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A GENERALIZATION OF DESCARTES' RULE

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A generalization of Descartes' rule*)

by

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ABSTRACT

Let f(z) be an exponential sum

$$f(z) = \sum_{k=1}^{m} c_k \exp(\alpha_k z), \qquad z \in \mathbb{C},$$

where $\alpha_k \in \mathbb{R}$, $c_k \in \mathbb{C}$ $(k = 1,\dots,m)$. We give upper bounds for the number of zeros of f in horizontal strips in the complex plane. By limit transition, these results are extended to exponential integrals

$$f(z) = \int_0^1 e^{zt} g(t)\mu(dt),$$

where g is a continuous complex-valued function and $\mu$ a positive measure on [0,1].

KEY WORDS & PHRASES: *Descartes' rule, zeros of functions, exponential polynomials, Laplace transform.*

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*) This report will be submitted for publication elsewhere.

## 1. INTRODUCTION

Let $f(z)$ be an exponential sum of the form

$$(1) \qquad f(z) = \sum_{k=1}^{m} c_k \exp(\alpha_k z), \qquad z \in \mathbb{C},$$

where $c_k, \alpha_k \in \mathbb{R}$ $(k = 1,\dots,m)$. We assume this sum to be ordered in such a way that $\alpha_1 < \alpha_2 < \dots < \alpha_n$ and $c_k \neq 0$ $(k = 1,\dots,m)$. Descarte's rule states that the number $N_{\mathbb{R}}(f)$ of zeros of f on the real axis does not exceed the number of sign changes in the sequence $\{c_1,c_2,\dots,c_m\}$ (see POLYA and SZEGÖ [1], chapter 5, problem 77). In this paper we present a generalization of this rule for complex $c_1,\dots,c_m$. We prove

$$(2) \qquad N_{\mathbb{R}}(f) \leq \frac{1}{\pi} \sum_{k=1}^{m-1} |\mathrm{Arg}(c_{k+1}/c_k)|,$$

where $\mathrm{Arg}(\zeta) \in (-\pi,\pi]$ is the principal value of the argument of $\zeta$.

We further use this result to derive an estimate for the imaginary parts of the complex zeros of f. We prove for instance that the number $N_E(f)$ of zeros of f in the strip

$$E = \{z \in C;\ 0 < \mathrm{Im}\, z < r\}$$

satisfies

$$(3) \qquad \pi N_E(f) \leq \sum_{k=1}^{m-1} |\mathrm{Arg}(c_{k+1}/c_k)| + r(\alpha_m-\alpha_1).$$

By limit transition, analogues of (2) and (3) are given for functions of the form

$$(4) \qquad f(z) = \int_0^1 e^{zt}\, g(t)\, \mu(dt),$$

where $g(t)$ is a continuous complex-valued function and $\mu$ a positive measure on $[0,1]$, such that $f(z)$ is not identically vanishing. By a simple transformation one can obtain corresponding results for

$$f(z) = \int_a^b e^{zt}\, \mu(dt)$$

where $(a,b) \subset \mathbb{R}$ and $\mu$ is a complex measure.

The method by which the above results are derived has been developed by the author in [2]. The method is explained in section 2; results from [2] are briefly sketched and some new ideas are inserted. In section 3 we apply this method to exponential sums (1). Finally, in section 4 we treat the exponential integral (4).

## 2. TOTAL VARIATION OF THE ARGUMENT OF A FUNCTION

Let $f(z)$ be an analytic complex-valued function in an open neighbourhood $\mathcal{O}$ of the interval $L = [a,b] \subset \mathbb{R}$. On $L$, $\mathrm{Im}(f'/f)$ is the restriction of an analytic function (see [2], Lemma 1). If $f(a)f(b) \neq 0$ we define

$$(5) \qquad A(a,b;f) = \int_a^b |\mathrm{Im}(f'(t)/f(t))|\,dt + \pi\, N_L(f),$$

where $N_L(f)$ is the number of zeros of $f$ in $L$. Here and everywhere else in this paper multiple zeros are counted according to their multiplicities. If $\arg f(t)$, the argument of $f$, is suitably defined, $A(a,b;f)$ appears to be the total variation of $\arg f(t)$. Since $f$ is analytic in $\mathcal{O}$, there is a $\theta > 0$ such that for $0 < \varepsilon < \theta$ the functions

$$f_\varepsilon(x) := f(x+i\varepsilon)$$

are analytic on $L$. The following properties of $A(a,b;f)$ hold.

LEMMA 1. *If* $f(a)f(b) \neq 0$ *then*

$$\lim_{\varepsilon\to 0} A(a,b;f_\varepsilon) = A(a,b;f).$$

PROOF. See [2], Lemma 2.

LEMMA 2. *Let* $f_1(z), f_2(z), \ldots$ *be a sequence of complex-valued functions, analytic and uniformly converging to* $f(z)$ *in* $\mathcal{O}$. *If* $f(a)f(b) \neq 0$, *then*

$$\lim_{k\to\infty} A(a,b;f_k) = A(a,b;f).$$

PROOF. Let K be a compact subset of $\mathcal{O}$, containing an open neighbourhood $\mathcal{O}'$ of $[a,b]$. Since $f(z)$ is analytic in $\mathcal{O}$, the number of zeros of f in K is finite. Hence there is a $\theta > 0$ such that for $0 < \varepsilon < \theta$ the functions $f_\varepsilon(x)$ have no zeros on $[a,b]$. For $0 < \varepsilon < \theta$ the functions $f_k'(x+i\varepsilon)$ converge to $f_\varepsilon'(x)$ and, since $f_\varepsilon(x) \neq 0$, the functions $1/f_k(x+i\varepsilon)$ converge to $1/f_\varepsilon(x)$ uniformly on $[a,b]$. Thus for $0 < \varepsilon < \theta$

$$\lim_{k\to\infty} A(a,b;f_k(x+i\varepsilon)) = A(a,b;f_\varepsilon).$$

By letting $\varepsilon \downarrow 0$ and applying Lemma 1, the lemma is proved.

LEMMA 3. *Let* f, g *be analytic in* $\mathcal{O}$. *If* $f(a)f(b)g(a)g(b) \neq 0$, *then*

$$A(a,b;fg) \leq A(a,b;f) + A(a,b;g).$$

PROOF. Straightforward (see [2], Lemma 3).

For complex $z \neq 0$, we define Arg(z), the principal value of the argument of z, in such a way that $-\pi < \mathrm{Arg}\, z \leq \pi$. Observe that the function $|\mathrm{Arg}\,(z)|$ is continuous for all $z \in \mathbb{C}$, $z \neq 0$. We now prove our main theorem.

THEOREM 1. *Let* f *be analytic in* $\mathcal{O}$. *If* $f(a)f'(a)f(b)f'(b) \neq 0$, *then*

$$A(a,b;f) \leq A(a,b;f') + \psi(a) - \psi(b),$$

where

$$\psi(x) = |\mathrm{Arg}(f'(x)/f(x))|. \tag{6}$$

PROOF. See [2], Theorem 1, but omit the estimate $\psi(b) - \psi(a) \geq -\pi$. The only new element needed is the observation that

$$\lim_{\varepsilon\to 0} \psi(a+i\varepsilon) = \psi(a); \quad \lim_{\varepsilon\to 0} \psi(b+i\varepsilon) = \psi(b),$$

which follows from $f(a)f(b)f'(a)f'(b) \neq 0$.

## 3. EXPONENTIAL SUMS

In this section we consider functions

$$f(z) = \sum_{k=1}^{m} c_k \exp(\alpha_k z), \tag{7}$$

where $\alpha_k \in \mathbb{R}$, $0 \neq c_k \in \mathbb{C}$ $(k = 1,\ldots,m)$. Without loss of generality we will assume in this section that $\alpha_1 < \alpha_2 < \ldots < \alpha_m$. We prove the following result.

THEOREM 2. *Let* $f(z)$ *be given by* (7). *For all* $a,b \in \mathbb{R}$ *such that* $f(a)f(b) \neq 0$

$$A(a,b;f) \leq \sum_{k=1}^{m-1} |\mathrm{Arg}(c_{k+1}/c_k)|.$$

PROOF. The theorem is proved by induction on m. If $m = 1$ the theorem is trivial, since $A(a,b;c_1 \exp(\alpha_1 x)) = 0$. Suppose that $m > 1$. By lemma 3, $A(a,b;f(x)\exp(-\alpha_1 x)) = A(a,b;f(x))$. We may thus assume without loss of generality that $\alpha_1 = 0$. Now we can choose arbitrarily large numbers $t \in \mathbb{R}$ such that $f(t)f'(t)f(-t)f'(-t) \neq 0$. By Theorem 1 we have

$$A(a,b;f) \leq \lim_{t\to\infty} A(-t,t;f) \leq \lim_{t\to\infty}(A(-t,t;f')+\psi(-t)+\psi(t)). \tag{8}$$

Here $\psi(x) = |\mathrm{Arg}(f'(x)/f(x)|$, according to (6). Since $\alpha_1 = 0$,

$$f'(x) = \sum_{k=2}^{m} c_k\,\alpha_k \exp(\alpha_k z).$$

Hence

$$\lim_{t\to\infty}\psi(t) = \lim_{t\to\infty}\left|\mathrm{Arg}\left(\frac{c_m\alpha_m e^{\alpha_m t}(1+\theta_1(t))}{c_m e^{\alpha_m t}(1+\theta_2(t))}\right)\right|$$

where $\theta_1(t)$ and $\theta_2(t)$ tend to zero if $t \to \infty$. So $\lim\limits_{t\to\infty}\psi(t) = 0$. Similarly

$$\lim_{t\to\infty}\psi(-t) = \lim_{t\to\infty}\left|\mathrm{Arg}\left(\frac{c_2\alpha_2 e^{-\alpha_2 t}(1+\theta_3(t))}{c_1(1+\theta_4(t))}\right)\right| = |\mathrm{Arg}(c_2/c_1)|.$$

So (8) becomes

$$A(a,b;f) \leq \lim_{t\to\infty} A(-t,t;f') + |\mathrm{Arg}(c_2/c_1)|. \tag{9}$$

By the induction hypothesis we have

$$A(-t,t;f') \leq \sum_{k=2}^{m-1} |\mathrm{Arg}(\alpha_{k+1}c_{k+1}/\alpha_k c_k)| = \sum_{k=2}^{m-1} |\mathrm{Arg}(c_{k+1}/c_k)|, \tag{10}$$

since the exponential sum $f'$ has only m-1 terms. On inserting (10) in (9) the desired result is obtained.

Since $\pi\, N_{\mathbb{R}}(f) \le \lim_{t\to\infty} A(-t,t;f)$, Theorem 2 yields (2). This shows that Theorem 2 is a pure generalization of Descartes' rule. The next result combines Theorem 2 with a method of WILDER [3].

<u>THEOREM 3</u>. *Let* $f(z)$ *be given by* (7). *The number* $N_E(f)$ *of zeros of* $f$ *in the strip*

$$E = \{z \in \mathbb{C};\ r \le \operatorname{Im} z \le s\}$$

*satisfies*

$$|2\pi N_E(f)-(s-r)(\alpha_m-\alpha_1)| \le \sum_{k=1}^{m-1} \left|\left(\frac{c_{k+1}\exp(ir\alpha_{k+1})}{c_k\exp(ir\alpha_k)}\right)\right| + \tag{11}$$

$$+ \left|\operatorname{Arg}\left(\frac{c_{k+1}\exp(is\alpha_{k+1})}{c_k\exp(is\alpha_k)}\right)\right|.$$

<u>PROOF</u>. Choose a number $\varepsilon$ with $0 < \varepsilon < 1$. There is a $T > 0$ such that

$$\begin{aligned} f(x+iy) &= c_m \exp(\alpha_m(x+iy))(1+\theta_1(x,y)) \\ f(-x+iy) &= c_1 \exp(\alpha_1(-x+iy))(1+\theta_2(x,y)) \\ f'(x+iy) &= c_m\alpha_m \exp(\alpha_m(x+iy))(1+\theta_3(x,y)) \\ f'(-x+iy) &= c_1\alpha_1 \exp(\alpha_1(-x+iy))(1+\theta_4(x,y)), \end{aligned} \tag{12}$$

where $|\theta_i(x,y)| < \varepsilon$ for $r \le y \le s$ and $x \ge T$ $(i = 1,\dots,4)$. Since $\varepsilon < 1$, we find by Rouché's theorem that all zeros of $f$ in $E$ lie in the smaller set

$$E_1 = \{z \in \mathbb{C};\ r \le \operatorname{Im} z \le s, |\operatorname{Re} z| \le T\}.$$

Denote by $\delta E_1$ the (positively oriented) boundary of $E_1$. Under the assumption that there are no zeros of $f$ on the boundary of $E$ and thus on $\delta E_1$, we have

$$2\pi N_E(f) = 2\pi N_{E_1}(f) = -i\int_{\delta E_1}\left(\frac{f'(z)}{f(z)}\right)dz = \operatorname{Im}\left(\int_{\delta E_1}\frac{f'(z)}{f(z)}\,dz\right) =$$

$$= \operatorname{Im}\int_{-T}^{T}\left(\frac{f'(x+ir)}{f(x+ir)} - \frac{f'(x+is)}{f(x+is)}\right)dx + \operatorname{Re}\left(\int_r^s\left(\frac{f'(T+iy)}{f(T+iy)} - \frac{f'(-T+iy)}{f(-T+iy)}\right)dy\right).$$

By (12)

$$\mathrm{Re}\left(\int_r^s \frac{f'(T+iy)}{f(T+iy)}\,dy\right) = \mathrm{Re}\left(\int_r^s \alpha_m\left(\frac{1+\theta_1(T,y)}{1+\theta_2(T,y)}\right)dy\right) = (s-r)\alpha_m(1+\theta_5),$$

where $|\theta_5| \le \frac{1+\varepsilon}{1-\varepsilon} - 1 = \frac{2\varepsilon}{1-\varepsilon}$. Similarly

$$\mathrm{Re}\left(\int_r^s \frac{f'(-T+iy)}{f(-T+iy)}\,dy\right) = (s-r)\alpha_1(1+\theta_6),$$

where $|\theta_6| \le 2\varepsilon/(1-\varepsilon)$. So we find after some simple estimations

$$(13) \qquad |2\pi N_E(f)-(s-r)(\alpha_m-\alpha_1)| \le A(-T,T;f(x+ir))+A(-T,T;f(x+is))+\theta_7,$$

where $|\theta_7| \le 4\varepsilon/(1-\varepsilon)$. Now

$$f(x+ir) = \sum_{k=1}^m c_k \exp(ir\alpha_k) \exp(\alpha_k x)$$

$$f(x+is) = \sum_{k=1}^m c_k \exp(is\alpha_k) \exp(\alpha_k x)$$

and a straigthforward application of Theorem 2 transforms (13) into

$$(14) \qquad |2\pi N_E(f) - (s-r)(\alpha_m-\alpha_1)| \le$$

$$\le \sum_{k=1}^{m-1} \left|\mathrm{Arg}\left(\frac{c_{k+1}\exp(ir\alpha_{k+1})}{c_k\exp(ir\alpha_k)}\right)\right| + \sum_{k=1}^{m-1} \left|\mathrm{Arg}\left(\frac{c_{k+1}\exp(is\alpha_{k+1})}{c_k\exp(is\alpha_k)}\right)\right| + \theta_7.$$

Letting $\varepsilon \downarrow 0$ the theorem is proved under the assumption that no zeros of $f$ lie on the boundary of $E$. We now drop this assumption. There is a $t > 0$ such that for $0 < \rho < t$ the strips

$$E(\rho) := \{z \in \mathbb{C};\ r-\rho \le \mathrm{Im}\, z \le s+\rho\}$$

have no zeros on their boundaries. We thus find that $N_E(f) = N_{E(\rho)}(f)$ for $0 < \rho < t$, whereas

$$|2\pi N_{E(\rho)}(f) - (s-r+2\rho)(\alpha_m-\alpha_1)| \le$$

$$\le \sum_{k=1}^{m-1} \left|\operatorname{Arg}\left(\frac{c_{k+1}\exp(i(r-\rho)\alpha_{k+1})}{c_k\exp(i(r-\rho)\alpha_k)}\right)\right| + \left|\operatorname{Arg}\left(\frac{c_{k+1}\exp(i(s+\rho)\alpha_{k+1})}{c_k\exp(i(s+\rho)\alpha_k)}\right)\right| .$$

By letting $\rho \downarrow 0$ and observing that in the above inequality both terms are continuous functions of $\rho$ the theorem is proved completely.

The following lemma is useful for the evaluation of (11).

LEMMA 4. *For complex* $y,z \neq 0$ *we have*

$$|\operatorname{Arg}(yz) \le |\operatorname{Arg}(y)| + |\operatorname{Arg}(z)| \le 2\pi - |\operatorname{Arg}(y/z)|.$$

PROOF. Since $\operatorname{Arg}(\zeta) \in (-\pi,\pi]$, we have either

$$\operatorname{Arg}(yz) = \operatorname{Arg}(y)+\operatorname{Arg}(z)$$

or

$$|\operatorname{Arg}(y) + \operatorname{Arg}(z)| > \pi.$$

In both cases we trivially have

$$|\operatorname{Arg}(yz)| \le |\operatorname{Arg}(y)| + |\operatorname{Arg}(z)|.$$

The following relations are easily checked

$$|\operatorname{Arg}(-\zeta)| = \pi - |\operatorname{Arg}(\zeta)|; \ |\operatorname{Arg}(1/\zeta)| = |\operatorname{Arg}(\zeta)|.$$

Hence

$$|\operatorname{Arg}(y)| + |\operatorname{Arg}(z)| = 2\pi-(|\operatorname{Arg}(-y)| + |\operatorname{Arg}(-1/z)|) \le 2\pi - |\operatorname{Arg}(y/z)|.$$

From Theorem 3 we now derive

COROLLARY 1. *Let* f *and* E *be as in Theorem* 3. *If* $r < 0 < s$, *then*

$$\pi N_E(f) \le \sum_{k=1}^{m} |\operatorname{Arg}(c_{k+1}/c_k)| + (s-r)(\alpha_m-\alpha_1).$$

PROOF. We have by Lemma 4

$$\left|\mathrm{Arg}\left(\frac{c_{k+1}\exp(ir\alpha_{k+1})}{c_k\exp(ir\alpha_k)}\right)\right| \le \left|\mathrm{Arg}\left(\frac{c_{k+1}}{c_k}\right)\right| + \left|\mathrm{Arg}(\exp(ir(\alpha_{k+1}-\alpha_k)))\right| \le$$

$$\le \left|\mathrm{Arg}\left(\frac{c_{k+1}}{c_k}\right)\right| + |r|(\alpha_{k+1}-\alpha_k) = \left|\mathrm{Arg}\left(\frac{c_{k+1}}{c_k}\right)\right| - r(\alpha_{k+1}-\alpha_k).$$

By inserting these and similar estimates in (11) the corollary follows.

COROLLARY 2. *Let* f, E *be as in Theorem* 3. *Then*

$$|2\pi N_E(f)-(s-r)(\alpha_m-\alpha_1)| \le 2\pi(m-1) - \sum_{k=1}^{m-1} |\mathrm{Arg}(\exp(i(s-r)(\alpha_{k+1}-\alpha_k)))|.$$

PROOF. By Lemma 4

$$\left|\mathrm{Arg}\left(\frac{c_{k+1}\exp(ir\alpha_{k+1})}{c_k\exp(ir\alpha_k)}\right)\right| + \left|\mathrm{Arg}\left(\frac{c_{k+1}\exp(is\alpha_{k+1})}{c_k\exp(is\alpha_k)}\right)\right| \le$$

$$\le 2\pi - |\mathrm{Arg}(\exp(i(s-r)(\alpha_{k+1}-\alpha_k)))|$$

giving the desired inequality by insertion in (11).

COROLLARY 3. *Let* f, E *be as in Theorem* 3. *If* $(s-r)(\alpha_{k+1}-\alpha_k) < \pi$ *for* $k = 1,\ldots,m-1$, *then*

$$N_E(f) \le m-1.$$

PROOF. By Corollary 2 and the fact that $\mathrm{Arg}(\exp(i(s-r)(\alpha_{k+1}-\alpha_k))) = (s-r)(\alpha_{k+1}-\alpha_k)$ we find

$$|2\pi N_E(f)-(s-r)(\alpha_m-\alpha_1)| \le 2\pi(m-1)- \sum_{k=1}^{m-1} (s-r)(\alpha_{k+1}-\alpha_k).$$

Hence,

$$2\ N_E(f)-(s-r)(\alpha_m-\alpha_1) \le 2\pi(m-1)-(s-r)(\alpha_m-\alpha_1).$$

## 4. EXPONENTIAL INTEGRALS

Let g be a not identically vanishing continuous complex function on the interval $I=[0,1]$ and let $\mu$ be a positive measure on I. We consider not identically vanishing functions of the form

$$(15) \qquad f(z) = \int_0^1 e^{zt} g(t)\mu(dt).$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ with $0 = x_0 < x_1 < \ldots < x_n = 1$ be a partition of I and put $I_k := [x_{k-1}, x_k)$ for $k = 1, \ldots, n-1$; $I_n := [x_{n-1}, x_n]$. We now construct the exponential sums

$$(16) \qquad F_P(z) = \sum_{k=1}^n e^{x_k z} g(x_k)\mu(I_k)$$

and observe that

$$|F_P(z) - f(z)| = |\sum_{k=1}^n \int_{I_k} (e^{x_k z} g(x_k) - e^{tz} g(t))\mu(dt)| \le$$

$$\le \mu(I) \max_{k=1,\ldots,n} \max_{t \in I_k} |e^{x_k z} g(x_k) - e^{tz} g(t)|.$$

Since $g(t)e^{zt}$ is uniformly continuous on I, the following lemma is immediately proved by letting $\max_k |x_k - x_{k-1}| \to 0$.

<u>LEMMA 5</u>. *Let* $D \subset \mathbb{C}$ *be bounded. Then there is a sequence* $P_1, P_2, \ldots$ *of partitions of* I *such that* $\lim_{k\to\infty} (F_{P_k}(z)) = f(z)$ *uniformly for* $z \in D$.

We thus have by Lemma 2 that if $(a,b) \subset \mathbb{R}$ such that $f(a)f(b) \neq 0$

$$A(a,b;f) \le \sup_P A(a,b;F_P).$$

Let the sequence $y_1, \ldots, y_m$ be derived from $x_1, \ldots, x_n$ by deleting those $x_k$ for which $g(x_k)\mu(I_k) = 0$. Then by Theorem 2 if P is chosen such that $\{y_1, \ldots, y_m\} \neq \emptyset$,

$$A(a,b;F_P) \le \sum_{k=1}^{m-1} |\mathrm{Arg}(g(y_{k+1})/g(y_k))|.$$

Now define

$$A^*(g) = \sup_P \sum_{k=1}^{m-1} |\mathrm{Arg}(g(y_{k+1})/g(y_k))|,$$

the supremum being taken over all partitions P. If arg(g), the argument of g, is defined as a right-continuous function, left continuous whenever $g(x) \neq 0$ and such that $\lim_{\varepsilon \downarrow 0} |\arg(g(x+\varepsilon)) - \arg(g(x-\varepsilon))| \leq \pi$ whenever $g(x) = 0$, then $A^*(g)$ is the total variation of $\arg(g(x))$ on $[0,1]$.

As an immediate consequence of Lemma 2, Theorem 2 and Lemma 5 we have the following theorem

THEOREM 4. *Let* $f(z)$ *be given by* (15). *For all* $a < b \in \mathbb{R}$ *such that* $f(a)f(b) \neq 0$

$$A(a,b;f) \leq A^*(g).$$

PROOF. Straigthforward.

COROLLARY 4. *The number* $N_{\mathbb{R}}(f)$ *of real zeros of* f *satisfies*

$$N_{\mathbb{R}}(f) \leq \frac{1}{\pi} A^*(g).$$

In order to estimate the number of zeros of f in a horizontal strip, we derive the following lemma, somewhat similar to Lemma 2.

LEMMA 6. *Let* $f_1, f_2, \ldots$ *be a sequence of analytic complex-valued functions uniformly converging to* f *in a compact set* $K \subset \mathbb{C}$. *If* f *has no zeros on the boundary of* K, *then there is an* $N > 0$ *such that for* $k > N$ *the functions* f *and* $f_k$ *have the same number of zeros in* K.

PROOF. Denote the boundary of K by $\delta K$. There is an $\varepsilon > 0$ such that $|f(z)| > \varepsilon$ for $z \in \delta K$. There is an $N > 0$ such that for $k > N$ and $z \in \delta K$

$$|f_k(z)-f(z)| < \varepsilon.$$

So by Rouché's theorem, $f_k(z)$ and $f(z)$ have the same number of zeros in K.

We now derive the following theorem

THEOREM 5. *Let* $f(z)$ *be given by* (15) *and let* $r,s \in \mathbb{R}$ *such that* $r < 0 < s$. *Then the number* $N_E(f)$ *of zeros of* f *in the strip*

$$E = \{z \in \mathbb{C};\ r \leq \operatorname{Im} z \leq s\}$$

*satisfies*

$$\pi N_E(f) \leq A^*(g) + s-r.$$

PROOF. Suppose that f has no zeros on the boundary of E. Choose $T > 0$ such that f has no zeros on the boundary of $E_1$, where

$$E_1 = \{z \in C;\ r \leq \operatorname{Im} z \leq s;\ |\operatorname{Re}(z)| \leq T\}.$$

By Lemma's 5 and 6

$$N_{E_1}(f) = \lim_{k\to\infty} N_{E_1}(F_{P_k}).$$

By Corollary 1, since $y_m - y_1 \leq 1$ for all partitions P

$$N_{E_1}(F_{P_k}) \leq N_E(F_{P_k}) \leq A^*(g) + s-r.$$

By letting $T \to \infty$ the theorem follows. If f has zeros on the boundary of E, there is a $\delta_1 > 0$ such that for $0 < \delta < \delta_1$ f has no zeros on the boundary of $E_\delta$, where

$$E_\delta = \{z \in \mathbb{C};\ r-\delta \leq \operatorname{Im} z \leq s+\delta\}.$$

Now $N_E(f) \leq N_{E_\delta}(f) \leq A^*(g) + s-r+2\delta$.

By letting $\delta \downarrow 0$ the theorem is proved completely.

## 5. REMARKS

1. Let $a,b \in \mathbb{C}$, $\Gamma$ the rectilinear segment running from a to b, $\mu$ a positive measure and g a complex-valued continuous functions on $\Gamma$. If

$$f(z) = \int_\Gamma g(\zeta)e^{z\zeta}\mu(d\zeta),$$

then

$$f(z) = e^{az}\int_0^1 g(a+(b-a)t)e^{(b-a)tz}\mu^*(dt) = e^{az}h((b-a)z).$$

Here $\mu^*$ is a positive measure on [0,1] derived from $\mu$. We can apply the results of section 4 to h and so derive corresponding results for f.

2. The representation (15) is not unique; if h is a positive $\mu$-measurable function on [0,1],

$$\int_0^1 e^{zt} g(t)\mu(dt) = \int_0^1 e^{zt}(g(t)/h(t))\mu^*(dt),$$

where $\mu^*$, defined by $\mu^*([\alpha,\beta]) = \int_\alpha^\beta h(t)\mu(dt)$, is again a positive measure. We can thus somewhat alleviate the condition that g is continuous, by letting $h \to \infty$ at the points of discontinuity of g, h continuous everywhere else.

3. The method in this paper is not limited to exponential sums and integrals; for instance Dirichlet series

$$\sum_{k=1}^{\infty} c_k \exp(-\lambda_k x)$$

and infinite integrals

$$\int_0^\infty g(t)\, e^{-zt} dt$$

can be represented as limits of exponential sums. Proceeding as in section 4, analogues of Theorem 4 can be derived for these functions. Compare PÓLYA & SZEGÖ [1], Chapter 5, section 1, especially problems 78 and 80.

## 6. REFERENCES


1. G. PÓLYA and G. SZEGÖ, *Aufgaben und Lehrsätze ans der Analysis II*, Springer Verlag, Berlin, 1971.

2. M. VOORHOEVE, *On the oscillation of exponential polynomials*, Math. Z. 151, (1976), 277-294.

3. C.E. WILDER, *Expansion problems of ordinary linear differential equations*, Trans. Amer. Math. Soc. 18 (1917), 415-442.