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A coherent embedding of an arbitrary topological space in a semi-regular space

by

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§1. Main result

In this paper a proof is given that any topological space can be coherently embedded as a closed subspace of a semi-regular space. The embedding preserves several topological properties, namely: connectedness, T_0, T_1 and T_2 separation, and the weight, when the weight is infinite.

An embedding of a non-semi-regular space in a semi-regular one can never be dense.

§2. Basic definitions and propositions

2.1 Definition:

The topological space $(X_0,\mathcal{T}_0)$ is coherently embedded in $(Y,\mathcal{T})$ provided that Y is the union of a collection of mutually disjoint subsets, each of which (as a subspace) is homeomorphic with $(X_0,\mathcal{T}_0)$.

2.2 Definition:

A topological space is semi-regular if there exists an open base $\mathcal{B}$ for the topology, consisting of regular open sets. A set is regular open if it is equal to the interior of its closure ($V = V^{-0}$).

2.3 Proposition:

A dense subspace of a semi-regular space is semi-regular.

Proof: Let $(W,\mathcal{T})$ be a semi-regular space, and let $(E,\mathcal{T}')$ be a dense subspace with the relative topology. Let V be a regular open subset of W, and let $V \cap E = U$. Suppose that $U^{-0} \neq U$. Then $U^{-0} \supset U$; so there exists some $q \in U^{-0} \setminus U$. Thus q is an interior point of U^{-} and there exists a basic neighbourhood $O_q \in \mathcal{T}$ such that $q \in O_q$ and $(O_q \cap E) \subset U^{-}$.

q cannot be an interior point of V^- because $V^{-0} = V$ and $q \notin U$. Thus $O_q \setminus V^-$ is a non empty open set. No point of $O_q \setminus V^-$ can be in E, because $O_q \cap E \subset \overline{U} \subset V^-$, so we have found a non-empty open set in $W \setminus E$, which contradicts the assumption that E is dense in W.

Conclusion: If V is regular open in $(W,\mathcal{T})$ then $V \cap E$ is regular open in $(E,\mathcal{T}')$.

We have a base for $(W,\mathcal{T})$ consisting of regular open sets. Thus the restriction of the sets to E is a regular open base for $(E,\mathcal{T}')$, and so $(E,\mathcal{T}')$ is semi-regular.

2.4 Definition:

The weight of a topological space $(X,\mathcal{T})$ is the least cardinal number $\underline{m}$ such that there exists a base for the topology $\mathcal{T}$ with cardinality $\underline{m}$.

2.5 Proposition:

A space $(X,\mathcal{T})$ is semi-regular provided that it has a subbase of regular open sets.

Proof: Take two regular open sets U and V. $U = U^{-0}$; $V = V^{-0}$.

$U \cap V \subset (U \cap V)^{-0}$ and

$(U \cap V)^- \subset U^- \cap V^-$; so $(U \cap V)^{-0} \subset (U^- \cap V^-)^0 = U^{-0} \cap V^{-0} = U \cap V$.

Thus $U \cap V = (U \cap V)^{-0}$.

Hence if we take a regular open subbase, all finite intersections will form a regular open base.

2.6 Example:

To obtain a good idea of what is going to happen in the next section, consider the following example.

Let I_0 be the closed interval $[0,1]$ on the real line, let Q be the set of all rational numbers, and let $Q_0 = Q \cap I_0$.

Consider the collection:

$\mathcal{S} = \{U_a \mid a \in I_0\} \cup \{U_b \mid b \in I_0\} \cup \{Q_0\}$, where

$U_a = \{x \mid x \in I_0; x > a\}$ and $U_b = \{x \mid x \in I_0; x < b\}$.

$\mathcal{S}$ considered as a subbase generates a topological space $(I_0, \mathcal{T}_0)$, which is not semi-regular, since the closure of any basic open set is a closed interval whose interior is an open interval. Such an interval can never be contained in a set of the form: $Q_0 \cap (a,b)$ and all sets of this form are open neighbourhoods of each of their points.
We will now coherently embed this space in a semi-regular space. For that purpose we take the set product of I_0 and another closed interval $[0,1] = I_1$ and we look at the following sets in $I_0 \times I_1$.

(i) $U_a = \{(x,y) \mid x \in I_0;\ a < x;\ y \in I_1\};\ (a \in I_0)$

(ii) $\tilde{U}_b = \{(x,y) \mid x \in I_0;\ x < b;\ y \in I_1\};\ (b \in I_0)$

(iii) $V_c = \{(x,y) \mid x \in I_0;\ y \in I_1;\ c < y\};\ (c \in I_1)$

(iv) $\tilde{V}_d = \{(x,y) \mid x \in I_0;\ y \in I_1;\ d > y\};\ (d \in I_1)$

(v) $W_e = \{(x,e) \mid x \in Q_0\};\ (e \in I_1)$

Now we use as a subbase for a new topology:

$$\mathcal{S}_{01} = \{U_a\} \cup \{\tilde{U}_b\} \cup \{V_c\} \cup \{\tilde{V}_d\} \cup \{W_e\}.$$

We used the one extra coordinate for making Q_0 a regular open subbasic element. To show that this trial is successful we should show that

a) I_0 is coherently embedded. It is obvious that any subset with a fixed second coordinate and a free first coordinate is homeomorphic with $(I_0, \mathcal{T}_0)$.

b) The space has a subbase of regular open sets. In fact:

bα) U_a is regular open for all a

bβ) $\tilde{U}_b$ is regular open for all b

bγ) V_c is regular open for all c

bδ) $\tilde{V}_d$ is regular open for all d

bε) W_e is regular open for all e.

bα) $U_a^- = \{(x,y) \mid x \in I_0;\ y \in I_1;\ a \leq x\}$

$U_a^{-0} = \{(x,y) \mid x \in I_0;\ y \in I_1;\ a < x\} = U_a$. So U_a is regular open.

bβ) Similar to bα).

bγ) $V_c^- = \{(x,y) \mid x \in I;\ y \in I_1;\ c < y\} \cup \{(x,c) \mid ;\ x \in I_0 \setminus Q_0\}$
Thus $V_c^{-0} = \{(x,y) \mid x \in I_0;\ y \in I_1;\ c < y\} = V_c$. So V_c is regular open.

bδ) Similar to bγ).

bε) $W_e^- \subset \{(x,e) \mid x \in I_0\}$. (This inclusion is actually equality). Clearly the right hand set is closed, and the only open set contained in it is W_e. Therefore $W_e = W_e^{-0}$. So we have found a regular open subbase, and from 2.5 it now follows that this space is semi-regular.

2.7 Remark:

Example 2.6 demonstrates a well defined case of a coherent embedding. We used one extra coordinate. It is obvious that this embedding saves the Hausdorff axiom because the product topology is Hausdorff and the topology of this space is stronger than that of the product.

Even the connectedness is saved. The first I_0 was connected, and so all of its homeomorphic images are connected. For an irrational point $x_0 \in I_0$ the set $\{(x_0,y) \mid y \in I_1\}$is homeomorphic with the usual topology on $[0,1]$ and so is connected.
It turns out that it is always possible to make a given subbasic element regular open by introducing one extra coordinate, and if there are more non-regular open subbasic sets the extra coordinates can be introduced and handled in such a way that they don't disturb one another.

The only difficulty that is left is the preserving of the weight. In the above example I_0 satisfies the second axiom of countability, but the cardinality of the set of all W_e's is continuous, because of the continuous cardinality of the set I_1. However, when we use as the coordinate space the rational numbers, we lose connectedness.

Therefore we are looking for a space which satisfies the following properties:

1) Hausdorff
2) A countable number of points
3) A countable base. (Second axiom of countability)
4) Connected
5) Semi-regular.

We close this section with a description of a topological space, <u>derived</u> from an example of R.H. Bing [Proc. Am. Math. Soc. <u>4</u> (1953) page 474: A connected countable Hausdorff space], that satisfies these axioms.

2.8 <u>Example</u>

Let M be the set of all rational points in the upper half plane; i.e. $M = \{(x,y) \mid x \in Q;\ y \in Q;\ y \geq 0\}$.

Now we define a topology $\mathcal{H}$ for M with the help of a countable local base $\mathfrak{R}$. For every point $(a,b) \in M$ and for every $\varepsilon > 0$; $\varepsilon \in Q$ we define the basic open set $U_{ab\varepsilon}$ to be the set of all points, both of whose 60° projections on the x-axis lie in the union of two real line ε neighbourhoods, having the 60° projections of (a,b) as mid-points; i.e.

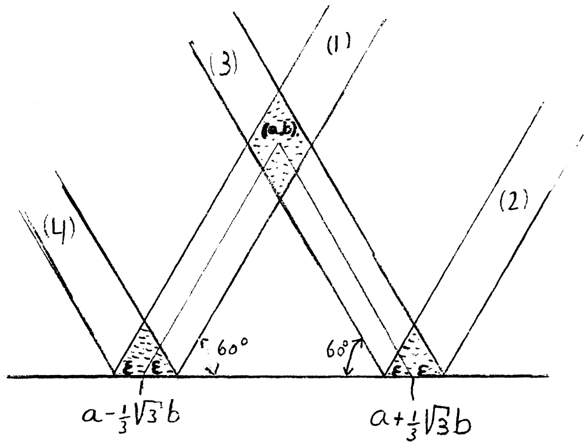

fig. 1

$$U_{ab\varepsilon} = \Big[\underbrace{\{(x,y) \mid |a-x-\tfrac{1}{3}\sqrt{3}(b-y)| < \varepsilon\}}_{(1)} \cup \underbrace{\{(x,y) \mid |a-x+\tfrac{1}{3}\sqrt{3}(b+y)| < \varepsilon\}}_{(2)}\Big] \cap$$
$$\cap \Big[\underbrace{\{(x,y) \mid |a-x+\tfrac{1}{3}\sqrt{3}(b-y)| < \varepsilon\}}_{(3)} \cup \underbrace{\{(x,y) \mid |a-x-\tfrac{1}{3}\sqrt{3}(b+y)| < \varepsilon\}}_{(4)}\Big].$$

Then the set $\mathfrak{R}' = \{U_{ab\varepsilon} \mid a \in Q;\ b \in Q;\ \varepsilon \in Q;\ b \geq 0;\ \varepsilon > 0\}$ is a base for the topological space $(M,\mathcal{H})$.

$(M,\mathcal{H})$ has the following properties:

1e) $(M,\mathcal{H})$ <u>has a countable number of points</u>.

2e) $(M,\mathcal{H})$ <u>has a countable base</u>.

3e) $(M,\mathcal{H})$ is <u>Hausdorff</u>. On any line intersecting the x-axis at an angle of 60° there lies at most one point of the space, so no pair of points (a,b) and (p,q) has one projection in common. Thus it is possible to choose ε to be less than the half of the difference of the two nearest projections, and we are sure that the basic open neighbourhoods $U_{ab\varepsilon}$ and $U_{pq\varepsilon}$ don't intersect.

4e) $(M,\mathcal{H})$ is <u>connected</u>. The closure of $U_{ab\varepsilon}$ is:

$U^{-}_{ab\varepsilon} = \{(x,y) \mid \min (|a - x \pm \frac{1}{3}\sqrt{3}\ (b \pm y)| \leq \varepsilon\}$.

(In the figure this is exactly the union of (1), (2), (3) and (4) with their boundary points). It is evident that each two closed neighbourhoods intersect.

5e) $(M,\mathcal{H})$ is <u>semi-regular</u>. It is obvious that the only interior points of $U^{-}_{ab\varepsilon}$ must have both projections in the intervals $(a - \frac{1}{3}\sqrt{3}\ b - \varepsilon,\ a - \frac{1}{3}\sqrt{3}\ b + \varepsilon)$ and $(a + \frac{1}{3}\sqrt{3}\ b - \varepsilon,$ $a + \frac{1}{3}\sqrt{3}\ b + \varepsilon)$; and the only points satisfying this are precisely those lying in $U_{ab\varepsilon}$; so $U_{ab\varepsilon}$ is regular open. Thus $(M,\mathcal{H})$ is semi-regular.

Now it is possible to prove the general theorem by using a generalisation of the method, described in example 2.6, with the help of coordinate spaces as described in 2.8 instead of compact real line intervals.

§3. <u>The general embedding theorem</u>

3.1 <u>Theorem</u>

Any topological space $(X_0,\mathcal{T}_0)$ can be coherently embedded as a closed subspace in a semi-regular space $(Y,\mathcal{T})$.

The embedding preserves the following properties:

T_0, T_1, T_2 and connectedness. Furthermore, the weight of $(X_0,\mathcal{T}_0)$ will be preserved if it is infinite.

3.2 <u>The construction of the overspace</u>

Let $\mathcal{S}_0$ be an open subbase of $(X_0,\mathcal{T}_0)$ with minimal cardinality. Let $\mathcal{N}$ be the set of all non-regular open elements of $\mathcal{S}_0$; and let $\mathcal{N}' = \mathcal{N} \cup \{0\}$.

For all $\alpha \in \mathcal{N}$ let $(X_\alpha,\mathcal{T}_\alpha)$ be a topological space homeomorphic with $(M,\mathcal{H})$ (cf. 2.8).

Now we form the topological product $(Y,\mathcal{T}') \overset{\text{def}}{=} \prod_{\alpha\in\mathcal{N}'} (X_\alpha,\mathcal{T}_\alpha)$.

The product space $\prod_{\alpha\in\mathcal{N}} (X_\alpha,\mathcal{T}_\alpha)$ is a Hausdorff space, since the product of Hausdorff spaces is Hausdorff. Thus $(Y,\mathcal{T}')$ will be T_0, T_1 or T_2, if $(X_0,\mathcal{T}_0)$ is T_0, T_1 or T_2, respectively.

Furthermore $(Y,\mathcal{T}')$ will be connected if $(X_0,\mathcal{T}_0)$ is connected.

The weight of $(M,\mathcal{H})$ is $\aleph_0$.

Let $k(\mathcal{S}_0)$ be the weight of $(X_0,\mathcal{T}_0)$ and let $k(\mathcal{N})$ be the cardinality of $\mathcal{N}$.

Then (provided that $k(\mathcal{S}_0)$ is infinite) the weight of $(Y,\mathcal{T}')$ will be:

$$k(\mathcal{S}_0) + \aleph_0\ k(\mathcal{N}) \leq k(\mathcal{S}_0) + \aleph_0 \cdot k(\mathcal{S}_0) = k(\mathcal{S}_0);$$

so the weight of $(Y,\mathcal{T}')$ is $k(\mathcal{S}_0)$ and is equal to the weight of $(X_0,\mathcal{T}_0)$.

For future reference, we now describe a subbase for $\mathcal{T}'$.

Let $U_\gamma = \{(x_\alpha)_{\alpha\in\mathcal{N}'} \mid x_0 \in \gamma\}$ $(\gamma \in \mathcal{S}_0)$ and let $\mathcal{S}_U = \{U_\gamma \mid \gamma \in \mathcal{S}_0\}$.

For each $\beta\in\mathcal{N}$ and for each $U_{ab\varepsilon} \subset M \simeq X_\beta$ we can define a set $V_{\beta ab\varepsilon} = \{(x_\alpha)_{\alpha\in\mathcal{N}'} \mid x_\beta \in U_{ab\varepsilon}\}$, $\mathcal{S}_V = \{V_{\beta ab\varepsilon} \mid \beta\in\mathcal{N};\ (a,b) \in X_\beta;\ \varepsilon \in Q;\ \varepsilon > 0\}$.

Then $\mathcal{S}_U \cup \mathcal{S}_V$ is an open subbase for $\mathcal{T}'$, $\mathcal{S}_U$ has cardinality $k(\mathcal{S}_0)$, and $\mathcal{S}_V$ has cardinality $k(\mathcal{N}) \cdot \aleph_0$.

Thus the topology $(Y,\mathcal{T}')$ has all of the required properties except semi-regularity.

To obtain that we use a stronger subbase for the open sets.

For each $\beta\in\mathcal{N}$ and each $e \in X_\beta$ we define:

$$W_{\beta e} = \{(x_\alpha)_{\alpha\in\mathcal{N}'} \mid x_0 \in \beta;\ x_\beta = e\} \quad \text{and}$$

$$\mathcal{S}_W = \{W_{\beta e} \mid \beta\in\mathcal{N};\ e \in X_\beta\}.$$

We now can generate a topology $\mathcal{T}$ on Y with the subbase $\mathcal{S} = \mathcal{S}_U \cup \mathcal{S}_V \cup \mathcal{S}_W$.
The cardinality of $\mathcal{S}_W$ is again $k(\mathcal{N}) \cdot \aleph_0$ and so the cardinality of $\mathcal{S}$ is $k(\mathcal{S}_0) + k(\mathcal{N}) \cdot \aleph_0 + k(\mathcal{N}) \cdot \aleph_0$; and this is equal to $k(\mathcal{S}_0)$ if $k(\mathcal{S}_0)$ is infinite.

Since $\mathcal{T}$ is a finer topology than $\mathcal{T}'$ it will be T_0, T_1 or T_2 if $\mathcal{T}'$ is T_0, T_1 or T_2, respectively.
So the weight and the separation axioms are preserved.

Pick and fix some $p \in \prod_{\alpha\in\mathcal{N}} X_\alpha$; $p = (p_\beta)_{\beta\in\mathcal{N}}$.
Then the set $X_{0p} = \{(x_\alpha)_{\alpha\in\mathcal{N}} \mid x_\beta = p_\beta \ \forall \beta\in\mathcal{N}\}$, considered as a subspace of $(Y,\mathcal{T})$ is homeomorphic with $(X_0,\mathcal{T}_0)$ because subbasic open subsets of X_{0p} have the following forms:
$U_\gamma \cap X_{0p} \cong \gamma$ in X_0.
$V_{\beta ab\varepsilon} \cap X_{0p} = X_0$ or $\emptyset$ in X_0, depending upon whether or not $p_\beta \in U_{ab\varepsilon}$.
$W_{\beta e} \cap X_{0p} \cong \beta$ or $\emptyset$ in X_0 depending upon whether or not $p_\beta = e (\in X_\beta)$.
Clearly X_{0p} is closed in $(Y,\mathcal{T})$ and the embedding is coherent, since $Y = \bigcup\{X_{0p} \mid p$ arbitrary in $\prod_{\alpha\in\mathcal{N}} X_\alpha\}$ and sets of the form X_{0p} are mutually disjoint.

Thus it remains only to show that $(Y,\mathcal{T})$ is semi-regular and is connected if $(X_0,\mathcal{T}_0)$ is connected.

3.3 Proof of semi-regularity

By proposition 2.5 it is sufficient to describe a subbase for $\mathcal{T}$, consisting of regular open elements.

If $\gamma\in\mathcal{N}$ it is obvious that $U_\gamma = \bigcup\{W_{\gamma e} \mid e \in X_\gamma\}$ so we don't need this U_γ's in a subbase for $\mathcal{T}$.
Thus if we define: $\mathcal{S}_U^* = \{U_\gamma \mid \gamma \in \mathcal{S}_0 \setminus \mathcal{N}\}$, we find a subbase, $\mathcal{S}^* = \mathcal{S}_U^* \cup \mathcal{S}_V \cup \mathcal{S}_W$, which is equivalent with $\mathcal{S}$. Hence it is sufficient to prove that

i) U_γ is regular open for every $U_\gamma \in \mathcal{S}_U^*$;
ii) $V_{\beta ab\varepsilon}$ is regular open for every $V_{\beta ab\varepsilon} \in \mathcal{S}_V$;
iii) $W_{\beta e}$ is regular open for every $W_{\beta e} \in \mathcal{S}_W$.

Proof of i): Let $\gamma_0 \in \mathcal{S}_0 \setminus \mathcal{N}$.

Clearly $U^-_{\gamma 0} \subset \{(x_\alpha)_{\alpha\in\mathcal{N}} \mid x_0 \in \gamma_0^-\}$.

Let O be a basic open set contained in $U^-_{\gamma_0}$.

Then it can be written as: $\mathcal{O} = \{\bigcap_{i=1}^{k} (U_{\gamma_i})\} \cap \{\bigcap_{i=k+1}^{m} V_{\beta_i a_i b_i \varepsilon_i}\} \cap$

$\cap \{\bigcap_{i=m+1}^{n} W_{\beta_i e_i}\}$.

($\mathcal{O}$ is basic, and so it is the intersection of a finite number of subbasic elements).

$\mathcal{O} \subset U^-_{\gamma 0}$; so $\pi_0(\mathcal{O}) \subset \gamma_0^-$.

This can only be true if $(\bigcap_{i=1}^{k} \gamma_i) \cap (\bigcap_{i=m+1}^{n} \beta_i) \subset \gamma_0^-$;

γ_0 was regular open, and this intersection is an open subset of γ_0^- and so it is an open subset of γ_0.

Hence $\pi_0(\mathcal{O}) \subset \gamma_0$ which implies that $\mathcal{O} \subset U_{\gamma 0}$.

Proof of ii): Let $V_0 = V_{\beta_0 a_0 b_0 \varepsilon_0} \in \mathcal{S}_V$, and assume that $\mathcal{O}$ is a basic open set contained in V_0^-.

$\mathcal{O}$ is contained in V_0^- and so $\mathcal{O} \cap V_0 \neq \emptyset$. (A)

Again $\mathcal{O}$ can be written as:

$$\mathcal{O} = \{\bigcap_{i=1}^{k} U_{\gamma_i}\} \cap \{\bigcap_{i=k+1}^{m} V_{\beta_i a_i b_i \varepsilon_i}\} \cap \{\bigcap_{i=m+1}^{n} W_{\beta_i e_i}\}.$$

We distinguish two cases:

a) $\beta_{i_0} = \beta_0$ for some $i_0 \geq m+1$

b) $\beta_i = \beta_0$ for no $i \geq m+1$.

Case a): When $e_{i0} \notin U_{a_0 b_0 \varepsilon_0}$ (cf. 2.8).

$\mathcal{O} \cap V_0 = \emptyset$, contradicting A.

So e_{i0} must be contained in $U_{a_0 b_0 \varepsilon_0}$; but that means that $W_{\beta_i e_i} \subset V_0$ for $i = i_0$ and hence $\mathcal{O} \subset V_0$.

Case b): $\mathcal{O} \subset V_0^-$: thus $\mathcal{O} \subset \pi^{-1}_{\beta_0} (U^-_{a_0 b_0 \varepsilon_0})$.

Hence there must exist a set of integers $\mathcal{J}$ such that $\forall_j \in \mathcal{J}$, $k+1 \leq j \leq m$ and $\beta_j = \beta_0$; and $\bigcap_{j\in\mathcal{J}} U_{a_j b_j \varepsilon_j} \subset U^-_{a_0 b_0 \varepsilon_0}$.

In 2.8 we proved that $U_{a_0 b_0 \varepsilon_0}$ is regular open in M and hence

$\bigcap_{j \in J} U_{a_j b_j \varepsilon_j} \subset U_{a_0 b_0 \varepsilon_0}$.

This implies that $\mathcal{O} \subset \pi_{\beta_0}^{-1} (U_{a_0 b_0 \varepsilon_0}) = V_0$ so $\mathcal{O} \subset V_0$.

Both cases a) and b) imply that $\mathcal{O} \subset V_0$; thus V_0 is regular open.

Proof of iii): $W_{\beta_0 e_0} \in \mathcal{T}_W$; $\bar{W}_{\beta_0 e_0} \subset \{(x_\alpha)_{\alpha \in \mathcal{N}} \mid x_{\beta_0} = e_0\}$.

Let $\mathcal{O}$ be a basic open set contained in $W_{\beta_0 e_0}$; then $\pi_{\beta_0} \mathcal{O} = \{e_0\}$.

But for the base described, the only element that is contained in $\{(x_\alpha)_{\alpha \in \mathcal{N}} \mid x_{\beta_0} = e_0\}$ must be contained in $W_{\beta_0 e_0}$, and hence $W_{\beta_0 e_0}$ is regular open.

The cases i), ii), and iii) together prove that $\mathcal{T}^*$ is a subbase consisting of regular open sets.
Hence $(Y, \mathcal{T})$ is semi-regular.

3.4 Proof of connectedness

Assume that $(X_0, \mathcal{T}_0)$ is connected.

a) If $p = (p_\alpha)_{\alpha \in \mathcal{N}}$ and $q = (q_\alpha)_{\alpha \in \mathcal{N}}$, are points such that $p_\alpha = q_\alpha \; \forall \; \alpha \in \mathcal{N}$, then p and q lie in a subspace

$$\{(x_\alpha)_{\alpha \in \mathcal{N}} \mid x_\alpha = p_\alpha \; \forall \alpha \in \mathcal{N}\}$$

which is homeomorphic with $(X_0, \mathcal{T}_0)$; and so is connected. Thus p and q lie in a connected subset of Y.

b) Let $t = (t_\alpha)_{\alpha \in \mathcal{N}}$ and $u = (u_\alpha)_{\alpha \in \mathcal{N}}$ be points of Y such that $t_\alpha = u_\alpha$ for all except one coordinate $\beta \in \mathcal{N}$. Then it is possible to choose $r, s \in Y$ such that

$$r_0 \in X_0, \; r_0 \notin \beta, \text{ and } r_\alpha = t_\alpha \; \forall \alpha \in \mathcal{N}$$

and such that

$$s_0 = r_0 \text{ and } s_\alpha = u_\alpha \; \forall \; \alpha \in \mathcal{N}.$$

Now the subspace

$$\{(x_\alpha)_{\alpha \in \mathcal{N}} \mid \forall \; \alpha \in \mathcal{N} \setminus \{\beta\} \; x_\alpha = t_\alpha; \; x_0 = r_0\}$$

is homeomorphic with $(X_\beta, \mathcal{T}_\beta) \simeq (M, \mathcal{U})$; and hence it is a connected subset containing both r and s. By a), t and r lie in a connected subset and u and s lie in a connected subset each of which intersect the connected subset containing r and s. Thus t and u lie in a connected subset of Y.

c) If p and q are points of Y which differ in only a finite number of coordinates, we can construct a finite chain of points between p and q such that every member of the chain has all except one coordinate in common with its predecessor and with its successor. Then by applying a) and b) we can obtain a chain of connected subsets whose union is connected and which contains both p and q.

d) Now suppose that $(Y, \mathcal{T})$ is not connected. Then there exist two non empty open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that $\mathcal{O}_1 \cup \mathcal{O}_2 = Y$ and $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$.

Let $(p_\alpha)_{\alpha \in \mathcal{N}} = p \in \mathcal{O}_1$ and let $(r_\alpha)_{\alpha \in \mathcal{N}} = r \in \mathcal{O}_2$. Since $\mathcal{O}_2$ is open, there must exist some basic open set B of the form:

$$B = \left(\bigcap_{i=1}^{k} U_{\gamma_i}\right) \cap \left(\bigcap_{i=k+1}^{m} V_{\beta_i a_i b_i \varepsilon_i}\right) \cap \left(\bigcap_{i=m+1}^{n} W_{\beta_i e_i}\right)$$

such that $r \in B \subset \mathcal{O}_2$.

Let $\Gamma = \bigcup \{\beta_i \mid k+1 \leq i \leq n\}$ and let q be the element of Y such that

$$q_0 = r_0,\ q_\alpha = r_\alpha \text{ for } \alpha \in \Gamma, \text{ and } q_\alpha = p_\alpha \text{ for } \alpha \in \mathcal{N} \setminus \Gamma.$$

Clearly $q \in B$ and p and q differ in only finite number of coordinates. Thus, by c), p and q lie in a connected subset of Y. But $p \in \mathcal{O}_1$ and $q \in \mathcal{O}_2$, contradicting our assumption.