STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

AFDELING ZUIVERE WISKUNDE

ZW 1967-006

Minimality of subbases and bases of topological spaces

by

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December 1967

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§1. Introduction

In this report the letters $\mathcal{O}, \mathcal{S}, \mathcal{A}, \mathcal{B}, \mathcal{O}, \mathcal{U}, \mathcal{W}, \mathcal{V}$, etc. denote collections of subsets of a given space. The collection of finite intersections of sets taken from $\mathcal{A}$ is denoted by $\mathcal{A}^{\wedge}$; the family of arbitrary unions of sets from $\mathcal{A}$ is denoted by $\mathcal{A}^{\vee}$.
We define the operator γ on the collections of subsets of a set by $\gamma(\mathcal{A}) = (\mathcal{A}^{\wedge})^{\vee}$.

We say that a set A is generated by $\mathcal{A}$ in case $A \in \gamma(\mathcal{A})$.

Let $\{X, \mathcal{O}\}$ be a topological space. As is well known, a subbase $\mathcal{S}$ for the topology $\mathcal{O}$ is a collection of subsets of X for which $\gamma(\mathcal{S}) = \mathcal{O}$. This report treats subbases which are minimal in the sense that no proper subcollection of $\mathcal{S}$ generates the topology $\mathcal{O}$.

A subminispace is a topological space which has a minimal subbase.

The ordinary subbases in general topology are seldom minimal in the sense defined above. However, it can be shown that each metric space is a subminispace.
The class of subminispaces is closed under the taking of disjoint topological unions and topological products. Any arbitrary topological space X can be embedded as a clopen subset of a subminispace Y so that $Y \setminus X$ is discrete. It is also possible to embed X as an open dense subset in a subminispace Y. These elementary properties on subminimality will be proved in §2.
We also construct some examples of topological spaces that are not subminispaces, one of which is a completely regular space [see §4]. Finally, it can be shown that any topological space can be embedded densely or clopen in a space that is not a subminispace [see §5].

One can define analogously the class of minispaces:
A collection $\mathcal{B}$ of subsets of X is called a minimal base for the topology $\mathcal{O}$ if $\mathcal{O} = \mathcal{B}^{\vee}$ and if no proper subcollection $\mathcal{V} \subset \mathcal{B}$ is a base for $\mathcal{O}$. A topological space is called a minispace if it possesses a minimal base.

The class of minispace does not prove to be very important however. Each T_1-minispace is a discrete space and in the case of T_0-spaces, the topology is completely determined by a partial ordering on X and a subset

$C \subset X$ of "central points". It is true however, that each T_0-space can be densely embedded in a T_0-minispace [see §3].
Although the definitions of minimal base and minimal subbase seem to be connected, the resulting classes of spaces are completely independent. The proof of this independence of the two classes will be given in §5.

The proof of the theorem that each metric space is a subminispace will appear in a separate report. See [1].

§2. Elementary properties of minimal subbases and subminispaces.

Definition 1: A subbase $\mathcal{S}$ is called a minimal subbase iff there exists no proper subset $\mathcal{S}'$ of $\mathcal{S}$ with $\gamma(\mathcal{S}') = \gamma(\mathcal{S})$.

Definition 2: A topological space $\{X, \mathcal{O}\}$ is called a subminispace iff there exists a minimal subbase $\mathcal{S}$ for the topology $\mathcal{O}$.

Proposition 1: The following are equivalent:
1) $\mathcal{S}$ is a minimal subbase.
2) for each $U \in \mathcal{S}$, $\gamma(\mathcal{S} \setminus \{U\}) \neq \gamma(\mathcal{S})$.
3) for each $U \in \mathcal{S}$, $U \notin \gamma(\mathcal{S} \setminus \{U\})$.

proof: 1) → 2) is trivial.
The following well known properties of γ are used in that which follows:

a) for each $\mathcal{A}, \mathcal{B}$ $\mathcal{A} \subset \mathcal{B} \Rightarrow \gamma(\mathcal{A}) \subset \gamma(\mathcal{B})$
b) for each $\mathcal{A}$ $\quad \mathcal{A} \subset \gamma(\mathcal{A})$
c) for each $\mathcal{A}$ $\quad \gamma^2(\mathcal{A}) = \gamma(\mathcal{A})$.

2) → 3). Let $U \in \mathcal{S}$, and suppose that $U \in \gamma(\mathcal{S} \setminus \{U\})$.
Then $U \in \gamma(\mathcal{S} \setminus \{U\})$ and $\mathcal{S} \setminus \{U\} \subset \gamma(\mathcal{S} \setminus \{U\})$ by b) and so $\mathcal{S} \subset \gamma(\mathcal{S} \setminus \{U\})$.
By a) and c), $\gamma(\mathcal{S}) \subset \gamma^2(\mathcal{S} \setminus \{U\}) = \gamma(\mathcal{S} \setminus \{U\}) \subset \gamma(\mathcal{S})$ and hence $\gamma(\mathcal{S}) = \gamma(\mathcal{S} \setminus \{U\})$, which contradicts 2).

3) → 1). Let $\mathcal{S}'$ be a proper subcollection of $\mathcal{S}$.
Then there exists a set $U \in \mathcal{S} \setminus \mathcal{S}'$ which implies that $U \notin \gamma(\mathcal{S} \setminus \{U\})$. By b) $U \in \gamma(\mathcal{S})$ and by a) $\gamma(\mathcal{S}') \subset \gamma(\mathcal{S} \setminus \{U\})$. From this it follows that $\gamma(\mathcal{S}') \neq \gamma(\mathcal{S})$.

The characterization 3) of minimality of a subbase means that a subbase is minimal iff no element of it can be generated by the others. This illustrates some kind of "Boolean independence" of the elements of a minimal subbase.

We have the following corollary:

Corollary 1: If $\mathcal{S}$ is a minimal subbase, then $\emptyset \notin \mathcal{S}$.

proof: $\emptyset \in \gamma(\mathcal{S} \setminus \{\emptyset\})$.

Remark: If we accept the convention that $\bigcap_{\alpha\in\emptyset} U_\alpha = X$, then in the same way one can prove that $X \notin \mathcal{S}$ if $\mathcal{S}$ is minimal. In this case, a space is indiscrete iff it has an empty subbase for the topology.

Proposition 2: Each finite space X is a subminispace.

proof: If X is finite, then any subbase $\mathcal{S}$ for its topology is finite. Therefore there exists a subcollection $\mathcal{S}' \subseteq \mathcal{S}$ which is a subbase of X while no proper subcollection of $\mathcal{S}'$ is a subbase.

Remark: In the proof of prop. 2, the finiteness of $\mathcal{S}'$ is essential. Any subbase for the real line consisting of open halflines $(-\infty, a)$ or (b, ∞) contains a proper subcollection that is a subbase for the real line. Hence it is not generally true that any subbase for a subminispace (the real line is a metric space and therefore a subminispace) contains a subcollection that is a minimal subbase.

Proposition 3: If $\{X_\alpha, \mathcal{O}_\alpha\}_{\alpha\in I}$ is a collection of subminispaces, then the topological product $\prod_{\alpha\in I} X_\alpha$ is a subminispace.

proof: If $\prod_{\alpha\in I} X_\alpha = \emptyset$, then the proposition is trivial. Hence we suppose that $\prod_{\alpha\in I} X_\alpha \neq \emptyset$.

For each $\alpha\in I$, let $\mathcal{S}_\alpha$ be a minimal subbase for the space X_α.
Let $\mathcal{S}$ be the collection $\{\pi_\alpha^{-1}(U_\alpha) \mid U_\alpha \in \mathcal{S}_\alpha, \alpha\in I\}$.
We show that $\mathcal{S}$ is a minimal subbase for the product topology.

Suppose that $\mathcal{S}'$ is a subcollection of $\mathcal{S}$ which is still a subbase for the product topology on X. Then the collection $\mathcal{S}'_\alpha = \{O \subset X_\alpha \mid \pi_\alpha^{-1}(O) \in \mathcal{S}'\}$ is a subcollection of $\mathcal{S}_\alpha$ which is still a subbase for X_α. Indeed, if U is an arbitrary open subset of X_α then $\pi_\alpha^{-1}(U)$ is a union of members $(\mathcal{S}')^\wedge$. Because the α'th projection of a member of $(\mathcal{S}'_\alpha)^\wedge$ is $\emptyset$, X or a member of $(\mathcal{S}'_\alpha)^\wedge$ it follows that U is always a member of $\gamma(\mathcal{S}'_\alpha)$.
The collection $\mathcal{S}_\alpha$ is a minimal subbase for X_α, so we have $\mathcal{S}_\alpha = \mathcal{S}_\alpha'$ for each α, and consequently $\mathcal{S}_\alpha = \mathcal{S}'_\alpha$. The proposition now follows.

From propositions 2 and 3, we have the following corollary:

<u>Corollary 2</u>: Any Cantor space (product of topological doublets) is a subminispace.

<u>Proposition 4</u>: If $\{X_\alpha, \mathcal{O}_\alpha\}_{\alpha \in I}$ is a collection of subminispaces, then the disjoint topological union is a subminispace.

<u>proof:</u> We may suppose that $X_\alpha \cap X_\beta = \emptyset$ for $\alpha \neq \beta$.
For each α, let $\mathcal{S}_\alpha$ be a minimal subbase for the space $\{X_\alpha, \mathcal{O}_\alpha\}$. Then $\bigcup_{\alpha \in I} \mathcal{S}_\alpha$ is a minimal subbase for the disjoint union of $\{X_\alpha, \mathcal{O}_\alpha\}_{\alpha \in I}$.

<u>Corollary 3</u>: Each discrete space is a subminispace. (It is the topological union of finite spaces).
The space of the irrational numbers is a subminispace. As is well-known this space is homeomorfic with a countable product of countable spaces.

<u>Remark</u>: In the example of the space of integers $\mathbb{Z}$, we can construct for each $k \geq 1$ a minimal subbase consisting of sets each containing k-points. Take for example $\mathcal{S}_k = \{O_u | O_u = \{u, u+1, \ldots, u+k-1\}, u \in \mathbb{Z}\}$.
If we take O_u from $\mathcal{S}_k$, we get a topology in which $u-1 \in \{u\}^-$ and $u+k \in \{u+k-1\}^-$. Hence $\mathcal{S}_k$ is clearly a minimal subbase.
The subbase consisting of the sets $A_u = \{t \in \mathbb{Z} | t \geq u\}$ and $B_u = \{t \in \mathbb{Z} | t \leq u\}$ is also an example of a minimal subbase for the space $\mathbb{Z}$.

A subbase of this type in a totally ordered space is not necessarily minimal. For example, it is not possible to construct a minimal subbase for the real line consisting of open half lines. However we have the following proposition:

<u>Proposition 5</u>: The spaces W_α consisting of all ordinals less then α with the usual topology is a subminispace.

<u>proof</u>: In this case there exists a subbase $\mathcal{S}$ consisting of open sets of the form $(0, \beta)$ and (β, α) which is minimal.
Take $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, where
$\mathcal{S}_1 = \{(0, \beta) \big| 0 < \beta < \alpha, \beta \text{ is not a limit ordinal}\}$
$\mathcal{S}_2 = \{(\beta, \alpha) \big| 0 < \beta + 1 < \alpha\}$

It is well-known that the collection $\mathcal{S}_1' \cup \mathcal{S}_2$ where $\mathcal{S}_1' = \{(0, \beta) | 0 < \beta < \alpha\}$ is a subbase for the space W_α.
If γ is a limit ordinal and $\gamma < \alpha$, then
$(0, \gamma) = \bigcup\{(0, \beta) | \beta < \gamma, \beta$ no limit ordinal$\}$.

Hence $\mathcal{S}_1' \subset \gamma(\mathcal{S}_1)$ and so $\mathcal{S}$ is a subbase.

The minimality of $\mathcal{S}$ follows from the following two relations:

I. If $(0, \beta) \in \mathcal{S}_1$, then there is a β' with $\beta = \beta' + 1$.
In the topology $\gamma(\mathcal{S} \setminus \{(0, \beta)\})$, there does not exist an open set that contains β' and not β: $(0, \beta) \notin \gamma(\mathcal{S} \setminus \{(0, \beta)\})$.

II. If $(\beta, \alpha) \in \mathcal{S}_2$, then there does not exist in the topology $\gamma(\mathcal{S} \setminus \{(\beta, \alpha)\})$ an open set that contains $\beta + 1$ and not β: $(\beta, \alpha) \notin \gamma(\mathcal{S} \setminus \{(\beta, \alpha)\})$.

The following proposition illustrates the fact that subminimality is not a hereditary property.

Proposition 6a: Any topological space X can be embedded as a clopen subspace of a subminispace Y such that each point of $Y \setminus X$ is an isolated point.

Proposition 6b: Any topological space X can be embedded as an open dense subspace of a subminispace Y.

proof: Let $\mathcal{S}$ be an arbitrary open subbase for the topological space X which does not contain the empty set. Then we take for Y the set $Y = X \cup (\mathcal{S} \times \{0, 1\})$.

We define the collections $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ by:

$\mathcal{S}_1 = \{U \cup \{(U, 0)\} | U \in \mathcal{S}\}$

$\mathcal{S}_2 = \{U \cup \{(U, 1)\} | U \in \mathcal{S}\}$

$\mathcal{S}_3 = \{\{(U,0), (U, 1)\} | U \in \mathcal{S}\}$

For each $S \in \mathcal{S}$ we have $S = (S \cup \{(S, 0)\}) \cap (S \cup \{(S, 1)\})$.
If $V \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$, then either $V \cap X = \emptyset$ or $V \cap X \in \mathcal{S}$.

ad a): We consider the topological space $\{Y, \gamma(\mathcal{S}^*)\}$, where $\mathcal{S}^* = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$.
As for each $V \in \mathcal{S}^*$ $V \cap X = \emptyset$ or $V \cap X \in \mathcal{S}$ the relative topology on X as a

subspace of Y is exactly the topology generated by $\mathcal{S}$ and therefore X is topologically embedded in Y. From the fact that $X = \bigcup\{U | U \in \mathcal{S}\}$ and the fact that each $U \in \mathcal{S}$ is open in Y, we have that X is an open subset of Y. As $Y \setminus X = \bigcup\{U | U \in \mathcal{S}_3\}$ is open we have also that X is a closed subset of Y.

The minimality of the subbase $\mathcal{S}^*$ follows from the fact that both sets containing the point (S, 0), resp. (S, 1), are needed to generate the open set $\{(S, 0)\}$, resp. $\{(S, 1)\}$.

<u>ad. b)</u>: We consider the topological space $\{Y, \gamma(\mathcal{S}^{**})\}$, where $\mathcal{S}^{**} = \mathcal{S}_1 \cup \mathcal{S}_2$.

As before we see that X is embedded as an open subspace of Y. But now $V \cap X \in \mathcal{S}$ for each $V \in \mathcal{S}^{**}$; since $\emptyset \notin \mathcal{S}$ we have that for each $V \in \mathcal{S}^{**}$, $V \cap X \neq \emptyset$. The intersection of two different elements of $\mathcal{S}^{**}$ either is empty or is a subset of X. Hence X is a dense subspace of Y.

The minimality of the subbase $\mathcal{S}^{**}$ follows from the fact that each point of $Y \setminus X$ is contained in exactly one subbase element which together form the subbase $\mathcal{S}^{**}$.

<u>Corollary 4</u>: If we assume that there exists a topological space that is not a subminispace (such spaces will indeed be constructed in section 4) we see that the property of subminimality is not inherited by arbitrary subspaces, open subspaces, closed subspaces, clopen subspaces, or dense subspaces.

Without proof we mention the following proposition.

<u>Proposition 7</u>: Each metric space is a subminispace.

The proof appears in a separate report. See [1].

§3. Minimal Bases

Definition 1: A base $\mathcal{B}$ for a topology is called a minimal base iff there exists no proper subcollection $\mathcal{B}' \subset \mathcal{B}$ such that $\mathcal{B}'$ is a base for the topology generated by $\mathcal{B}$.

Definition 2: A topological space $\{X, \mathcal{O}\}$ is called a minispace iff there exists a minimal base for the topology $\mathcal{O}$.

In this section we restrict ourselves to T_0-spaces.

Proposition 1: If $\mathcal{B}$ is a minimal base for a space X, then for each $O \in \mathcal{B}$ there exists a point $x_0 \in O$ such that each neighborhood of x_0 contains O; O is the smallest neighborhood of x_0.

proof: Suppose, on the contrary, that for each $x \in O$, there exists a basic neighborhood $U(x) \in \mathcal{B}$ of x which is properly contained in O. Then we have $O = \bigcup_{x \in O} U(x)$ and consequently $\mathcal{B} \setminus \{O\}$ is a base for X which is properly contained in $\mathcal{B}$. This contradicts the minimality of $\mathcal{B}$.

Definition 3: An open set O is called a central open set if it is the smallest neighborhood of some point $x \in O$.

In a T_0-space, a central open set cannot be the smallest neighborhood of two different points. Hence we can define:

Definition 4: The point x is called the central point of a central open set O if O is the smallest neighborhood of x.

A point is called a central point if it is the central point of some central open set. It is easy to see that in a T_0 space there is a one to one correspondence between central open sets and central points.
As it is impossible to generate a central open set by taking the union of other open sets, we have the following propositions and corollaries:

Proposition 2: If O is a central open set of a space, then $O \in \mathcal{B}$ for each base $\mathcal{B}$ for the topology of the space.

Corollary 1: Each base $\mathcal{B}$ contains all central open sets of the topology.

Corollary 2: A base consisting of central open sets is a minimal base.

Combining with prop. 1 we have the following characterization:

Corollary 3: In a minispace there exists exactly one minimal base $\mathcal{B}$.

Corollary 4: Each finite space is a minispace.

proof: In a finite space each point is a central point. The corresponding central open sets form a base for the topology.

As an Example of a T_0-minispace, we take the following "partially ordered topology", which proves to generate the whole class of T_0-minispaces.

Example: Let X be an arbitrary set and let $\leq$ be a partial ordering on X. Let C be a subset of X such that the following conditions are fulfilled:

1) $\forall_{a \in X} \exists_{b \in C}\ b \leq a$

2) $\forall_{a,b \in C} \forall_{x \in X} [a \leq x \text{ and } b \leq x \Rightarrow \exists_{c \in C} [c \leq x, \text{ and } a \leq c \text{ and } b \leq c]]$

3) $\forall_{x,y \in X} [(\forall_{c \in C} [c \leq x \Rightarrow c \leq y]) \iff x \leq y]$.

1) means that any element of x has an $\leq$-predecessor in C.

2) means that any element that is $\leq$-preceded by a and b has a $\leq$-predecessor in C that is also $\leq$-preceded by a and b.

3) means that x $\leq$-precedes y if and only if all the $\leq$-predecessors of x also are $\leq$-predecessors of y.

Now we define a base $\mathcal{B}$ for a topology $J^C_\leq$ by
$\mathcal{B} = \{O(c) | c \in C\}$ where $O(c) = \{x | c \leq x\}$.

By 1) and 2) $\mathcal{B}$ is a base for some topology on X and by 3) this topology is a T_0-topology. It is easy to see that all points $c \in C$ are central points and that the $O(c)$ are the corresponding central open sets. Hence $\{X, J^C_\leq\}$ is a minispace.

Now we show that any T_0-minispace has the form described above.

Let $\{X, \mathcal{O}\}$ be a T_0-minispace. Let C be the set of all central points in X. Define a partial ordering $\leq$ on X by:

$$x \leq y \iff \forall_{0 \in \mathcal{O}} \; [x \in 0 \Rightarrow y \in 0].$$

It is easy to verify that $\leq$ has the properties of a partial ordering. Now we have that if O(c) is the central open set corresponding to the central point c, then:

$$c \leq x \iff x \in O(c).$$

Since $\{O(c) \mid c \in C\}$ is the minimal base, we have that.

$$\forall_{c \in C} \; [c \leq x \Rightarrow c \leq y] \Rightarrow \forall_{c \in C} \; [x \in O(c) \Rightarrow y \in O(c)] \Rightarrow \ldots$$
$$\ldots \forall_{0 \in \mathcal{O}} \; [x \in 0 \Rightarrow y \in 0] \Rightarrow x \leq y.$$

Hence condition 3) is fulfilled. It is easy to verify 1) and 2) because these are formulations of the basic properties of a base for a topology. Thus we see that $\{X, \mathcal{O}\}$ is homeomorphic with the space $\{X, J^C_{\leq}\}$.

So we have proved the following characterization:

<u>Proposition 4</u>: A space $\{X, \mathcal{O}\}$ is a minispace if and only if it is homeomorphic to a space $\{X, J^C_{\leq}\}$.

<u>Corollary 5</u>: Let $\{X, J^C_{\leq}\}$ be a T_1-minispace, then $\{X, J^C_{\leq}\}$ is discrete.

<u>proof</u>: If $\{X, J^C_{\leq}\}$ is a T_1-space, then $x \leq y$ iff $x = y$. From this it follows that $C = X$ (each point of X is central) and $O(x) = \{x\}$. Hence the minimal base consists of all singletons of X. Thus the space is discrete.

<u>Proposition 5</u>: a) Each open subspace of a minispace is a minispace.
b) Each closed subspace of a minispace is a minispace.

<u>proof</u>: a) If $\{X, J^C_{\leq}\}$ is a minispace and O is an open subset of X, then $O = \bigcup\{O(c) \mid c \in C \cap O\}$.
Hence, the collection $\{O(c) \mid c \in C \cap O\}$ is a base for the relative topology on O consisting of central open sets, which implies that O is a minispace.

b) If $\{X, J^C_{\leq}\}$ is a minispace and G is a closed subset of X, then we have that for each central point $c \in C$, $c \in G \Longleftrightarrow O(c) \cap G \neq \emptyset$.
Because the intersection of a central open set with G is empty or contains the central point corresponding to this set, we have that the collection $\{G \cap O(c) \mid c \in C \cap G\}$ is a base for the relative topology on G, consisting of sets that are central open in G. Hence, this is a minimal base.

Remark: A subspace of a minispace is not always a minispace. We give the following example.

Example: Take for X be the set $\mathbb{R}$ of all real numbers. Let $\leq$ be the usual ordering on $\mathbb{R}$, and take for C the set $\mathbb{Q}$ of rational numbers. It is clear that the conditions 1), 2) and 3) are fulfilled, hence $\{\mathbb{R}, J^{\mathbb{Q}}_{\leq}\}$ is a minispace. But in the subspace consisting of the irrational numbers, there are no central open sets.

Proposition 6: a) The disjoint topological union of a collection of minispaces is a minispace.
b) the box-product of a collection of minispaces is a minispace.

proof: a) Let $\mathcal{B}_\alpha$ be a minimal base for X_α, then $\bigcup_{\alpha \in I} \mathcal{B}_\alpha$ is a minimal base for $\bigcup_{\alpha \in I} X_\alpha$.
b) Let $\mathcal{B}_\alpha$ be a minimal base for X_α. Then $\mathcal{B}_\alpha = \{O_\alpha(c_\alpha) \mid c_\alpha \in C_\alpha\}$, where C_α is the collection of central points of X_α.
It is easy to see that in the box-product topology a point $x = (x_\alpha)_{\alpha \in I}$ is central if and only if $x_\alpha \in C_\alpha$ for each $\alpha \in I$.
In this case the central open set of $(x_\alpha)_{\alpha \in I}$ is exactly the product-set $\prod_{\alpha \in I} O(x_\alpha)$. Hence, the collection $\mathcal{B} = \{\prod_{\alpha \in I} O_\alpha \mid O_\alpha \in \mathcal{B}_\alpha \ \ \forall_{\alpha \in I}\}$ is a minimal base for the box-product topology.

Remark: The topological product of minispaces is not always a minispace. We give the following:

Example: The Cantorspace is the product of a countable collection of doublets (which are clearly minispaces) and contains no central open set.

Proposition 7: Each T_0-space can be densely embedded in a T_0-minispace.

proof: Let $\mathcal{B}$ be an arbitrary base for the topology $\mathcal{O}$ of the space X such that $\emptyset \notin \mathcal{B}$. If each set of $\mathcal{B}$ is central open, the space was already a minispace and there is nothing to prove. Hence we suppose that $\mathcal{B}$ contains open sets which are not central open.
We take for each non central open set $B \in \mathcal{B}$ a point $c_B \notin X$ such that $c_B \neq c_{B'}$ for $B \neq B'$
Let $\hat{X}_{\mathcal{B}} = X \cup \{c_B | B \in \mathcal{B}, \text{ B not central open}\}$.
For each O, O open in X we define:

$$\hat{O} = O \cup \{c_B | B \in \mathcal{B},\ B \subset O\}.$$

Now we have: 1) $\hat{\emptyset} = \emptyset$ for $\emptyset \notin \mathcal{B}$

2) $\hat{O}_1 \cap \hat{O}_2 = \widehat{O_1 \cap O_2}$

3) $O_1 \subset O_2 \iff \hat{O}_1 \subset \hat{O}_2$

4) $\hat{O} = \bigcup\{\hat{B} | B \in \mathcal{B},\ B \subset O\}$ for $B \subset O \Rightarrow c_B \in \hat{B} \subset \hat{O}$.

Thus the collection $\{\hat{O} | O \in \mathcal{O}\}$ is a base for some topology on $\hat{X}$.

By 4) we see that $\{\hat{B} | B \in \mathcal{B}\}$ is also a base for this topology. From the definition it follows that $c_B \in \hat{O} \iff B \subset O \iff \hat{B} \subset \hat{O}$.
Hence c_B is a central point and $\hat{B}$ is its central open set. Thus $\hat{X}$ is a minispace.
As $\hat{B} \cap X = B \neq \emptyset$ for any $B \in \mathcal{B}$, X is densely embedded in the space $\hat{X}$.
From the fact that $\hat{O} \cap X = O$, we see that the embedding is also a topological embedding.

Proposition 8: a) A minispace X is compact if and only if there exists a finite set of central points $\{c_1, \ldots, c_k\}$ such that $X = \bigcup_{j=1}^{k} O(c_j)$.
b) A minispace X has a countable base if and only if the set of central points is countable.

proof: Trivial.

Remark: A quotient space of a minispace is not always a minispace.
We have the following:

Example: Let X_1 and X_2 be two disjoint copies of the real line $\mathbb{R}$. Let

$\leq$ be the usual ordering on both X_1 and X_2 and let $\mathbb{Q}_1$ and $\mathbb{Q}_2$ be the sets of national numbers of X_1 and X_2.

On X_1 we take the mini-topology $J_{\leq}^{Q_1}$; on X_2 we take the topology $J_{\leq}^{(X_2 \setminus Q_2)}$.
Let X be the disjoint topological union of X_1 and X_2; then X is a minispace. The quotient space of X which is formed by identifying the two copies of a single real number x for each $x \in \mathbb{R}$ is not a minispace. Its topology consists of the open half lines $\{(a, \infty) \mid a \in \mathbb{R}\}$.

§4. Examples of spaces without minimal subbases

As has been shown in the previous section, it is not hard to find topological spaces that are not minispaces (each non-discrete T_1 space is an example). In this section we construct three examples of spaces that are not subminispaces. For each example the non-existence of a minimal subbase can be derived by "set-theoretical argumentation". The first two examples are minispaces. The third example is a completely regular space which consists of a "large" discrete open subset converging to an accumulation point.

Definition 1: A directed doublet $\vec{D}$ is atopological space consisting of two points $\{0, 1\}$ with the topology: $\{\emptyset, \{1\}, \{0, 1\}\}$.

Example 1: The box-product of a countable collection of directed doublets is not a subminispace.

Remark: By §3, 4 prop. 6b and cor. 4, this space is a minispace.

proof: We denote the box-product by $X = \prod_{k=1}^{\infty} \vec{D}_k$ and its topology by $\mathcal{O}$. Each point $x \in X$, $x = (x_k)_{k=1}^{\infty}$, where $x_k = 0$ or 1, is a central point and its central open set $O(x)$ consists of the point $y = (y_k)_{k=1}^{\infty}$, with $y_k \geq x_k$ for each $k \in \mathbb{N}$.

If we define: $N_x = \{k \in \mathbb{N} | x_k = 0\}$, then we have

$$y \in O(x) \iff N_y \subset N_x.$$

It is easy to see that there exists a one to one mapping ϕ from $2^{\mathbb{N}}$ onto X defined by:

$$\phi(A) = x \quad \text{iff} \quad N_x = A$$

We have that if $N_x \cap N_y = N_z$, then $O(x) \cap O(y) = O(z)$.

Let us suppose that $\mathcal{S}$ is a minimal subbase for the space $\{X, \mathcal{O}\}$. We derive a contradiction by the arguments following below.

1: Because $\mathcal{S}$ is a subbase, $\mathcal{S}^{\wedge}$ is a base and therefore $\mathcal{S}^{\wedge}$ contains the minimal base $\mathcal{B}_0 = \{O(x) | x \in X\}$.

2: If $S \in \mathcal{S}$, then there exists a point $x \in X$ such that $O(x) \notin (\mathcal{S} \setminus \{S\})^{\wedge}$.

For if we suppose on the contrary that $\mathcal{B}_0 \subset (\mathcal{S}\setminus\{S\})^\wedge$ then $\mathcal{S}\setminus\{S\}$ is also a subbase. This contradicts the minimality of $\mathcal{S}$.

3: If $O(x) \not\subset (\mathcal{S}\setminus\{S\})^\wedge$ and $k, l \in \mathbb{N}$, $k, l \notin N_x$ and if we define y and z by: $y = \phi(N_x \cup \{k\})$, resp. $z = \phi(N_x \cup \{l\})$, then we have that $O(x) = O(y) \cap O(z)$. So it is impossible that simultaneously $O(x) \in (\mathcal{S}\setminus\{S\})^\wedge$ and $O(y) \in (\mathcal{S}\setminus\{S\})^\wedge$.

4: If $\mathbb{N}\setminus N_x$ is infinite, where x is some point of X such that $O(x) \notin (\mathcal{S}\setminus\{S\})^\wedge$, then there exists an infinite set $\{n_k\}_{k=1}^\infty$ such that for each $m \in \mathbb{N}$ the central open set $O(x_m)$, where $x_m = \phi(N_x \cup \{n_1, \ldots, n_m\})$, is not contained in $(\mathcal{S}\setminus\{S\})^\wedge$. This follows by induction from 3.

5: If $O(x) \notin (\mathcal{S}\setminus\{S\})^\wedge$, then $O(x) \subset S$.
We have that $O(x) \in \mathcal{S}^\wedge$ and $O(x) \notin (\mathcal{S}\setminus\{S\})^\wedge$; hence $O(x) = S_1 \cap \ldots \cap S_k$ where $S = S_j$ for some j, $j = 1, \ldots, k$.

6: If $O(x) \notin (\mathcal{S}\setminus\{S\})^\wedge$ and if A is an infinite subset of $\mathbb{N}$, then there exists an infinite subset $B = \{n_k\}_{k=1}^\infty$, $B \subset A$, such that for each m, $O(x_m) \subset S$, where $x_m = \phi(N_x \cup \{n_1, \ldots, n_m\})$.
For if $A \cap N_x$ is infinite, we may take $B = A \cap N_x$.
If $A \cap N_x$ is finite, then $A\setminus N_x$ is infinite and we construct as in 4 an infinite subset $B = \{n_k\}_{k=1}^\infty$ of $A\setminus N_x$ such that for each m, $O(x_m) \notin (\mathcal{S}\setminus\{S\})^\wedge$, where $x_m = \phi(N_x \cup \{n_1, \ldots, n_m\})$. From 5 it follows that $O(x_m)$ is contained in S.

7: Let $p = \{1\}_{k=1}^\infty \in X$. Then $O(p) = \{p\}$. As $O(p) \in \mathcal{S}^\wedge$, there exist elements $S_1, \ldots, S_k \in \mathcal{S}$ such that $O(p) = S_1 \cap \ldots \cap S_k$. By 2 we have that there exist points $q_1, \ldots, q_k$ such that $O(q_i) \notin (\mathcal{S}\setminus\{S_2\})^\wedge$.

8: Let $A(B, l)$ denote the statement:
"B is an infinite subset of $\mathbb{N}$, $B = \{n_j\}_{j=1}^\infty$ such that for each $m \in \mathbb{N}$, we have $O(x_m^l) \subset S_l$, where $x_m^l = \phi(N_{q_l} \cup \{n_1, \ldots, n_m\})$".
It is clear that for B', B infinite, $B' \subset B$ the implication: $A(B, m) \Rightarrow A(B', m)$ holds.

9: Now we have by 6:

a) There exists an infinite subset B_1 of $\mathbb{N}$ such that $A(B_1, 1)$ is true.

b) If $A(B_l, l)$ is true for some $l < k$, then there exists an infinite subset $B_{l+1} \subset B_l$ such that $A(B_{l+1}, l+1)$ is true.

10: By induction we conclude that there exists an infinite subset B_k of N such that $A(B_k, k)$ is true. Because of the fact that $B_k \subset B_l$ for $l < k$, we have that $A(B_k, l)$ is true for $l = 1, \ldots, k$.

Let $B_k = \{n_j\}_{j=1}^{\infty}$ and let m be some natural number $m \geq 1$.
Let $r = \phi(\{n_1, \ldots, n_m\})$ and let $r_l = \phi(N_{q_l} \cup \{n_1, \ldots, n_m\})$ for $l = 1, \ldots, m$.
Now we have $O(r) \subset O(r_l) \subset S_l$ for each $l = 1, \ldots, m$; hence $O(r) \subset \bigcap_{l=1}^{k} S_l = \{p\}$.

Since $r \neq p$ and $r \in O(r)$, this gives the desired contradiction.

Example 2: This is a subspace Y of the space X from example 1.
$Y = \{x \in X \mid N_x \text{ is finite}\}$.
As $Y = \bigcup\{O(y) \mid y \in Y\}$, we see that Y is an open subspace of a minispace and therefore by prop. 5a, §3, Y also is a minispace. The minimal base is a countable base.

The fact that Y is not a subminispace can be proved by repeating the argument given in the proof of example 1.

Example 3: Let $\underline{m}$ be a cardinal, such that $\underline{m}$ is not the sum of a countable collection of smaller cardinals, and let A be some set with cardinality $\underline{m}$. The space X will consist of the set $A \times A$ and a single point $\infty \notin A \times A$. We define a topology $\mathcal{O}$ on X by means of the following neighborhood bases:

a) For each point $(x, y) \in A \times A$, the set $\{(x, y)\}$ is open.
b) A subset $U \subset X$ is a neighborhood of the point ∞ iff $\infty \in U$ and for each $x \in A$ the set $\{y \in A \mid (x, y) \notin U\}$ has a cardinal number $< \underline{m}$.

This topological space is not a subminispace.

proof: From this definition and the choice of $\underline{m}$, we have that the intersection of a countable collection of neighborhoods of ∞ is again a neighborhood of ∞. As each closed subset not containing ∞ is clopen, the space is normal and therefore completely regular.

1) The point ∞ has no neighborhood base with cardinal $\leq \underline{m}$. For if $\{B_x\}_{x \in A}$ is a collection of neighborhoods of ∞, we choose for each $x \in A$ a point $y_x \in \{y \in A \mid (x, y) \in B_x\}$. Now the set $U = X \setminus \{y_x\}_{x \in A}$ is a neighborhood of ∞ such that $B_x \not\subset U$ for all $x \in A$.

2) Let $\underline{w}$ be the weight of $\{X, \mathcal{O}\}$. From 1) we conclude that $\underline{w} > \underline{m}$. Let $\mathcal{S}$ be a subbase for the space $\{X, \mathcal{O}\}$, then we have card $(\mathcal{S}) \geq \underline{w} > \underline{m}$.

3) There are $\underline{m} \times \underline{m} = \underline{m}$ isolated points in the space $\{X, \mathcal{O}\}$. For each of these points there is a finite collection of elements from $\mathcal{S}$ that generates the singletons consisting of these points. The union of all these collections is a subcollection $\mathcal{S}_1 \subset \mathcal{S}$ with card$(\mathcal{S}_1) \leq \underline{m}$. Hence we conclude from 2) that card $(\mathcal{S} \setminus \mathcal{S}_1) > \underline{m} > \aleph_0$.

4) Suppose that $\mathcal{S}$ is a minimal subbase. Because each set not containing ∞ is contained in $\gamma(\mathcal{S}_1)$, we deduce that each set in $\mathcal{S} \setminus \mathcal{S}_1$ is a neighborhood of ∞.
Now let $O \in \mathcal{S} \setminus \mathcal{S}_1$ and let V be some neighborhood of ∞. Now we have $V \subset O \Rightarrow V \notin \gamma(\mathcal{S} \setminus \{O\})$. For if $V \in \gamma(\mathcal{S} \setminus \{O\})$, we know that $O = V \cup (O \setminus V)$ and therefore $V \in \gamma(\mathcal{S} \setminus \{O\})$ because of the fact that $O \setminus V \in \gamma(\mathcal{S}_1)$. This contradicts the fact that $\mathcal{S}$ is a minimal subbase.

5) Now we take a countable collection $\{O_k\}_{k=1}^{\infty}$ from $\mathcal{S} \setminus \mathcal{S}_1$.
As $\bigcap_{k=1}^{\infty} O_k$ is a neighborhood of ∞, we know that there exists a finite intersection $S_1 \cap \ldots \cap S_k = U$ with $\infty \in U \subset \bigcap_{k=1}^{\infty} O_k$ and $S_1, \ldots, S_k \in \mathcal{S}$. By 4) we have that $U \notin \gamma(\mathcal{S} \setminus \{O_k\})$ for $k = 1, 2, \ldots$. Hence each O_k appears as some S_j for $j = 1, \ldots, n$. This is a contradiction.

Remark: In the proof of example 3, the condition that $\underline{m}$ is not the sum of a countable collection of smaller cardinals is essential for step 5). It remains an open question whether or not this condition is essential for the space having no minimal subbase. For example the case where $\underline{m} = \aleph_0$ remains open.

Remark: The space X of example 3 is the quotient space of a space Y which is the disjoint topological union of $\underline{m}$ copies of the space Z defined as follows: $Z = A \cup \{w\}$, where w is a point not contained in A; the topology on Z is generated by a subbase containing all the singletons contained in A and all the sets U containing w with card$(Z \setminus U) < \underline{m}$.

It is clear that the space generated by identifying all the endpoints w_x from the union $Y = \bigcup_{x \in A} Z_x$ is homeomorphic with the space X. The space Z is a subminispace. For ,let α be the first ordinal with card(α) = $\underline{m}$ and let W_α be the space of all ordinals $\leq \alpha$ with its usual topology. If we take the subspace W'_α of W_α consisting of α and all the ordinals $< \alpha$ that are no limit-ordinal, then it is easy to see that W'_α is homeomorphic with Z. The subbase constructed in the proof of prop. 5, §2, is also a minimal subbase for the space W'_α.
We see that a quotientspace of a subminispace is not always a subminispace.

§5. On the independence of the existence of minimal bases and subbases.

Proposition 1: The properties "minimality" and "subminimality" are independent.

proof: We have the following examples:

	minimal base	minimal subbase
Discrete space, finite space	+	+
Cantor space, non discrete metric space	-	+
Example 1, 2 (§4)	+	-
Example 3 (§4)	-	-

Proposition 2: Each topological space can be embedded in a space that does not have a minimal subbase,

a) as an open dense subspace

b) as a clopen subspace.

proof: a) Let $\{X, \mathcal{O}\}$ be the topological space given in Example 1, §4. Let $\{Y, \mathcal{V}\}$ be an arbitrary topological space. We may assume $Y \cap X = \emptyset$. Now we define the "directed topological union" $X \vec{\cup} Y$. This is a topological space consisting of the set $X \cup Y$ and a topology that is defined by:

$$O \text{ is open in } X \vec{\cup} Y \iff \begin{cases} O \cap X \in \mathcal{O} & \text{and} \quad Y \subset O \\ O \cap X = \emptyset & \text{and} \quad O \cap Y \in \mathcal{V}. \end{cases}$$

Because each open set which has a non-empty intersection with X contains the whole set Y, we conclude that a minimal subbase $\mathcal{S}$ for the space $X \vec{\cup} Y$ would induce a minimal subbase for the space X, namely the subbase $\mathcal{S}^*$ consisting of the intersections $S \cap X$ where $S \in \mathcal{S}$ and $S \cap X \neq \emptyset$. Therefore $X \vec{\cup} Y$ is not a subminispace.

b) Let $\{Y, \mathcal{V}\}$ be an arbitrary topological space with weight $\underline{w}$. Let $\underline{m}$ be a cardinal number $\geq \underline{w}$ such that $\underline{m}$ is not the sum of a countable collection of cardinals $< \underline{m}$.

Now we construct the space $\{X,\mathcal{O}\}$ constructed in Example 3 (§4) with this cardinal number $\underline{m}$. Then the disjoint topological union $X \cup Y$ is not a subminispace.
This can be proved by taking some subbase $\mathcal{S}$ for $X \cup Y$. As $\mathcal{S}^\wedge$ is a base there exists a collection $\mathcal{W} \subset \mathcal{S}^\wedge$ which is a base for the topology of Y as a clopen subspace of $X \cup Y$, such that card $(\mathcal{W}) \leq \underline{w} \leq \underline{m}$.
Then there exists a subcollection $\mathcal{S}_2 \subset \mathcal{S}$ with card $(\mathcal{S}_2) \leq \underline{m}$ and $\mathcal{W} \subset \gamma(\mathcal{S}_2)$. Let $\mathcal{S}_3$ be choosen so that $\{\{x\} \mid x \in X,\ x \neq \infty\} \subset \gamma(\mathcal{S}_3)$ and card $(\mathcal{S}_3) \leq \underline{m}$. Put $\mathcal{S}_1 = \mathcal{S}_2 \cup \mathcal{S}_3$, then we conclude as before that card $(\mathcal{S} \setminus \mathcal{S}_1) > \underline{m}$. The proof is completed as in section 4.

Corollary 1: As the "directed sum" of two minispaces is again a minispace (as can easily be verified), we have by prop. 7 (§3):
Each topological space can be embedded as a dense subset of a minispace that is not a subminispace.

proof: First we embed the space in a minispace (prop. 7, §3) and secondly we embed this minispace in a non-subminispace as above.

Remark: As the proof of prop. 2a) also works if we take example 2 (§4) in stead of example 1 (§4), we may choose an embedding in a compact or a non-compact space as we like.

Remark: From examples 2 and 3 (§4) we conclude that neither of the properties of complete regularity, or the second countability axiom, are sufficient to imply the existence of a minimal subbase. However both properties together imply metrizability and therefore subminimality.

Concluding we have the following invariants:

	minimal base	minimal subbase
Topological unions	+	+
Topological products	-	+
Box-products	+	-
Open subsets	+	-
Closed subsets	+	-
Dense subsets	-	-
Quotient spaces	-	-

REFERENCE

[1]. P. van Emde Boas; Any metric space has a minimal subbase.
Report ZW - 1967 - 007, Mathematisch Centrum, Amsterdam.