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ON A CONJECTURE OF ERDÖS (I)

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ABSTRACT

This note contains a partial solution of a conjecture of ERDÖS. See Amer. Math. Monthly, Vol. 56 (1949) p. 343, Advanced Problem 4347.

KEY WORDS & PHRASES: *Inequalities, sums of powers of integers.*

ON A CONJECTURE OF ERDÖS (I)

by

J. van de Lune

0. INTRODUCTION

In [1] ERDÖS proposed the following problem: Prove that if m and n are positive integers satisfying

$$(0.0)\qquad (1-\frac{1}{m})^n > \tfrac{1}{2} > (1-\frac{1}{m-1})^n$$

then

$$(0.1)\qquad 1^n + 2^n + \dots + (m-2)^n < (m-1)^n$$

and

$$(0.2)\qquad 1^n + 2^n + \dots + m^n > (m+1)^n.$$

Show also that

$$(0.3)\qquad 1^n + 2^n + \dots + (m-1)^n < m^n$$

in infinitely many instances and that

$$(0.4)\qquad 1^n + 2^n + \dots + (m-1)^n > m^n$$

in infinitely many instances.

It appears that this problem has never been solved or even partially solved.

In this note ERDÖS' conjecture will be proved partially by showing that (0.1) is true indeed. In addition to this it will be shown (also by elementary means) that

$$(0.5) \qquad 1^n + 2^n + \ldots + (m+1)^n > (m+2)^n$$

for all m and n satisfying (0.0).

In a forthcoming paper by the present author and H.J.J. te RIELE the remaining inequalities (0.2), (0.3) and (0.4) will be discussed.

In that paper it will be shown (by less elementary means) that, roughly speaking, (0.2) and (0.3) are true with probability 1.

Finally, we must confess that we have not yet succeeded in finding even one single pair (m,n) satisfying (0.0) such that either (0.2) or (0.3) is false. Also, we do not have an example in which (0.4) is true.

1. PREPARATIONS

If $n = 1$ then (0.0) reads

$$(1.1) \qquad 1 - \frac{1}{m} > \tfrac{1}{2} > 1 - \frac{1}{m-1}$$

which is equivalent to

$$(1.2) \qquad 2 < m < 3.$$

Since m is a positive integer (1.2) is impossible so that we cannot have $n = 1$.

Hence, from now on we will assume that $n \geq 2$.

LEMMA 1.1. *If* $n \geq 2$ *then there is precisely one* $m(=m(n))$ *satisfying* (0.0), *namely*

$$(1.3) \qquad m(n) = [\lambda(n)] + 1$$

where

$$(1.4) \qquad \lambda(n) = \frac{1}{1-2^{-1/n}},$$

and [ ] is the greatest integer function.

PROOF. It is easily seen that (0.0) is equivalent to

$$(1.5) \qquad \lambda(n) < m < \lambda(n) + 1.$$

Since $2^{1/n}$ is irrational for $n \geq 2$ it follows from (1.4) that the positive number $\lambda(n)$ is irrational so that there is precisely one positive integer $m(n)$ between $\lambda(n)$ and $\lambda(n) + 1$. □

Since $\lambda(n)$ is increasing it follows from (1.5) that

$$(1.6) \qquad m(n) > \lambda(n) \geq \lambda(2) = 3.41\ldots$$

so that

$$(1.7) \qquad m(n) \geq 4, \qquad (\forall n \geq 2).$$

LEMMA 1.2. *For any pair* (m,n) *of positive integers we have*

$$(1.8) \qquad \sigma_m(n) < \frac{m^n(2m+1)^{n+1}}{(2m+1)^{n+1} - (2m-1)^{n+1}}$$

where

$$\sigma_m(n) = \sum_{k=1}^{m} k^n.$$

PROOF. We proceed by induction. It is clear that (1.8) holds true for $m = 1$ and all $n \in \mathbb{N}$. Assume that (1.8) is still true for $m = 1,\ldots,M$ and all

$n \in \mathbb{N}$. Then we have

$$(1.10) \qquad \sigma_{M+1}(n) = (M+1)^n + \sigma_M(n) < (M+1)^n + \frac{M^n(2M+1)^{n+1}}{(2M+1)^{n+1} - (2M-1)^{n+1}}$$

so that it suffices to show that for all $n \in \mathbb{N}$

$$(1.11) \qquad (M+1)^n + \frac{M^n(2M+1)^{n+1}}{(2M+1)^{n+1} - (2M-1)^{n+1}} \le \frac{(M+1)^n(2M+3)^{n+1}}{(2M+3)^{n+1} - (2M+1)^{n+1}} .$$

Putting $x = \frac{1}{2M}$ we arrive at the equivalent inequality

$$(1.12) \qquad (1+2x)^n + \frac{(1+x)^{n+1}}{(1+x)^{n+1} - (1-x)^{n+1}} \le \frac{(1+2x)^n(1+3x)^{n+1}}{(1+3x)^{n+1} - (1+x)^{n+1}} .$$

After crossmultiplication and some simplifications it turns out that it suffices to prove that

$$(1.13) \qquad (1+2x)^n\{(1+x)^{n+1} - (1-x)^{n+1}\} \ge (1+3x)^{n+1} - (1+x)^{n+1}$$

or, equivalently, that

$$(1.14) \qquad \frac{(1+x)^{n+1} - (1-x)^{n+1}}{x} \ge \frac{(1+\frac{x}{1+2x})^{n+1} - (1-\frac{x}{1+2x})^{n+1}}{\frac{x}{1+2x}} .$$

Since

$$(1.15) \qquad \frac{(1+x)^{n+1} - (1-x)^{n+1}}{x} = 2 \sum_{r=0}^{[\frac{n}{2}]} \binom{n+1}{2r+1} x^{2r}$$

it follows that the left hand side of (1.14) is an increasing function of x on $\mathbb{R}^+$.

Observing that

$$(1.16) \qquad x > \frac{x}{1+2x} , \qquad (x>0)$$

it follows that (1.14) holds true, proving the lemma. □

For a geometrical interpretation of lemma 1.2 see [3].

2. PROOF OF (0.1).

THEOREM 2.1. *If* $n \geq 2$ *and* m *is determined by* (0.0) *then*

$$1^n + 2^n + \ldots + (m-2)^n < (m-1)^n. \tag{2.1}$$

PROOF. In order to prove (2.1) we may just as well show that

$$1^n + 2^n + \ldots + (m-2)^n + (m-1)^n = \sigma_{m-1}(n) < 2.\ (m-1)^n. \tag{2.2}$$

In view of (1.7) and Lemma 1.2 it suffices to show that

$$\frac{(m-1)^n (2m-1)^{n+1}}{(2m-1)^{n+1} - (2m-3)^{n+1}} \leq 2.\ (m-1)^n \tag{2.3}$$

which may be reduced to

$$1 + \frac{2}{2m-3} \geq 2^{\frac{1}{n+1}}. \tag{2.4}$$

Since $m(n) < \lambda(n) + 1$ we have

$$1 + \frac{2}{2m-3} > 1 + \frac{2}{2(\lambda(n)+1) - 3} = \tag{2.5}$$

$$= 1 + \frac{2}{-1 + \frac{2}{1-2^{-1/n}}} = 1 + 2.\ \frac{1-2^{-1/n}}{1+2^{-1/n}}\ .$$

Hence, it suffices to show that

$$1 + 2.\ \frac{1-2^{-1/n}}{1+2^{-1/n}} \geq 2^{\frac{1}{n+1}} = 2^{\frac{1/n}{1+1/n}}\ , \tag{2.6}$$

or, putting $x = \frac{1}{n}$, that

$$1 + 2.\ \frac{1-2^{-x}}{1+2^{-x}} \geq 2^{\frac{x}{1+x}}\ . \tag{2.7}$$

Now observe that (2.7) is equivalent to

$$(2.8) \qquad 1 + 2\left(-1+\frac{2}{1+2^{-x}}\right) \geq 2^{\frac{x}{1+x}}$$

which may also be written as

$$(2.9) \qquad -1 + \frac{4}{1+2^{-x}} \geq 2^{1-\frac{1}{1+x}}$$

or, equivalently, as

$$(2.10) \qquad 1 \geq \left\{\tfrac{1}{2} + 2^{-(1+x)}\right\} \left\{\tfrac{1}{2} + 2^{-\frac{1}{1+x}}\right\}.$$

Hence, putting $t = 1 + x$, it certainly suffices to show that

$$(2.11) \qquad 1 \geq \left\{\tfrac{1}{2} + 2^{-t}\right\} \left\{\tfrac{1}{2} + 2^{-1/t}\right\}$$

for all $t \in (1,\frac{3}{2}]$.

Since the right hand side of (2.11) is positive for $t > 0$ we may just as well show that

$$(2.12) \qquad \phi(t) = \log(\tfrac{1}{2}+2^{-t}) + \log(\tfrac{1}{2}+2^{-1/t}) \leq 0$$

for all $t \in (1,\frac{3}{2}]$.

Since

$$(2.13) \qquad \phi(1) = \log(\tfrac{1}{2}+\tfrac{1}{2}) + \log(\tfrac{1}{2}+\tfrac{1}{2}) = 0$$

it suffices to show that

$$(2.14) \qquad \phi'(t) \leq 0 \qquad \text{for } 1 < t \leq \frac{3}{2}.$$

Observing that

$$(2.15) \qquad \phi'(t) = -\frac{2^{-t}\log 2}{\frac{1}{2}+2^{-t}} + \frac{2^{-1/t}\frac{\log 2}{t^2}}{\frac{1}{2}+2^{-1/t}}$$

we will be through if we can show that

$$(2.16)\qquad \frac{\frac{1}{t^2}\,2^{-1/t}}{\frac{1}{2}+2^{-1/t}} \le \frac{2^{-t}}{\frac{1}{2}+2^{-t}}\,, \qquad (1<t\le\tfrac{3}{2})$$

or, equivalently, that

$$(2.17)\qquad \frac{1/t}{2+2^{1/t}} \le \frac{t}{2+2^{t}}\,, \qquad (1<t\le\tfrac{3}{2}).$$

Since $t > 1$ we have $\frac{1}{t} < t$ so that it suffices to show that the function

$$(2.18)\qquad \psi(t) = \frac{t}{2+2^{t}}\,, \qquad (t>0)$$

is increasing on the interval $\frac{2}{3} \le t \le \frac{3}{2}$.
Since

$$(2.19)\qquad \psi'(t) = \frac{2+2^{t}-t2^{t}\log 2}{(2+2^{t})^{2}}$$

we will be finished if we can show that

$$(2.20)\qquad 2 + 2^{t} - t.2^{t}\log 2 > 0, \qquad (\tfrac{2}{3}\le t\le\tfrac{3}{2})\,,$$

or, equivalently, that

$$(2.21)\qquad 2^{1-t} + 1 > t\log 2, \qquad (\tfrac{2}{3}\le t\le\tfrac{3}{2}).$$

Now observe that for $t \le \frac{3}{2}$ we have

$$(2.22)\qquad 2^{1-t} + 1 \ge 2^{-\frac{1}{2}} + 1 > \frac{3}{2}$$

and

$$(2.23)\qquad t\log 2 \le \frac{3}{2}\log 2 < \frac{3}{2}\,,$$

completing the proof. □

3. ANOTHER INEQUALITY

In this section we will prove the following

THEOREM 3.1. *If* $n \geq 2$ *and* m *is determined by* (0.0) *then*

$$(3.1) \qquad 1^n + 2^n + \dots + (m+1)^n > (m+2)^n.$$

First we prove some lemmas.

LEMMA 3.1. *For any pair* (m,n) *of positive integers we have* (*compare* [3])

$$(3.2) \qquad \sigma_m(n) > \frac{m^{n+1}(m+1)^n}{(m+1)^{n+1} - m^{n+1}} .$$

PROOF. It is easily verified that (3.2) holds true for $m = 1$ and all $n \in \mathbb{N}$. Assume that (3.2) is still true for $m = 1,\dots,M$ and all $n \in \mathbb{N}$. Then we have

$$(3.3) \qquad \sigma_{M+1}(n) = (M+1)^n + \sigma_M(n) > (M+1)^n + \frac{M^{n+1}(M+1)^n}{(M+1)^{n+1} - M^{n+1}}$$

so that it suffices to show that

$$(3.4) \qquad (M+1)^n + \frac{M^{n+1}(M+1)^n}{(M+1)^{n+1} - M^{n+1}} \geq \frac{(M+1)^{n+1}(M+2)^n}{(M+2)^{n+1} - (M+1)^{n+1}}$$

for all $n \in \mathbb{N}$.

It is easily verified that (3.4) is equivalent to

$$(3.5) \qquad \frac{(M+1)^n}{(M+1)^{n+1} - M^{n+1}} \geq \frac{(M+2)^n}{(M+2)^{n+1} - (M+1)^{n+1}} .$$

Writing $x = \frac{1}{M+1}$, (3.5) may be reduced to

$$(3.6) \qquad \frac{1}{1 - (1-x)^{n+1}} \geq \frac{(1+x)^n}{(1+x)^{n+1} - 1}$$

or, equivalently, to

$$(3.7) \qquad (1+x)^{n+1} - 1 \geq (1+x)^n - (1-x)^{n+1}(1+x)^n.$$

Now multiply by $1 + x$ and simplify in order to arrive at the equivalent inequality

$$(3.8) \qquad \frac{(1+x)^{n+1} - 1}{x} \geq \frac{1 - (1-x^2)^{n+1}}{x^2} \quad .$$

Since for any fixed $n \in \mathbb{N}$ the function x^{n+1} is convex on $\mathbb{R}^+$ it follows immediately that (3.8) holds true for $0 < x = \frac{1}{M+1} \leq \frac{1}{2}$, completing the proof. □

<u>LEMMA 3.2</u>. *If* $x > 0$ *and* $n \in \mathbb{N}$ *then*

$$(3.9) \qquad 1 + \frac{x}{n+1} > (1+x)^{\frac{1}{n+1}} \quad .$$

<u>PROOF</u>. This is an immediate consequence of Bernoulli's inequality

$$(3.10) \qquad (1+a)^{n+1} > 1 + (n+1)a, \quad (n\in\mathbb{N}\,;a>0). \quad \square$$

<u>LEMMA 3.3</u>. *For any* $x > 0$ *we have*

$$(3.11) \qquad \frac{1}{x} - \frac{1}{2} < \frac{1}{e^x-1} < \frac{1}{x} - \frac{1}{2} + \frac{x}{12} \; .$$

<u>PROOF</u>. It is well known that (see [2; p.378])

$$(3.12) \qquad \frac{1}{e^x-1} - \frac{1}{x} + \frac{1}{2} = \sum_{r=1}^{\infty} \frac{2x}{x^2+4r^2\pi^2}$$

for all real $x \neq 0$.

Hence, if $x > 0$ it follows that

$$(3.13) \qquad \frac{1}{e^x-1} - \frac{1}{x} + \frac{1}{2} > 0$$

and

$$(3.14)\qquad \frac{1}{e^x-1} - \frac{1}{x} + \frac{1}{2} < \sum_{r=1}^{\infty} \frac{2x}{4r^2\pi^2} = \frac{x}{2\cdot\pi^2} \sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{x}{2\cdot\pi^2} \cdot \frac{\pi^2}{6} = \frac{x}{12}$$

completing the proof. □

Proof of theorem 3.1.

In order to prove (3.1) we may just as well show that

$$(3.15)\qquad 1^n + 2^n + \dots + (m+1)^n + (m+2)^n > 2\cdot(m+2)^n$$

or that

$$(3.16)\qquad \sigma_{m+2}(n) > 2\cdot(m+2)^n.$$

In view of lemma (3.1) it suffices to show that

$$(3.17)\qquad \frac{(m+2)^{n+1}(m+3)^n}{(m+3)^{n+1} - (m+2)^{n+1}} \geq 2\cdot(m+2)^n$$

which is easily seen to be equivalent to

$$(3.18)\qquad \frac{m+2}{m+3} \geq 2 - 2\left(\frac{m+2}{m+3}\right)^{n+1}$$

which in its turn may be rewritten as

$$(3.19)\qquad 2\left(1-\frac{1}{m+3}\right)^{n+1} \geq 1 + \frac{1}{m+3}\ .$$

Hence, since $m(n) > \lambda(n)$, it suffices to show that

$$(3.20)\qquad 2\left(1-\frac{1}{\lambda(n)+3)}\right)^{n+1} \geq 1 + \frac{1}{\lambda(n)+3}$$

or, equivalently, that

$$(3.21)\qquad 2^{\frac{1}{n+1}}\left(1-\frac{1}{\lambda(n)+3}\right) \ge \left\{1+\frac{1}{\lambda(n)+3}\right\}^{\frac{1}{n+1}}.$$

Hence, in view of lemma 3.2, it suffices to show that

$$(3.22)\qquad 2^{\frac{1}{n+1}}\left(1-\frac{1}{\lambda(n)+3}\right) \ge 1+\frac{1}{(n+1)(\lambda(n)+3)}$$

which may also be written as

$$(3.23)\qquad 2^{\frac{1}{n+1}} - 1 \ge \frac{1}{\lambda(n)+3}\left\{2^{\frac{1}{n+1}}+\frac{1}{n+1}\right\}$$

or as

$$(3.24)\qquad 3+\lambda(n) \ge \frac{2^{\frac{1}{n+1}}+\frac{1}{n+1}}{2^{\frac{1}{n+1}}-1}.$$

Using the definition of $\lambda(n)$ it is easily seen that (3.24) is equivalent to

$$(3.25)\qquad 3+\frac{1}{2^{\frac{1}{n}}-1} \ge \frac{1+\frac{1}{n+1}}{2^{\frac{1}{n+1}}-1}$$

so that, in view of lemma 3.3, it suffices to show that

$$(3.26)\qquad 3+\left(\frac{n}{\log 2}-\frac{1}{2}\right) \ge \left(1+\frac{1}{n+1}\right)\left(\frac{n+1}{\log 2}-\frac{1}{2}+\frac{\log 2}{12(n+1)}\right)$$

which may be reduced to

$$(3.27)\qquad 3-\frac{2}{\log 2} \ge \frac{1}{n+1}\cdot\left\{\frac{\log 2}{12}-\frac{1}{2}+\frac{\log 2}{12(n+1)}\right\}.$$

Since

$$(3.28)\qquad 3-\frac{2}{\log 2} > 0,$$

and

$$\frac{\log 2}{12} - \frac{1}{2} + \frac{\log 2}{12(n+1)} \le \frac{\log 2}{12} - \frac{1}{2} + \frac{\log 2}{36} = \frac{\log 2}{9} - \frac{1}{2} < 0 \tag{3.29}$$

it follows that (3.27) holds true, completing the proof of theorem 3.1. □

REFERENCES

[1] ERDÖS, P., *Advanced Problem 4347*, Amer. Math. Monthly Vol. 56 (1949) p. 343.

[2] KNOPP, K., *Theory and application of infinite series*, Blackie & Son Ltd., London (1928).

[3] VAN DE LUNE, J., *Monotonic approximation of integrals in relation to some inequalities for sums of powers of integers*, Mathematical Centre, Amsterdam, Report ZW 39/75 (1975).

ADDENDUM. During the preparation of this note R. TIJDEMAN notified the author that (0.1) may be proved much simpler as follows.
If

$$\left(1-\frac{1}{m-1}\right)^n < \frac{1}{2}$$

then also

$$\left(1-\frac{1}{k}\right)^n < \frac{1}{2}\,, \qquad (1 \le k < m-1),$$

so that

$$(k-1)^n < \frac{1}{2}\,k^n, \qquad (1 \le k < m-1).$$

Hence

$$(m-2)^n < \frac{1}{2}\,(m-1)^n,$$

$$(m-3)^n < \frac{1}{2}\,(m-2)^n < \frac{1}{2^2}\,(m-1)^n,$$

$$(m-4)^n < \frac{1}{2}\,(m-3)^n < \frac{1}{2^2}\,(m-2)^n < \frac{1}{2^3}\,(m-1)^n$$

.............etc. ....

so that certainly

$$1^n + 2^n + \ldots + (m-2)^n < (m-1)^n \cdot \sum_{r=1}^{\infty} \frac{1}{2^r} = (m-1)^n,$$

proving (0.1). However, it seems that this beautiful trick does not apply to any of the other inequalities in ERDÖS' conjecture.