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SOME LOTTO NUMBERS FROM AN EXTENSION OF TURÁN'S THEOREM

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Some lotto numbers from an extension of Turán's theorem

by

A.E. Brouwer

ABSTRACT

Turán's theorem implies that the minimum possible number of edges of a graph G on 2m points with maximum stable set of size two is m(m-1), and that this minimum is attained only when G is the union of two m-cliques. Here we show that when G has no more than m^2-2 edges still the conclusion holds that G is the union of two cliques (but not necessarily of equal size). As a corollary the lotto numbers L(n,3,3,2) are determined. (I.e., the minimum number of lotto forms of size three on has to complete in order to be sure of two correct answers when three numbers are drawn.)

KEY WORDS & PHRASES: *lotto number, Turán*

1. VARIATIONS ON A THEME OF TURÁN

Turán showed that if G is a graph with independence number $\alpha(G) \le t$ on n vertices then (writing $m := \frac{n}{\lfloor t \rfloor}$) G has at least

$$T(n,t) := ((m+1)t-n)\binom{m}{2} + (n-mt)\binom{m+1}{2}$$

$$= t\binom{m}{2} + (n-mt)m$$

edges, and if equality holds then G is a union of complete graphs K_m or K_{m+1}. We can slightly strengthen this result by showing that if G has no more than $T(n,t) + m-2$ edges then G is spanned by the union of t cliques. First an auxiliary result:

<u>THEOREM 1</u>. *Let* G *be a graph on* n *points with at most* $T(n,t) + m-1$ *edges. Then* $V(G)$ *can be written as union of* t *cliques in at most one way.*

<u>PROOF</u>. Suppose $V(G) = \bigcup_{i=1}^{t} C_i = \bigcup_{i=1}^{t} C_i'$ are two partitions of $V(G)$ into t cliques. If $C_j \subsetneq C_j'$ for some j, and $|C_j| = m-s$ then $\Sigma_{i=1}^{t} \binom{|C_i|}{2} \ge T(n,t) + \binom{s+1}{2}$ and C_j' contains at least $m-s$ edges not contained in any C_i, so that

$$\binom{s+1}{2} + m-s \le m-1, \qquad \text{a contradiction.}$$

Next, if $C_j \neq C_i'$ for all i, then C_j is covered by some cliques C_i' none of which is contained within C_j, so that again each point of C_j is adjacent to at least one point not in C_j, and we reach the same contradiction. Hence the two partitions were the same. $\square$

<u>REMARK</u>. The bound is best possible: if G_0 is the graph with $\alpha(G_0) = t$ and $T(n,t)$ edges, and G is obtained from G_0 by joining a fixed point to all the points of an m-clique (not containing that point) then G has $T(n,t)+m$ edges and can be written as a union of t cliques in two ways.

<u>THEOREM 2</u>. *Let* G *be a graph on* n *points such that* $\alpha(G) \le t$, *and* $n > 2t$. *If* G *has no more than* $T(n,t) + m-2$ *edges (where* $m := \frac{n}{\lfloor t \rfloor}$*) then* $V(G)$ *can be written as union of* t *cliques. Moreover, this bound is best possible.*

PROOF. Induction on n. Let G be a graph with $\alpha(G) \le t$ and at most $T(n,t)+m-2$ edges, and assume that $V(G)$ is not the union of t cliques. Then $m > 2$. If x is a point of G with valency at least m, then let G_x be the graph induced by G on $V(G)\setminus\{x\}$. By hypothesis of induction $V(G_x)$ can be written as union of t cliques: $V(G_x) = \cup_{i=1}^t C_i(x)$. If x and y are two vertices with valency at least m, and G_{xy} is the graph induced by G on $V(G)\setminus\{x,y\}$ then $V(G_{xy}) = \cup_i (C_i(x)\setminus\{y\}) = \cup_i (C_i(y)\setminus\{x\})$. But by Theorem 1 there is at most one partition of $V(G_{xy})$ into cliques, hence w.l.o.g. we have $C_i(x)\setminus\{y\} = C_i(y)\setminus\{x\}$ for all i. If $x \in C_{i_0}(y)$ and $y \in C_{i_1}(x)$ and $i_0 \ne i_1$ then $V(G) = C_{i_0}(y) \cup \cup_{i\ne i_0} C_i(x)$ is a partition of $V(G)$ into t cliques. Hence $i_0 = i_1$. Now let z be a third point with valency at least m. Again we find $C_i(x)\setminus\{z\} = C_i(z)\setminus\{x\}$ for all i and $x \in C_j(z) \rightarrow z \in C_j(x) \rightarrow z \in C_j(y) \rightarrow y \in C_j(z)$ so that x and y are adjacent. But then $V(G) = (C_{i_0}(x) \cup \{x\}) \cup \cup_{i\ne i_0} C_i(x)$ is the required partition. It follows that G contains at most two points with valency at least m. In particular G does not contain cliques of size m+1 (since $m \ge 2$). Since by Turán's theorem G contains at least $T(n,t)$ edges, and G cannot contain exactly $T(n,t)$ edges, there is at least one point x with valency at least m. Now either $n = mt+1$ and $|C_i(x)| = m$ for all i, or $n = mt$ and $|C_i(x)| = m$ for $i \ne i_0$, $|C_{i_0}(x)| = m-1$. In the first case at most one point can be adjacent to x, a contradiction. In the second case it follows in the same way that x is adjacent to each point of C_{i_0}, so that $V(G)$ is again partitionable into t cliques.

To see that the bound is best possible: let G be the union of a point x and t cliques C_i $(1\le i\le t)$ each of size $\lfloor\frac{n-1}{t}\rfloor$ or $\lceil\frac{n-1}{t}\rceil$, where x is joined to all points of $C_1\setminus\{x_1\} \cup C_2\setminus\{x_2\}$ (where $x_i \in C_i$, $i = 1,2$), and x_1 is joined to x_2. If $|C_1| = \lfloor\frac{n-1}{t}\rfloor$ and $|C_2| = \min_{i\ge 2} |C_i|$ then G has $T(n,t)+m-1$ edges, but is not spanned by t cliques. □

REMARK. If $n < 2t$ then no graph G with $\alpha(G) \le t$ has no more than $T(n,t) + m-2$ edges, while for $n = 2t$ the ladder graph is the only example. In these cases the bound of the theorem is not best possible: for $n = 4$, $t = 2$ any G is union of two K_2's; for $n = 6$, $t = 3$ the smallest counterexample is the pentagon plus isolated point (Seidel graph) with $T(6,3)+2 = 5$ edges.

THEOREM 3. *Let* G *be a graph on* $n = 2m+1 > 5$ *points, without loops but possibly with repeated edges, such that* $\alpha(G) \le 2$ *and all valencies are even. Then* G *has at least* $m^2 + \lceil\frac{m}{2}\rceil$ *edges, with equality iff* G *is the union of an* m-*clique and an* (m+1)-*clique (where in one of the cliques the edges of a complete matching are repeated in order to make the valencies even).*

PROOF. Let $m > 3$ and suppose G has at most $m^2 + \lceil\frac{m}{2}\rceil \le m^2 + m-2 = T(n,2) + m-2$ edges. By Theorem 2 V(G) is union of two cliques: $K_{2s} \cup K_{n-2s}$, and in order to make all valencies even we need at least s more edges. Hence G has at least $4s^2 - s(2n-1) + \frac{1}{2}n(n-1)$ edges, which is minimal for $s = \lceil\frac{m}{2}\rceil$ and then reduces to $m^2 + \lceil\frac{m}{2}\rceil$.

For $m = 3$ we have to look into some more detail: Let G be a graph on 7 points with at most 11 edges, all valenceies even and with $\alpha(G) \le 2$. If some point is isolated, then the remaining points form a K_6 which has 15 edges, impossible. If some point has valency 6 then G has at least $6 + T(6,2) = 12$ edges, impossible. Hence only valencies two and four occur, and at least three points (say x_1, x_2, x_3) have valency two. The points not connected to x_1 form a K_4, hence x_2 and x_3 are connected to x_1, and for the same reason is x_2 adjacent to x_3. But now V(G) is union of a K_3 and a K_4. □

REMARK. For $m = 2$ the conclusion of the theorem does not hold, as is shown by the pentagon.

If $n = 2m$ then when m is odd, $G = K_m \cup K_m$ has T(n,2) edges and all valencies even; when m is even $G = K_{m-1} \cup K_{m+1}$ has $T(n,2) + 1$ edges and is optimal.

2. LOTTO NUMBERS

DEFINITION. (Covering numbers). C(t,k,v) is the minimum number of k-subsets of a v-set such that each of its t-subsets is covered by (i.e., subset of) at least one of these k-sets.

FORT & HEDLUND [1] showed that for all n

$$C(2,3,n) = \left\lceil\frac{n}{3}\left\lceil\frac{n-1}{2}\right\rceil\right\rceil,$$

which is the only result we shall need here.

DEFINITION. (Lotto numbers). $L(n,k,\ell,t)$ is the minimum cardinality of a collection of k-subsets of an n-set such that for each ℓ-subset of this n-set there is a k-set in the collection that has at least t elements with it in common.

The purpose of this note is to prove

THEOREM 4.

$$L(4m,3,3,2) = C(2,3,2m-1) + C(2,3,2m+1)$$

$$L(4m+2,3,3,2) = 2\cdot C(2,3,2m+1)$$

$$L(2m+1,3,3,2) = C(2,3,m) + C(2,3,m+1).$$

PROOF. Obviously the right hand sides are upper bounds for the left hand sides: e.g., $L(2m+1,3,3,2) \le C(2,3,m) + C(2,3,m+1)$, since if $X = Y_1 \cup Y_2$, $X = 2m+1$, $|Y_1| = m$, $|Y_2| = m+1$ and we cover all pairs in Y_1 and Y_2 and somebody chooses a triple T in X, then T has at least two points in one the Y_i and hence intersects one of the chosen triples in a pair.
Remains to show that we cannot do better. Let us first compute the right hand sides explicitly: for $n = 12t + r$ $(0 \le r \le 11)$ we find

$$\text{R.H.S.} = \begin{cases} 12t^2 - 2t + 1 & \text{for } r = 0 \\ 12t^2 + t & \text{for } r = 1 \\ 12t^2 + 2t & \text{for } r = 2 \\ 12t^2 + 5t + 1 & \text{for } r = 3 \\ 12t^2 + 6t + 1 & \text{for } r = 4 \\ 12t^2 + 9t + 2 & \text{for } r = 5 \\ 12t^2 + 10t + 2 & \text{for } r = 6 \\ 12t^2 + 13t + 4 & \text{for } r = 7 \\ 12t^2 + 14t + 5 & \text{for } r = 8 \\ 12t^2 + 17t + 7 & \text{for } r = 9 \\ 12t^2 + 18t + 8 & \text{for } r = 10 \\ 12t^2 + 21t + 10 & \text{for } r = 11. \end{cases}$$

(This follows from

$$C(2,3,6t+r) = \begin{cases} 6t^2 - 3t + 1 & \text{for } r = -1 \\ 6t^2 & \text{for } r = 0 \\ 6t^2 + t & \text{for } r = 1 \\ 6t^2 + 4t + 1 & \text{for } r = 2 \\ 6t^2 + 5t + 1 & \text{for } r = 3 \\ 6t^2 + 8t + 3 & \text{for } r = 4 \\ 6t^2 + 9t + 4 & \text{for } r = 5 \\ 6t^2 + 12t + 6 & \text{for } r = 6.) \end{cases}$$

Now when we have a lotto system with $L(n,3,3,2)$ triples, and we replace each triple by the three pairs it contains, then we obtain a graph G on n points with all valencies even and such that it has an edge in any triple, i.e., $\alpha(G) \leq 2$. This proves that $L(n,3,3,2) \geq \frac{1}{3} T(n,2)$. For $n \equiv 0 \pmod 4$ we can improve this to $L(n,3,3,2) > \frac{1}{3} T(n,2)$ since equality would mean that G was the union of two $K_{\frac{1}{2}n}$'s, but in that case G has odd valency. For odd $n = 2m+1 > 5$ we can use Theorem 3 to get $L(n,3,3,2) \geq \frac{1}{3}(m^2 + \lceil \frac{m}{2} \rceil)$. These considerations prove that $L(n,3,3,2)$ is at least the expression given in the theorem except for $n \equiv 9$ or $10 \pmod{12}$.

For $n = 12t+9$ and $m = 6t+4$ we find

$$L(n,3,3,2) \geq \frac{1}{3}(m^2 + \frac{1}{2}m) = 12t^2 + 17t + 6.$$

If equality would hold, G was union of $(K_{6t+4}$ + matching) and K_{6t+5}. But by construction G is a union of triangles, so K_{6t+5} must be a union of triangles. Since it has $\binom{6t+5}{2} = 18t^2 + 27t + 10 \not\equiv 0 \pmod 3$ edges, this is not the case. Hence $L(12t+9,3,3,2) = 12t^2 + 17t + 7$.

For $n = 12t+10$ we find

$$L(n,3,3,2) \geq \lceil \frac{1}{3} T(n,2) \rceil = \lceil \frac{1}{3}(36t^2 + 54t + 20) \rceil = 12t^2 + 18t + 7.$$

If equality would hold, G had $T(n,2) + 1$ edges and by Theorem 2 is either $K_{6t+4} \cup K_{6t+6}$ (impossible because of odd valencies) or $K_{6t+5} \cup K_{6t+5}$ with one extra edge (repeated or not). But we saw already that K_{6t+5} is not a union of triangles. Hence $L(12t+10,3,3,2) = 12t^2 + 18t + 8$.
Finally for $1 \leq n \leq 5$ the theorem is verified immediately. □

REMARK 1. In [2] NOVÁK studied the problem of finding the minimal cardinality of a collection C of triples such that any further triple intersects one of the triples in C in a pair, while no two triples in C have a pair in common. This produces similar results. (Obviously $|C| \geq L(n,3,3,2)$, and strict inequality holds when for the optimal lotto system the graph G has repeated edges.)

REMARK 2. Surely it is possible to determine $L(n,3,\ell,2)$ for general ℓ in the same way. This will yield (for $n \geq \ell$):

$$L(n,3,\ell,2) = \sum_{i=1}^{\ell-1} C(2,3,n_i)$$

where the n_i are determined by

(i) $$n = \sum_{i=1}^{\ell-1} n_i$$

(ii) $$|n_i - n_j| \leq 2 \qquad \text{for } 1 \leq i < j \leq \ell-1$$

(iii) $$n_i \equiv 1 \pmod 2 \qquad \text{for } i > 1.$$

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