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J. DE VRIES

THE LOCAL WEIGHT OF AN EFFECTIVE LOCALLY COMPACT TRANSFORMATION GROUP

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The local weight of an effective locally compact transformation group

J. de Vries

Abstract. It is known that a locally compact topological group G is metrizable whenever G acts continuously and effectively on a separable metrizable space. In this note we prove a generalization of this fact and, in addition, we give two applications. First, we prove that for any locally compact Hausdorff topological group G the Hilbert dimension of $L^2(G)$ is equal to the weight of G. This was previously known only for compact groups. Second, we are able to give a very simple proof of another known result, namely that the least cardinal number of an approximate unit for $L^1(G)$ is equal to the local weight of G.

1. Notation and conventions

A left semitopological group is a group G endowed with a topology such that the mapping $t \longmapsto ts : G \to G$ is continuous for each $s \in G$. A right semitopological group is defined in an obvious way, and a semitopological group is a group endowed with a topology such that G is both a left and a right semitopological group. A topological group is a semitopological group G in which the function $(s,t) \longmapsto st^{-1}: G \times G \to G$ is continuous. The identity of a group G is always denoted by e_G or e.

A transformation group is a triple (G,X,π) where G is a left semitopological group, X is a topological space, and $\pi : G \times X \to X$ is a function satisfying the following conditions:

(i) $\forall x \in X$: $\pi(e,x) = x$.

(ii) $\forall x \in X$, $\forall (s,t) \in G \times G$: $\pi(s,\pi(t,x)) = \pi(st,x)$.

(iii) π is separately continuous, that is:

$\forall x \in X$, the function $\pi_x: t \longmapsto \pi(t,x): G \to X$ is continuous,

$\forall t \in G$, the function $\pi^t: x \longmapsto \pi(t,x): X \to X$ is continuous.

Observe, that $t \longmapsto \pi^t$ is a morphism of groups from G into the group $G(X)$ of all homeomorphisms of X onto itself. A transformation group (G,X,π) is said to be <u>effective</u> whenever this morphism is injective, that is, if and only if

$$\forall t \in G: t \neq e \Longrightarrow \exists x \in X: \pi(t,x) \neq x.$$

A transformation group (G,X,π) is said to be <u>free</u>, whenever

$$\forall t \in G: t \neq e \Longrightarrow \forall x \in X: \pi(t,x) \neq x.$$

For any topological space X the following cardinal numbers are unambiguously defined (where $|A|$ denotes the cardinality of the set A):

the <u>weight</u> of X:

$w(X) := \min\{|\mathcal{U}| \mid \mathcal{U}$ is an open base for $X\}$;

the <u>local weight of X at</u> $x \in X$:

$\chi(x,X) := \min\{|\mathcal{V}| \mid \mathcal{V}$ is a local base at $x\}$;

the <u>local weight</u> of X:

$\chi(X) := \sup\{\chi(x,X) \mid x \in X\}$,

the <u>density</u> of X:

$d(X) := \min\{|A| \mid A$ is dense in $X\}$

and the <u>Lindelöf degree</u> of X:

$L(X) := \min\{\aleph_0 \mid$ each open covering of X has a subcovering of cardinality $\aleph_0\}$.

It is easy to see that for any topological space X the following inequality holds:

$$(1.1) \qquad w(X) \geq d(X).\ \chi(X).$$

This equality may be strict. A standard example is the Sorgenfrey space S, that is the real line $\mathbb{R}$ endowed with the half-open interval topology: $\chi(S) = d(S) = \aleph_0$, but $w(S) = 2^{\aleph_0} > \aleph_0$.

It follows from [1], Theorem 1.1.6 that for any topological space X,

(1.2) $\quad L(X) \leq w(X)$.

It is an easy exercise to show that in any topological group G the equality

(1.3) $\quad w(G) = \chi(G) \,.\, L(G)$

holds. Notice that here

(1.4) $\quad \chi(G) = \min\{|\mathcal{B}| \mid \mathcal{B} \text{ is a local base at } e\}$,

because G is homogenous. However, left semitopological groups are homogenous as well, so that there (1.4) holds too.

2. Main result

Our starting point is a well-known lemma.

2.1. LEMMA. *Let Y be a topological Hausdorff space and suppose that some $x_0 \in Y$ has a compact neighbourhood. Then $\chi(x_0,Y) \leq |\mathcal{B}|$ for any set $\mathcal{B}$ of neighbourhoods of x_0 such that $\cap\mathcal{B} = \{x_0\}$.*

PROOF. Cf. [3], 28.70(a).

2.2. THEOREM. *Let G denote a locally compact Hausdorff left semitopological group. Then for any effective transformation group (G,X,π) with X a Hausdorff space, the inequality*

(2.1) $$\chi(G) \leq d(X).\ \chi(X)$$

holds. For any free transformation group (G,X,π) with X a T_1-space one has $\chi(G) \leq \min_{x\in X} \chi(x,X)$.

PROOF. Let A be a dense subset of X having cardinality $d(X)$. In addition, for each $x \in X$ let $\mathcal{B}_x$ denote a local base at x with $|\mathcal{B}_x| = \chi(x,X)$. For every $x \in X$ and $V \in \mathcal{B}_x$ there is a neighbourhood $U(x,V)$ of e in G such that $\pi(t,x) \in V$ for each $t \in U(x,V)$, because the mapping $t \longmapsto \pi(t,x): G \to X$ is continuous at e. Now suppose that (G,X,π) is effective and that X is a Hausdorff space. Then

$$(2.2) \qquad \bigcap_{a\in A} \bigcap_{V\in\mathcal{B}_a} U(a,V) = \{e\}$$

Indeed, let $t \in G$, $t \neq e$. For some $a \in X$ we have $\pi^t a \neq a$, and we may assume that $a \in A$ because π^t is a continuous mapping of the Hausdorff space X in itself, and A is dense in X. Now there is some $V \in \mathcal{B}_a$ such that $\pi(t,a) \notin V$. Consequently, $t \notin U(a,V)$, and (2.2) is proved.
It follows from (2.2) and the preceding lemma, that

$$\chi(G) \leq |A|. \sup_{a\in A} |\mathcal{B}_a| \leq d(X). \chi(X).$$

In the case of a free action of G on a T_1-space X, (2.2) may be replaced by

$$\forall a \in X: \bigcap_{V\in\mathcal{B}_a} U(a,V) = \{e\},$$

from which it follows that $\chi(G) \leq \min_{a\in X} |\mathcal{B}_a|$.

2.3 COROLLARY. (Cf. [4], 2.11). A locally compact Hausdorff topological group which acts effectively on a separable first countable Hausdorff space, or which acts freely on a first countable T_1-space is metrizable.

PROOF. In both cases, $\chi(G) \leq \aleph_0$. Now use [3], 8.3.

REMARK. In Section 4 we shall give some examples to show that local compactness and effectiveness in Theorem 1 are essential, even if we consider transformation groups (G,X,π) with G a topological group and π a continuous function on $G \times X$. Also one might ask whether Corollary 2.3 holds for locally compact left semitopological groups. We discuss this problem in Section 4.

3. Applications

Let G be a locally compact Hausdorff topological group. Fix a right-invariant Haar-measure μ on G; for any μ-integrable function f we will write $\int_G f(t)\, dt$ instead of $\int_G f\, d\mu$

Let $L^1(G)$ denote the space of all equivalence classes of complex valued functions on G which are μ-measurable and μ-integrable, and let $L^2(G)$ denote the space of μ-measurable functions f on G such that $|f|^2 \in L^1(G)$. Recall that $L^1(G)$ can be given the structure of a Banach algebra, defining addition and scalar multiplication pointwise, and multiplication by

$$(3.1) \qquad (f*g)(s) = \int_G f(t)\, g(t^{-1}s)\, dt$$

for every $f,g \in L^1(G)$ and μ-almost every $s \in G$ (convolution). The norm on $L^1(G)$ is, as usual, defined by $\|f\|_1 = \int_G |f(t)|\, dt$ $(f \in L^1(G))$.

A <u>(right) approximate unit</u> in $L^1(G)$ is a net $\{f_\alpha\}_{\alpha\in A}$ in $L^1(G)$ such that

$$\forall f \in L^1(G): \lim_{\alpha\in A} f*f_\alpha = f.$$

It follows from [3], 20.27 that $L^1(G)$ has such an approximate unit: for any local base $\mathcal{B}$ at e in G there is a right approximate unit in $L^1(G)$ with $\mathcal{B}$ as its directed set. Hence the least cardinal number κ of a directed set defining an approximate unit for $L^1(G)$ satisfies the inequality $\kappa \leq \chi(G)$. However, we have always $\kappa = \chi(G)$. Since $L^1(G)$ has a unit if and only if G is discrete ([3], 20.25), this equality is trivial for discrete groups. For non-discrete groups G, a proof is indicated in [3] , 28.70(b). We shall present now an essential simplification of that proof, which was inspired by the method of proving Theorem 2.2 (it is not just a corollary of Theorem 2.2).

3.1. PROOF OF $\kappa \geq \chi(G)$ (G non-discrete).
Define a function $\pi: G \times L^1(G) \to L^1(G)$ by

$$(3.2) \qquad \pi(s,f)\,(t) = f(ts) \qquad (s,t \in G;\ f \in L^1(G)).$$

Then π is separately continuous (cf. [4], 20.1 and 20.4) and $(G, L^1(G), \pi)$ is a transformation group. It is easy to see that this transformation group is effective. Indeed, for any $t \in G$, $t \neq e$, there is some continuous function f on G with compact support such that

$$\int_G |f(s) - f(st)|\, ds > 0,$$

i.e. $\pi^t f \neq f$ for some $f \in L^1(G)$.
Now let $\{f_\alpha\}_{\alpha \in A}$ be any approximate unit for $L^1(G)$. Since we assumed G to be non-discrete, A is infinite. For any $\alpha \in A$ and $n \in \mathbb{N}$ there is a neighbourhood $U(\alpha,n)$ of e in G such that $||\pi(s,f_\alpha) - f_\alpha||_1 < \frac{1}{n}$ for every $s \in U(\alpha,n)$.

Let $s \in \cap\{U(\alpha,n) \mid (\alpha,n) \in A \times \mathbb{N}\}$. Then it is clear, that $\pi^s f_\alpha = f_\alpha$ for every $\alpha \in A$. Consider any $f \in L^1(G)$. It follows immediately from the definitions (3.1) and (3.2) that for all $\alpha \in A$

$$\pi^s(f * f_\alpha) = f * \pi^s f_\alpha = f * f_\alpha,$$

hence

$$f = \lim_{\alpha\in A} f * f_\alpha = \lim_{\alpha\in A} \pi^s(f * f_\alpha) = \pi^s f.$$

Consequently, $\pi^s f = f$ for every $f \in L^1(G)$, so that $s = e$ by effectiveness. We have proved, that $\cap\{U(\alpha,n) \mid (\alpha,n) \in A \times \mathbb{N}\} = \{e\}$, whence $\chi(G) \leq |A|.\aleph_0 = |A|$ by Lemma 2.1. It follows that $\chi(G) \leq \kappa$.

Next we consider the space $L^2(G)$. This space can be given the structure of a Hilbert space; the inner product $(..|..)$ in $L^2(G)$ is defined by

$$(f \mid g) = \int_G f(t)\, \overline{g(t)}\, dt \qquad (f,g\in L^2(G))$$

The <u>Hilbert dimension</u> of $L^2(G)$, that is the cardinality of an orthonormal base, will be denoted by $\delta[L^2(G)]$. It is easy to see that

$$d[L^2(G)] = w[L^2(G)] = \delta[L^2(G)], \tag{3.3}$$

provided $L^2(G)$ is not finite dimensional. Notice that $\delta[L^2(G)] \geq \aleph_0$ if and only if G is infinite. This is well-known, but it follows also from the fact that G is finite if and only if $L(G) < \aleph_0$, and from the following lemma.

3.2 LEMMA *For any locally compact Hausdorff topological group G one has $\delta[L^2(G)] \geq L(G)$.*

PROOF. For finite groups the result is trivial, so we may suppose that G is infinite. There is a family $\mathcal{W}$ of pairwise disjoint, non-empty open subsets of G such that $|\mathcal{W}| = L(G)$. Indeed, if G is σ-compact (i.e. $L(G) = \aleph_0$), take $\mathcal{W} = \{U_n \setminus \bar{U}_{n+1} \mid n \in \mathbb{N}\}$ for some suitable sequence $\{U_n \mid n \in \mathbb{N}\}$ of neighbourhoods of e, and if G is not σ-compact, take for $\mathcal{W}$ the family of all left cosets of an open, σ-compact subgroup of G (cf. [5]).

For each $W \in \mathcal{W}$, let f_W be a continuous function with a compact support contained in W, $f_W \geq 0$ and $\int_G f_W(t)^2\,dt = 1$. Then $\{f_W \mid W \in \mathcal{W}\}$ is an orthonormal subset of $L^2(G)$, so that $\delta[L^2(G)] \geq |\mathcal{W}| = \mathcal{L}(G)$.

3.3 LEMMA. _For any locally compact Hausdorff topological group_ G _one has_ $\delta[L^2(G)] \geq \chi(G)$.

PROOF. Define an action π of G on $L^2(G)$ by formula (3.2) (with $f \in L^2(G)$ instead of $f \in L^1(G)$). Then $(G, L^2(G), \pi)$ is an effective transformation group (the proof is similar to that for $(G, L^1(G), \pi)$ in 3.1). Now for infinite groups G, the desired inequality follows immediately from (3.3) and Theorem 2.2. For finite groups the inequality is trivial, because then $\chi(G) = 1$.

3.4 THEOREM. _For any locally compact Hausdorff topological group_ G _the equality_

$$w(G) = \delta[L^2(G)]$$

is valid.

PROOF. It is clear that the equality holds for finite groups, so we may assume that G is infinite. In that case the proof of Theorem 24.15 in [3] shows that $d[L^2(G)] \geq w(G)$, so that $\delta[L^2(G)] \leq w(G)$ by (3.3) (this argument is also used in [3], Theorem 28.2). The converse inequality follows immediately from (1.3) and the Lemma's 3.2 and 3.3 (in [3], 28.2 the equality $\delta[L^2(G)] \geq w(G)$ is obtained only for compact groups G, using representation theory).

3.5 COROLLARY. _A locally compact Hausdorff topological group_ G _is separable and metrizable if and only if_ $L^2(G)$ _is separable_.

4. Examples

Our first example shows that local compactness in Theorem 2.2 is essential, even if we consider effective or free actions of abelian topological Hausdorff groups on compact spaces.

4.1 EXAMPLE. Let A be a set and for each $\alpha \in A$, let $(G_\alpha, X_\alpha, \pi_\alpha)$ be a transformation group such that G_α is a non-discrete topological group and π_α is simultaneously continuous (i.e. continuous on $G_\alpha \times X_\alpha$ with the producttopology). Define G to be the group $\Pi\{G_\alpha \mid \alpha \in A\}$ endowed with the <u>box-topology</u>, that is the topology in which each point $(x_\alpha)_{\alpha\in A}$ in G has as a neighbourhood base all sets of the form $\Pi\{V_\alpha \mid \alpha \in A\}$ with V_α a neighbourhood of x_α in G_α for each $\alpha \in A$. Then G is a topological group, and $\chi(G) \gneq |A|$ (the proof is similar to the proof that $|\mathbb{R}| > \aleph_0$). Let X be the ordinary topological product $\Pi\{X_\alpha \mid \alpha \in A\}$, and define π: $G \times X \to X$ in such a way that

$$\pi(t,x)_\alpha = \pi_\alpha(t_\alpha, x_\alpha) \qquad\qquad ((t,x) \in G \times X)$$

for each $\alpha \in A$. Then π is continuous, and it is easy to see that (G,X,π) is a transformation group. In addition, (G,X,π) is effective (free) if and only if each $(G_\alpha, X_\alpha, \pi_\alpha)$ is effective (free). Finally, observe that

$$\chi(X) \leq |A| . \sup_{\alpha\in A} \chi(X_\alpha); \; d(X) \leq |A| \sup_{\alpha\in A} d(X_\alpha)$$

(These inequalities are quite trivial. It is known that for Hausdorff spaces with at least two points one has

$$\chi(X) = |A| . \sup_{\alpha\in A} \chi(X_\alpha) \text{ and } d(X) = \log |A| . \sup_{\alpha\in A} d(X_\alpha),$$

where $\log |A| := \min\{\beta \mid 2^\beta \geq |A|\}$.)

To get the desired example, take for A a countable, infinite set and, for every $\alpha \in A$, $G_\alpha = X_\alpha = \mathbb{T}$, the torus group, and π_α = multiplication in $\mathbb{T}$. Then G is an abelian topological Hausdorff group and

$\chi(G) > \aleph_0$, whereas X is a compact Hausdorff space such that $\chi(X) = d(X) = \aleph_0$. Notice that (G,X,π) is free.

The following example shows that effectiveness cannot be left out from Theorem 2.2, even when π is simultaneously continuous and G is a compact Hausdorff topological group.

4.2 EXAMPLE. Let G be a non-metrizable compact Hausdorff topological group (so that $\chi(G) > \aleph_0$), and let $\phi: G \times \mathbb{T} \to \mathbb{T}$ be the projection mapping, $\phi(s,t) = t$ $(s \in G, t \in \mathbb{T})$. The function $\tilde{\phi}: (G \times \mathbb{T}) \times \mathbb{T} \to \mathbb{T}$, defined by

$$\tilde{\phi}((s,t),x) = \phi(s,t).x = t.x$$

is continuous, and $(G\times\mathbb{T},\mathbb{T},\tilde{\phi})$ is a transformation group. Here $G \times \mathbb{T}$ is a compact Hausdorff topological group, $\mathbb{T}$ is a compact Hausdorff space, but

$$\chi(G\times\mathbb{T}) = \max\{\chi(G),\chi(\mathbb{T})\} = \chi(G) > \aleph_0 = \chi(\mathbb{T}).\ d(\mathbb{T}).$$

4.3. Concerning Corollary 2.3 we want to remark, that the essential part of its proof is that a first countable Hausdorff topological group is metrizable. The proof of this fact fails for left semitopological groups, so one may ask whether Corollary 2.3 is true for locally compact Hausdorff left semitopological groups. Incidentally, it is true for locally compact Hausdorff semitopological groups because of the fact that such groups are topological groups by a well-known theorem of Ellis [2]. For the same reason, the Corollary is true for compact Hausdorff left semitopological groups (Indeed, if such a group G acts effectively on a Hausdorff space X by means of a separately continuous function π, then $t \longmapsto \pi^t$ is a homeomorphism of G into $G(X)$, the latter space endowed with the topology of pointwise convergence. Since $f \longmapsto f\circ g$ and $f \longmapsto g\circ f$ are continuous mappings of $G(X)$ into itself for every $g \in G(X)$, it follows that multiplication in G is separately continuous.

Hence G is a topological group by Ellis' theorem). Observe, that we did not show that a compact Hausdorff left semitopological group G is metrizable if $\chi(G) \leq \aleph_0$. So there are, in fact, two open problems:

(i) Is a locally compact Hausdorff left semitopological group G metrizable whenever it acts effectively (freely) on a separable first countable Hausdorff space (resp. a first countable T_1-space) by means of a separately continuous action ?

(ii) Is such a group G metrizable whenever $\chi(G) \leq \aleph_0$?

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