a fixed (but arbitrary) infinite-dimensional (I-D) normed linear space (NLS) which is homeomorphic ($\cong$) to F^ω or F^ω_f for some NLS F, where F^ω denotes the countable infinite product of F by itself and $F^\omega_f \subset F^\omega$ denotes the subset consisting of all points having at most finitely many non-zero coordinates. The following is our main theorem which classifies, up to homotopy type, all metric absolute neighborhood retracts (ANR) of type (Zq, 1).

<u>Theorem 1.</u> <u>Let</u> Y, Y' <u>be metrizable connected</u> ANR <u>of type</u> ($\mathbb{Z}q$, 1) <u>and let</u> $e \in \pi_1(Y)$, $e' \in \pi_1(Y')$ <u>be generators. Then there is a homotopy</u> equivalence $h : Y \to Y'$ such that $h_\#(e) = e'$.

The case for q = 1 is rather well known (see for example, the Collorary following Theorem 15 of Palais ([7])). This Theorem 1 may be viewed as a generalization of it. With this in mind, we assume from here on that $q > 1$. It is not well known that E-manifolds can be classified by their homotopy types ([4],[5]) and the same is true in the C^∞-category for separable C^∞- Hilbert manifolds ([3], [6]). We

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Proposition 1.

(A) Each homotopy equivalence between E-manifolds is homotopic to a homeomorphism.

(B) Each homotopy equivalence between separable C^∞-Hilbert manifolds is homotopic to a C^∞-diffeomorphism.

Since all E-manifolds are ANR, applying Theorem 1 and Proposition 1 we obtain the following theorem which classifies all metrizable connected E-manifolds (or C^∞-Hilbert manifolds) of type $(\mathbb{Z}q, 1)$.

Theorem 2. (Classification) Let M and M_1 be metrizable E-manifolds of type $(\mathbb{Z}q, 1)$ and let $e \in \pi_1(M_1)$, $e_1 \in \pi_1(M_1)$ be generators. Then there is a homeomorphism $h : M \to M_1$ such that $h_\#(e) = e_1$.

Let l_2 denote the separable Hilbert space of all square summable complex sequences and S denote its unit sphere. For any $q > 1$, define a fixed point free periodic homeomorphism $\alpha : S \to S$ of period q by

$$\alpha(z_0, z_1, \ldots) = (e^{2\pi i/q} z_0, e^{2\pi i/q} z_1, \ldots).$$

Then α induces (by restrictions) periodic homeomorphisms $\alpha_n : S^{2n-1} \to S^{2n-1}$ of period q, where S^{2n-1} is the unit sphere of the 2n-dimensional complex space C^n. The inductive limit of $\{S^{2n-1}/\alpha_n\}_{n\geq 1}$ (S^{2n-1}/α_n the orbit spaces), denoted by $\varinjlim S^{2n-1}/\alpha_n$, is a CW-complex of type $(Zq, 1)$. Hence, by means of Theorem 1, we obtain

Theorem 3. Let M be a metrizable connected E-manifold of type $(\mathbb{Z}q, 1)$, then M has the same homotopy as $\varinjlim S^{2n-1}/\alpha_n$.

Let M be as above with $q > 1$ a prime number. The universal covering space $\tilde{M}$ of M is a homotopically trivial E-manifold such that the projection $p : \tilde{M} \to M$ is a q-folds covering map. By Proposition 1(A), $\tilde{M} \cong E$. Let $\beta : \tilde{M} \to \tilde{M}$ be any fixed point free period q homeomorphism (β always exists, see [9]). Then the orbit space $\tilde{M}/\beta$ is an E-manifold of type $(Zq, 1)$. By Theorem 2 there is a homeomorphism $h : \tilde{M}/\beta \to M$ which then induces a fibre homeomorphism $h_* : \tilde{M} \to \tilde{M}$. Let $\beta_* = h_* \circ \beta \circ h_*^{-1}$. We obtain the following theorem.

Theorem 4. (Representation) *Let M be a metrizable connected E-manifold of type* ($\mathbb{Z}q$, 1), $q > 1$ *a prime number. Then there is a q-folds covering projection* $p : E \to M$ *and a fixed point free periodic homeomorphism* $\beta_*: E \to E$ *of period* q *such that* β_* *induces a homeomorphism* $\beta_0 : E/\beta_0 \to M$ *for which* $\beta_0 \circ p_0 = p \circ \beta_*$.

Added in proof. For the sake of completion we mention here that Theorem 1 is true for $q = 0$ ($\mathbb{Z}_0 = \mathbb{Z}$) and it is not difficult to show (using the universal covering space of Y and Lemma 1 of this paper) that Y has the homotopy type of a circle.

2. Aplication to periodic homeomorphisms and other results

Throughout this section let $q > 1$ denote a prime number.

Theorem 5. (Conjugation) *Let* $\beta, \beta_1 : E \to E$ *be fixed point free periodic homeomorphisms of period* q. *Then there is a homeomorphism* $h_0 : E \to E$ *such that* $h_0 \circ \beta = \beta_1 \circ h_0$.

Moreover, if $E = l_2$ *and* β, β_1 *are* C^∞*-smooth, we may choose* h_0 *to be a* C^∞*-diffeomorphism.*

Proof. The C^0 case. Let $b \in E$ and suppose $\lambda, \lambda_1 : ([0,1]) \to (E,b)$ are maps (preserving base points) such that $\lambda(1) = \beta(b)$ and $\lambda_1(1) = \beta_1(b)$. Let $p : E \to E/\beta$, $p_1 : E \to E/\beta_1$ denote the projections. Then $e = [p\circ\lambda] \in \pi_1(E/\beta)$ and $e_1 = [p_1\circ\lambda_1] \in \pi_1(E/\beta_1)$ are generators. It follows from theorem 2 that there is a homeomorphism $h : (E/\beta, p(b)) \to (E/\beta_1, p_1(b))$ such that $h_\#(e) = e_1$. The function h then induces a (fibre) homeomorphism $h_0 : (E,b) \to (E,b)$ such that $p_1 \circ h_0 = h \circ p$ and $h_0 \circ \beta(b) = \beta_1 \circ h_0(b)$. For each $x \in E$, since $\{h_0(x), h_0 \circ \beta(x)\} \subset p_1^{-1}(h\circ p(x))$, there is an $1 \le i \le q$ for which $h_0 \circ \beta(x) = \beta_1^i \circ h_0(x)$. Let $A_i = \{x \in E : h_0 \circ \beta(x) = \beta_1^i \circ h_0(x)\}$. We easily verify that each A_i is closed and $\{A_i\}$ are pairwise disjoint. Since E is connected and $A_1 \neq \emptyset$, hence $A_1 = E$. The C^∞ case follows exactly the same considerations using Theorem 1 and Proposition 1(B).

The above theorem is our principle application. In the following we state, without proof, several other consequences which are essentially corollaries of Theorem 5. We refer to [12] for their proofs. Suppose $X \cong X \times E$, a subset Y of X is said to be E-deficient if there is a homeomorphism $h : X \to X \times E$ such that $h(Y) \subset X \times \{0\}$. Let H denote the Hilbert space of all square complex sequences indexed by an infinite abstract set $I(H)$. Note that $H \cong H^{\omega}$ ([1]).

Theorem 6. (Homeomorphism Extension) *Let $A \subset H$ be a closed H-deficient subset. Then each period n homeomorphism $\beta : A \to A$ extends to a period n homeomorphism $\tilde{\beta}$ on H such that $\tilde{\beta}(x) = x$ if and only if $\beta(x) = x$.*

The proof of Theorem 6 is independent of Theorem 5 and is essentially an elementary application of [2.- Theorem 1]. Note that in [12 - Theorem 7] we assume n is a prime, which is irrelevant.

Theorem 7. (Closed Imbeddings) *Suppose X is a space which can be imbedded as a closed subset of a Hilbert space H. Then for any two fixed point free period q homeomorphisms β, β_1 on X, H respectively, there is a closed imbedding $m : X \to H$ satisfying $m \circ \beta = \beta_1 \circ m$.*

Moreover, if X is a connected H-manifold, we may choose m so that m(X) is a submanifold of H.

Theorem 8. (Negligible Subsets) *Let $K_1, K_2, \dots$ be closed H-deficient subsets of H. Suppose $\beta, \beta_1 : H \to H$ are fixed point free periodic homeomorphisms of period q for which $\beta(K) = K$, where $K = \bigcup_{i \geq 1} K_i$, then there is a homeomorphism: $H \to H \backslash K$ satisfying $m \circ \beta = \beta_1 \circ m$.*

For any space X, let G(X) denote the space of homeomorphisms on X (of X onto itself) equipped with the compact-open topology. Note that G(X) is a group under composition. Let $G_0(X)$ denote the subspace consisting of all periodic homeomorphisms and $G_n(X) = \{\beta \in G_0(X) :$ period $(\beta) = n\}$.

Theorem 9. (Homeomorphism spaces are contractible) For $k \geq 0$, each $G_K(E)$ is contractible and there is a contraction $\{\phi_t\} : G(E) \to G(E)$ such that $\{\phi_t|_{G_k(E)}\}$, $k \geq 0$, is a contraction for $G_k(E)$.

In [12 - Corollary 3] it is proved that for $E \cong E^{\omega}$, the group $G(E)$ is simple, in the sense that $G(E)$ contains no non-trivial proper normal subgroup. For each fixed k, the collection of all finite composition of members in $G_k(E)$ clearly forms a non-trivial normal subgroup of $G(E)$. Hence we have

Theorem 10. (Periodic Stability) Suppose $E \cong E^{\omega}$. Then for any $h \in G(E)$ and any $k \geq 0$, there are $h_1, \ldots, h_i \in G_k(E)$ such that $h = h_i \circ \ldots \circ h_2 \circ h_1$.

3. Proof of Theorem 1

We say two maps $f,g : X \to Y$ are homotopic relative $A \subset X$, written $f \sim g$ rel (A), if there is a homotopy $\{\lambda_t\}$ joining f and g such that $\lambda_t(a) = \lambda_0(a)$ for all $a \in A$, $t \in [0,1]$. Let $\alpha : S \to S$ and $\alpha_n : S^{2n-1} \to S^{2n-1}$ be defined as before. To give a proof of Theorem 1, we need

Lemma 1. Let X, Y be metric spaces with X compact. Let $A \subset X$ be closed. Then for each map $g : X \to Y \times l_2$, there is a map $\tilde{g} : X \to Y \times l_2$ such that $\tilde{g} \sim g$ rel (A) and for $x \neq y$, $\tilde{g}(x) = \tilde{g}(y)$ only if $\{x,y\} \in A$.

Proof. (Technically we have to assume $g|_A$ is not one-to-one.) Note that the above statement implies $\tilde{g}|_A = g|_A$. Without loss of generality, we may write $Y \times l_2$ as $Y \times l_2 \times l_2 \times l_2$ and suppose $g(A) \subset Y \times l_2 \times \{0\} \times \{0\}$. Let $h : X \to l_2$ be an imbedding sucht that all coordinates of each $h(x)$ are positive. Let $\lambda : X \to [0,1]$ and $\lambda_1 : Y \times l_2 \to [0,1]$ be maps satisfying $\lambda^{-1}(0) = A$ and $\lambda_1^{-1}(0) = g(A)$. Define $\tilde{g} : X \to (Y \times l_2) \times l_2 \times l_2$ by $\tilde{g}(x) = (g(x), \lambda(x)h(x), \lambda_1(g(x))h(x))$. By the linear structure on l_2, $\tilde{g} \sim g$ rel(A).

Lemma 2. (*The Key Lemma*) *Let* X *be a metric* AR. *Suppose for some metric* ANR Y, *there is a* q-*fold covering projection* $p : X \to Y \times l_2$. *Let* $\lambda : ([0,1], 0) \to (Y \times l_2, b)$ *be a map such that* $[\lambda]$ *generates* $\pi_1(Y \times l_2, b)$. *Denote the lifting* $([0,1], 0) \to (X, b_0)$ *by* $\tilde{\lambda}$. *Let* $b_1 = \tilde{\lambda}(1)$. *Then there are imbeddings* $f_n : (S^{2n-1}, a_0) \to (X, b_0)$ *such that* (1) $f_1 \circ \alpha_1(a_0) = b_1$, (2) *for all* $n \geq 1$, $f_{n+1}|_{S^{2n-1}} = f_n$ *and* (3) $p \circ f_n(x) = p \circ f_n \circ \alpha_n(x)$ *for all* x.

Proof. Exactly the same as Lemma 1 of [12]. Note that the setting in [12] is for covering projection $p : E \to M$. We observe (1) the only property of E we need is E being an AR and (2) Lemma 2 of [12] may be replaced by Lemma 1 of this paper.

Proof of Theorem 1. Fix any $b \in Y$ and $a_0 \in S^1$. The universal covering space X of $Y \times l_2$ (with respect to base point $\tilde{b} = (b, 0)$) is a connected metrizable AR ([7 - Theorem 5 and 15]) for which the projection $p : X \to Y \times l_2$ is a q-folds covering map. Let $b_0 \in p^{-1}(\tilde{b})$ and let $\lambda : ([0,1], 0) \to (Y \times l_2, \tilde{b})$ be a map such that $[\lambda] = j_{\#}(e)$, where $j : Y \to Y \times \{0\} \subset Y \times l_2$ is the inclusion. λ lifts to a map $\tilde{\lambda} : ([0,1], 0) \to (X, b_0)$. Denote $b_1 = \tilde{\lambda}(1)$. Let $f_n : (S^{2n-1}, a_0) \to$ $\to (X, b_0)$ be imbeddings satisfying (1) - (3) of Lemma 2. $\{f_n\}$ induces (in a natural way) one-to-one maps $\tilde{f} : (\varinjlim S^{2n-1}, a_0) \to (X, b_0)$ and $f : \varinjlim (S^{2n-1}/\alpha_n) \to Y \times l_2$ satisfying $\tilde{f} \circ \alpha_1(a_0) = \tilde{b}_0$ and $p \circ \tilde{f} = f \circ p_0$, where $p_0 : \varinjlim S^{2n-1} \to \varinjlim S^{2n-1}/\alpha_n$ is the natural projection. It may be routinely verified that f is a weak homotopy equivalence. Hence by [7 - Theorem 14] and by Whitehead [10 - Theorem 1], f is a homotopy equivalence. Repeating the whole process for Y' we obtain the following diagram:

$$
\begin{array}{ccccccccc}
 & & (X, b_0) & \xleftarrow{\tilde{f}} & (\varinjlim S^{2n-1}, a_0) & \xrightarrow{\tilde{f}'} & (X', b_0') & & \\
 & & \downarrow p & & \downarrow p_0 & & \downarrow p' & & \\
(Y, b) & \xrightarrow{j} & (Y \times l_2, b) & \underset{g}{\overset{f}{\rightleftarrows}} & (\varinjlim S^{2n-1}/\alpha_n, a) & \xrightarrow{f} & (Y' \times l_2, \tilde{b}') & \underset{j_1}{\overset{j'}{\leftrightarrows}} & (Y', b')
\end{array}
$$

where g, j_1 are respectively homotopy inverses of f and j' (j_1 being obtained by shrinking I_2 to 0). In particular, $\tilde{f}'$ satisfies $\tilde{f}' \circ \alpha_1(a_0) = \tilde{b}_0'$, where $\tilde{b}_0'$ is the end point $\tilde{\lambda}'(1)$ of a map $\tilde{\lambda}' : ([0,1], 0) \to (X', b_0')$ for which $[p' \circ \tilde{\lambda}'] = j_{\#}'(e')$. Let $h = j_1 \circ f' \circ g \circ j$. Then

$h \circ \lambda = j_1 \circ f' \circ g \circ j \circ \lambda = j_1 \circ f' \circ g \circ p \circ \tilde{\lambda} \sim j_1 \circ f' \circ g \circ p \circ \lambda 1$,

where $\lambda_1 : ([0,1], 0) \to (X, b_0)$ is a map such that $\lambda_1 \sim \tilde{\lambda}$ rel (0,1) and $\lambda_1([0,1]) \subset f(\lim S^{2n-1})$. Thus

$h \circ \lambda \sim j_1 \circ f' \circ g \circ f \circ p_0 \circ \tilde{f}^{-1} \circ \lambda_1 \sim j_1 \circ f' \circ p_0 \circ \tilde{f}^{-1} \circ \lambda_1 =$

$= j_1 \circ p' \circ \tilde{f}' \circ \tilde{f}^{-1} \circ \lambda_1$. Since $\tilde{f}' \circ \tilde{f}^{-1}(\tilde{b}_0) = \tilde{b}_0'$, it follows that $h_{\#}(e) = e'$.

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