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Any Metric space has a Minimal Subbase

by

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§1. Introduction

Consider the real line with its usual (Euclidian) topology, and a subbase $\mathcal{S}$ for this topology consisting of open half-lines;

so $$\mathcal{S} = \{(a, \infty) \mid a \in A_L\} \cup \{(-\infty, b) \mid b \in A_R\}.$$

It is easy to prove that for $\mathcal{S}$ to be a subbase, the sets A_L and A_R must be dense subsets of the real line. From this fact it follows that for each $a \in A_L$ resp. A_R:

$(a, \infty) = \cup\{(b, \infty) \mid b \in A_L,\ b > a\}$ resp. $(-\infty, a) = \cup\{(-\infty, b) \mid b \in A_R,\ b < a\}$.

Hence, if we remove an arbitrary set (a, ∞) or $(-\infty, b)$ from the subbase $\mathcal{S}$, then the resulting collection of open sets remains a subbase for the topology.

If we consider the subspace $\mathbb{Z}$ of the integers, then the situation is different. Let $\mathcal{S}$ be the following subbase:

$$\mathcal{S} = \{(-\infty, k) \cap \mathbb{Z} \mid k \in \mathbb{Z}\} \cup \{(k, \infty) \cap \mathbb{Z} \mid k \in \mathbb{Z}\}.$$

If we remove some set $(-\infty, k)$ from $\mathcal{S}$, then the resulting collection is a subbase for a new topology in which any open set containing $k-1$ also contains k. In fact the subbase $\mathcal{S}$ is a <u>minimal subbase</u> in the following sense:

There exists no proper subcollection of $\mathcal{S}$ that is a subbase for the topology generated by $\mathcal{S}$.

As we have seen the discrete space Z possesses a minimal subbase.
It seems that the real line does not possess such a subbase, but in fact we only have shown that there exists no minimal subbase for the real line consisting of open half-lines.

In this report it will be shown that an arbitrary metric space possesses a subbase which is minimal in the sense defined above.

In the proof we shall construct a minimal subbase starting from a σ-discrete base for the topology. Furthermore, we shall use the notion of a minimal neighborhood subbase for a point p; this we define to be a collection $\mathcal{A}$ of neighborhoods of that point p with the property that the family of finite intersections of $\mathcal{A}$ is a neighborhood base for p, and that no proper subcollection of $\mathcal{A}$ generates a neighborhood base of p.

In this report the characters $\mathcal{U}, \mathcal{A}, \mathcal{B}, \mathcal{V}$ etc. will denote collections of subsets of a given topological space X. The collection of all finite intersections of sets taken from $\mathcal{A}$ will be denoted by $\mathcal{A}^{\wedge}$; their arbitrary union by $\mathcal{A}^{\vee}$.

By definition $\gamma(\mathcal{A}) = (\mathcal{A}^{\wedge})^{\vee}$.

The expression "A is generated by $\mathcal{A}$" will express the fact that $A \in \gamma(\mathcal{A})$.

In a metric space we denote the open ε-neighborhood of the point p by $U_{\varepsilon}(p)$.

Except for the general properties of section 2 and the example in section 4 all spaces considered are metric.

A <u>subminispace</u> is a topological space that possesses a minimal subbase.

In section 2 we state some elementary properties on minimal subbases, the proofs of which appear in a separate report [1]. Section 3 contains the proof that each metric space is a subminispace. Section 4 gives the construction of a completely regular space that possesses no minimal subbase.

§2. <u>Some elementary properties on minimal subbases.</u>

The proofs of the propositions stated in this section appear in a separate report; see [1].

<u>prop. 1</u>: A subbase $\mathcal{S}$ is minimal if and only if for each $S \in \mathcal{S}$, $S \notin \gamma(\mathcal{S} \setminus \{S\})$.

A space that possesses a minimal subbase will be called a <u>subminispace</u>.

<u>prop. 2</u>: The topological product of subminispaces is a subminispace.

<u>prop. 3</u>: The disjoint topological union of subminispaces is a subminispace.

From <u>prop. 2</u> and <u>prop. 3</u> it follows directly (given the fact that each finite space is a subminispace) that each Cantorspace, each discrete space, and each product of discrete spaces (for example the space of the irrational numbers) is a subminispace.

prop. 4: Any topological space (not necessarily metric) can be embedded in a subminispace a) as a clopen subset where the complement consists of isolated points, b) as an open dense subset.

From prop. 4 it follows that the property "subminimality" is not inherited by open, closed or dense subsets. We depend here on the existence of a space that is not a subminispace, which will indeed be provided in section 4.

§3. Minimal neighborhood subbases

Definition: A collection $\mathfrak{A}$ of subsets of a topological space X is called a neighborhood subbase for the point $p \in X$ if $\mathfrak{A}^\wedge$ is a neighborhood base for p.

A neighborhood subbase $\mathfrak{A}$ for p is called a minimal neighborhood subbase for p if there exists no proper subcollection $\mathfrak{A}'$ of $\mathfrak{A}$ such that $\mathfrak{A}'$ is a neighborhood subbase for p.

With this notion it is possible to "localize" the notion of a minimal subbase. We have the following proposition:

prop. 5: A subbase $\mathcal{S}$ of a topological space is a minimal subbase, if and only if for each $S \in \mathcal{S}$ there exists a point $p \in S$, such that the collection $\mathcal{S}'_{(p,S)} = \{U \in \mathcal{S} | p \in U,\ U \neq S\}$ is not a neighborhood subbase for p.

proof: $\Rightarrow$ Let $\mathcal{S}$ be a minimal subbase; then $S \in \mathcal{S} \Rightarrow S \notin \gamma(\mathcal{S} \setminus \{S\})$.
Now we have that $(\mathcal{S} \setminus \{S\})^\wedge$ is a base for the topology $\gamma(\mathcal{S} \setminus \{S\})$; hence $S \notin \gamma(\mathcal{S} \setminus \{S\})$ means that there exists a point $p \in S$ such that there is no set $U \in (\mathcal{S} \setminus \{S\})^\wedge$ with $p \in U \subset S$.
This implies that there exists no open set in $(\mathcal{S}'_{(p,S)})^\wedge$ that is contained in S.
This proves the fact that $\mathcal{S}'_{(p,S)}$ is not a neighborhood subbase for p.

$\Leftarrow$
Let $S \in \mathcal{S}$. Then there exists a point $p \in S$, such that $\mathcal{S}'_{(p,S)}$ is not a neighborhood subbase of p. This means that there exists an open set containing p which is not a neighborhood of p in the topology $\gamma(\mathcal{S} \setminus \{S\})$. This proves the fact that $\gamma(\mathcal{S} \setminus \{S\}) \neq \gamma(\mathcal{S})$.
Since S was taken arbitrarily from $\mathcal{S}$, this means that $\mathcal{S}$ is a minimal subbase.

Lemma: Let p be a point of a metric space M and let 0 be an open set containing p. Then there exists a minimal neighborhood subbase $\mathcal{O}_p$ consisting of open subsets of 0. If p is not an isolated point we may assume that $0 = \bigcup\{U \mid U \in \mathcal{O}_p\}$.

proof: Suppose first that p is an isolated point. Then we take $\mathcal{O}_p = \{\{p\}\}$ and the proof is trivial. Therefore, we suppose in the following that $p \in \overline{0 \setminus \{p\}}$.
Choose a sequence $\{x_i\}_{i=1}^{\infty}$ of points from 0 such that:

1) if $\alpha_i = \rho(p, x_i)$ then $\{\alpha_i\}_{i=1}^{\infty}$ is a monotone descending sequence with converges to zero.

2) $\overline{U_{\alpha_1}(p)}$ is contained in 0.

Let V_k be $U_{\alpha_{2k}}(p)$, and put $V_0 = 0$.
Now $0 \setminus \bar{V}_1 \neq \emptyset$ and for each k $V_k \setminus \bar{V}_{k+1} \neq \emptyset$, since $x_1 \in 0 \setminus \bar{V}_1$ and $x_{2k+1} \in V_k \setminus \bar{V}_{k+1}$.
It is easy to see that $\{V_i\}_{i=1}^{\infty}$ is a neighborhood base for p.
Now we take $W_1 = V_1$, and $W_k = V_k \cup (0 \setminus \bar{V}_{k-1})$ for $k \geq 2$.
Then $\{W_i\}_{i=1}^{\infty} = \mathcal{O}_p$ is a minimal neighborhood subbase for p.
This can be proved the following way.
In the first place $V_k = \bigcap_{j=1}^{k} W_j$ for each k, hence $\mathcal{O}_p$ is a neighborhood base for p.
From the construction of the W_i's it follows that each set in $(\{W_i\}_{i=1}^{\infty} \setminus \{W_k\})^{\wedge}$ contains the non empty set $V_{k-1} \setminus \bar{V}_k$, hence no proper subcollection of $\mathcal{O}_p$ is a neighborhood subbase for p. Thus $\mathcal{O}_p$ is minimal.

It is easy to see that $0 = \bigcup_{k=1}^{\infty} W_k$.

It is useful to remark that for each point $q \in 0$, $q \neq p$, the intersection of all $U \in \mathcal{O}_p$ with $q \in U$ is an open set.

Theorem: Each metric space possesses a minimal subbase.

proof: Let $\{X, \rho\}$ be a metric space. From the metrization theorem of Bing (see [2]) it follows that there exists a σ-discrete open base $\mathcal{B} = \bigcup_{k=1}^{\infty} \mathcal{B}_k$ for the topology such that $\mathcal{B}_k$ is discrete for each k.

From the base $\mathcal{B}$ we construct by induction, for each natural number k, a collection of open sets $\mathcal{S}_k$ and a discrete closed subset D_k of X, such that the following conditions are satisfied:

1) $D_k \supset D_{k-1}$; D_k is a discrete and closed subset of X.
   Put $X_k = X \setminus D_k$, $X_0 = X$.
2) $\mathcal{S}_k \supset \mathcal{S}_{k-1}$; for each $S \in \mathcal{S}_k$, $S \notin \mathcal{S}_{k-1}$ implies that S is an open subset of X_{k-1}.
3) Each $S \in \mathcal{S}_k$ is either a set consisting of one isolated point, or else there exists a point $p \in D_k$ such that $\mathcal{S}_k$ is a neighborhood subbase for p, and $\mathcal{S}_k \setminus \{S\}$ is not a neighborhood subbase for p.
4) $\bigcup_{n=1}^{k} \mathcal{B}_n \subset \gamma(\mathcal{S}_k)$.
5) For each $y \in X_k$ the intersection $X_k \cap (\bigcap \{S \mid S \in \mathcal{S}_k,\ y \in S\})$ is an open set.

construction: For k = 0 we take $D_k = \emptyset$ and $\mathcal{S}_k = \emptyset$; then 1), 2), 3), 4) and 5) are fulfilled.

Now we suppose that the construction is performed for $k \leq n$.

We construct $\mathcal{S}_{n+1}$ and D_{n+1} in the following way:

Let 0 be an open set of $\mathcal{B}_{n+1}$. Then there are two possibilities:

I. $0 \cap X_n$ only consists of isolated points. In this case we put
   $\mathcal{S}_{n+1}(0) = \{\{x\} \mid x \in 0 \cap X_n\}$, and $D_{n+1}(0) = \emptyset$.

II. There exists a non-isolated point $x_0 \in 0 \cap X_n$. Then we take
   $D_{n+1}(0) = \{x_0\}$.

Let V be the intersection $V = X_n \cap 0 \cap (\cap \{S \in \mathcal{S}_n | x_0 \in S\})$, then V is an open neighborhood of x_0 by 5). As in the proof of the Lemma we construct a neighborhood base $\{U_j\}_{j=1}^{\infty}$ of x_0 such that $\bar{U}_1 \in V$ and $V \setminus \bar{U}_1 \neq \emptyset$;

$$\bar{U}_{j+1} \subset U_j \quad \text{and} \quad U_j \setminus \bar{U}_{j+1} \neq \emptyset \quad \text{for each } j.$$

We now define $\mathcal{S}_{n+1}(0) = \{U_1\} \cup \{((X_n \cap 0) \setminus \bar{U}_j) \cup U_{j+1}\}_{j=1}^{\infty}$.

Now we see that $\mathcal{S}_{n+1}(0)$ is a minimal neighborhood subbase for x_0 consisting of open sets contained in $X_n \cap 0$ such that their union equals $X_n \cap 0$, and that for each point $y \in X_n \cap 0$, $y \neq x_0$, the intersection of all $U \in \mathcal{S}_{n+1}(0)$ containing y is open.

Since $\mathcal{B}_{n+1}$ is a discrete collection of open sets, we can perform this construction for each $0 \in \mathcal{B}_{n+1}$ simultaneously. Now we define:

$$D_{n+1} = D_n \cup (\cup \{D_{n+1}(0) | 0 \in \mathcal{B}_{n+1}\}) \quad \text{and} \quad \mathcal{S}_{n+1} = \mathcal{S}_n \cup (\cup \{\mathcal{S}_{n+1}(0) | 0 \in \mathcal{B}_{n+1}\}).$$

It is easy to check that with this construction the conditions 1), 2) 3), 4) and 5) are fulfilled.

Now let $\mathcal{S}^*$ be the union $\bigcup_{k=1}^{\infty} \mathcal{S}_k$. It is clear that

$\mathcal{B} = \bigcup_{k=1}^{\infty} \mathcal{B}_k \subset \gamma(\bigcup_{k=1}^{\infty} \mathcal{S}_k) = \gamma(\mathcal{S}^*)$; hence $\mathcal{S}^*$ is a subbase.

Each set in $\mathcal{S}^*$ is either a singleton consisting of an isolated point or an element which is contained in a minimal neighborhood subbase for some point x in some D_k.

Let $\mathcal{S}_1^*$ be the collection of all singletons in $\mathcal{S}^*$ and $\mathcal{S}_2^* = \mathcal{S}^* \setminus \mathcal{S}_1^*$.

If $S \in \mathcal{S}_2^*$ there exists a k such that $S \in \mathcal{S}_k$ but $S \notin \mathcal{S}_{k-1}$.

Then there exists a point $x \in D_k$ such that $\mathcal{S}_k \setminus \{S\}$ is not a neighborhood subbase for x. But for each $U \in \mathcal{S}^* \setminus \mathcal{S}_k$ we have $x \notin U$; hence $\mathcal{S}^* \setminus \{S\}$ again is not a neighborhood subbase for x.

Let $\mathcal{S}_3^*$ be the set of all singletons $\{x\} \in \mathcal{S}_1$ such that $\{x\} \notin \gamma(\mathcal{S}_2^*)$. Then it is clear that $\{x\} \notin \gamma(\mathcal{S}^* \setminus \{x\})$, hence $\mathcal{S}^* \setminus \{x\}$ is not a neighborhood subbase for x. Now we form the union $\mathcal{S} = \mathcal{S}_2^* \cup \mathcal{S}_3^*$.

It is clear that $\mathcal{S}$ is a subbase for the space X and by <u>prop. 5</u> we have that $\mathcal{S}$ is a minimal subbase, which completes the proof of the theorem.

§4. An example of a non-subminimal space.

By adjoining one non-isolated point to a discrete space we construct an example of a space which has not a minimal subbase. It is clear that the resulting space is a normal space.

Let A be a set with card(A) $= \aleph_1$ and let $<$ be a well ordering for A, such that each proper $<$-section is countable. Now we form the product $A \times A$. Let ∞ be a point not contained in $A \times A$; then we define the set X by $X = A \times A \cup \{\infty\}$.
We define a topology on X by means of the following open subbase $\mathcal{S}$:
$\mathcal{S} = \{\{x\} \mid x \in A \times A\} \cup \mathcal{V}(\infty)$, where

$$\mathcal{V}(\infty) = \{U \mid \infty \in U \text{ and } \forall_{a \in A} \exists_{t(a) \in A} \forall_{b \in A} [t(a) < b \Rightarrow (a, b) \in U]\}.$$

So a neighborhood of ∞ is a set U that contains from each set $\{a\} \times A$ a tailpiece.
In this topology the intersection of a countable collection of neigborhoods of ∞ is again a neighborhood of ∞.
The weight of this space is greater then $\aleph_1$, as can be concluded from a "diagonal construction" in a "neighborhood base of ∞" with cardinality $\aleph_1$.
Now let $\mathcal{S}$ be a subbase for the topology. Then card($\mathcal{S}$) $> \aleph_1$.
Dor each $x \in A \times A$ there exists a finite subset $\mathcal{S}(x)$ of $\mathcal{S}$ such that $\bigcap\{S \mid S \in \mathcal{S}(x)\} = \{x\}$.
Let $\mathcal{S}_1 = \mathcal{S} \setminus \bigcup\{\mathcal{S}(x) \mid x \in A \times A\}$. Then card($\mathcal{S}_1$) $> \aleph_1$.

If $\mathcal{S}$ is a minimal subbase then we may conclude:
For each $S \in \mathcal{S}_1$, $\mathcal{S} \setminus \{S\}$ is not a neighborhood subbase for ∞.
Hence if $S_1 \cap \ldots \cap S_k \subset S$, $S \in \mathcal{S}_1$, and $S_1, \ldots, S_k \in \mathcal{S}$ then $S = S_j$ for some $j = 1, \ldots, k$.

Now we take a countable collection $\{S_j\}_{j=1}^{\infty}$ from $\mathcal{S}_1$.
We have that $\bigcap_{j=1}^{\infty} S_j$ is a neighborhood of ∞; hence there exists a finite collection $U_1, \ldots, U_k$ such that $\infty \in U_1 \cap \ldots \cap U_k \subset \bigcap_{j=1}^{\infty} S_j$.

From this we conclude that $S_j = U_{n_j}$ for $1 \leq n_j \leq k$ which gives a contradiction.

REFERENCES

[1] P. van Emde Boas, Minimality of subbases and bases of topological spaces, report ZW 1967-006 mathematisch Centrum, Amsterdam.

[2] J.L. Kelley, General Topology, New York 1955.