stichting
mathematisch
centrum

MC

AFDELING TOEGEPASTE WISKUNDE TW 245/83 OKTOBER
(DEPARTMENT OF APPLIED MATHEMATICS)

H.A. LAUWERIER

BIFURCATION OF A MAP AT RESONANCE 1:4

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, Kruislaan 413, Amsterdam, The Netherlands.

The Mathematical Centre, founded 11 February 1946, is a non-profit institution for the promotion of pure and applied mathematics and computer science. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 39AXX,58F14, 58F21, 34C05,92A15

Bifurcation of a map at resonance 1:4

by

H.A. Lauwerier

ABSTRACT

The maps considered here find their origin in a discrete model of population dynamics of the logistic type and containing a delay term. The main theme is to obtain full understanding of the various computer plots in the case that the multipliers of the equilibrium state are close to $\pm i$. Shown is how the theory of normal forms can be used here as an effective tool. For a discussion of the possible limit cycles the mapping is compared to a corresponding flow. A few new results have been obtained giving additional information for this hitherto incompletely known case.

KEY WORDS & PHRASES: *non-linear difference equations, bifurcation, logistic equation, normal forms*

1. INTRODUCTION

The problems considered in this paper were motivated originally by computer experiments on some models of population dynamics or epidemiology. In particular we considered the recurrent relation

$$(1.1)\qquad x_{n+1} = ax_n(1-(1-b)x_n-bx_{n-1})$$

for $a > 0$ and $0 \le b \le 1$. This equation generalizes the by now famous logistic difference equation

$$(1.2)\qquad x_{n+1} = ax_n(1-x_n)$$

and the delayed logistic equation

$$(1.3)\qquad x_{n+1} = ax_n(1-x_{n-1})$$

studied by POUNDERS and ROGERS [1] and by ARONSON et al. [2].

Of particular interest is the "arithmetic mean" case

$$(1.4)\qquad x_{n+1} = ax_n(1-\tfrac{1}{2}x_n-\tfrac{1}{2}x_{n-1}),$$

which will be written as the iterative planar map

$$(1.5)\qquad \begin{cases} x_{n+1} = y_n, \\ y_{n+1} = ay_n(1-\tfrac{1}{2}x_n-\tfrac{1}{2}y_n). \end{cases}$$

For $1 < a \le 3$ the non-trivial fixed point $x = y = 1-1/a$ is stable but if we increase a somewhat beyond 3 the fixed point bifurcates into a stable 4-cycle and an unstable 4-cycle as shown in fig.6.1. Further increase of a yields a sequence of bifurcations in the spirit of the Feigenbaum scenario. These phenomena will be discussed in a forthcoming paper. Here we consider only the local behaviour at a critical bifurcation point.

The bifurcation can be described by using the multipliers λ, $\bar{\lambda}$ of the fixed point. A simple calculation shows that

(1.6) $\lambda + \bar{\lambda} = (3-a)/2, \quad \lambda\bar{\lambda} = (a-1)/2$

so that a = 3 gives $\lambda = i$, a well-known case of strong resonance. For a slightly different value of a, λ and $\bar{\lambda}$ are situated on a circle in the λ-plane according to (cf. fig.1.1)

(1.7) $\lambda\bar{\lambda} + (\lambda+\bar{\lambda}) = 1.$

We observe that λ leaves the unit circle at i in a direction which makes an angle of $\pi/4$ with respect to the normal.

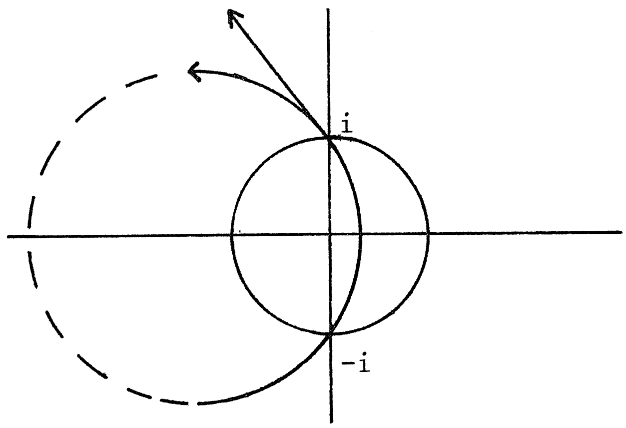


fig. 1.1

The next experiment was the behaviour of the two-parameter map corresponding to (1.1)

(1.8)
$$\begin{cases} x_{n+1} = y_n \\ y_{n+1} = ay_n(1-bx_n-(1-b)y_n) \end{cases}$$

close to the bifurcation point $\lambda = i$. This map has the same non-trivial fixed point x = y = 1 - 1/a. Its multipliers satisfy

$$(1.9)\qquad \begin{cases} a = 2-(\lambda+\bar{\lambda}) + \lambda\bar{\lambda}, \\ b(a-1) = \lambda\bar{\lambda}. \end{cases}$$

The parameters enable us to move λ away from i in any direction. If we take $\lambda = i(1+\mu)$ with e.g. $\mu = 0.01$ we do not find periodic cycles but we get an invariant curve that looks like a deformed Hopf circle as shown in fig. 6.2.

These and similar experiments stimulated us to explore such bifurcation phenomena in a more systematic way. The case of resonance of order four has not been fully explored up to the present. What is known goes back essentially to Arnold and can be found at the end of his book, now available in english [3]. Additional work has been done by WAN [5] and IOOSS [4]. An excellent survey of what is known has been given recently by WHITLEY [8].

In this paper we have attempted to show how the theory of normal forms can be applied to a specific map such as (1.8), and more generally to the family of maps described by

$$(1.10)\qquad \begin{cases} x' = y, \\ y' = Ax + By + Cx^2 + Dxy + Ey^2, \end{cases}$$

where $A = -\lambda\bar{\lambda}$, $B = \lambda + \bar{\lambda}$.

Throughout this paper computer experiments have been a guiding principle for further theoretical investigations. Most of Arnolds' findings have been confirmed. However, as regards the nature of the stable bifurcated 4-cycle the situation appears to be more complicated than as suggested by Arnold. We have succeeded in obtaining supplementary information in the lesser known cases.

We have used purely analytical techniques, perturbation methods etcetera in a bit formal way. In practice this means that everything is expanded in powers of μ and that only those terms are written down as needed in further calculations.

In section 2 we show how the map (1.10) can be reduced to the normal form

$$(1.11) \qquad z' = \lambda z - Qz^2\bar{z} + S\bar{z}^3 + \ldots ,$$

where

$$(1.12) \qquad \lambda = i(1+\mu e^{i\phi}).$$

The perturbation variables μ,ϕ can be considered as local polar coordinates at the bifurcation point i in the λ-plane. The unstable side of the unit circle is determined by

$$(1.13) \qquad -\pi/2 \leq \phi \leq \pi/2,$$

so that $\phi = 0$ indicates the normal direction. We need the values of Q and S only for $\mu = 0$. Shown is how Q(0) and S(0) can be obtained from the coefficients of the map (1.10). Using an appropriate similarity transformation in the z-plane it can be arranged that S = 1.

Accordingly, in section 3 we consider the normal map

$$(1.14) \qquad z' = \lambda z - Qz^2\bar{z} + \bar{z}^3$$

where

$$(1.15) \qquad Q = ce^{i\gamma} = p + iq.$$

The following, for the most part well-known, results are obtained.

Case $c > 1$

If $q > -1$ there exists a sector

$$(1.16) \qquad \mathrm{Max}(\gamma - \tfrac{\pi}{2} - \delta, -\tfrac{\pi}{2}) < \phi < \mathrm{Min}(\gamma - \tfrac{\pi}{2} + \delta, \tfrac{\pi}{2}), \quad \delta = \arcsin 1/c$$

where the fixed point bifurcates into two 4-cycles. The outer cycle is stable, the inner cycle is unstable. According to a theorem by Wan which gives a local result for this case with $q > 0$ there exists a limit cycle outside the sector for sufficiently small μ. The stable cycle, if present, may consist of either foci or nodes. No simple condition for this has been found contrary to a statement in Arnolds' book (l.c. p.306). In the special case $\phi = \gamma - \pi/2$ our condition is as follows. There are foci if

$$\frac{3-|\cos\gamma|}{1+|\cos\gamma|} < c < \frac{3+|\cos\gamma|}{1-|\cos\gamma|}, \qquad 0 < \gamma < \pi. \tag{1.17}$$

and nodes otherwise

<u>Case $c < 1$</u>

For all values of ϕ there exists a single unstable 4-cycle. Moreover for some values of c,γ and ϕ there appears to be a limit cycle, i.e. an invariant attracting orbit.

In section 5 the possible limit cycles are studied in great detail. For $\mu \to 0$ the mapping (1.14), or better its four times iterated version can be replaced by the corresponding flow

$$\dot{z} = ze^{i\phi} + iQz^2\bar{z} - i\bar{z}^3, \tag{1.18}$$

or

$$\begin{cases} \dfrac{\dot{r}}{r} = \cos\phi - (q+\sin 4\theta)r^2, \\ \dot{\theta} = \sin\phi + (p-\cos 4\theta)r^2, \end{cases} \tag{1.19}$$

in polar coordinates.

For $c \to \infty$ a limit cycle is obtained for which

$$\frac{1}{r^2} = \frac{q}{\cos\phi} + \frac{\sin 4\theta\ \cos\phi - 2\omega\ \cos 4\theta}{\cos^2\phi + 4\omega^2} + O(\tfrac{1}{c}) \tag{1.20}$$

where

$$\omega = \frac{p}{q}\cos\phi + \sin\gamma. \tag{1.21}$$

This gives a circular shape for large values of c and a rounded square if c is moderate. In the special case $\phi = \gamma - \pi/2$ this predicts a square limit cycle for $q \approx 3 + 2\sqrt{2}$.

A limit cycle may exist in the region between the origin and the unstable quadruplet of fixed points. In order to derive sufficient conditions guaranteeing the existence of such a limit cycle the normal form (1.18) is rewritten as

$$\frac{\dot{z}}{z} = r^2 e^{i\phi}\{\frac{1}{r^2} - \frac{1}{R^2} + ie^{-i\beta}(1-e^{-4i\theta})\}, \tag{1.22}$$

where $z = r \exp i\theta$.

The unstable fixed points are given by $r = R$ and $4\theta = 0 \pmod{2\pi}$. The parameters R,β,ϕ can be used instead of p,q,ϕ or c,γ,ϕ as the independent parameters. We note that $c > 1$ corresponds to $2R^2 \sin\beta > 1$. Using the Poincaré Bendixson theorem the following results have been obtained.

<u>THEOREM</u>. *There exists a limit cycle in the following two cases* $(0<\beta<\pi/2)$

(i)
$$\sin\beta - \cos\beta < \frac{1}{2R^2} < 1,$$
$$\phi > \frac{\pi}{4} + \arctan\frac{2R^2\sin\beta}{2R^2\cos\beta+1},$$

(ii)
$$1 < \frac{1}{2R^2} < \sin\beta + \cos\beta,$$
$$\phi > \frac{\pi}{4} + \arctan\frac{1-2R^2\cos\beta}{2R^2\sin\beta}.$$

In both cases there exist symmetric situations to be obtained by $\phi \to -\phi$, $\beta \to \pi-\beta$.

We note that the first case covers both $c > 1$ and $c \leq 1$. This theorem gives only sufficient conditions. Experiments show that a limit cycle exists up to the critical situation when the unstable manifold of a fixed point coalesces with the stable manifold of the next fixed point. In that situation the limit cycle is like a four-pointed star (cf. fig.6.8).

In section 6 a number of illustrations and applications are given. In particular the logistic delay map (1.8) is considered. Shown is how the theory can be used to predict or to explain the various bifurcation phenomena. Computer obtained illustrations for this case are given in figs. 6.1 to 6.6. A next example is the Maynard Smith map mentioned in Whitleys' paper [8]. An illustration of the corresponding flow is given in fig. 6.9. A list is given of interesting cases of the general map (1.10) for the subcase C = 0. For the case D = 3, F = −2 fig. 6.7 and 6.8 show a.o. the formation of a nice starlike limit cycle.

The behaviour of the corresponding flow is illustrated in figs. 6.9 to 6.11. Of particular interest is the flow pattern for the case $p = q = \frac{1}{2}$, $\phi = \pi/3$ which just fails to form a limit cycle (the value $\phi = 4\pi/9$ would do it).

Two appendices have been added giving additional information. Appendix B gives a brief survey of the main results and formulae for the general mapping (1.10).

2, NORMAL FORMS

We consider the following quadratic map

$$\begin{cases} x' = y, \\ y' = Ax + By + Cx^2 + Dxy + Ey^2, \end{cases} \tag{2.1}$$

where the fixed point x = y = 0 has multipliers λ and $\bar{\lambda}$ close to $\pm i$. We write

$$\lambda = i(1+\mu e^{i\phi}) \tag{2.2}$$

when μ is very small. Throughout this section all calculations will be carried out up to $O(\mu)$. This means that the coefficients of quadratic and higher terms can be taken at $\mu = 0$. Of course A and B follow from (2.2) as

$$\begin{cases} A = -\lambda\,\bar{\lambda} = -(1+2\mu\cos\phi+\mu^2), \\ B = \lambda + \bar{\lambda} = -2\mu\,\sin\phi. \end{cases} \tag{2.3}$$

The substitution

$$z = x - iy \tag{2.4}$$

brings (2.1) into its pre-normal form

$$(2.5) \qquad z' = \lambda z + A_2(z,\bar{z}),$$

where

$$(2.6) \qquad A_2(z,\bar{z}) = a_{20}z^2 + a_{11}z\bar{z} + a_{02}\bar{z}^2.$$

with

$$(2.7) \qquad \begin{cases} 4ia_{20} = C + iD - E, \\ 2ia_{11} = C + E, \\ 4ia_{02} = C - iD - E. \end{cases}$$

The reduction of (2.5) into the normal form

$$(2.8) \qquad z' = \lambda z - Qz^2\bar{z} + S\bar{z}^3 + \ldots$$

is effected by a coordinate transformation

$$(2.9) \qquad \begin{cases} z_0 = z + P_2(z,\bar{z}) + P_3(z,\bar{z}) + \ldots \\ z_0' = z' + P_2(z',\bar{z}') + P_3(z',\bar{z}') + \ldots \end{cases}$$

where

$$(2.10) \qquad P_k(z,\bar{z}) = \sum_{i+j=k} p_{ij}z^i\bar{z}^j .$$

The actual calculations become very simple if one realises that substitution of (2.9) in the normal form (2.8) written with z_0 and z_0' will bring back the pre-normal form (2.5). By elementary bookkeeping we obtain

$$(2.11) \qquad \begin{cases} a_{20} + \lambda^2 p_{20} = \lambda p_{20}, \\ a_{11} + \lambda\bar{\lambda} p_{11} = \lambda p_{11}, \\ a_{02} + \bar{\lambda}^2 p_{02} = \lambda p_{02}, \end{cases}$$

so that with $\lambda = i$ and using (2.7)

$$(2.12)\qquad \begin{cases} 4(1+i)p_{20} = C + D + iE, \\ 2(1-i)p_{11} = C + iE, \\ 4(1+i)p_{02} = C - D + iE. \end{cases}$$

Similarly we obtain

$$(2.13)\qquad -Q = a_{21} + 2\lambda a_{11}p_{20} + (\lambda\bar{a}_{11}+\bar{\lambda}a_{20})p_{11} + 2\bar{\lambda}\bar{a}_{02}p_{02},$$

and

$$(2.14)\qquad S = a_{03} + \bar{\lambda}a_{02}p_{11} + 2\bar{\lambda}\bar{a}_{20}p_{02},$$

where we have added the necessary coefficients of a possible third order term $A_3(z,\bar{z})$ in (2.5) if this were needed.

However, (2.13) and (2.14) will only be used here for $a_{21} = a_{03} = 0$, $\lambda = i$ with (2.7) and (2.12). We find

$$(2.15)\qquad (1-i)Q = i|a_{11}|^2 + 2|a_{02}|^2 - (2+i)a_{20}a_{11},$$

and

$$(2.16)\qquad (1-i)S = (2a_{02}+ia_{11})a_{02}.$$

In actual calculations it is advisable to determine a_{20}, a_{11}, a_{02} by means of (2.7) and to substitute the values found in (2.15) and (2.16).

The map (2.1) is invertible for $C = 0$. In that case we have explicitly

$$(2.17)\qquad \begin{cases} 16Q = (D^2-3DE-2E^2) + i(D^2+DE), \\ \\ 16S = (D^2+DE-2E^2) + i(D^2-3DE). \end{cases}$$

It will turn out that the normal form (2.8) is somewhat special for $|Q| = |S|$.

It is perhaps of interest to note that for $C = 0$ this happens only for $D = 0$ and $E = 0$.

The general expressions are

$$(2.18)\qquad \begin{cases} 16Q = (D^2-2E^2-3CD-6CE-3DE)+i(6C^2+D^2+CD+2CE+DE), \\ 16S = (D^2-2E^2-3CD+2CE+DE)+i(-2C^2+D^2+CD+2CE-3DE). \end{cases}$$

EXAMPLES.

1. For the mixed delay logistic map (1.5) we have $C = 0$, $D = E = -3/2$. This gives

$$Q = \frac{9}{32}(-2+i), \qquad S = -\frac{9}{32}i.$$

2. For the Maynard Smith map (cf. [8])

$$(x,y) \to (y, \varepsilon y+\mu-x^2)$$

we have at the bifurcation $\varepsilon = 0$, $\mu = 3/4$ the coefficients $C = -1$, $D = E = 0$.
This gives

$$Q = 3i/8, \qquad S = -1/8.$$

3. FIXED POINTS

In this section we consider the behaviour of the planar map written in its complex normal form as

$$(3.1)\qquad z' = \lambda z - Qz^2\bar{z} + S\bar{z}^3,$$

where

$$(3.2)\qquad \lambda = i(1+\mu e^{i\phi}).$$

By a suitable similarity transformation it can be arranged that $S = 1$.
We write

$$(3.3)\qquad Q = ce^{i\gamma} = p + iq.$$

In polar coordinates $z = r \exp i\theta$ the complex map (3.1) can be written in real form as

$$(3.4) \quad \begin{cases} r'^2 = (1+2\mu\cos\phi)r^2 - 2(q+\sin 4\theta)r^4 + \dots \\ \theta' = \theta + \frac{\pi}{2} + (p-\cos 4\theta)r + \dots \end{cases}$$

Assuming the existence of a period 4 cycle we obtain from (3.4) the conditions

$$(3.5) \quad \begin{cases} \mu \cos\phi = (q+\sin 4\theta)r^2, \\ \mu \sin\phi = (-p+\cos 4\theta)r^2. \end{cases}$$

Elimination of r^2 gives

$$(3.6) \quad \cos(4\theta+\phi) = c \cos(\phi-\gamma)$$

which shows that we have to consider two generic cases $c > 1$ and $c < 1$.

<u>Case $c > 1$</u>

From (3.6) we obtain the necessary condition

$$(3.7) \quad |\cos(\phi-\gamma)| \le 1/c.$$

If the auxiliary angle β is defined by

$$(3.8) \quad \cos\beta = c\cos(\phi-\gamma), \qquad 0 \le \beta \le \pi,$$

there are two possible sets of cyclic points determined by

$$(3.9) \quad 4\theta = -\phi \pm \beta \quad (\mathrm{mod}\ 2\pi).$$

From (3.5) we obtain two possible values for the radius

$$(3.10) \quad \frac{\mu}{r^2} = -c\sin(\phi-\gamma) \pm \sin\beta.$$

If the two values are indicated by r_+ and r_- we have

(3.11) $$\frac{\mu^2}{r_+^2 r_-^2} = c^2-1.$$

This means that we have either two cycles or none depending on the sign of $\sin(\phi-\gamma)$. By (3.7) a double sector is defined as $|\gamma \pm \pi/2| < \delta$ where δ is given by

(3.12) $$\delta = \arcsin 1/c.$$

For $\sin(\gamma-\phi) > 0$ there are two cycles what reduces the double sector to the single one

(3.13) $$\gamma - \pi/2 - \delta < \phi < \gamma - \pi/2 + \delta.$$

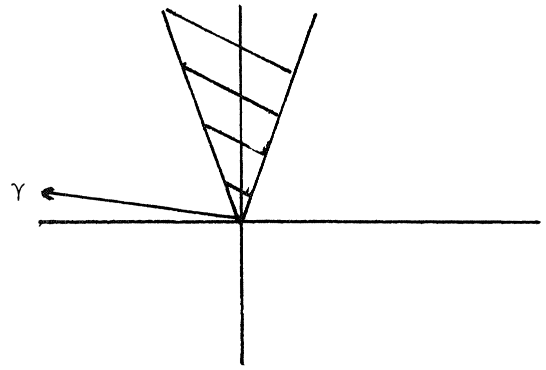
However, ϕ should still be in the interval $(-\pi/2, \pi/2)$. So there are various possibilities of partial overlap and non-overlap. A simple calculation shows that for $q < -1$ there is no overlap, for $|q| < 1$ a partial overlap and for $q > 1$ a full sector in $(-\pi/2, \pi/2)$. The various cases are indicated sketchwise below.

Result

For $c > 1$ there are two cycles of period 4 for those values of ϕ satisfying

(3.14) $$\text{Max}(\gamma - \frac{\pi}{2} - \delta, -\frac{\pi}{2}) < \phi < \text{Min}(\gamma - \frac{\pi}{2} + \delta, \frac{\pi}{2}).$$

At the boundary the cycles coalesce. For other values of ϕ there are no cycles.



$\pi - \delta < \gamma < \pi + \delta$, $-1 < q < 1$

fig. 3.1

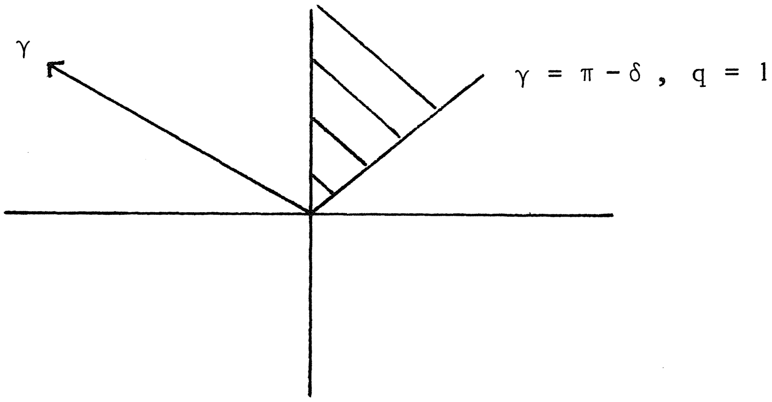


fig. 3.2

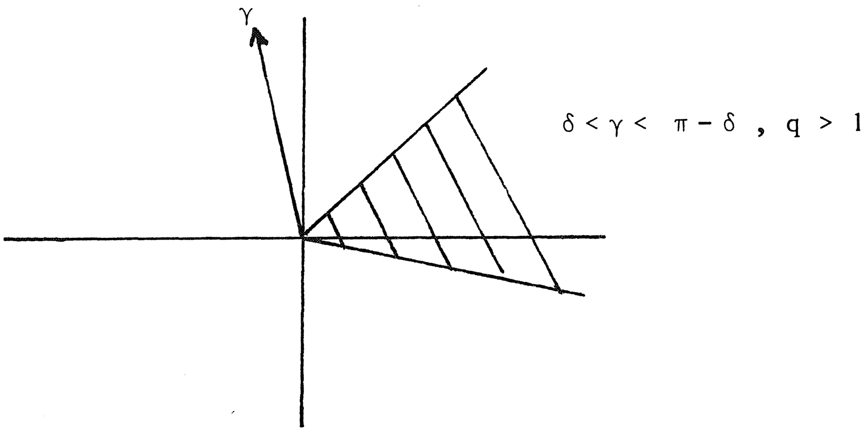


fig. 3.3

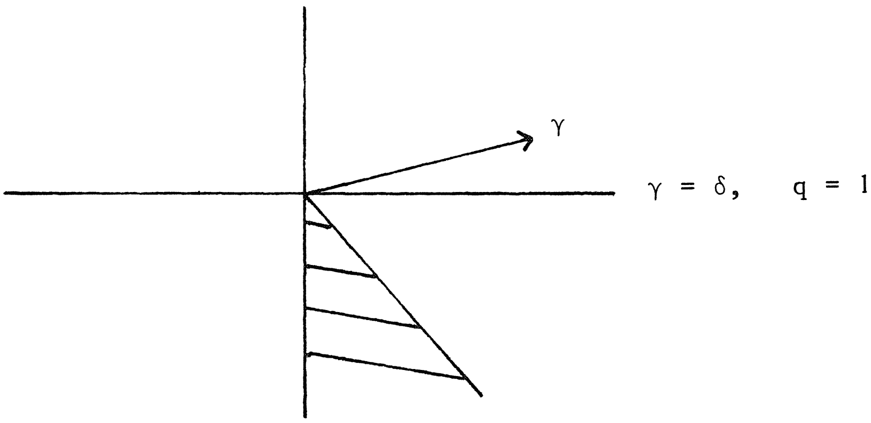
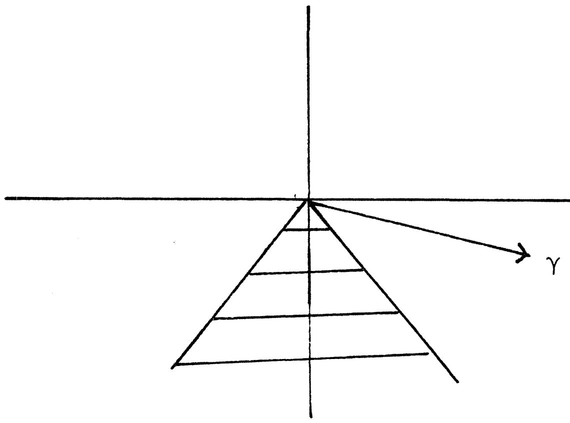
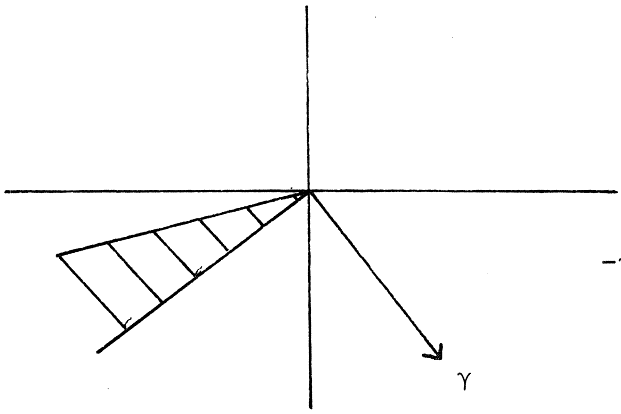


fig. 3.4



$-\delta<\gamma<\delta$, $-1<q<1$

fig. 3.5



$-\pi+\delta<\gamma<-\delta$, $q<-1$

fig. 3.6

<u>Case c < 1</u>

The relation (3.11) shows that

$$\mu/r_+^2 > 0, \qquad \mu/r_-^2 < 0.$$

Accordingly for all ϕ there is only a single 4-cycle for which

$$(3.15) \qquad \begin{cases} 4\theta = -\phi + \beta, \\ \mu/r^2 = -c\sin(\phi-\gamma) + \sin\beta. \end{cases}$$

We might say that the second cycle has become imaginary but that for $\mu < 0$, in the stable region of the unit circle, we have a real cycle of period 4, unstable of course. It will turn out that the first cycle is also unstable.

Results

For $c < 1$ there is a cycle of period 4 on either side of the bifurcation point $\mu = 0$.

Case $c = 1$

The relation (3.6) gives either $4\theta = -\gamma$ or $4\theta = \gamma - 2\phi$. In the first case we have $r = \infty$ so that one 4-cycle has disappeared into infinity, at least within the present approximation. In the second case we obtain

$$(3.16) \qquad \frac{\mu}{r^2} = 2\sin(\gamma-\phi),$$

which means a 4-cycle in the sector $\gamma - \pi < \phi < \gamma$.

Result

For $c = 1$ there is a sector

$$\mathrm{Max}(\gamma-\pi, -\pi/2) < \phi < \mathrm{Min}(\gamma, \pi/2)$$

for which there is a single 4-cycle. Outside this sector no 4-cycle is present.

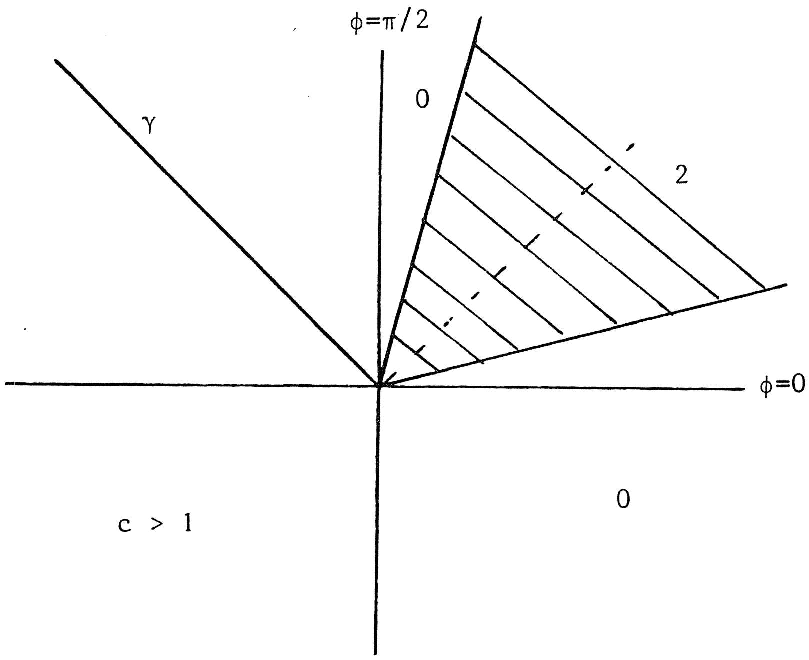


fig. 3.7

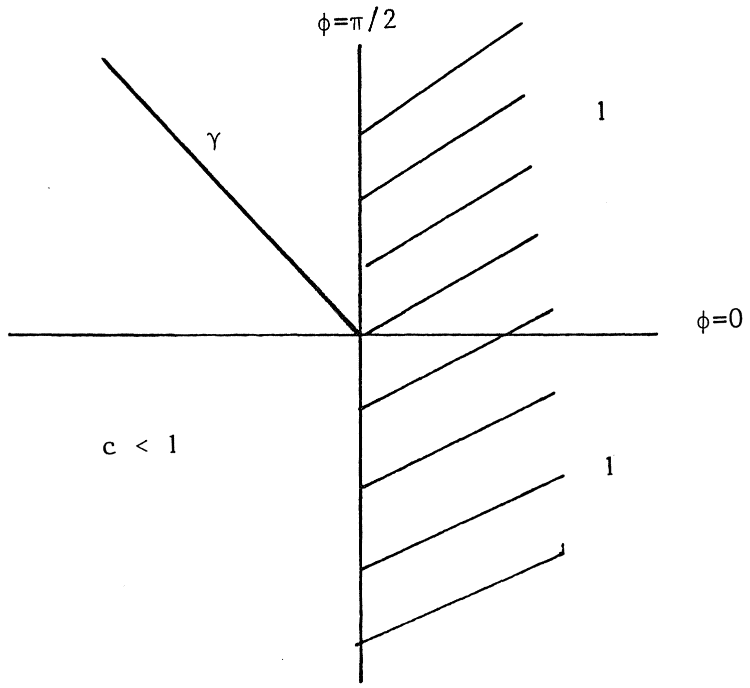


fig. 3.8

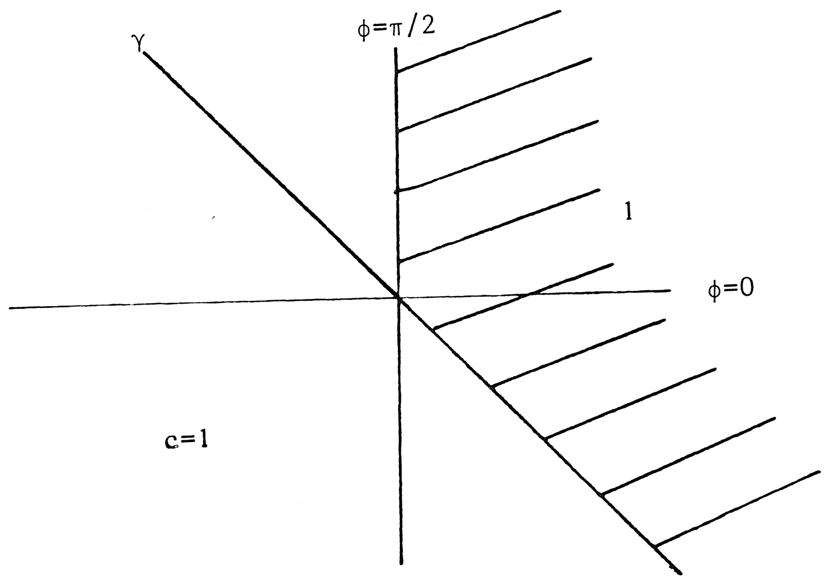


fig. 3.9

4. STABILITY

In this section we consider the nature of the possible cyclic fixed points of the map (3.1). This is essentially a technical problem involving lengthy calculations. Since ordinary fixed points are simpler to work with we replace (3.1) by its four times iterated version

$$z' = \lambda^4 z - \lambda^3(1+|\lambda|^2+|\lambda|^4+|\lambda|^6)Qz^2\bar{z}+(\lambda^3+\lambda^2\bar{\lambda}^3+\lambda\bar{\lambda}^6+\bar{\lambda}^9)S\bar{z}^3 +\ldots \tag{4.1}$$

or in the usual first order approximation

$$z' = (1+4\mu e^{i\phi})z + 4ice^{i\gamma}z^2\bar{z} - 4i\bar{z}^3. \tag{4.2}$$

The fixed points of this map are of course determined by (3.5), (3.9), (3.10), (3.12) and (3.13).

The stability condition for a fixed point is

$$-1 + |\sigma_1+\sigma_2| < \sigma_1\sigma_2 < 1 \tag{4.3}$$

where σ_1,σ_2 are its multipliers. This condition can be obtained from (4.2) in the following way.

Let

$$z' = F(z,\bar{z}) \tag{4.4}$$

be a planar map in conjugate complex variables

<u>Case $c > 1$</u>

Using (3.10) and (3.11) we may write

$$(1-\sigma_1)(1-\sigma_2) = 128(c^2-1)r^2(r^2-(r_+^2+r_-^2)/2) + \ldots . \tag{4.11}$$

This shows at once that the inner fixed points ($r=r_+$) are unstable and the outer fixed points ($r=r_-$) are stable. The latter fixed point, can be either a node or a focus depending on the sign of (4.8). However, the expression for (4.8) in terms of ϕ and Q is rather complicated contrary to a statement made by Arnold (cf. [3] p.306). By way of illustration we consider the special case $\phi = \gamma - \pi/2$ where ϕ is in the middle of the admissible sector. In that case we have $0 < \gamma < \pi$ and next

$$(4.12) \qquad \beta = \pi/2, \qquad \mu/r^2 = c-1$$

so that

$$(4.13) \qquad \sigma_1 + \sigma_2 - 2 = 8(c+1)r^2 \sin \gamma$$

and

$$(4.14) \qquad (1-\sigma_1)(1-\sigma_2) = 128(c-1)r^4 .$$

coordinates $z = x + iy$, $\bar{z} = x - iy$. Its real representation follows from

$$(4.5) \qquad F(z,\bar{z}) = f(x,y) + ig(x,y).$$

The multipliers of a fixed point follow from

$$(4.6) \qquad \begin{cases} \sigma_1\sigma_2 = f_x g_y - f_y g_x, \\ \sigma_1\sigma_2 = f_x + g_y. \end{cases}$$

These relations can easily be translated into

$$(4.7) \qquad \begin{cases} \sigma_1\sigma_2 = |F_z|^2 - |F_{\bar{z}}|^2, \\ \sigma_1\sigma_2 = 2 \operatorname{Re} F_z. \end{cases}$$

They enable us to derive the stability condition at once from (4.2).

The question whether a stable fixed point is a node or a focus can then be decided by inspecting the sign of

(4.8) $$(\sigma_1-\sigma_2)^2.$$

From (4.7) and (4.2) we obtain

(4.9) $$\sigma_1+\sigma_2 = 2 + 8(\mu\cos\phi-2cr^2\sin\gamma) +\ldots$$

and

(4.10) $$\frac{1}{128}(1-\sigma_1)(1-\sigma_2) = (c^2-1)r^4 + \mu cr^2 \sin(\phi-\gamma) +\ldots$$

the sign of which determines the stability.
This gives

$$(\sigma_1-\sigma_2)^2 = (\sigma_1+\sigma_2-2)^2-4(1-\sigma_1)(1-\sigma_2)$$

$$= 64r^4\{(c+1)^2\sin^2\gamma-8(c-1)\}$$

(4.15) $$= \mathrm{sgn}\left(c-\frac{3+\cos\gamma}{1-\cos\gamma}\right)\left(c-\frac{3-\cos\gamma}{1+\cos\gamma}\right).$$

In the Q-plane where c and γ are polar coordinates the curves c = 1 and

(4.16) $$c = \frac{3+\cos\gamma}{1-\cos\gamma}, \qquad c = \frac{3-\cos\gamma}{1+\cos\gamma}$$

are the boundaries of regions where the outer fixed points are nodes or foci as indicated in fig. 4.1.

<u>Case $c < 1$</u>

Using (3.12) we have in a similar way

(4.17) $$(1-\sigma_1)(1-\sigma_2) = -\,128\mu r^2 \sin\beta$$

which means instability.

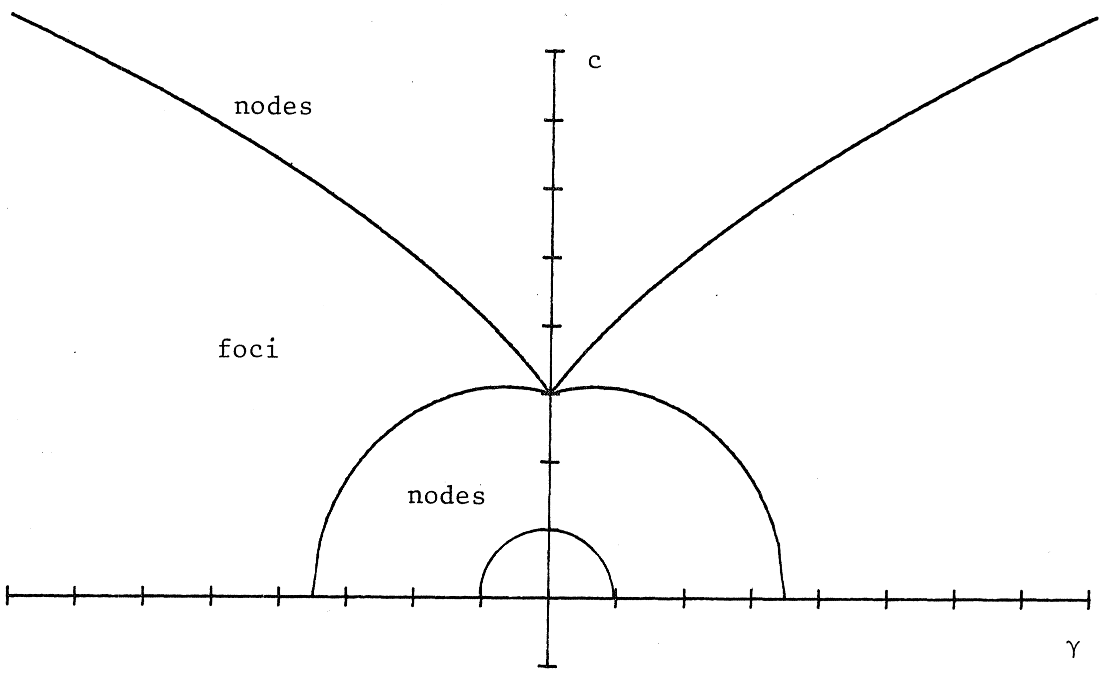


fig. 4.1

5. LIMIT CYCLES

According to numerical experiments the mapping

$$(5.1) \qquad z' = \lambda z - Qz^2\bar{z} + \bar{z}^3$$

where λ and Q are given by (3.2) and (3.3) may have an attracting invariant curve in a number of different cases. Some partial results have been obtained by Arnold (l.c. p.298, 304-310,314), WAN [5] and others [6], [7]. Here it will be discussed what can be obtained by explicit analytic methods and the Poincaré-Bendixson theorem as the main tool.

If μ is very small successive points z_0, z_4, z_8 ... will be very close and trace out an almost continuous curve. This is a sufficient motivation to approximate the map by a flow.

The four times iterate of (5.1) given by (4.2) passes into a flow by introducing a time scale by

$$(5.2) \qquad \dot{z} = \lim_{\mu \to 0} \frac{z_4 - z_0}{4\mu} .$$

Then we obtain

(5.3) $$\dot{z} = e^{i\phi}z + (ip-q)z^2\bar{z} - i\bar{z}^3$$

which will be our primary object of study. In polar coordinates r,θ this can be written by using

(5.4) $$\dot{z}/z = \dot{r}/r + i\dot{\theta}$$

as

(5.5) $$\begin{aligned}\dot{r}/r &= \cos\phi - (q+\sin 4\theta)r^2,\\ \dot{\theta} &= \sin\phi + (p-\cos 4\theta)r^2.\end{aligned}$$

The general idea is as follows. If $z = 0$ is the only fixed point and if $z = \infty$ is repelling in all directions a limit cycle is likely. If $z = 0$ is surrounded by a quadruplet of unstable fixed points there is a possibility of a limit cycle in between. Of course this does not exclude the existence of another quadruplet of fixed points farther away. So this situation may happen both for $c < 1$ and $c > 1$. The simplest result is as follows.

THEOREM 5.1. *For* $q > 1$ *there exists a unique limit cycle satisfying*

(5.6) $$\frac{\cos\phi}{q+1} < r^2 < \frac{\cos\phi}{q-1}\ .$$

For $q < -1$ *no limit cycle exists.*

PROOF. The possibility of two limit cycles can be excluded by using the divergence theorem for the equivalent flow $\dot{z} = F(z,\bar{z})$ where

$$\dot{z} = e^{i\phi}/z + (-q+ip)z - i\bar{z}^2/z^2.$$

Using

$$\text{div } F = 2\ \text{Re}\ \frac{\partial F}{\partial z}$$

we obtain

$$-q + \sin 4\theta$$

which has a fixed sign.

On the circle $r^2 = \cos\phi/(q+1)$ the flow is apparently outgoing since $\dot{r}/r > 0$.

On the circle $r^2 = \cos\phi/(q-1)$ the flow is ingoing. Therefore there exists a limit cycle satisfying (5.6),

For $q < -1$ we have always $\dot{r} > 0$ which forbids the existence of a limit cycle. □

Assuming the existence of a limit cycle where $z(t)$ is a periodic function of time t we may solve the first equation of (5.5) for a given function $\theta(t)$. A straightforward calculation gives the following asymptotic representation of the limit cycle

$$\frac{1}{r^2} = \frac{q}{\cos\phi} + 2\int_0^t e^{-2\tau\cos\phi}\sin(4\theta(t-\tau))d\tau \tag{5.7}$$

for $t \to \infty$.

In order to get the shape of the limit cycle we may consider the asymptotic behaviour for $c \to \infty$. From (5.5) it follows that

$$r^2 \approx \frac{\cos\phi}{q}, \qquad \theta \approx \omega t \tag{5.8}$$

with

$$\omega = \frac{p}{q}\cos\phi + \sin\phi. \tag{5.9}$$

Next we may put

$$\frac{1}{r^2} = \frac{q}{\cos\phi}\left(1 + \frac{A\cos 4\theta + B\sin 4\theta}{c} + O\left(\frac{1}{c^2}\right)\right). \tag{5.10}$$

Substitution in

$$\frac{dr}{rd\theta} = \frac{-(q+\sin 4\theta)+\cos\phi/r^2}{p-\cos 4\theta+\sin\phi/r^2} \tag{5.11}$$

gives

$$2A\sin 4\theta - 2B\cos 4\theta = \frac{qA\cos 4\theta + qB\sin 4\theta - c\sin 4\theta}{p+q\tan\phi}$$

from which A and B can be determined. Eventually we arrive at the following expression

(5.12) $$\frac{1}{r^2} = \frac{q}{\cos\phi} + \frac{\sin 4\theta\cos\phi - 2\omega\cos 4\theta}{\cos^2\phi + 4\omega^2} + O(\frac{1}{c}).$$

For $p \to \infty$ and $q = O(1)$ we obtain from (5.10) and (5.12) the following simple special result

(5.13) $$\frac{1}{r^2} = \frac{q}{\cos\phi}(1 - \frac{1}{2p}\cos 4\theta + \ldots).$$

The curves described by (5.10) and (5.13) are almost circular if c is large but are more like a square with rounded corners if c is moderate. By way of illustration fig.5.1 shows the graph of

(5.14) $$\frac{1}{r^2} = \frac{3}{2} - \frac{1}{2}\cos 4\theta.$$

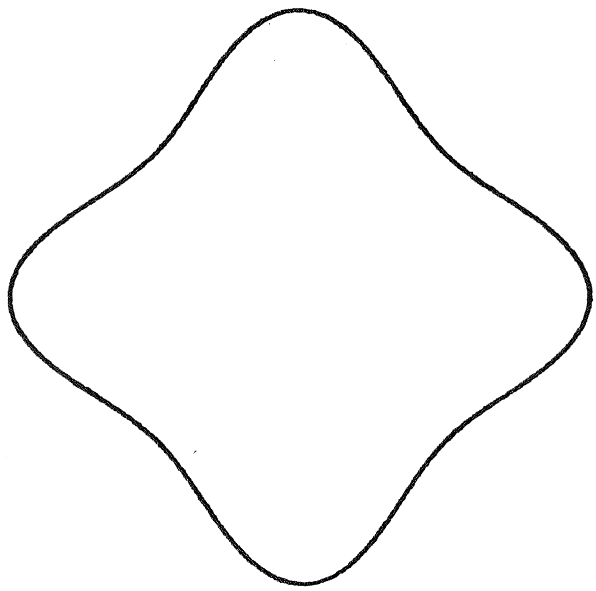


fig. 5.1.

Assuming the existence of at least a quadruplet of unstable fixed points we may write (5.3) as

(5.15) $$\frac{\dot{z}}{z} = r^2 e^{i\phi}\{(\frac{1}{r^2} - \frac{1}{R^2}) + i(e^{-i\beta} - e^{-i(4\theta+\phi)})\},$$

where

(5.16) $$\cos\beta = c\cos(\phi-\gamma)$$

and

(5.17) $$\frac{1}{R^2} = \sin\beta - c\sin(\phi-\gamma).$$

The unstable fixed points are given by

(5.18) $$4\theta = \beta - \phi, \qquad r = R.$$

Technically it is helpful to position the unstable fixed points at $\theta = 0$, $\pi/2$, π and $3\pi/2$. This can be effected by rotating the flow field by a suitable angle. Thus we replace (5.15) by

(5.19) $$\frac{\dot{z}}{z} = r^2 e^{i\phi}\{(\frac{1}{r^2} - \frac{1}{R^2}) + ie^{-i\beta}(1-e^{-4i\theta})\}.$$

We may consider R,ϕ and β as independent parameters. Then the values of c and γ (or p and q) can be derived from

(5.20) $$\begin{cases} c^2 = 1 - \dfrac{2\sin\beta}{R^2} + \dfrac{1}{R^4}, \\ \tan(\phi-\gamma) = \tan\beta - \dfrac{1}{R^2\cos\beta} \end{cases}.$$

We note that for any values of R,ϕ and β suitable values of c and γ can be found.

If $2R^2\sin\beta > 1$ we are in the case $c < 1$ so that there are no stable fixed points. However, if $2R^2\sin\beta < 1$ we obtain a stable quadruplet in the positions determined by

(5.21) $$4\theta = -2\beta, \qquad \frac{1}{r^2} = \frac{1}{R^2} - 2\sin\beta.$$

Our plan is to apply the Poincaré-Bendixson theorem to the square formed by connecting the unstable fixed points by straight lines. The idea is that under certain still unknown conditions a branch of the unstable manifold of one fixed point is spiralling inwards leaving the other fixed points at its outside. Apparently there exists a critical situation where a branch of the unstable manifold of one fixed point becomes a branch of the stable manifold of the next fixed point. Such a situation is obtained experimentally for a map in the case (1.10) with $C = 0$, $D = 3$, $E = -2$ (cf. fig. 6.8). In this critical situation the limit cycle is formed in rotational symmetry by four arcs connected in a non-differentiable way.

In order to apply the Poincaré-Bendixson theorem we have to consider the flow along one side of the square, $x+y = R$, or in polar coordinates

$$(5.22) \qquad \frac{1}{r} = \frac{\sin\theta+\cos\theta}{R}, \qquad 0 \le \theta \le \pi/2.$$

Substitution of this in (5.15) gives

$$(5.23) \qquad \psi = \arg \dot{z}/z = \phi + \arg\,(1-2R^2 e^{-i(2\theta+\beta)}).$$

The condition for inward flow is

$$(5.24) \qquad \frac{3\pi}{4} < \psi + \theta < \frac{7\pi}{4}$$

with the convention $0 \le \psi < 2\pi$.

Let us try the special case $\theta = (\pi-\beta)/2$. Then $\psi = \phi$ and from (5.24)

either $\qquad \phi > \frac{\pi}{4} + \frac{\beta}{2}$ or $\phi < -\frac{3\pi}{4} + \frac{\beta}{2}$.

The general situation can be dealt with as follows. In view of the symmetry we consider only the first case. We know already that $\phi > \pi/4 + \beta/2$ which has only a meaning for $0 \le \beta < \pi/2$.

We write (5.23) as

$$(5.25) \qquad \psi = \phi + \pi/2 - \theta + \arg(A+i\beta)$$

where

$$(5.26) \qquad \begin{cases} A = 2R^2\sin(\theta+\beta) + \sin\theta, \\ B = 2R^2\cos(\theta+\beta) - \cos\theta. \end{cases}$$

The condition of inflow at all points of the side of the square can now be written as

(5.27) $$\phi > \frac{\pi}{4} - \text{Min}\arg(A+iB).$$

A geometrical picture shows that for $0 < \theta < \pi/2$ the point $A+iB$ follows an arc of an ellipse which has the origin as its centre. For $2R^2 > 1$ the orientation is clockwise, for $2R^2 < 1$ anti-clockwise. For $2R^2 = 1$ the ellipse is degenerated into a line. Accordingly we have two cases.

If $2R^2 > 1$ the minimum of the argument is obtained for $\theta = \pi/2$ so that

(5.28) $$\phi > \frac{\pi}{4} + \arctan\frac{2R^2\sin\beta}{2R^2\cos\beta+1}.$$

Since we must have $\phi < \pi/2$ this condition is only useful for

(5.29) $$2R^2\sin\beta < 2R^2\cos\beta + 1.$$

If $2R^2 < 1$ the minimum is attained at $\theta = 0$ so that

(5.30) $$\phi > \frac{\pi}{4} + \arctan\frac{1-2R^2\cos\beta}{2R^2\sin\beta}.$$

This condition is meaningful as long as

(5.31) $$1 - 2R^2\cos\beta < 2R^2\sin\beta.$$

The results obtained so far can be summarised as follows. We consider β with $0 < \beta < \pi/2$ as the primary independent parameter.

(i) $$\sin\beta - \cos\beta < \frac{1}{2R^2} < \sin\beta.$$

This is the case $c < 1$. There exists a limit cycle for ϕ satisfying the condition (5.28).

(ii) $$\sin\beta < \frac{1}{2R^2} < 1.$$

This is the case $c > 1$ with nine fixed points. There exists a limit cycle for ϕ satisfying (5.28)

(iii) $$1 < \frac{1}{2R^2} < \sin\beta + \cos\beta.$$

Again this is the case $c < 1$ with nine fixed points. There exists a limit cycle under the condition (5.30).

6. ILLUSTRATIONS AND APPLICATIONS

A. Our starting-point has been the map (1.5). If the non-trivial fixed point is taken on the origin of a translated coordinate system the map can be written as

(6.1) $$\begin{cases} x' = y, \\ y' = -\frac{a-1}{2}x + \frac{3-a}{2}y - \frac{a}{2}(xy+y^2). \end{cases}$$

Accordingly to (2.20) and (2.21) the corresponding normal form is

(6.2) $$z' = \lambda z - Qz^2\bar{z} + S\bar{z}^3$$

where

(6.3) $$\begin{cases} \lambda = i(1+\mu \exp\frac{\pi i}{4}), \\ 32Q = 9(-2+i), \quad 32S = -9i. \end{cases}$$

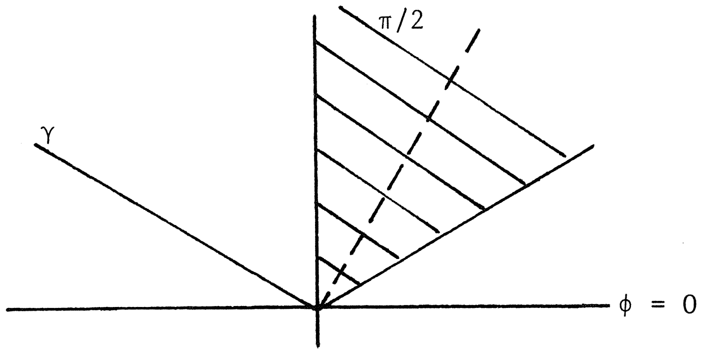
Therefore using the notation of section 3 we have

(6.4) $$c = \sqrt{5}, \qquad \gamma = \pi - \arcsin\frac{1}{\sqrt{5}} = 2.6779.$$

The sector with the two quadruplets of period 4 fixed points is determined by

(6.5) $2\gamma - 3\pi/2 < \phi < \pi/2.$

For the present value $\phi = \pi/4$ this case is realised with $\beta = 3\pi/4$ as sketched below



From (3.9) and (3.10) the positions of the fixed points are obtained as

(6.6) $$4\theta = -\pi, \quad \frac{\mu}{r_-^2} = \sqrt{2}.$$

For the stable set

and

(6.7) $$4\theta = \frac{\pi}{2}, \quad \frac{\mu}{r_+^2} = 2\sqrt{2}$$

for the unstable set.

However, since in section 3 the calculations were performed with $S = 1$ an overall rotation by $-\pi/8$ and a central multiplication by $\sqrt{32/9}$ has still to be applied (cf. Appendix B). This brings e.g. the unstable set in the positions

(6.8) $$4\theta = 0, \quad r_+^2 = \frac{8\mu\sqrt{2}}{9}.$$

As a curiosity we note that in the particular case (6.1) the unstable cycle can be obtained in an explicit way. This fact may be discovered by returning to the original version (1.4). For a 4-cycle we should have

$$(6.9)\quad \begin{cases} 2C = aB(2-B-A), \\ 2D = aC(2-C-B), \\ 2A = aD(2-D-C), \\ 2B = aA(2-A-D). \end{cases}$$

Obviously a solution can be obtained by setting $A = C = 2/a$. Then a simple calculation shows that

$$(6.10)\qquad B = \frac{a-1+\sqrt{(a+1)(a-3)}}{a} \quad ,$$

and for D a similar expression with the minus root.
For (6.1) this gives the cycle

$$(6.11)\qquad \frac{3-a}{a} \to \frac{\sqrt{(a+1)(a-3)}}{a} \to \frac{3-a}{a} \to \frac{-\sqrt{(a+1)(a-3)}}{a} \quad .$$

In order to compare this with the previously obtained values we take

$$(6.12)\qquad a = 3 + 2\mu\sqrt{2}$$

which gives the right correspondence up to $O(\mu^{\frac{1}{2}})$.

It is not possible to obtain an explicit solution for the stable cycle. However, the system (6.9) can be solved by expanding A,B,C,D in powers of $\mu^{\frac{1}{2}}$

$$(6.13)\qquad A = A_1\mu^{\frac{1}{2}} + A_2\mu + A_3\mu^{3/2} + \ldots,$$

etcetera. An elementary, but a bit complicated, computation gives the following results:

$$(6.14)\qquad A_1 = 0, \quad B_1 = 2^{7/4}/3, \quad C_1 = 0, \quad D_1 = -2^{7/4}/3$$

for the unstable cycle, in agreement with (6.11) and

$$(6.15)\qquad A_1 = \frac{2}{3}\sqrt{2\sqrt{2}-2}, \quad B_1 = \frac{2}{3}\sqrt{2\sqrt{2}+2}, \quad C_1 = -A_1, \quad D_1 = -B_1$$

for the stable cycle.
In fig. 6.6 we have made a plot of the case $\mu = 0.005$, $\phi = \pi/5$ for which the theory predicts a limit cycle. Its shape is approximately given by (5.13). Translated back from flow to map, which means an extra scale factor $2\sqrt{\mu}$, we obtain

(6.16) $$r^2 \approx \frac{9}{8}\mu\cos\phi(1+\tfrac{1}{4}\sin 4(\theta+\pi/8))$$

in good agreement with the experimental result.

B. As has been noted in section 2 the Maynard Smith map is determined by

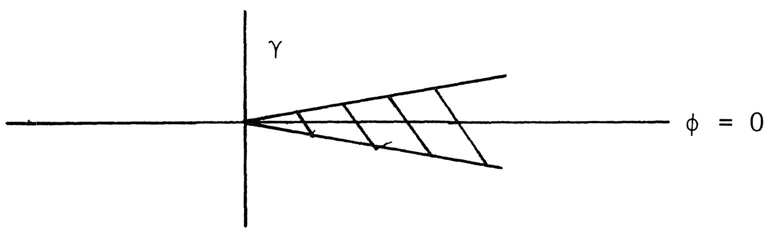
(6.17) $$\begin{cases} x' = y \\ y' = -\lambda\bar{\lambda}x + (\lambda+\bar{\lambda})y - x^2. \end{cases}$$

According to (2.20) and (2.21) the normal form is (6.2) with

$$8Q = 3i, \quad 8S = -1.$$

This gives (see sketch below)

$$c = 3, \quad \gamma = \pi/2, \quad \delta = 0.3398.$$



For values of ϕ in the sector

$$-\delta < \phi < \delta$$

the unstable cycle is determined by

$$(6.18)\quad \begin{cases} \dfrac{\mu}{r_+^2} = 3\cos\phi + \sqrt{1-9\sin^2\phi}, \\ 4\theta = -\phi + \arccos(3\sin\phi), \end{cases}$$

and the stable cycle by

$$(6.19)\quad \begin{cases} \dfrac{\mu}{r_-^2} = 3\cos\phi - \sqrt{1-9\sin^2\phi}, \\ 4\theta = -\phi-\arccos(3\sin\phi). \end{cases}$$

For $\phi = 0$ the map is degenerated. After a suitable rotation the normal form can be written as

$$(6.20)\qquad z' = i(1+\mu)z - 3iz^2\bar{z} - i\bar{z}^3$$

or in real coordinates

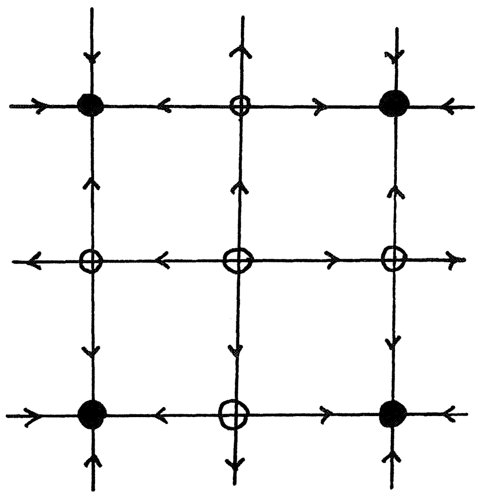
$$(6.21)\quad \begin{cases} x' = -(1+\mu)y + 4y^3, \\ y' = (1+\mu)x - 4x^3. \end{cases}$$

The coordinates get separated after a single iteration. This shows that the lines $x = \pm\sqrt{\mu}/2$ and $y = \pm\sqrt{\mu}/2$ are the stable and unstable manifolds of the cyclic fixed points separating the various flow domains. This is illustrated below for the corresponding flow

$$(6.22)\qquad \dot{z} = z - 3z^2\bar{z} - \bar{z}^3$$

or

$$(6.23)\quad \begin{cases} \dot{x} = x(1-x^2) \\ \dot{y} = y(1-y^2) \end{cases}$$



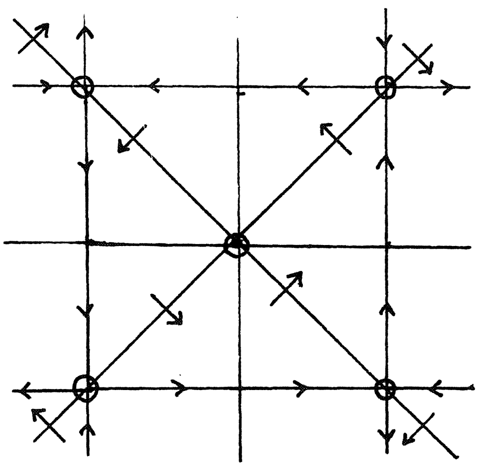
A related degenerate case is obtained for $\phi = \pi/2$, $p = 1$, $q = 0$. The corresponding flow is

$$\dot{z} = iz - iz^2\bar{z} + i\bar{z}^3 \tag{6.23}$$

or

$$\begin{cases} \dot{x} = -y(1-4x^2), \\ \dot{y} = x(1-4y^2). \end{cases} \tag{6.24}$$

There are five unstable fixed points here as sketched below



Obviously the flow pattern is conservative with orbits given by

$$(1-4x^2)(1-4y^2) = Cx^2y^2.$$

C. An endless number of maps of the kind

$$\begin{cases} x' = y, \\ y' = -\lambda\bar{\lambda}x + (\lambda+\bar{\lambda})y + Cx^2 + Dxy + Ey^2, \end{cases}$$

may be considered showing the various possibilities of bifurcation 1:4. Below we list a few possibilities in the special case C = 0.

D	E	16Q	16S	c	γ^0	p	q	δ^0
1	1	-4+2i	-2i	$\sqrt{5}$	153	-1.79	0.89	27
2	1	-4+6i	4-2i	1.61	124	-0.90	1.33	38
3	1	-2+12i	10	1.22	99	-0.19	1.03	55
1	-1	2	-2+4i	0.32	0	2	0	-
2	-1	8+2i	10i	0.82	14	0.80	0.20	-
3	-1	16+6i	4+18i	0.93	21	0.87	0.33	-
3	-2	19+3i	-5+27i	0.70	9	0.69	0.11	-
1	-2	-1-i	-9+7i	0.12	-135	-0.08	-0,08	-

By way of illustration a few plots are given in fig.6.7, 6.8 for the case D = 3, E = -2. In fig.6.7 hyperbolic flow and the formation of a limit cycle is shown. In fig.6.8 the limit cycle is shown together with an inward four-step orbit spiralling towards the limit cycle.

D. As an illustration of the formation of a limit cycle in the corresponding flow we consider the flow (5.1) written in polar coordinates as

$$\begin{cases} \dot{r}/r = \cos\phi - (q+\sin 4\theta)r^2, \\ \dot{\theta} = \sin\phi + (p-\cos 4\theta)r^2. \end{cases}$$

In fig.6.9 the case p = 0, q = 3 is considered for $\phi = \pi/4$ corresponding to a perturbation of the Maynard Smith map. According to the asymptotic theory the limit cycle can be approximated by

$$\frac{1}{r^2} = \sqrt{2}\left(3+\frac{1}{5}\sin 4\theta - \frac{2}{5}\cos 4\theta\right).$$

Agreement between theory and experiment is very good.

In fig.6.10 we consider the case $p = q = \frac{1}{2}$ with $\phi = \pi/3$. According to the theory the unstable fixed points are in the positions

$$r = 1.35, \quad 4\theta = -0.23$$

as shown in the plot. The orbits starting at the x-axis, say with $0 < x < 1$, are seen to make a few revolutions before disappearing into infinity along one of the unstable manifolds of the fixed points. Apparently they just fail to form a limit cycle. Theory predicts a limit cycle if ϕ is close to $\pi/2$, "if more twist is added to the flow pattern". Experiments, not shown here, give the formation of a limit cycle already for $\phi = 4\pi/9$.

In fig.6.11 again $p = q = \frac{1}{4}$ is taken but with $\phi = 0$. This case can be integrated explicitly as shown in Appendix A. The full separatrix is shown as the union of the stable and unstable manifolds of the five fixed points.

APPENDIX A

The flow (5.3) written in polar coordinates as (5.5) gives the following differential equation determining the phase curves

$$(1) \qquad \frac{dr}{rd\theta} = \frac{\cos\phi-(q+\sin 4\theta)r^2}{\sin\phi+(p-\cos 4\theta)r^2}$$

which cannot be integrated explicitly in the general case. However, for $\phi = 0$ it reduces to a linear d.e. of the form

$$(2) \qquad \frac{dr^2}{d\theta} = -r^2A(\theta) + B(\theta)$$

the solution of which is

$$(3) \qquad r^2 = e^{-\int Ad\theta}\{\int e^{\int Ad\theta}B(\theta)d\theta + \text{constant}\}.$$

We distinguish the three cases $|p| > 1$, $p^2+q^2 = 1$ and $|p| > 1$. For simplicity we take $p > 0$.

Case $p > 1$

A simple calculation shows that

$$(4) \qquad \int A(\theta)d\theta = \tfrac{1}{2}\log\frac{s^2+\alpha^2}{s^2+1} + \frac{q}{\sqrt{p^2-1}}\arctan\frac{s}{\alpha}$$

where

$$(5) \qquad \alpha = \sqrt{\frac{p-1}{p+1}}, \qquad s = \tan 2\theta.$$

This shows that

$$(6) \qquad \int_0^{2\pi} A(\theta)d\theta = \frac{4\pi q}{\sqrt{p^2-1}}$$

so that

$$(7) \qquad \int_0^{\theta} A(\tau)d\tau = \frac{2q\theta}{\sqrt{p^2-1}} + \text{periodic function.}$$

The structure of the solutions (3) is now sufficiently clear and apparently for $\theta \to \infty$ all orbits spiral away into infinity.

Case $p^2+q^2 = 1$

From (1) we obtain by simple trigonometry

$$\frac{dr^2}{d\theta} = -2r^2\cot(2\theta-\gamma/2) + \frac{1}{\sin(2\theta+\gamma/2)\sin(2\theta-\gamma/2)} \tag{8}$$

which can be integrated immediately giving

$$2r^2\sin(2\theta-\gamma/2) = \log|\tan(\theta+\gamma/4)| + \text{constant.} \tag{9}$$

The four lines $4\theta = \gamma$ connect the origin to the unstable fixed points as unstable manifolds of the origin and stable manifolds of the four nodal points. The four lines $4\theta = -\gamma$, again unstable manifolds of the origin lead to the four infinite fixed points. The curved part of the separatrix through the fixed point at $\theta = \gamma/4$, its unstable manifold, is determined by

$$2r^2\sin(2\theta-\gamma/2) = \log\left|\frac{\tan(\theta+\gamma/4)}{\tan\gamma/2}\right|. \tag{10}$$

This formule is used to draw the flow pattern of fig.6.11 for the particular case $\gamma = \pi/4$.

Case $0 < p < 1$

Writing $p = \cos\beta$ the flow equations can be written as

$$\begin{cases} \dfrac{\dot r}{r^3} = \dfrac{1}{r^2} - \dfrac{1}{R^2} + (\sin\beta-\sin 4\theta), \\ \dot\theta = (\cos\beta-\cos 4\theta)r^2. \end{cases} \tag{11}$$

This shows as in the previous case that the eight halflines $4\theta = \pm\beta$ are invariant manifolds. The lines $4\theta = \beta$ pass through the unstable fixed point.

The integration of (11) proceeds in a similar way as in the previous section. Again we may use (2) and (3). But now with $p = \cos\beta$ we have

(12) $$\int A(\theta)d\theta = \tfrac{1}{2}\log\frac{s^2-\alpha^2}{s^2+1} + \frac{q}{2\sin\beta}\log\frac{s-\alpha}{s+\alpha}$$

where

(13) $$\alpha = \tan\beta/2, \qquad s = \tan 2\theta.$$

Eventually we obtain

(14) $$\frac{2r^2}{\alpha^2+1} = \frac{(s+\alpha)^{\nu-\frac{1}{2}}}{(s-\alpha)^{\nu+\frac{1}{2}}}\sqrt{s^2+1}\int_{s_0}^{s}\frac{(\sigma-\alpha)^{\nu-\frac{1}{2}}}{(\sigma+\alpha)^{\nu+\frac{1}{2}}}\frac{d\sigma}{\sqrt{\sigma^2+1}} ,$$

with

(15) $$\nu = -\tfrac{1}{2} + \frac{1}{2R^2\sin\beta} .$$

We note that $-\frac{1}{2} < \nu < \frac{1}{2}$.

The unstable manifold of the four fixed points $s = \alpha$, the curved part of the complete separatrix, is given by (14) when the lower boundary s_0 is taken as α. The resulting flow pattern is as in the previous case.

APPENDIX B

In this Appendix a sort of recipe is given for the prediction of the bifurcation behaviour of the map

(1) $$\begin{cases} x' = y \\ y' = Ax + By + Cx^2 + Dxy + Ey^2 \end{cases}$$

close to resonance 1:4.

We have

(2) $$\lambda = i(1+\mu\exp i\phi), \qquad -\pi/2 < \phi < \pi/2$$

and

(3) $$\begin{cases} A = -\lambda\bar{\lambda} = -(1+2\mu\cos\phi+\mu^2), \\ B = \lambda + \bar{\lambda} = -\ 2\mu\sin\phi. \end{cases}$$

The normal form is

(4) $$z' = \lambda z - Qz^2\bar{z} + S\bar{z}^3$$

with

(5) $$z = x - iy = r \exp i\theta$$

and

(6) $$16Q = (D^2-2E^2-3CD-6CE-2DE) + i\ (6C^2+D^2+CD+2CE+DE)$$

(7) $$16S = (D^2-2E^2-3CD+2CE+DE)+i(-2C^2+D^2+CD+2CE-3DE).$$

We write

(8) $$c = |Q/S|, \quad \gamma = \arg Q,$$

and

(9) $$S = s \exp i\sigma.$$

<u>Case $|Q| > |S|$</u>

Define δ by

(10) $$\delta = \text{arc}\sin|S/Q|.$$

If for a given $\phi \in (-\pi/2,\pi/2)$

(11) $$\gamma-\pi/2-\delta < \phi < \gamma-\pi/2+\delta$$

then there is an inner unstable 4-cycle and an outer stable 4-cycle.

Define β by

(12) $$\cos\beta = c\,\cos(\phi-\gamma), \quad 0 \le \beta \le \pi.$$

The unstable 4-cycle is determined by

(13) $$\begin{cases} \dfrac{\mu}{r_+^2} = -|Q|\sin(\phi-\gamma) + |S|\sin\beta, \\ 4\theta = \sigma - \phi + \beta. \end{cases}$$

The stable 4-cycle is determined by

(14) $$\begin{cases} \dfrac{\mu}{r_-^2} = -|Q|\sin(\phi-\gamma) - |S|\sin\beta, \\ 4\theta = \sigma - \phi - \beta. \end{cases}$$

Case $|Q| < |S|$

Define β as in the previous case. There is only an unstable 4-cycle, determined as above.

For $|Q|/|S| \to \infty$ the asymptotic shape of a possible limit cycle is given by

(15) $$\frac{\mu s}{r^2} = \frac{\mathrm{Im}Q}{\cos\phi} + \frac{\sin(4\theta-\sigma)\cos\phi - 2\omega\cos(4\theta-\sigma)}{\cos^2\phi + 4\omega^2} + \ldots$$

where

(16) $$\omega = \frac{\cos(\phi-\gamma)}{\sin\gamma}.$$

The spacing of successive 4-iterates $z_0, z_4, z_8, \ldots$ on the limit cycle is approximately given by

(17) $$4\mu\omega r.$$

If there exists an unstable 4-cycle a limit cycle may exist in the region between these four points provided ϕ is close to $\pi/2$ or $-\pi/2$.

DESCRIPTION OF THE ILLUSTRATIONS

6.1. Model

$$\begin{cases} x' = y \\ y' = ay(1-\frac{x+y}{2}) \end{cases}$$

with a = 3.01

Scale $-0.3 < x-1+\frac{1}{a} < 0.3$

$$-2 < y-1+\frac{1}{a} < 0.2$$

Two orbits are shown both converging to the stable 4-cycle.
Note that four successive points $P_1P_2P_3P_4$ are always the corners of an only slightly deformed square. Successive points of the 4× iterated map $P_1P_5P_9\dots$ are tracing an almost continuous curve.

6.2. Model (6.1) with $\mu = 0.0005$, $\phi = 0$.
Scale -0.3, 0.3, -0.2, 0.2.
Start 0,0.15

The orbit converges to a limit cycle.

6.3. This and the next few plots show what may happen if μ is further increased. Again we consider the model (3.1) with the scale -1.5, 1.5, -1,1. For $\mu = 0.05$, $\phi = 0$ the limit cycle gets bumpy.

6.4. As in the previous plot. For $\mu = 0.06$, $\phi = 0$ the limit cycle disappears and degenerates into an attracting cycle of order 22.

6.5. Further increase of μ, here $\mu = 0.09$, shows a "strange" attractor. It can be proved that the invariant curve is analytic and of the form $x = f(t)$, $y = g(t)$ where f and g are entire functions of t.

6.6. The model (6.1) with $\mu = 0.005$, $\phi = \pi/5$
Scale - 0.3, 0.3,-0.2, 0.2.
The orbit starting at 0,0.15 converges to a star-shaped limit cycle.

6.7. The model (2.1) with C = 0, D = 3. E = -2 with $\mu = 0.005$, $\phi = 4\pi/9$.
Scale -0.3, 0.3, - 0.2, 0.2.
Shown is an orbit converging to a limit cycle and a few orbits escaping to infinity.

6.8. As previous with $\mu = 0.006$, $\phi = 4\pi/9$.
Scale -0.2, 0.2, $-0.4/3$, $0.4/3$.
Shown is a limit cycle with an inner orbit.

6.9. The flow (5.3) resp. (5.5) with $p = 0$, $q = 3$ and $\phi = \pi/4$.
Scale -3, 3, -2, 2.
A square-like limit cycle is formed.

6.10. The flow (5.3) resp. (5.5) with $p = q = \frac{1}{2}$ and $\phi = \pi/3$.
Scale $-3,3,-2,2$.
The flow just fails to form a limit cycle. Orbits are given for the starting-points $x = 0.04$, 0.1, 0.2, 0.3, 0.4, 0.5 on the y-axis.

6.11. The flow (5.3) resp. (5.5) with $p = q = 2^{-\frac{1}{2}}$ and $\phi = 0$.
Scale $-3,3,-2,2$.

REFERENCES

[1] POUNDERS, J.R. and Th.D. ROGERS, (1980), *The geometry of chaos: Dynamics of a nonlinear second-order difference equation*, Bull. Math. Biology 42, 551-597.

[2] ARONSON, D.G., et al. (1982), *Bifurcation from an invariant circle for two-parameter families of maps of the plane*, A computer-assisted study, Comm. Math. Phys. 3, 303-354.

[3] ARNOLD, V.I., (1983), *Geómetrical methods in the theory of ordinary differential equations*, (Translation of "Supplementary chapters of the theory of o.d.e.'s"). Springer, Grundlehren vol.250.

[4] IOOSS, G., (1979), *Bifurcation of maps and applications*, North-Holland, Mathematics Studies vol. 36.

[5] WAN, A.H., (1978), *Bifurcation into invariant tori at points of resonance*, Arch. Rat. Mech. Anal. 68, 343-357.

[6] NEISHTADT, A.J., (1978), *Bifurcations of the phase portrait of a system of equations occurring in the problem of the loss of stability of oscillations close to resonance* 1:4, Prikl. Math. Meh. 42, 830-840.

[7] BERZOVSKAYA, F.S., A.I. HIBNIK, (1980), *On the bifurcations of the separatrices in the problem of the loss of stability of oscillations close to resonance* 1:4, Prikl. Mat. Meh. 44, 938-943.

[8] WHITLEY, D., (1983), *Discrete dynamical systems in dimension one and two*, Bull. Lond. Math. Soc. 15, 177-217.

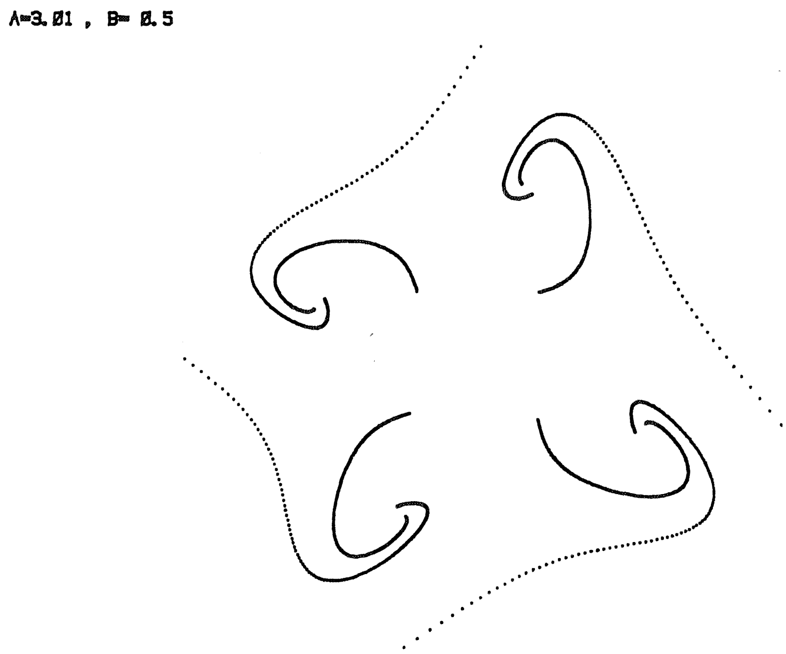


fig. 6.1

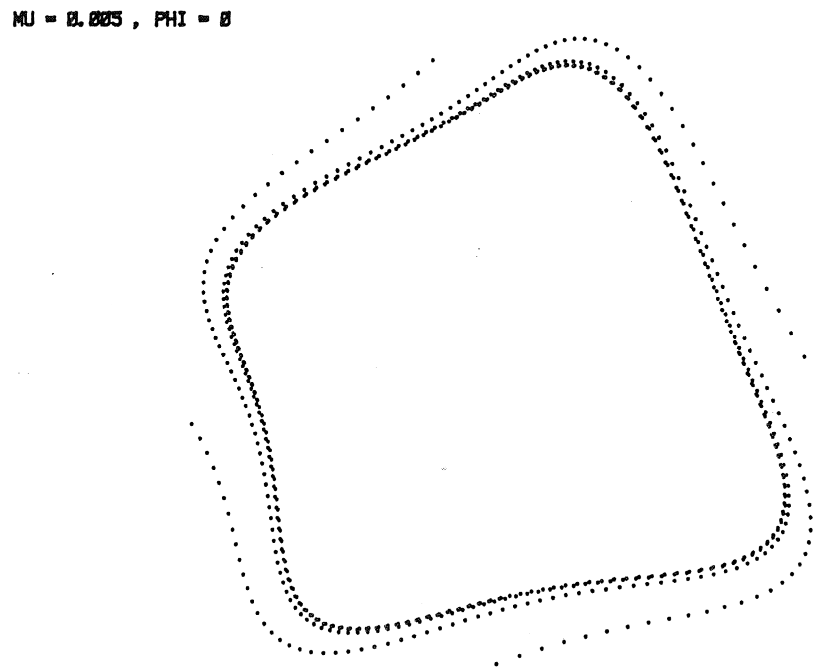


fig. 6.2

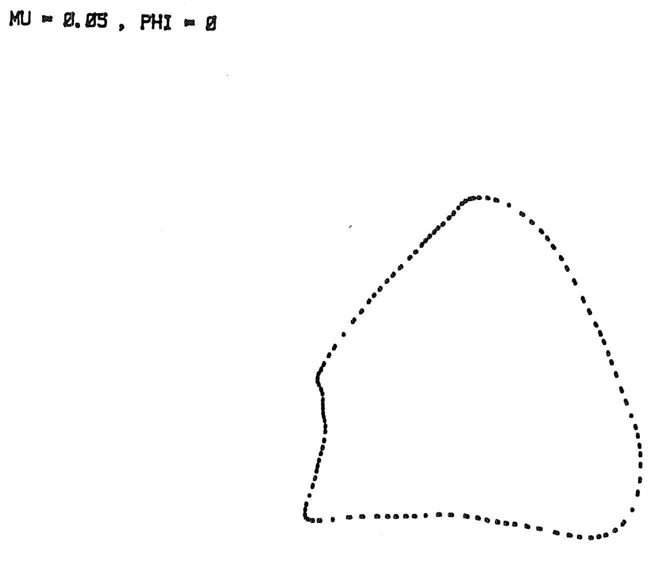


fig. 6.3

MU = 0.08 , PHI = 0

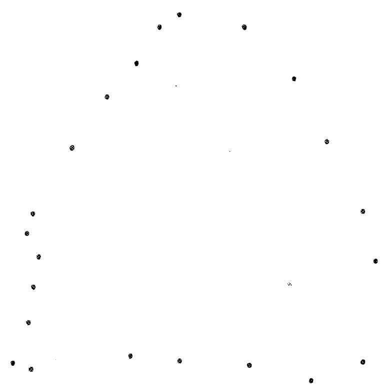


fig. 6.4

MU = 0.09 , PHI = 0

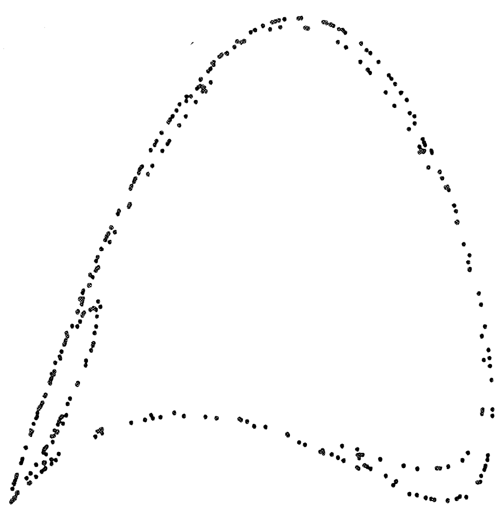


fig. 6.5

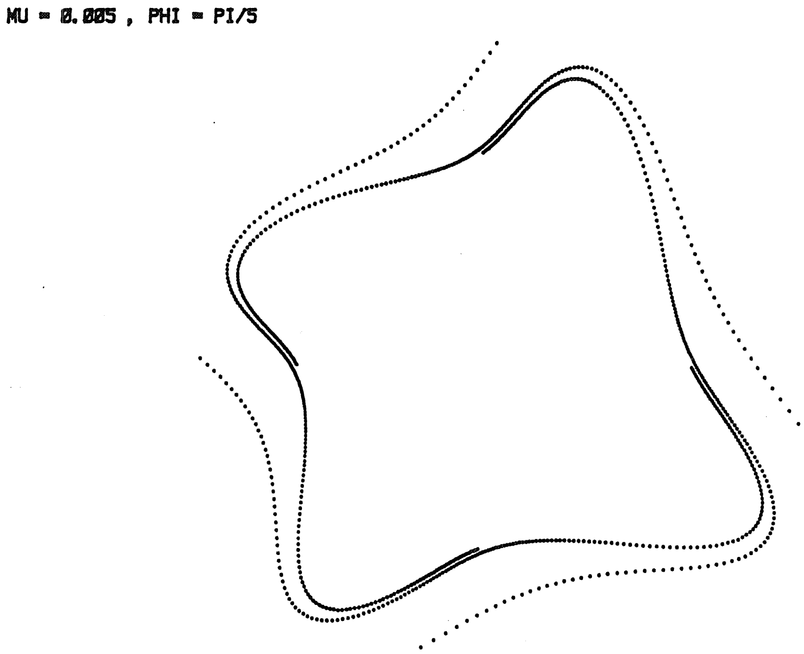


fig. 6.6

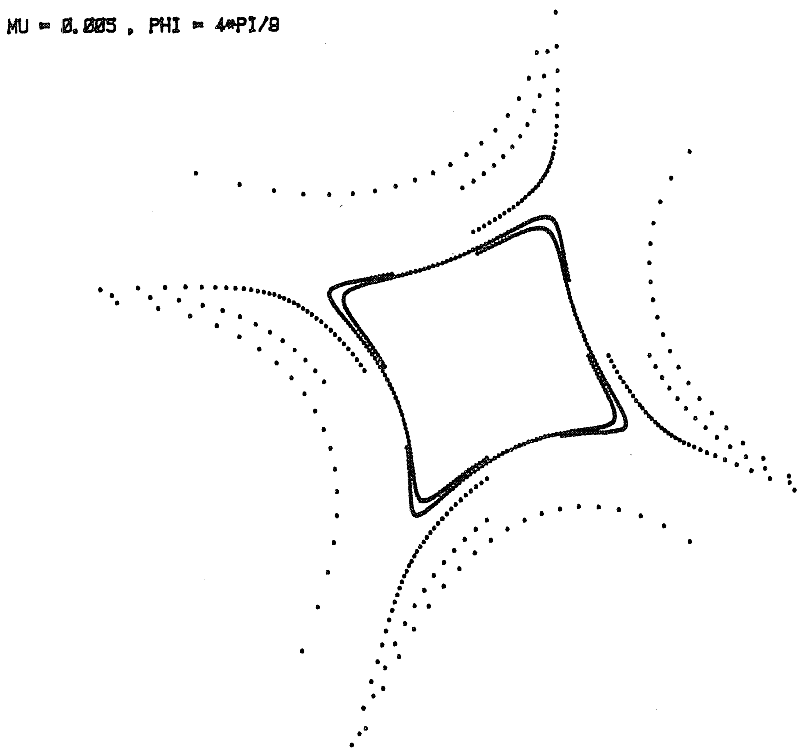


fig. 6.7

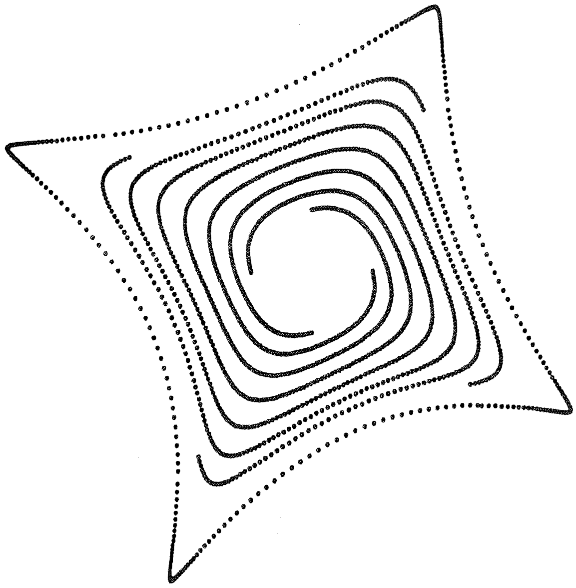


fig. 6.8

P = Ø , Q = 3 , PHI = PI/4

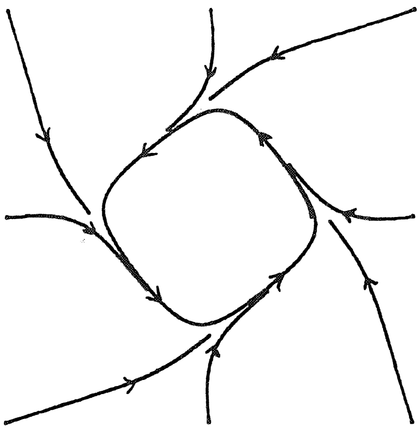


fig. 6.9

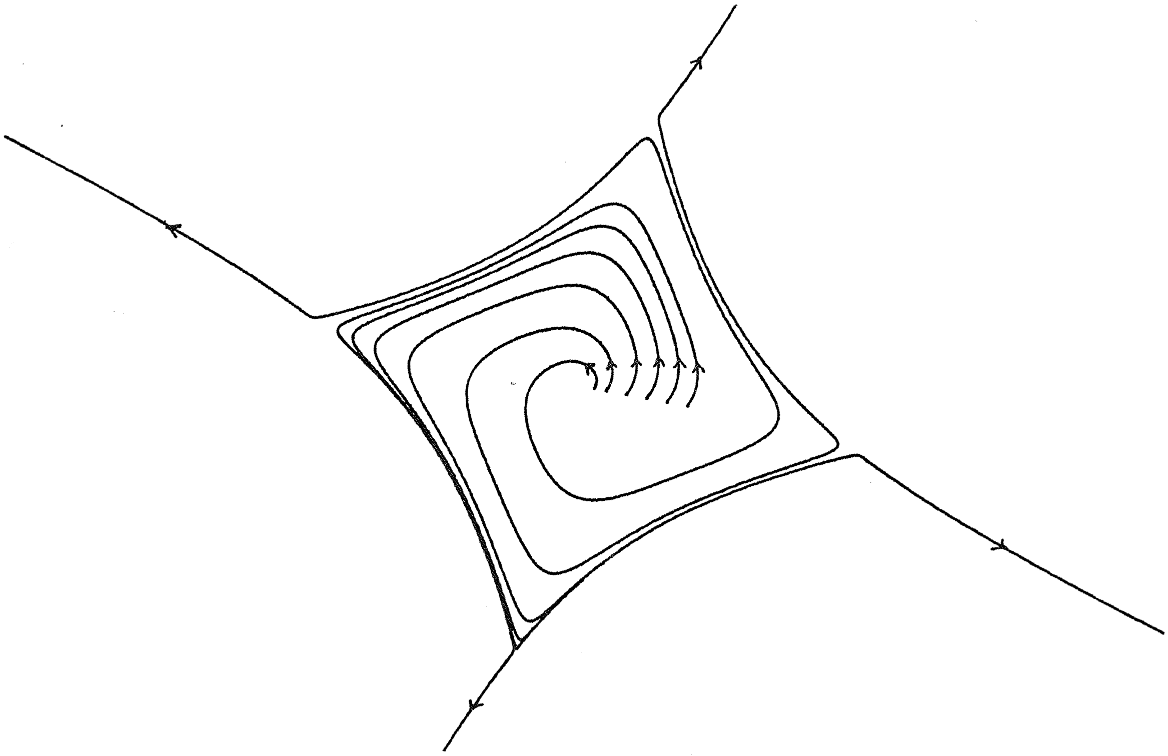


fig. 6.10

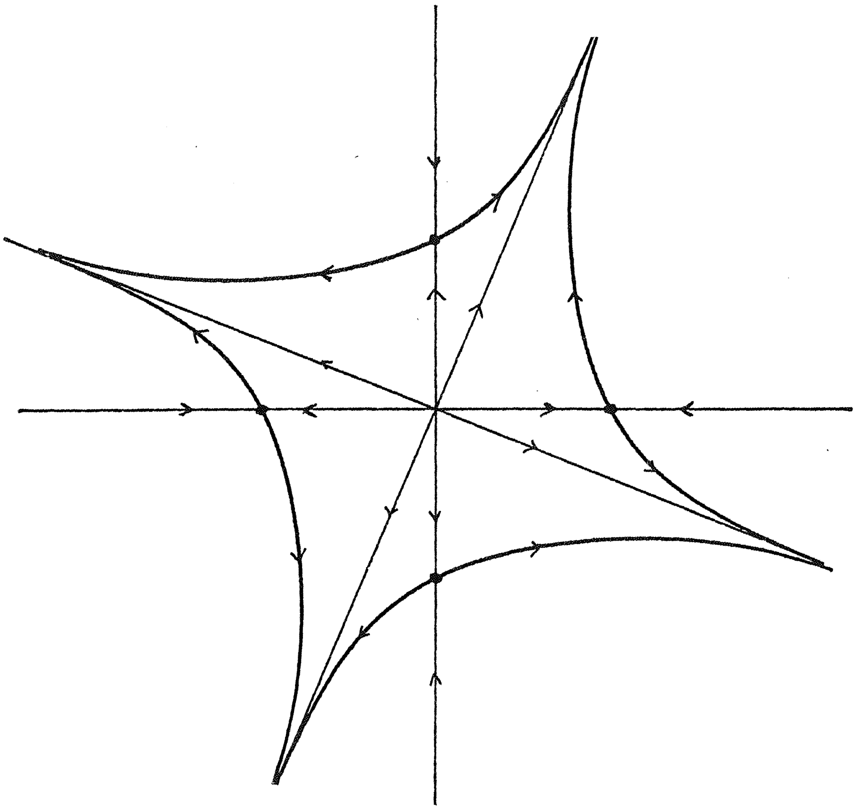


fig. 6.11