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THE UNIQUENESS OF THE NEAR HEXAGON ON 759 POINTS

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The uniqueness of the near hexagon on 759 points

by

A.E. Brouwer

ABSTRACT

We show that the unique near hexagon with $s = 2$ and $t = 14$ and $t_2 = 2$ is the one with the blocks of the Steiner system $S(5,8,24)$ as vertices and sets of three pairwise disjoint blocks as lines.

KEY WORDS & PHRASES: *Steiner system, near hexagon*

INTRODUCTION

A *near hexagon* is a partial linear space (X,L) such that

a. For any point $p \in X$ and line $\ell \in L$ there is a unique point on ℓ nearest p.

b. Every point is on at least one line.

c. The distance between any two points is at most three.

(The distances are measured in the point graph: $d(p,q) = 1$ iff p and q are collinear.)

A *regular* near hexagon with parameters (s,t,t_2) is a near hexagon such that each line contains 1+s points, and each point is in 1+t lines, and a point at distance 2 from a fixed point x_o is in $1+t_2$ lines containing a neighbour of x_o.

SHULT & YANUSHKA [1] showed that there are exactly eleven possibilities for the parameters of a regular near hexagon with s=2. For nine parameter sets the corresponding near hexagons have been classified completely. Here we settle one of the two remaining cases by showing that there is a unique regular near hexagon with parameters $(s,t,t_2) = (2,14,2)$. As SHULT & YANUSHKA indicate an example is given by the 759 blocks of the Steiner system S(5,8,24), where lines are triples of pairwise disjoint blocks. One finds that distance 0,1,2,3 in the point graph corresponds to blocks intersecting in 8,0,4,2 points, respectively. Here we prove that this is the only example. [Note that WITT [2] proved the uniqueness of S(5,8,24).] (The last open case is $(s,t,t_2) = (2,11,1)$, v = 729 where an example can be found from the ternary Golay code. Most likely this example is unique as well.)

1. STRUCTURE OF THE SYSTEM W.R.T. A POINT.

Let x_0 be any point of the near hexagon H.

Let $k_i = |\Gamma_i(x_0)| = |\{x | d(x,x_0) = i\}|$.

Then

$k_0 = 1$

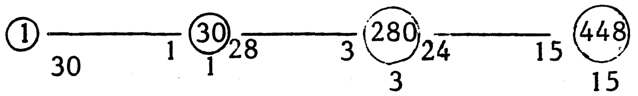
$k_1 = 30 \quad (= s(t+1))$

$k_2 = 280 \quad (= k_1.s.t/(t_2+1))$

$k_3 = 448 \quad (= k_2.s.(t-t_2)/t)$

so that $v = \sum k_i = 759$.

Diagram of the distance regular point graph:



It is an association scheme with intersection numbers (p^k_{ij}) where

$$(p^k_{oj})_{jk} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (p^k_{1j}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 30 & 1 & 3 & 0 \\ 0 & 28 & 3 & 15 \\ 0 & 0 & 24 & 15 \end{pmatrix}$$

$$(p^k_{2j}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 28 & 3 & 15 \\ 280 & 28 & 140 & 85 \\ 0 & 224 & 136 & 180 \end{pmatrix},$$

$$(p^k_{3j}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 24 & 15 \\ 0 & 224 & 136 & 180 \\ 448 & 224 & 288 & 252 \end{pmatrix}.$$

2. QUADS AND OVALS

Let us first recall some facts from SHULT & YANUSHKA [1]. Two points p,q at distance 2 determine a generalized quadrangle (possibly degenerated) $Q(p,q)$. If $\mu(p,q)$ is the set of common neighbours of p and q then $Q = Q(p,q)$ is the set of points with distance at most two to each point of $\{p,q\} \cup \mu(p,q)$. Any point adjacent to two points in Q is already inside Q. Points outside Q are of two types:

α) "classical type"

x is of classical type if there is a unique point $y \in Q$ closest to x. In this case $d(x,z) = d(x,y) + d(y,z)$ for each point $z \in Q$.

β) "Oval type"

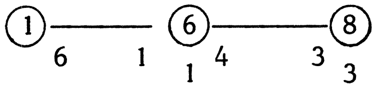
x is of oval type if the set of points in Q closest to x form an oval in Q, i.e., a set meeting each line of Q in exactly one point. In this case Q is regular and the oval has $1 + s_Q t_Q$ points.

Two quads (generalized quadrangles Q(p,q)) intersect in the empty set, a point, a line, or coincide.

Let us now apply this to our situation. Two points at distance 2 have 3 common neighbours, so our quads will be GQ(2,2)'s. Points at distance one from a quad are necessarily of classical type. In GQ(2,2) points at distance 2 occur, but in H distance 4 does not occur, so points at distance two from a quad are of oval type. No points have distance 3 from a quad. GQ(2,2) is unique up to isomorphism - an easy description is given by: vertices are the 15 unordered pairs of 6 objects, lines are formed by three pairwise disjoint pairs. We have v = b=15; in fact GQ(2,2) is self-dual.

There are six ovals, each containing five points, namely the sets of pairs containing a fixed object. Two nonadjacent points determine a unique oval, two ovals intersect in a unique point and each point is in two ovals.

Diagram of the quad:



Let us determine the structure of the system w.r.t. a quad. Let n_i be the number of points at distance i from Q. Then

$$n_0 = 15$$
$$n_1 = 360$$
$$n_2 = 384$$

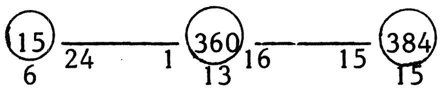
(for: given a point $x \in Q$, it is incident with 15 lines, 3 inside Q, 12 leave, so x has 24 neighbours outside Q, and there are 15.24 = 360 points adjacent to Q).

<u>CLAIM</u>. There are no lines disjoint from Q with 2 points adjacent to Q.

<u>PROOF</u>. Suppose xyz is a line disjoint from Q, and $x \sim x'$, $y \sim y'$ with $x',y' \in Q$. Then $d(x,y') = 2 = 1 + d(x',y')$, so $x' \sim y'$. If z has distance 2 to Q then z determines an oval O_z in Q containing the adjacent points x' and y'. Contradiction. □

Now by counting we see that if x is adjacent to Q then x is in 1 line intersecting Q, in 6 lines contained in $\Gamma_1(Q)$ (for: let $x \sim x' \in Q$. For each point $y' \in Q$ with $y' \sim x'$ we find three lines through x containing a neighbour of y', one of them intersecting Q and the other two in $\Gamma_1(Q)$, projecting onto the line x'y') and in 8 lines with 1 point in $\Gamma_1(Q)$ and 2 points in $\Gamma_2(Q)$. If $x \in \Gamma_2(Q)$ and x determines the oval $O = O_x$ in Q then the 15 common neighbours of x and some point of O are on distinct lines through x (by the previous claim). But $t+1 = 15$, so any line through x has 1 point in $\Gamma_1(Q)$ and two points in $\Gamma_2(Q)$.

Diagram of H w.r.t. a quad:



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3. TWO DISJOINT QUADS

CLAIM. Let Q and Q' be two disjoint quads. Then Q' contains 7 points at distance 1 from Q: a point x_0 and its six neighbours in Q'.

PROOF. Any line in Q contains 1 or 3 points at distance 1 from Q'. Let $Z = Q \cap \Gamma_1(Q')$. Then Z is a (possibly degenerate) generalized quadrangle: if ℓ is a line in Z and p a point in $Z \setminus \ell$ then there is a line m in Q containing p and intersecting ℓ. Since $|m \cap Z| \geq 2$ it follows that $m \subset Z$. Consequently we have the following possibilities:

(α) Z is an oval in Q.

(β) Z is the union of three concurrent lines.

(γ) Z is a GQ(2,1): a lattice with 9 points and 6 lines

(δ) Z = Q.

Let $Z' = Q' \cap \Gamma_1(Q)$. We saw above that adjacent points project to adjacent points, so Z' is isomorphic to Z.

Let us first rule out case (α).

Choose $x \in Q \setminus Z$. Then x determines an oval O_x in Q', and x is adjacent to exactly $|O_x \cap Z'|$ points of Z. But a point outside an oval is adjacent to three points of the oval, while two ovals intersect in 1 or 5 points. Contradiction.

The cases (γ) and (δ) are ruled out by counting. Let x_0 be a fixed point at distance 2 from the quad Q. Count the number of points adjacent to Q and at distance 2 from x_0 in two ways.

Let O be the oval in Q determined by x_0. Let $z \in Q$. If $z \notin O$ then $d(x_0,z) = 3$ and z has 15 neighbours at distance 2 from x_0. Three are in O, and the remaining 12 count. If $z \in O$ then $d(x_0,z) = 2$ and z has 3 neighbours at distance 2 from x_0. None of them is in Q.

Altogether we find $10.12 + 5.3 = 135$ points in $\Gamma_1(Q) \cap \Gamma_2(x_0)$.

On the other hand, consider quads Q' through x_0. There are 15 lines incident with x_0. and any two intersecting lines determine a quad Q', while Q' contains three lines through x_0. This shows that we have the structure of a STS(15) on lines and quads incident with x_0. (Later we shall see that in fact this STS(15) is PG(3,2).) In particular there are 35 quads incident with x_0. These quads are of three possible types:

a) intersecting Q

b) of type β: with 7 points adjacent to Q

c) of type γ: with 9 points adjacent to Q.

Let there be n_a,n_b,n_c quads of each type.

Then $n_a + n_b + n_c = 35$. Clearly $n_a = 5$. Now each point in $\Gamma_1(Q) \cap \Gamma_2(x_0)$ determines together with x_0 a unique quad Q'. Each quad of type a contains 3 such points (it has 1 point in Q and 6 points in $\Gamma_1(Q)$, 3 of which are adjacent to x_0), each quad of type b: 4 such points, and each quad of type c: 6 such points. Altogether we find

$$|\Gamma_1(Q) \cap \Gamma_2(x_0)| = 3n_a + 4n_b + 6n_c = 135.$$

Solving our equations yields $n_a = 5$, $n_b = 30$, $n_c = 0$, so quads of type γ do not exist.

In a similar way we dispose of type δ:

Let x be a point at distance one from Q. Count the number of points in $\Gamma_1(Q) \cap \Gamma_2(x)$ in two ways.

Let $z \in Q$. If $d(x,z) = 3$ then z has 15 neighbours at distance two from x, 3 in Q and 12 in $\Gamma_1(Q)$. There are 8 such points z. If $d(x,z) = 2$ then z has 3 neighbours in $\Gamma_2(x)$, one in Q and 2 in $\Gamma_1(Q)$. There are 6 such points z. If z is the unique neighbour of x in Q then z has 28 neighbours in $\Gamma_2(x)$, 6 in Q and 22 in $\Gamma_1(Q)$. Altogether we find $|\Gamma_1(Q) \cap \Gamma_2(x)| = 8.12 + 6.2 + 1.22 = 130$.

On the other hand, consider quads Q' through x. These are of five possible types:

a) intersecting Q in a line.

b) intersecting Q in a point.

c) contained in $\Gamma_1(Q)$.

d) with 7 points in $\Gamma_1(Q)$, where x is the point of intersection of the three lines on those 7 points.

e) with 7 points in $\Gamma_1(Q)$, where x is not the point of intersection.

Let there be n_a, n_b, n_c, n_d, n_e quads of each type.

Then $n_a + n_b + n_c + n_d + n_e = 35$.

Let $x \sim x' \in Q$. Each of the three lines through x' determines a quad of type a, so $n_a = 3$. The line xx' is in 7 quads, so $n_a + n_b = 7$ and $n_b = 4$. Counting points in $\Gamma_1(Q) \cap \Gamma_2(x)$ we find

$$130 = 6n_a + 4n_b + 8n_c + 0n_d + 4n_e.$$

Hence $2n_c + n_e = 24$, $n_c + n_d + n_e = 28$. Now vary the point x, so that n_c, n_d, n_e become functions of x. Let $\bar{n}_c, \bar{n}_d, \bar{n}_e$ be the average values.

For any quad of type β there is one point x for which it is of type d and six points x for which it is of type e. Consequently $\bar{n}_e = 6\bar{n}_d$. This yields $\bar{n}_c = 0$, $\bar{n}_d = 4$, $\bar{n}_e = 24$. But if $\bar{n}_c = 0$ then clearly $n_c = 0$ for each x. This shows that quads of type δ do not exist. □

4. THE GRAPH ON THE OVALS

Given two points p and q at distance 2 they determine a quad Q = Q(p,q) and inside Q an oval O = O(p,q). By counting one finds that there are exactly $\binom{24}{4}$ ovals, and our aim is to identify the set of ovals with the vertices of the Johnson scheme J(24,4). We use the following characterization (BROUWER [3]):

<u>THEOREM</u>. *Let* G *be a graph with* $v = \binom{24}{4}$ *vertices, regular of valency* $k = 80$, *where each edge is in* $\lambda = 22$ *triangles and any two nonadjacent vertices have at most 4 common neighbours. Then* G *can be labelled such that the vertices are the 4-subsets of a 24-set, and edges are pairs of 4-sets with*

3 points in common.

In order to apply the theorem we have to define adjacency between two ovals and to prove that $k = 80$, $\lambda = 22$, $\mu(x,y) \le 4$.

DEFINITION. Two ovals O and O' are called adjacent if $|O \cap O'| = |Q \cap Q'| = 1$, where Q and Q' are the quads containing O and O', respectively, and any two points of $O \cup O'$ have distance 2.

A. $k = 80$

Given an oval O in a quad Q, choose a point $x \in O$. Then x is in 35 quads, one is Q, 18 intersect Q in a line (for: each line is in 7 quads) and the remaining 16 intersect Q in $\{x\}$. Let Q' be one of these 16. Inside Q' the point x is in two ovals, O' and O''. Let $y \in O$. Then $d(y,Q') = 2$ (for if $y \sim z \in Q'$ then $d(y,x) = 1+d(z,x) = 2$, so $z \sim x$ and z has two neighbours in Q, so $z \in Q$, i.e. $z = x$, contradiction) and y determines an oval in Q', say O'. Now any point z of $O'\backslash\{x\}$ has distance 2 to x and y hence determines the oval $O_z = O$ in Q. We proved:

LEMMA. *Let* Q *and* Q' *be two quads intersecting in the point* x. *Then* x *is in ovals* O_1, O_2 *in* Q *and* O_1', O_2' *in* Q' *such that any two points in* $O_i \cup O_i'$ *have distance* 2 $(i = 1,2)$, *and any point in* $O_1\backslash\{x\}$ *has distance 3 to each point of* $O_2'\backslash\{x\}$ *(and similarly for* O_2 *and* O_1'*). [Thus:* $O_i \sim O_i'$ $(i = 1,2)$.]

Now the oval O contains 5 points, each point is in 16 quads Q' with $|Q' \cap Q| = 1$ and each quad Q' contains a unique oval $O' \sim O$. This shows that $k = 5.16.1 = 80$.

B. $\lambda = 22$.

Let O and O' be two adjacent ovals in quads Q and Q', respectively, where $Q \cap Q' = \{p\}$.

If $y \in O\backslash\{p\}$ and $z \in O'\backslash\{p\}$ then $d(y,z) = 2$ so that y and z determine a quad $Q'' = Q(y,z)$ and an oval $O'' = O(y,z)$. One sees immediately that $Q \cap Q'' = \{y\}$ (otherwise Q and Q'' intersect in a line ℓ, z has a neighbour u on ℓ, so $d(z,Q) = 1$ and $d(p,z) = 1 + d(u,p) = 2$ so $u \sim p$, $u \sim z$ and hence $u \in Q'$, so $u = p$, contradiction) and $Q' \cap Q'' = \{z\}$ and since $d(p,z) = 2$ it follows that $O \sim O'' \sim O'$. Thus we find $4.4 = 16$ common neighbours O'' not containing p.

Through p there are 6 quads Q'' intersecting both Q and Q' only in the point p (- count in the local STS(15) at p: there are 6 triples disjoint from a given pair of disjoint triples), and each quad Q'' contains a unique oval adjacent to O and a unique oval adjacent to O' - if we show that this is always the same oval then it follows that $\lambda = 16 + 6 = 22$ as desired.

<u>LEMMA</u>. *Let* O *be an oval,* $p \in O$ *and* y,z *two points at distance* 2 *to each point of* O. *Then either* $d(y,z) = 2$ *or* ($d(y,z) = 3$ *and there is a line* $\ell = py'z'$ *through* p *with* $y \sim y'$, $z \sim z'$).

<u>PROOF</u>. Let Q be the quad containing O. Then $d(y,Q) = d(z,Q) = 2$. If $y \sim z$ then each point of O has two points at distance 2 on the line yz, so the third point on this line is adjacent to each point of O, a contradiction.

Given y, there are 3 lines through p containing a neighbour of y. The third point z' on each of these lines has 16 neighbours in $\Gamma_2(Q)$, situated on 8 lines. If p is in the ovals O and O' inside Q then each of these 16 points has either O or O' as the corresponding oval in Q, but we just saw that adjacent points correspond to different ovals, so we find 8 neighbours of z' corresponding to O. Two of these are in the quad determined by y and p. Thus we find three choices for z' and for each z' six choices for z - 18 points altogether.

(Note that if y and z have distinct neighbours on a line ℓ and y,z and ℓ are not in a quad, then $d(y,z) = 3$.) In order to prove the lemma it suffices to show that there are 64 points in $\Gamma_2(Q)$ corresponding to O, 45 of which have distance 2 to a given point y.

But a given point in Q is at distance 2 from $24.6/3 = 128$ points in $\Gamma_2(Q)$. Let O_i (i=1,2,3,4,5,6) be the six ovals in Q and P_i ($1 \le i \le 6$) the six subsets of $\Gamma_2(Q)$ corresponding to O_i. Then $|P_i \cup P_j| = 128$ for all pairs i,j with $i \neq j$ and it follows that $|P_i| = 64$ for all i. Let $O_1 = O$. Now if $d(y,z) = 2$ then y and z determine a quad $Q' = Q(y,z)$. If Q' intersects Q then Q' is one of the five quads $Q(x,y)$ with $x \in O$. For each such quad $Q' \cap \Gamma_2(Q)$ is contained in $P_1 \cup P_j$ for some j and has the structure of $K_{4,4}$-matching (i.e. of the cube 2^3), so that Q' contains 3 points in $\Gamma_2(y) \cap P_1$.

If Q' is disjoint from Q (this is the case for the remaining 30 quads through y) then if u is the center of $Q' \cap \Gamma_1(Q)$ and $u \sim u' \in Q$ and $u' \in O_5 \cap O_6$, say, then $Q' \cap \Gamma_2(Q)$ has the structure of a cube with 2 points in each P_i $(1 \le i \le 4)$ and no points in $P_5 \cup P_6$.

Consequently Q' contains 1 point in $\Gamma_2(y) \cap P_1$. Altogether we find $5.3 + 30.1 = 45$ points z as desired. □

LEMMA. *Let* Q_i (i=1,2,3) *be three quads pairwise intersecting in the point* x. *Let* O_i *be an oval in* Q_i (i=1,2,3) *such that* $O_1 \sim O_2$ *and* $O_1 \sim O_3$. *Then* $O_2 \sim O_3$.

PROOF. Choose $y \in O_2 \setminus \{x\}$, $z \in O_3 \setminus \{x\}$. If $d(y,z) = 2$ we are done. Otherwise apply the previous lemma to find a line ℓ through x containing neighbours of y and z. But then $\ell \subset Q_2 \cap Q_3$, contradiction. □

This completes the proof of $\lambda = 22$.

C. $\mu \le 4$

As auxiliary result we need the following characterization of PG(3,2).

THEOREM. *Let* (X,ζ) *be an* STS(15) *such that each pair of disjoint lines is contained in a spread. Then* (X,ζ) = PG(3,2).

PROOF. By counting one sees that there are at least 28 spreads. From the data given by BUSSEMAKER & SEIDEL [4] on the number of spreads in each of the 80 distinct STS(15)'s we see that there are only two candidates; but inspection of one of them shows the existence of pairs of disjoint lines not in a spread. Hence (X,ζ) is PG(3,2). Note that in PG(3,2) any two disjoint lines are in exactly two spreads, and any three pairwise disjoint lines determine a unique spread. □

REMARK. It is an easy exercise to show that an STS(15) where any two disjoint lines are in at least two spreads, must be PG(3,2) (by showing that Pasch's axiom holds). For the above result however, I need the classification of all STS(15)'s.

COROLLARY. *In our near-hexagon, the 15 lines and 35 quads through a point form a* PG(3,2).

PROOF. Let Q_1 and Q_2 be two quads with $Q_1 \cap Q_2 = \{p\}$. Choose points $x_i \in Q_i$ such that $d(x_i,p) = 2$ $(i=1,2)$ and $d(x_1,x_2) = 2$. Let $Q = Q(x_1,x_2)$. Let O be the oval in Q determined by p. Then the five quads $Q(p,x)$ with $x \in O$ intersect pairwise in the point p. This shows that the local STS(15) at p satisfies the hypothesis of our theorem. □

Now let O and O' be two nonadjacent ovals. We must show that they have at most four common neighbours.

(i) Let O, O' be two ovals in the same quad Q. Then $\mu(O,O') = 0$.

(ii) Let O, O' be two ovals in quads Q, Q', respectively, where $Q \cap Q' = \ell$, a line. Suppose that the oval O'' contained in the quad Q'' is a common neighbour of O and O'. Then $Q'' \cap Q = Q'' \cap Q' = \{p\}$ and $p \in \ell$. Consequently, if $\mu(O,O') > 0$ then $O \cap \ell = O' \cap \ell = \{p\}$. Looking at the local PG(3,2) in p, we see that there are 8 quads Q'' intersecting both Q and Q' in p only (for: p is in 35 quads, 33 distinct from Q and Q'; 5 contain ℓ; 4 intersect both Q and Q' in a line other than ℓ; 16 intersect one of Q and Q' in a line, remain 8); if we call two such quads adjacent if they intersect in p only then the graph on these 8 quads is the union of two four-cycles. Now suppose $Q'' \sim Q_1''$. Then there is an oval $O_1'' \subset Q_1''$ such that $O_1'' \sim O''$. By the transitivity lemma in the previous section we find from $O_1'' \sim O'' \sim O$ that $O_1'' \sim O$ and likewise $O_1'' \sim O'$. This proves that if a quad contains a common neighbour of O and O' then so does any adjacent quad. Therefore the number of common neighbours of O and O' is 0, 4 or 8. But clearly, if $\bar{O}$ is the other oval through p in Q' then $\mu(O,O') + \mu(O,\bar{O}) = 8$ (each of the 8 quads contains a unique neighbour of O; this oval is adjacent to either O' or $\bar{O}$), so in order to prove that we have $\mu(O,O') = 4$ it suffices to prove $\mu(O,\bar{O}) \geq 1$.

To this end choose $x \in O$ and $\bar{x} \in \bar{O}$ with $x \neq p \neq \bar{x}$ and $d(x,\bar{x}) = 2$. (This is possible: $\bar{O}\backslash\{p\}$ contains 2 points at distance 2 and 2 points at distance 3 from x.) In $Q(x,\bar{x})$ the points x and $\bar{x}$ have three common neighbours; one is on ℓ. Let $y \notin \ell$ be a common neighbour of x and $\bar{x}$. Then $d(y,Q) = d(y,Q') = 1$. Let m be a line through y such that the two other points on m have distance 2 to both Q and Q'. (In fact there are 4 such lines).

Let $m = \{y,u,v\}$. Both u and v determine an oval through x in Q; let u be the point determining $O = O(x,p)$. Then $d(u,p) = 2$ so that u also determines $\bar{O} = O(\bar{x},p)$ in Q'. Consequently, $O(p,u)$ is a common neighbour of O and $\bar{O}$ as was to be proved.

(iii) Let O, O' be nonadjacent ovals in Q, Q' with $Q \cap Q' = \{p\}$.

If Q'' intersects both Q and Q' in a single point then either $p \in Q''$ or the points of intersection have distance 2 from p. If $p \notin O \cup O'$ then both O and O' contain three neighbours and two nonneighbours of p. Each pair of nonneighbours gives at most one Q'', so in this case $\mu(O,O') \leq 4$.

If $p \in O$, $p \notin O'$ then of the two points in O' nonadjacent to p one has distance 2 and the other distance 3 to each point of $O\backslash p$. Again $\mu(O,O') \in 4$.

Finally, if both O and O' contain p then each point of $O\backslash p$ has distance 3 to each point of $O'\backslash p$ so common neighbours can only be found in quads through p. But then in view of the lemma above $\mu(O,O') = 0$.

(iv) Let O, O' be ovals in Q, Q' with $Q \cap Q' = \emptyset$. Each oval O in Q has 2 or 4 points at distance 2 from Q', the latter case occurring exactly when $x \in O$, where x is the center of the set $Q \cap \Gamma_1(Q')$. If both O and O' contain 2 points at distance 2 from Q' respectively Q, then $\mu(O,O') \leq 4$.

If O contains 4 points at distance 2 from Q' then there is no point $z \in Q'$ at distance 2 from each point of O (for: let $x \sim x' \in Q$. Since $d(z,x) = 2$ it follows that $z \sim x'$ and so $d(z,Q) = 1$ and $\Gamma_2(z) \cap Q$ does not contain an oval) so that there cannot be an oval $O(y,z)$ intersecting both O and O' with $O \sim O(y,z) \sim O'$. Thus $\mu(O,O') = 0$ in this case.

This completes the proof of $\mu \leq 4$.

From now on we may assume that the ovals are labelled with quadruples from a 24-set such that adjacent ovals have labels with 3 elements in common.

5. CONSTRUCTION OF S(5,8,24)

Given the labelling of the ovals it is not difficult to find the Steiner system S(5,8,24). Repeating the same argument three times we find objects inside H labeled with triples, pairs and singletons from a 24-set. Then points can be labeled with 8-sets.

A. Look at the graph J(24,4), where adjacent quadruples have Johnson distance 1. Each edge is in two maximal cliques: a 5-clique and a 21-clique.

In J(24,4) the 5-cliques are the five quadruples in a 5-set and the 21-cliques are the 21 quadruples containing a given triple. Consequently we may label the 21-cliques with triples. In H the 5-cliques are the sets of five ovals through a fixed point p, where no quad not containing p intersects three of the ovals.

The 21-cliques are sets P of size 21 such that two points in P are at distance 2, and the oval they determine is contained in P. P together with its ovals has the structure of a projective place PG(2,4).

(For: let $O_1 \sim O_2$, $O_1 \cap O_2 = \{x\}$ and choose $y \in O_1 \backslash \{x\}$, $z \in O_2 \backslash \{x\}$. Let $Q = Q(y,z)$. Then $d(x,Q) = 2$ and x determines an oval O inside Q. For each point $u_i \in O$ we find an oval $O_i = O(x,u_i)$ $(i=1,2,3,4,5)$.

Now $\{O, O_i (1 \le i \le 5)\}$ is a 6-clique and hence contained in a 21-clique. This shows that an oval $\not\ni$ x intersecting two of the O_i intersects each of them.) Each oval is in four such 21-sets, and two 21-sets are disjoint, have a point or an oval in common, or coincide. In the sequel we write '21-set' instead of '21-set such as described under A'.

B. Look at the graph J(24,3) where adjacent triples have Johnson distance 1. Each edge is in two maximal cliques: a 4-clique and a 22-clique.

The 4-cliques are the four triples contained in a fixed quadruple. In H these correspond to the four 21-sets containing a fixed oval.

The 22-cliques are the 22 triples containing a fixed pair. In H these correspond to sets of twenty-two 21-sets, any two of them intersecting in an oval, where each of the 21 ovals in a 21-set is in exactly one other 21-set. Consequently each point in a 21-set is in five other 21-sets, so that each point of such a 22-clique is in six 21-sets and there are 77 points. In the sequel we shall call the 22-cliques '77-sets'. The incidence structure in a 77-set with the 21-sets as points and the points as blocks has S(2,5,21) as derived system, hence is the unique S(3,6,22) design. We label the 77-sets with pairs from 24 symbols.

C. Look at the graph J(24,2) where adjacent pairs intersect (the 'triangular graph' T(24)). Each edge is in two maximal cliques: a 3-clique and a 23-clique. The 3-cliques are the three pairs contained in a fixed triple. In H these correspond to the three 77-sets containing a fixed 21-set.

The 23-cliques are the 23 pairs containing a fixed symbol. In H these correspond to sets of twenty-three 77-sets, any two of them having a unique 21-set in common. Each of the twenty-two 21-sets in a 77-set is in exactly one other 77-set. Consequently each point in a 77-set is in six other 77-sets so that each point of such a 23-clique is in seven 77-sets and there are 253 points. In the sequel we shall call the 23-cliques '253-sets'. The incidence structure in a 253-set with 77-sets as points and points as blocks is the unique S(4,7,23) design. We label the 253-sets with 24 symbols.

D. Each point is in 8 253-sets.

For ovals, 21-sets, 77-sets and 253-sets we had that inclusion of sets was equivalent to inverse inclusion of labels (one implication by definition, the other by counting). If $x \in O$, O an oval with label $ijk\ell$, then $x \in O \subset P \subset A \subset B$ where P is the 21-set with label ijk, A is the 77-set with label ij and B is the 253-set with label i. This shows that any symbol occurring in the label of an oval containing x also is a symbol in the label of x. Thus the $70 = \binom{8}{4}$ ovals incident with x are labeled with the 4-subsets of the label of x.

Now we have identified the points of H with the blocks of S(5,8,24). In S(5,8,24) blocks intersect in either 0,2,4 or 8 points. Two blocks intersecting in 4 points are points of H in a common oval, i.e., points at distance 2. But knowledge of the graph with adjacencies $x \sim y$ when $d(x,y) = 2$ in H determines H itself: two adjacent points have 28 points at distance 2 from both of them, while two points at distance 3 have 85 points distance 2 from both of them.

This shows that adjacent points in H are disjoint blocks in S(5,8,24), and lines are triples of pairwise disjoint blocks.

This completes the uniqueness proof.

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