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ON THE WEAK LIMITS OF ELEMENTARY SYMMETRIC POLYNOMIALS

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In this paper we extend recent results of *Székely* and others on the weak limits of elementary symmetric polynomials $S_n^{(k_n)}(X_1,\ldots,X_n)$ in the case where the order k_n of the polynomials is proportional to the number of variables n.

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1. Introduction

In Székely (1982) it was shown that the normalized elementary symmetric polynomials

$$T_n^{(k_n)} := \{S_n^{(k_n)}(X_1,...,X_n) / \binom{n}{k_n}\}^{1/k_n} \tag{1.1}$$

are asymptotically normal for $n \to \infty$ if $X_1, X_2,...$ is an i.i.d. sequence of strictly positive random variables and if $k_n / n \to c$ for some constant c, $0<c<1$. More precisely

$$n^{\frac{1}{2}}(T_n^{(k_n)} - L_n) \overset{w}{\to} CN, \tag{1.2}$$

where N is standard normal, C and L_n are positive norming constants and L_n converges to a positive constant L. In a second paper, Móri & Székely (1982), a similar situation was investigated for random variables X_n of the form $P(X_n=1)=1-P(X_n=-1)=\frac{1}{2}$. This case is more delicate since terms cancel in the sum $S_n^{(k_n)}$. However, the authors succeeded in giving a complete analysis in this situation. In particular they proved that if $(2\pi)^{-1} arcsin(\sqrt{c})$ is irrational then

$$n^{\frac{1}{4}}(S_n^{(k_n)} / \binom{n}{k_n}^{\frac{1}{2}}) \overset{w}{\to} C_1 e^{N^2/4} \cos(2\pi U), \tag{1.3}$$

with U and N independent, U uniformly distributed on [0,1] and N standard normal.

Note the difference in magnitude of the random variables $S_n^{(k_n)} / \binom{n}{k_n}^{\frac{1}{2}}$ in the two cases:

$$\log(|S_n^{(k_n)} / \binom{n}{k_n}^{\frac{1}{2}}|) = nr_n + n^{\frac{1}{2}} V_n \quad \text{in (1.2)}$$

$$= -\tfrac{1}{4}\log n + W_n \quad \text{in (1.3)}$$

where V_n and W_n have nondegenerate limit distributions, and r_n converges to a constant.

All we shall do is to allow the variables X_n to vanish with positive probability. Thus we shall consider the case $X_n \geqslant 0$ and $P(X_n>0)=p$, and the case $P(X_n=1)=P(X_n=-1)=\frac{1}{2}P(X_n \neq 0)=\frac{1}{2}p$, both with $0<p<1$.
In the first case there are no substantial changes, as long as we assume that $0<c=\lim k_n/n<p$. In the second case, if $0<c<p<1$ and $n^{\frac{1}{2}}(k_n/n-c)$ converges, then

$$\log(|S_n^{(k_n)} / \binom{n}{k_n}^{\frac{1}{2}}|) = -ns_n + n^{\frac{1}{2}} W_n', \tag{1.4}$$

where s_n has a positive limit and W_n' has a nondegenerate normal limit distribution.

It would seem that the case $P(X_n=1)=P(X_n=-1)=\frac{1}{2}$ considered in Móri & Székely is exceptional. A slight disturbance of this distribution completely alters the limit behaviour. However, it is not known what the limit behaviour is for symmetrically distributed variables X_n other than those described above. In particular, it would be interesting to know what happens if X_n is uniformly distributed on the interval [-1,1] or if X_n is uniformly distributed over the points $-2,-1,1,2$. These cases cannot be handled by the technique developed in this paper.

2. Preliminaries

For a finite collection of random variables $X_1,...,X_n$ we define the elementary symmetric variables $S^{(k)}(X_1,...,X_n)$ as the sum over all subsets $E \subset \{1,...,n\}$ of size k of $\Pi_{j \in E} X_j$. Then

$$S^{(k)}(X_1,...,X_n) = \sum_{1 \leqslant i_1 < ... < i_k \leqslant n} X_{i_1}...X_{i_k}, \quad 1 \leqslant k \leqslant n.$$

Usually we start with an i.i.d. sequence $X_1,X_2,\ldots$ with common distribution function F and write $S_n^{(k)}$ for $S^{(k)}(X_1,\ldots,X_n)$. Note that $S^{(k)}(X_1,\ldots,X_n)$ is the coefficient of t^k in the expansion of the random polynomial $\Pi_{i=1}^n(1+tX_i)$.

$S^{(k)}(X_1,\ldots,X_n)/\binom{n}{k}$ is the mean value of the product $X_{i_1}\ldots X_{i_k}$ over all subsets $\{i_1,\ldots,i_k\}\subset\{1,\ldots,n\}$ of size k, and the statistic $T^{(k)}(X_1,\ldots,X_n)=\{S^{(k)}(X_1,\ldots,X_n)/\binom{n}{k}\}^{1/k}$ is homogeneous of degree 1: $T^{(k)}(rX_1,\ldots,rX_n)=rT^{(k)}(X_1,\ldots,X_n)$.

Considering limits of the statistics $S_n^{(k_n)}$ let us first take $k_n=k$ fixed and $n\to\infty$. Then the sequence $(S_n^{(k_n)})_n$ is a sequence of U-statistics of order k with kernel $h(x_1,\ldots,x_k)=x_1\ldots x_k$ (cf. Serfling (1980)). Since the fundamental paper of Hoeffding (1948) U-statistics have been studied intensively and their limit behavior (for fixed k) is well understood.

Our concern is with the case that $k_n\to\infty$ and $k_n/n\to c$ $(0\leqslant c\leqslant 1)$. In this case the norming constant $\binom{n}{k_n}^{1/k_n}$ in the definition of $T_n^{(k_n)}$, see (1.1), satisfies

$$\frac{1}{k_n}\log\binom{n}{k_n}=\phi\left(\frac{k_n}{n}\right)+\frac{1}{k_n}R(n,k_n)$$

where $\phi(x)=-x\log x-(1-x)\log(1-x)$ is bounded, continuous and nonnegative on $[0,1]$ and $R=0(\frac{1}{k}\log k)$ by Stirling's formula. If $k_n/n\to c\in(0,1)$ the exponent $1/k_n$ reduces the factor $1/\binom{n}{k_n}$ in the definition of $T_n^{(k_n)}$ to an innocuous constant.

Remark 2.1.

We shall investigate the limit behaviour of $S_n^{(k_n)}, n=1,2,\ldots,$ although all theorems are also valid for statistics $S_{n_j}^{(k_j)}, j=1,2,\ldots,$ where (k_j) and (n_j) are sequences of integers satisfying $k_j\to\infty$, $n_j\to\infty$, $1\leqslant k_j\leqslant n_j$ and $k_j/n_j\to c$. In fact this is used in sections 4 and 5 where the particular sequence $(n_j)=1,2,\ldots$ is replaced by a sequence of random integers $E_1,E_2,\ldots$.

3. The simple case: $P(X_n=1)=p=1-P(X_n=0)$.

Let $E_n=X_1+\ldots+X_n$ denote the number of nonzero variables X_j. The random variable E_n has a $Bin(n,p)$ distribution and $S_n^{(k)}=\binom{E_n}{k}$ since the product $X_{i_1}\ldots X_{i_k}$ vanishes unless all k variables equal 1. Then $T_n^{(k_n)}=L_n(E_n/n,k_n/n)$ where L_n is defined on a subset of $I^2=[0,1]\times[0,1]$ by

$$L_n(x,y)=\begin{cases}\left\{\binom{nx}{ny}/\binom{n}{ny}\right\}^{1/ny} & \text{if } x,y\in\{1/n,2/n,\ldots,1\} \text{ and } x\geqslant y\\ 0 & \text{if } 0\leqslant x<y\leqslant 1.\end{cases}\tag{3.1}$$

The functions L_n can be extended to functions on I^2 in a straightforward way.

Lemma 3.1. *Let L be the function on I^2 defined by*

$$L(x,y)=\begin{cases}\exp\{\frac{1}{y}(x\log x+(1-y)\log(1-y)-(x-y)\log(x-y))\} & \text{if } 0<y\leqslant x\leqslant 1\\ 0 & \text{if } 1\geqslant y>x\geqslant 0\\ x & \text{if } y=0\end{cases}\tag{3.2}$$

then for $\alpha<1$ *and all* $(x,y)\in I^2$

$$\lim_{n\to\infty} n^{\alpha}(L_n(x,y)-L(x,y)) = 0, \tag{3.3}$$

uniformly on sets $D_\delta = [0,1]\times[\delta,1]$,$\delta>0$.

By (3.3) if suffices to investigate $L(E_n/n, k_n/n)$ instead of $T_n^{(k_n)} = L_n(E_n/n, k_n/n)$. This results in the next two limit theorems for zero-one X_n.

Parts a) and c) of the next theorem can also be found in Székely (1974) where they are proved directly using Stirling's formula.

Theorem 3.2. *Let* $X_1, X_2, \ldots$ *be i.i.d. zero-one random variables with* $P(X_n=1)=p=1-P(X_n=0)$ $(0<p<1)$ *and let* (k_n) *be a sequence of integers with* $1\leqslant k_n\leqslant n$, $k_n\to\infty$ *and* $k_n/n\to c$ $(0\leqslant c\leqslant 1)$.

a) *If* $c<p$ *then*

$$T^{(k_n)}(X_1,\ldots,X_n)\to L(p,c), \text{ almost surely.}$$

b) *If* $c=p$ *then* $T^{(k_n)}(X_1,\ldots,X_n)$ *converges in distribution if and only if*

$$n^{\frac{1}{2}}(k_n/n-p)\to a, \text{ for some } a\in[-\infty,\infty].$$

Moreover, in case of convergence the limit variable T *is two valued,*

$$P(T=0) = 1-P(T=L(p,p)) = \Phi(a/(p(1-p))^{\frac{1}{2}}).$$

c) *If* $c>p$ *then there exists an almost surely finite random variable* N_0 *such that* $T^{(k_n)}(X_1,\ldots,X_n)=0$ *for all* $n\geqslant N_0$.

(Φ denotes the standard normal distribution function).

This theorem can be intuitively understood by viewing the process $(E_n/n, k_n/n, L(E_n/n, k_n/n))_n$ in I^3 as a random walk on the graph of L.

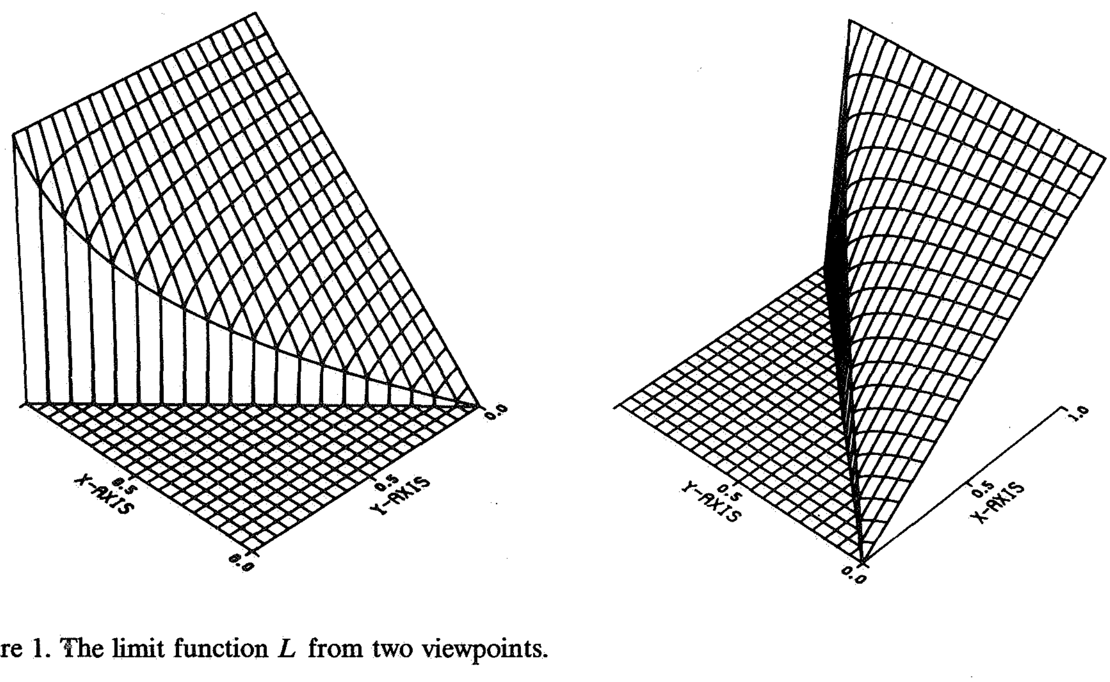

Figure 1. The limit function L from two viewpoints.

By lemma 3.1 for each sequence (b_n) of location constants the difference between the statistics

$n^{\frac{1}{2}}(T_n^{(k_n)}-b_n)$ and $n^{\frac{1}{2}}(L(E_n/n,k_n/n)-b_n)$ tends to zero almost surely. Therefore they have the same weak limits. Examining the second statistic we obtain the following weak convergence theorem.

Theorem 3.3. *Let N denote a standard normal random variable. Let $X_1,X_2,...$ be i.i.d. zero-one random variables with $P(X_n=1)=p=1-P(X_n=0)$ $(0<p<1)$ and let (k_n) be a sequence of integers with $1 \leqslant k_n \leqslant n$ and $k_n/n \to c$ $(0<c \leqslant p)$.*

a) *If $0<c<p$ then*

$$n^{\frac{1}{2}}(T^{(k_n)}(X_1,...,X_n)-L(p,k_n/n))$$

converges in distribution to

$$c^{-1}\log(p/(p-c))L(p,c)(p(1-p))^{\frac{1}{2}}N.$$

b) *If $c=p$ and $n^{\frac{1}{2}}(k_n/n-p) \to a \in (-\infty,\infty)$ then*

$$\frac{2n^{\frac{1}{2}}}{\log n}(T^{(k_n)}(X_1,...,X_n)-L(k_n/n,k_n/n))\iota(E_n-k_n)$$

converges in distribution to

$$p^{-1}L(p,p)((p(1-p))^{\frac{1}{2}}N-a)^{+}.$$

c) *If $c=p$ and $n^{\frac{1}{2}}(k_n/n-p) \to -\infty$ then*

$$\frac{-n^{\frac{1}{2}}}{\log(p-k_n/n)}(T^{(k_n)}(X_1,...,X_n)-L(p,k_n/n))$$

converges in distribution to

$$p^{-1}L(p,p)(p(1-p))^{\frac{1}{2}}N.$$

d) *If $c=p$ and $n^{\frac{1}{2}}(k_n/n-p) \to \infty$ then*

$$T^{(k_n)}(X_1,...,X_n) \to 0, \textit{ in probability.}$$

($\iota(x)=1$ if $x \geqslant 0$ and 0 otherwise, $x^{+}=x\,\iota(x)$).

4. Nonnegative X_n

Theorems 3.2 and 3.3 enable us to extend Halász & Székely's (1976) and Székely's (1982) results for strictly positive X_n to nonnegative X_n.

Suppose that $X_1,X_2,...$ are i.i.d. nonnegative random variables and that $X_n=Z_nY_n$ where $Z_1,Z_2,...$ is a sequence of i.i.d. zero-one random variables and $Y_1,Y_2,...$ is a sequence of i.i.d. strictly positive random variables. These two sequences are assumed to be independent. Let E_n denote the number of X_j in $X_1,...,X_n$ unequal to zero.

Let $1 \leqslant T_1<T_2<...$ be the indices n for which Z_n is strictly positive. The sequence $X_{T_1},X_{T_2},...$ is distributed like $Y_1,Y_2,...$ and $S^{(k)}(X_1,...,X_n)=S^{(k)}(X_{T_1},...,X_{T_{E_n}})$ for all ω. This gives us the following lemma which is crucial for the extensions in this section.

Lemma 4.1. *With the above notation the two sequences of random variables*

$$(S^{(k_n)}(X_1,...,X_n))_n \textit{ and } (S^{(k_n)}(Y_1,...,Y_{E_n}))_n$$

have the same distribution, i.e. each corresponding finite subset of the sequences has the same distribution.

It follows that

$$T_n^{(k_n)} \overset{d}{=} \left\{ \begin{bmatrix} E_n \\ k_n \end{bmatrix} \Big/ \begin{bmatrix} n \\ k_n \end{bmatrix} \right\}^{1/k_n} T^{(k_n)}(Y_1,...,Y_{E_n}) = T^{(k_n)}(Z_1,...,Z_n)T^{(k_n)}(Y_1,...,Y_{E_n}). \tag{4.1}$$

Note that (4.1) is the product of an elementary symmetric polynomial of zero-one random variables and a polynomial of a random number of strictly positive random variables. A combination of the results of section 3 and the results of Halász & Székely (1976) and Székely (1982) proves the following two theorems.

Theorem 4.2. *Let $X_1,X_2,...$ be i.i.d. nonnegative random variables with $P(X_1>0)=p>0$ and let (k_n) be a sequence of integers with $1 \leqslant k_n \leqslant n, k_n \to \infty$ and $k_n/n \to c$ $(0 \leqslant c \leqslant 1)$. Define Y_1 to be a random variable distributed like X_1 conditional on $X_1>0$.*
If $c<p$, assuming $EY_1<\infty$ for $c=0$ and $E\log(1+Y_1)<\infty$ for $0<c<p$, then we have

$$T^{(k_n)}(X_1,...,X_n) \to S(c), \textit{ almost surely.}$$

The limit constant $S(c)$ is defined by

$$S(c) = \begin{cases} EX_1 & \text{if } c=0 \\ c(1-c)^{(1-c)/c}\exp\{\frac{1}{c}(E\log(r_c+X_1)+(c-1)\log r_c)\} & \text{if } 0<c<p \\ c(1-c)^{(1-c)/c}\exp\{E\log Y_1\} & \text{if } c=p, \end{cases} \tag{4.2}$$

where for $0<c<p$ the constant r_c is the unique nonnegative root of the equation

$$Er/(r+X_1) = 1-c. \tag{4.3}$$

Theorem 4.3. *Let $X_1,X_2,...$ be i.i.d. nonnegative random variables with $P(X_1>0)=p>0$ and let (k_n) be a sequence of integers with $1 \leqslant k_n \leqslant n$ and $k_n/n \to c$ $(0<c\leqslant p)$. Let N denote a standard normal random variable and Y_1 a random variable distributed like X_1 conditional on $X_1>0$.*
If $0<c<p$, assuming $E\log^2(1+Y_1)<\infty$, then we have

$$n^{1/2}(T^{(k_n)}(X_1,...,X_n) - S(k_n/n)) \overset{w}{\to} CN,$$

where C is a positive constant.

When restricted to zero-one X_n these theorems give the $c<p$ parts of the theorems in section 3. In the appendix it is shown that the $c=p$ and $c>p$ parts also hold for nonnegative X_n. For p equal 1 they reduce to results of Halász and Székely for strictly positive X_n.

5. Three valued symmetric X_n

Let $Y_1,Y_2,...$ be i.i.d. random variables with common distribution $P(Y_1=1)=P(Y_1=-1)=\frac{1}{2}$ and $Z_1,Z_2,...$ i.i.d. zero-one random variables, independent of $Y_1,Y_2,...$. Taking $X_n=Z_nY_n$ we may draw the same conclusion as in Lemma 4.1. The next theorem for X_n with distribution $P(X_n=1)=P(X_n=-1)=\frac{1}{2}P(X_n\neq 0)=\frac{1}{2}p$ $(0<p<1)$ is obtained from Móri & Székely's (1982) results and our theorem 3.3 by examination of $S^{(k_n)}(Y_1,...,Y_{E_n})$.

Theorem 5.1. *Let N denote a standard normal random variable. Let $X_1,X_2,...$ be i.i.d. random variables with $P(X_1=1)=P(X_1=-1)=\frac{1}{2}P(X_1\neq 0)=\frac{1}{2}p$ $(0<p<1)$ and let (k_n) be a sequence of integers with $1\leqslant k_n \leqslant n$ and $k_n/n \to c$ $(0<c<1)$.*
If $0<c<p$ and $n^{\frac{1}{2}}(k_n/n-p)$ converges then

$$n^{\frac{1}{2}}(|S^{(k_n)}(X_1,\ldots,X_n)|^{1/k_n} / \binom{n}{k_n}^{1/2k_n} - L(p,k_n/n)^{\frac{1}{2}}) \tag{5.1}$$

converges in distribution to

$$\tfrac{1}{2}c^{-1}\log(p/(p-c))L(p,c)^{\frac{1}{2}}(p(1-p))^{\frac{1}{2}}N.$$

Note the absence of conditions on $(2\pi)^{-1}arcsin(\sqrt{c})$ and the different order of magnitude compared to Móri & Székely's theorem.

6. Proofs

6.1 Proofs of section 3.

The extension of the function L_n of formula (3.1) to a function on I^2 is achieved by interpreting the factorials in the binomial coefficients in (3.1) as gamma functions, using $n!=\Gamma(n+1)$. So we redefine L_n as

$$L_n(x,y) = \begin{cases} \left[\dfrac{\Gamma(nx+1)\Gamma(n(1-y)+1)}{\Gamma(n(x-y)+1)\Gamma(n+1)}\right]^{1/ny} & \text{if } 0<y\leqslant x\leqslant 1 \\ 0 & \text{if } 1\geqslant y>x\geqslant 0, \\ \exp(\psi(nx+1)-\psi(n+1)) & \text{if } y=0 \end{cases} \tag{6.1}$$

where the psi function as usual denotes the derivative of $\log\Gamma(x)$.
For the properties of the gamma- and psi function used in the next proof we refer to Abramowitz & Stegun (1965).

Proof of Lemma 3.1. Since both $L_n(x,y)$ and $L(x,y)$ are zero if $0\leqslant x<y\leqslant 1$ we restrict attention to points (x,y) with $0\leqslant y\leqslant x\leqslant 1$.
A straigthforward application of Stirling's formula for the gamma function yields

$$\log\Gamma(t+1) = t\log t+\tfrac{1}{2}\log(t+1)-t+R(t), \quad t\geqslant 0, \tag{6.2}$$

where R is a bounded function. Substituting (6.2) in (6.1) we find for $0<y\leqslant x\leqslant 1$

$$ny\log L_n(x,y) = ny\log L(x,y)+\tfrac{1}{2}\log\left[\frac{(nx+1)(n(1-y)+1)}{(n+1)(n(x-y)+1)}\right]+R_n(x,y),$$

with $|R_n(x,y)|\leqslant M$ for some $M>0$. Since

$$1 \leqslant \frac{(nx+1)(n(1-y)+1)}{(n+1)(n(x-y)+1)} \leqslant 1+ny$$

it follows that for $0<y\leqslant x\leqslant 1$

$$|\log L_n(x,y)-\log L(x,y)| \leqslant \frac{1}{ny}(\log(1+ny)+M), \tag{6.3}$$

and hence

$$\lim_{n\to\infty} n^{\alpha}(\log L_n(x,y)-\log L(x,y)) = 0,$$

uniformly on sets D_δ.
Because the values of both L_n and L are between zero and one by

$$x,y\in(0,1] \Rightarrow |x-y| \leqslant |\log x-\log y|$$

this implies

$$\lim_{n\to\infty} n^{\alpha}(L_n(x,y)-L(x,y)) = 0,$$

again uniformly on sets D_δ.

The convergence for $y=0$ is a consequence of

$$\psi(t) = \log t - \frac{1}{2t} + O(\frac{1}{t^2}),\ t\to\infty.$$

Proof of theorem 3.2. By Lemma 3.1 for $c>0$ and (6.3) for $c=0$ the difference between $T_n^{(k_n)} = L_n(E_n/n, k_n/n)$ and $L(E_n/n, k_n/n)$ tends to zero almost surely if $k_n\to\infty$. Therefore it suffices to study the limits of $L(E_n/n, k_n/n)$.

Parts a) and c) of the theorem follow from the continuity of L outside the diagonal. For part c) observe that the random variable $N_0 := \mathit{inf}\{n: E_j<k_j \text{ for all } j\geq n\}$ is almost surely finite.

In order to prove b) note that

$$L(E_n/n, k_n/n) = 0 \Leftrightarrow n^{\frac{1}{2}}(E_n/n-p) < n^{\frac{1}{2}}(k_n/n-p), \tag{6.4}$$

and by the monotonicity of L

$$L(E_n/n, k_n/n) \geq L(k_n/n, k_n/n) \Leftrightarrow n^{\frac{1}{2}}(E_n/n-p) \geq n^{\frac{1}{2}}(k_n/n-p).$$

So the distribution function of $L(E_n/n, k_n/n)$, F_n say, has a point mass in zero equal to

$$P(n^{\frac{1}{2}}(E_n/n-p)<n^{\frac{1}{2}}(k_n/n-p)), \tag{6.5}$$

and no mass in the interval $(0, L(k_n/n, k_n/n))$. Since $L(k_n/n, k_n/n)\to L(p,p)>0$ the fact that E_n is $Bin(n,p)$ distributed implies that if F_n converges in distribution the limit of $n^{\frac{1}{2}}(k_n/n-p)$ has to exist in $[-\infty, \infty]$.

Conversely suppose that this limit exists then for sufficiently large n we have $k_n/n>c'$ for each $0<c'<p$ and hence $L(E_n/n, k_n/n)<L(E_n/n, c')$. Since the right hand side of this inequality converges to $L(p,c')$ almost surely we have for all $t>L(p,c')$

$$\limsup_{n\to\infty} P(L(E_n/n, k_n/n) \geq t) \leq \lim_{n\to\infty} P(L(E_n/n, c')\geq t) = 0,$$

and by left continuity in y of L in the point (p,p) for all $t>L(p,p)$

$$\lim_{n\to\infty} P(L(E_n/n, k_n/n)\geq t) = 0.$$

Together with the convergence of (6.5) this proves b).

Proof of theorem 3.3. Recall that E_n is $Bin(n,p)$ distributed.
Part a) follows from the differentiability of L in the point (p,c). Note that in particular $\frac{\partial}{\partial x}L(x,y)=y^{-1}\log(x/(x-y))L(x,y)$ for $1>x>y>0$.

The more complex behaviour in the case $c=p$ is caused by the jump of L and by its infinite right hand partial derivative in x at the diagonal. The next expansion follows from the definition of L, see (3.2). Consider sequences of real numbers $(x_n),(y_n)$ and (z_n) such that $x_n\to p, y_n\to p, x_n\geq y_n$ for sufficiently large n and (z_n) is bounded. For such sequences we have for $n\to\infty$

$$(L(x_n+n^{-\frac{1}{2}}z_n, y_n)-L(x_n,y_n))\iota(x_n+n^{-\frac{1}{2}}z_n-y_n) =$$

$$\gamma_n(\log L(x_n+n^{-\frac{1}{2}}z_n, y_n)-\log L(x_n,y_n))\iota(x_n+n^{-\frac{1}{2}}z_n-y_n) =$$

$$\gamma_n y_n^{-1}(-n^{-\frac{1}{2}} z_n \log(x_n - y_n + n^{-\frac{1}{2}} z_n) + R_n + O(n^{-\frac{1}{2}} z_n))\iota(x_n + n^{-\frac{1}{2}} z_n - y_n),$$

where undefined values of the logarithm are set to zero, γ_n is chosen equal to $L(p,p)$ if $x_n + n^{-\frac{1}{2}} z_n < y_n$ and, by the mean value theorem, chosen between $L(x_n + n^{-\frac{1}{2}} z_n, y_n)$ and $L(x_n, y_n)$ such that

$$\gamma_n^{-1} = (\log L(x_n + n^{-\frac{1}{2}} z_n, y_n) - \log L(x_n, y_n)) / (L(x_n + n^{-\frac{1}{2}} z_n, y_n) - L(x_n, y_n))$$

otherwise. The remainder R_n equals

$$R_n = (x_n - y_n)(\log(x_n - y_n) - \log(x_n - y_n + n^{-\frac{1}{2}} z_n)).$$

Note that in particular $\gamma_n y_n^{-1} = p^{-1} L(p,p) + o(1)$, $n \to \infty$.
The assertions b) and c) of the theorem follow from two specific choices of sequences (x_n) and (y_n).
Taking x_n and y_n equal to k_n / n gives

$$\frac{n^{\frac{1}{2}}}{\log n}(L(k_n / n + n^{-\frac{1}{2}} z_n, k_n / n) - L(k_n / n, k_n / n))\iota(z_n) =$$

$$(p^{-1} L(p,p) + o(1))(\tfrac{1}{2} z_n + o(1))\iota(z_n) =$$

$$\tfrac{1}{2} p^{-1} L(p,p) z_n^{+} + o(1),\ n \to \infty$$

for all bounded sequences (z_n). Substituting $Z_n = n^{\frac{1}{2}}(E_n / n - k_n / n)$ for z_n and using $Z_n \xrightarrow{w} (p(1-p))^{\frac{1}{2}} N + a$ proves b).

Part c) follows similarly from the choice $x_n = p$, $y_n = k_n / n$ and $Z_n = n^{\frac{1}{2}}(E_n / n - p)$.
Part d) is immediate from (6.4).

6.2 Proofs of section 4.

Clearly the limit constant $S(c)$, see (4.2), depends on the distribution of X_n. However this constant is also defined for the variables Z_n and Y_n since both are nonnegative. To avoid misunderstanding denote their corresponding limit constants by $S_z(c)$ and $S_y(c)$, and that of X_n by $S_x(c)$. Note that $S_z(c) = L(p,c)$ and that $S_y(c)$ is the limit constant in Halász & Székely (1976).

Proof of theorem 4.2. The following lemma deals with the random sample size in the second term of the statistic (4.1).

Lemma 6.2.1. *If* $c < p$, *assuming* $EY_1 < \infty$ *for* $c = 0$ *or* $E\log(1+Y_1) < \infty$ *for* $0 < c < p$, *then we have*

$$T^{(k_n)}(Y_1, \ldots, Y_{E_n}) \to S_y(c/p),\ \textit{almost surely.}$$

Proof. It suffices to prove the lemma for the specific probability space $(\Omega, \mathcal{F}, P)$, with $\Omega = \Omega_z \times \Omega_y$, where Ω_z and Ω_y denote copies of the set of sequences of real numbers. $P = P_z \times P_y$, where P_z and P_y are the probabilities on Ω_z and Ω_y induced by the sequences $Z_1, Z_2, \ldots$ and $Y_1, Y_2, \ldots$ and $\mathcal{F}$ is the Borel σ-field on Ω.
Represent an element ω of Ω as $\omega = (\omega_z, \omega_y) = (z_1, z_2, \ldots; y_1, y_2, \ldots)$ and define the coordinate functions $\tilde{Z}_i$ and $\tilde{Y}_i$ by

$$\tilde{Z}_i(\omega) = z_i, \tilde{Y}_i(\omega) = y_i.$$

Next consider the almost surely defined function V_n,

$$V_n(\omega) = T^{(k_n)}(\tilde{Y}_1(\omega), \ldots, \tilde{Y}_{E_n(\omega_z)}(\omega)),$$

where $E_n(\omega_z)$ denotes the number of ones in the first n components of $\omega_z = (z_1, z_2, \ldots)$. With these definitions the random variables $\tilde{Z}_j, \tilde{Y}_j$ and V_n have the distributions of Z_j, Y_j and $T^{(k_n)}(Y_1, \ldots, Y_{E_n})$.

The proof of a) and b) is now just an application of Fubini's theorem. By the strong law of large numbers the set $\{\omega_z \in \Omega_z: k_n / E_n(\omega_z) \to c/p\}$ has P_z probability one. Therefore by Halász & Székely's theorem for positive random variables we have for P_z almost all ω_z

$$P_y(\{\omega_y \in \Omega_y: \lim_{n\to\infty} V_n(\omega_z,\omega_y) = S_y(c/p)\}) = 1.$$

Writing $P(\{\omega: \lim_{n\to\infty} V_n(\omega) = S_y(c/p)\})$ as a repeated integral with respect to dP_y and dP_z then completes the proof.

A first consequence is the following relation between the constants S_x, S_z and S_y, which follows from (4.1).

$$S_x(c) = S_z(c)S_y(c/p) = L(p,c)S_y(c/p), \text{ for } 0 \leq c < p. \tag{6.6}$$

It is immediate from the definitions that this relation also holds for $c = p$.

Next observe that the limit behaviour of the first term of the product (4.1) is covered by theorem 3.2., while the second term is treated in the previous lemma.
Since $0 \leq T^{(k_n)}(Z_1,...,Z_n) \leq 1$ the difference between the statistic (4.1) and $T^{(k_n)}(Z_1,...,Z_n)S_y(c/p)$ tends to zero almost surely. By part a) of theorem 3.2 we have then proved almost sure convergence of the statistics (4.1) to the limit constant $L(p,c)S_y(c/p)$ which equals $S_x(c) = S(c)$ by (6.6).

Proof of theorem 4.3. By lemma 4.1 we have $S^{(k_n)}(X_1,...,X_n)^{1/k_n} \stackrel{d}{=} S^{(k_n)}(Y_1,...,Y_{E_n})^{1/k_n}$. The following lemma is used to derive the weak limit of the latter statistic from Székely's (1982) weak limit theorem for strictly positive variables.

Lemma 6.2.2. *Let $X_{k,n}$, $k = 1,...,n$ denote a triangular array of random variables and let $E_1,E_2,...$ be a sequence of integer valued random variables satisfying*
- *E_n is independent of $X_{1,n},...,X_{n,n}$*
- *$\mu_n := EE_n \sim np, n \to \infty$ $(p > 0)$*
- *$\sigma_n := stdev E_n = o(n)$, $n \to \infty$.*

Let c be a constant, $(0 < c < p)$. Suppose that $\alpha(k,n)$ are positive affine transformations such that $k_n \sim cn, n \to \infty$ and $e_n \sim pn, n \to \infty$ imply

$$\alpha^{-1}(k_n,e_n)X_{k_n,e_n} \stackrel{w}{\to} X \tag{6.7}$$

for some random variable X, then for any sequence (k_n) with $k_n \sim cn, n \to \infty$

$$\alpha^{-1}(k_n,E_n)X_{k_n,E_n} \stackrel{w}{\to} X.$$

Moreover, if additionally there exist positive affine transformations $\gamma(k_n,n)$ and a random affine transformation β such that

$$\gamma^{-1}(k_n,n)\alpha(k_n,E_n) \stackrel{w}{\to} \beta, \tag{6.8}$$

then

$$\gamma^{-1}(k_n,n)X_{k_n,E_n} \stackrel{w}{\to} \beta X,$$

with β and X independent.
(By a positive affine transformation α we mean that there exist a_0 and $a_1 > 0$ such that $\alpha(x) = a_0 + a_1 x$).

Proof. Let $F_{k,n}$ denote the distribution function of $\alpha^{-1}(k,n)X_{k,n}$,

$$F_{k,n}(x) = P(\alpha^{-1}(k,n)X_{k,n} \leq x),$$

and F the distribution function of X.

By the independence of E_n we have

$$P(\alpha^{-1}(k,E_n)X_{k,E_n} \leq x) = E_{E_n}F_{k,E_n}(x) = E_{E_n^*}F_{k,\mu_n+\sigma_n E_n^*}(x).$$

Writing $G_n^x(t) = F_{k_n,\mu_n+\sigma_n t}(x)$, with x a fixed continuity point of F, by (6.7) we have for every bounded sequence (t_n) that $G_n^x(t_n)$ converges to $F(x)$. Therefore $G_n^x(t)$ converges to $F(x)$ uniformly on bounded t-intervals, and since E_n^* is tight and $G_n^x(t)$ bounded we conclude

$$P(\alpha^{-1}(k_n,E_n)X_{k_n,E_n} \leq x) = E_{E_n^*}G_n^x(E_n^*) \to F(x), \tag{6.9}$$

which proves the first part of the lemma.

In order to prove the second part rewrite the affine transformation $\gamma^{-1}(k_n,n)\alpha(k_n,E_n)$ as $x \to \beta_{0n}(E_n^*) + \beta_{1n}(E_n^*)x$ and β as $x \to \beta_0 + \beta_1 x$. Consider the joint distribution of $\beta_{0n}(E_n^*)$, $\beta_{1n}(E_n^*)$ and $Z_n := \alpha^{-1}(k_n,E_n)X_{k_n,E_n}$. Let (x_0,x_1,y) be a continuity point of the distribution of (β_0,β_1,X), then

$$P(\beta_{0n}(E_n^*) \leq x_0,\ \beta_{1n}(E_n^*) \leq x_1, Z_n \leq y) = E_{E_n^*}I_{A_n}(E_n^*)G_n^y(E_n^*),$$

where A_n denotes the set $\{t : \beta_{0n}(t) \leq x_0, \beta_{1n}(t) \leq x_1\}$. By the tightness of E_n^*, (6.8) and (6.9) this probability converges to

$$P(\beta_0 \leq x_0, \beta_1 \leq x_1)P(X \leq y).$$

Hence the continuous function $\gamma^{-1}(k_n,n)X_{k_n,E_n} = \beta_{0n}(E_n^*) + \beta_{1n}(E_n^*)Z_n$ of $(\beta_{0n}(E_n^*), \beta_{1n}(E_n^*), Z_n)$ converges weakly to $\beta_0 + \beta_1 X$, which proves the second part.

First note that a $Bin(n,p)$ distributed random variable satisfies the conditions imposed on E_n in the previous lemma. Taking $X_{k,n}$ equal to $S^{(k)}(Y_1,...,Y_n)^{1/k}$ we have by Szekely's weak limit theorem, see (1.2),

$$e_n^{\frac{1}{2}} C_y^{-1}(c/p)(X_{k_n,e_n} / \binom{e_n}{k_n}^{1/k_n} - S_y(k_n/e_n)) \xrightarrow{w} N_1,$$

if $k_n \sim cn, n \to \infty$ and $e_n \sim pn, n \to \infty$ $(0 < c < p)$. Here $C_y(\cdot)$ denotes the asymptotic standard deviation in (1.2) as a function of c, and N_1 is a standard normal random variable. Thus condition (6.7) is satisfied with X equal N_1 and

$$\alpha(k,n)(x) = S_y(k/n)\binom{n}{k}^{1/k} + C_y(c/p)n^{-\frac{1}{2}}\binom{n}{k}^{1/k} x.$$

Next define

$$\gamma(k,n)(x) = S_x(k/n)\binom{n}{k}^{1/k} + n^{-\frac{1}{2}}\binom{n}{k}^{1/k} x.$$

Condition (6.8) is dealt with in the following lemma.

Lemma 6.2.3.

$$\gamma^{-1}(k_n,n)\alpha(k_n,E_n)(x) \xrightarrow{w} p^{-\frac{1}{2}}L(p,c)C_y(c/p)x + D(p,c)(p(1-p))^{\frac{1}{2}}N_2, \tag{6.10}$$

where N_2 is a standard normal random variable and $D(s,t)$ denotes the partial derivative with respect to s of the function $L(s,t)S_y(t/s)$.

Proof. Rewrite $\gamma^{-1}(k,n)\alpha(k,E_n)$ as follows (use (6.6)).

$$\gamma^{-1}(k,n)\alpha(k,E_n) =$$

$$(n/E_n)^{\frac{1}{2}}L_n(E_n/n, k/n)C_y(c/p)x +$$

$$n^{\frac{1}{2}} S_y(k/E_n)\{L_n(E_n/n,k/n)-L(E_n/n,k/n)\} +$$
$$n^{\frac{1}{2}}\{L(E_n/n,k/n)S_y(k/E_n)-L(p,k/n)S_y(k/np)\}.$$

Replacing k by k_n the first term converges almost surely to $p^{-\frac{1}{2}}L(p,c)C_y(c/p)x$. Recalling $0<c<p$ the second term vanishes almost surely by lemma 3.1. Writing $W(s,t)$ for $L(s,t)S_y(t/s)$, $0<t<s<1$, the third term equals

$$n^{\frac{1}{2}}(W(E_n/n,k_n/n)-W(p,k_n/n)).$$

By dominated convergence arguments the function W can be shown to have a continuous partial derivative in s, $D(s,t)$ say. The expression of this partial derivative is not very instructive and is therefore omitted. By the mean value theorem the third term converges weakly to

$$D(p,c)(p(1-p))^{\frac{1}{2}}N_2.$$

Together these arguments prove (6.10).

Since all conditions of lemma 6.2.2 are fulfilled we have

$$\gamma^{-1}(k_n,n)S^{(k_n)}(Y_1,...,Y_{E_n})^{1/k_n} \overset{w}{\to} p^{-\frac{1}{2}}L(p,c)C_y(c/p)N_1+D(p,c)(p(1-p))^{\frac{1}{2}}N_2$$

with N_1 and N_2 independent, which proves theorem 4.3.

6.3. Proofs of section 5.

Proof of theorem 5.1. Writing $S_n^{(k)}$ for $S^{(k)}(Y_1,...,Y_n)$ Móri & Székely's (1982) theorem 2 states that if $k_n\to\infty$ and $n-k_n\to\infty$

$$\left[\frac{k_n(n-k_n)}{n}\right]^{\frac{1}{4}} S_n^{(k_n)} \Big/ \begin{bmatrix} n \\ k_n \end{bmatrix}^{\frac{1}{2}} -(2/\pi)^{1/4}\exp(S_n^{(1)2}/4n)\cos(-\tfrac{1}{2}k_n\pi+S_n^{(1)}\arcsin((k_n/n)^{\frac{1}{2}}) \overset{P}{\to} 0.$$

For $0<c<p$ we may replace n by E_n, which gives

$$\left[\frac{k_n(E_n-k_n)}{E_n}\right]^{1/4} S_{E_n}^{(k_n)} \Big/ \begin{bmatrix} E_n \\ k_n \end{bmatrix}^{\frac{1}{2}} - \tag{6.11}$$
$$(2/\pi)^{1/4}\exp(S_{E_n}^{(1)2}/4E_n)\cos(-\tfrac{1}{2}k_n\pi+S_{E_n}^{(1)}\arcsin((k_n/E_n)^{\frac{1}{2}})) =$$
$$A_n - B_n \overset{P}{\to} 0.$$

Our aim is to show

$$n^{\frac{1}{2}}\left[(|S_{E_n}^{(k_n)}| / \begin{bmatrix} E_n \\ k_n \end{bmatrix}^{\frac{1}{2}})^{1/k_n}-1\right] \overset{P}{\to} 0 \tag{6.12}$$

since this implies that the difference between

$$n^{\frac{1}{2}}\left[(|S_{E_n}^{(k_n)}| / \begin{bmatrix} n \\ k_n \end{bmatrix}^{\frac{1}{2}})^{1/k_n} - L(p,k_n/n)^{\frac{1}{2}}\right] \tag{6.13}$$

and

$$n^{\frac{1}{2}}\left[\left[\begin{bmatrix} E_n \\ k_n \end{bmatrix} / \begin{bmatrix} n \\ k_n \end{bmatrix}\right]^{1/2k_n} - L(p,k_n/n)^{\frac{1}{2}}\right] = \tag{6.14}$$

$$n^{\frac{1}{2}}\left[T^{(k_n)}(Z_1,...,Z_n)^{\frac{1}{2}}-L(p,k_n/n)^{\frac{1}{2}}\right]$$

also vanishes in probability. The weak limit of (6.14) is easily derived from theorem 3.3.a). Since the statistics (5.1) and (6.13) have equal distributions the theorem is then proved by checking the limit of (6.14) against the limit claimed in the theorem.

In order to prove (6.12) we need the following three lemma's.

Lemma 6.3.1. *If (V_n), (W_n) and (E_n) are sequences of random variables such that*
- $(V_n,W_n)\stackrel{w}{\rightarrow}(V,W)$
- E_n *is independent of* (V_n,W_n)
- $E_n\rightarrow\infty$, *almost surely*
- $(E_n-a_n)/b_n\stackrel{w}{\rightarrow}E$,

then

$$(V_{E_n},W_{E_n},(E_n-a_n)/b_n)\stackrel{w}{\rightarrow}(V,W,E),$$

with E independent of V and W.

The proof is similar to the proof of lemma 6.2.2. and is therefore omitted.

Lemma 6.3.2. *Let (C_n) and (D_n) be sequences of random variables such that the sequence $C_n \bmod 2\pi$, $n = 1,2,...$ has only finitely many possible values and that D_n converges in distribution to a continuously distributed limit variable D. Then $\log(|\cos(C_n+D_n)|)$ is bounded in probability.*

Proof. Denote the finitely many possible values of $C_n \bmod 2\pi$, $n = 1,2,...$ by $c_1,...,c_m$.
Since for each $i = 1,...,m$ the random variable $|\cos(c_i+D_n)|$ converges to a continuously distributed limit $|\cos(c_i+D)|$ we have

$$\limsup_{n\rightarrow\infty} P(\log(|\cos(C_n+D_n)|)<-M) \leqslant$$
$$\sum_{i=1}^{m}\lim_{n\rightarrow\infty} P(\log(|\cos(c_i+D_n)|)<-M) =$$
$$\sum_{i=1}^{m}\lim_{n\rightarrow\infty} P(|\cos(c_i+D_n)|<e^{-M})\rightarrow 0 \text{ if } M\rightarrow\infty.$$

Hence $\log(|\cos(C_n+D_n)|)$ is bounded in probability.

Lemma 6.3.3. *If (X_n) is a sequence of nonnegative random variables then $n^{\frac{1}{2}}\log X_n\stackrel{P}{\rightarrow}0$ implies $n^{\frac{1}{2}}(X_n-1)\stackrel{P}{\rightarrow}0$.*

Proof. The quotient $|x-1|/|\log x|$ is bounded in a neighbourhood of $x=1$.

The first step in proving (6.12) is to show that $\log(|B_n|)$ is bounded in probability. Write

$$\log(|B_n|) = \tfrac{1}{4}\log(2/\pi)+S_{E_n}^{(1)2}/4E_n+R_n, \qquad (6.15)$$

where R_n denotes $\log(|\cos(-\frac{1}{2}k_n\pi+S_{E_n}^{(1)}\arcsin((k_n/E_n)^{\frac{1}{2}}))|)$. By lemma 6.3.1 and the central limit theorem we have

$$(E_n^{-\frac{1}{2}}S_{E_n}^{(1)},n^{-\frac{1}{2}}(p(1-p))^{-\frac{1}{2}}(E_n-np))\stackrel{w}{\rightarrow}(N_1,N_2),$$

where N_1 and N_2 are independent standard normal random variables. This implies that the second term in (6.15) is bounded in probability. Since the first term is a constant we next focus our attention on R_n.

We distinguish two cases.

Firstly let $\alpha = (2\pi)^{-1}\arcsin((c/p)^{\frac{1}{2}})$ be rational. Write R_n as $\log(|\cos(C_n + D_n)|)$ with

$$C_n = -\tfrac{1}{2}k_n\pi + 2\pi\alpha S_{E_n}^{(1)}$$

and

$$D_n = E_n^{-\frac{1}{2}} S_{E_n}^{(1)} (E_n/n)^{\frac{1}{2}} n^{\frac{1}{2}} (\arcsin((k_n/E_n)^{\frac{1}{2}}) - \arcsin((c/p)^{\frac{1}{2}})).$$

Note that by the assumption that $n^{\frac{1}{2}}(k_n/n - c)$ converges, to a constant b say, we have

$$n^{\frac{1}{2}}(k_n/E_n - c/p) \xrightarrow{w} p^{-2}(b + c(p(1-p))^{\frac{1}{2}} N_2)$$

and hence

$$D_n \xrightarrow{w} D = N_1 p^{\frac{1}{2}} \tfrac{1}{2}(\frac{c}{p}(1-\frac{c}{p}))^{-\frac{1}{2}} p^{-2}(b + c(p(1-p))^{\frac{1}{2}} N_2).$$

Since $S_{E_n}^{(1)}$ is integer valued the conditions of lemma 6.3.2. are satisfied and R_n is bounded in probability.

Secondly suppose that α is irrational. Let N_1, N_2 and U be independent random variables with N_1 and N_2 standard normal and U uniformly distributed on [0,1]. It is shown by Móri & Székely that

$$(n^{-\frac{1}{2}} S_n^{(1)}, \{\alpha S_n^{(1)}\}) \xrightarrow{w} (N_1, U),$$

where $\{\cdot\}$ denotes the fractional part. Lemma 6.3.1 thus implies

$$(E_n^{-\frac{1}{2}} S_{E_n}^{(1)}, \{\alpha S_{E_n}^{(1)}\}, n^{-\frac{1}{2}}(p(1-p))^{-\frac{1}{2}}(E_n - np)) \xrightarrow{w} (N_1, U, N_2). \tag{6.16}$$

Next write R_n as $\log(|\cos(C_n' + D_n')|)$ with

$$C_n' = -\tfrac{1}{2}k_n\pi$$

and

$$D_n' = 2\pi\{\alpha S_{E_n}^{(1)}\} + D_n.$$

By (6.16) we have $D_n' \xrightarrow{w} 2\pi U + D$. Hence in this case the conditions of lemma 6.3.2 are satisfied as well.

Thus in both cases R_n is bounded in probability, implying the same for $\log(|B_n|)$.

The proof of (6.12) is completed by observing that by (6.11) we have $|A_n| - |B_n| \xrightarrow{P} 0$, and consequently that $\log(|A_n|)$ is also bounded in probability. Since therefore

$$\frac{n^{\frac{1}{2}}}{k_n}\log(|S_{E_n}^{(k_n)}| / \binom{E_n}{k_n}^{\frac{1}{2}}) = \frac{n^{\frac{1}{2}}}{k_n}\log(|A_n|) - \frac{n^{\frac{1}{2}}}{4k_n}(\log k_n + \log(E_n - k_n) - \log E_n) \xrightarrow{P} 0,$$

the condition of lemma 6.3.3 is satisfied for $X_n = (|S_{E_n}^{(k_n)}| / \binom{E_n}{k_n}^{\frac{1}{2}})^{1/k_n}$ and the conclusion gives (6.12).

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Appendix

A.1 The complete version of theorem 4.2.

Theorem *Let $S(c)$ denote the limit constant defined in (4.2). Under the conditions of theorem 4.2 we distinguish the following cases.*

a) *If $c<p$, assuming $EY_1<\infty$ for $c=0$ and $E\log(1+Y_1)<\infty$ for $0<c<p$, then we have*

$$T^{(k_n)}(X_1,...,X_n) \to S(c), \text{ almost surely.}$$

b) *If $c=p$, assuming $E\log Y_1<\infty$, then $T^{(k_n)}(X_1,...,X_n)$ converges in distribution if and only if*

$$n^{\frac{1}{2}}(k_n/n-p) \to a, \text{ for some } a \in [-\infty,\infty].$$

Moreover, in case of convergence the limit variable T is two valued if $0<p<1$,

$$P(T=0) = 1-P(T=S(p)) = \Phi(a/(p(1-p))^{\frac{1}{2}}),$$

and if $p=1$

$$P(T=S(1)) = 1.$$

c) *If $c>p$ then there exists an almost surely finite random variable N_0 such that $T^{(k_n)}(X_1,...,X_n)=0$ for all $n \geq N_0$.*

Proof. The proof is similar to that of part a) which was given in section 6. A $c=p$ part of lemma 6.2.1 can be stated as follows.

If $c=p$, assuming $E|\log Y_1|<\infty$, then we have

$$(T^{(k_n)}(Y_1,...,Y_{E_n}) - S_y(1))\iota(E_n-k_n) \to 0, \text{ almost surely.} \qquad \text{(A.1)}$$

This is proved similarly to the $c<p$ part by observing

$$P_y(\{\omega_y \in \Omega_y : \lim_{n\to\infty}(V_n(\omega_z,\omega_y)-S_y(1))\iota(E_n-k_n)=0\}) = 1$$

for P_z almost all ω_z.

Since $0 \leq T^{(k_n)}(Z_1,...,Z_n) \leq \iota(E_n-k_n)$ the difference between the statistic (4.1) and $T^{(k_n)}(Z_1,...,Z_n)S_y(c/p)$ tends to zero almost surely. Theorem 3.2 then completes the proof.

A.2 The complete version of theorem 4.3.

Theorem. *Under the conditions of theorem 4.3 we distinguish the following cases.*

a) *If $0<c<p$, assuming $E\log^2(1+Y_1)<\infty$, then we have*

$$n^{\frac{1}{2}}(T^{(k_n)}(X_1,...,X_n)-S(k_n/n)) \xrightarrow{w} CN,$$

where C is a positive constant.

b) *If $c=p$, assuming* $\mathrm{var}(\log Y_1)<\infty$ *and $n^{\frac{1}{2}}(k_n/n-p)\to a\in(-\infty,\infty)$, then*

$$\frac{2n^{\frac{1}{2}}}{\log n}(T^{(k_n)}(X_1,...,X_n)-L(k_n/n,k_n/n)S_y(1))\iota(E_n-k_n)$$

converges in distribution to

$$p^{-1}S_x(p)((p(1-p))^{\frac{1}{2}}N-a)^+.$$

c) *If $c=p$, assuming* $\mathrm{var}(\log Y_1)<\infty$ *and $n^{\frac{1}{2}}(k_n/n-p)\to-\infty$, then*

$$\frac{-n^{\frac{1}{2}}}{\log(p-k_n/n)}(T^{(k_n)}(X_1,...,X_n)-L(p,k_n/n))S_y(1))$$

converges in distribution to

$$p^{-1}S_x(p)(p(1-p))^{\frac{1}{2}}N.$$

d) *If $c=p$ and $n^{\frac{1}{2}}(k_n/n-p)\to\infty$ then*

$$T^{(k_n)}(X_1,...,X_n)\to 0, \textit{ in probability.}$$

Proof. Part a) was already proved in section 6.

In order to prove b) write

$$\frac{2n^{\frac{1}{2}}}{\log n}\iota(E_n-k_n)(T^{(k_n)}(Z_n,...,Z_n)T^{(k_n)}(Y_1,...,Y_{E_n})-L(k_n/n,k_n/n)S_y(1)) =$$

$$\frac{2n^{\frac{1}{2}}}{\log n}\iota(E_n-k_n)(T^{(k_n)}(Z_1,...,Z_n)-L(k_n/n,k_n/n))T^{(k_n)}(Y_1,...,Y_{E_n})+$$

$$\frac{2n^{\frac{1}{2}}}{\log n}\iota(E_n-k_n)(T^{(k_n)}(Y_1,...,Y_{E_n})-S_y(1))L(k_n/n,k_n/n).$$

By theorem (3.3) and (A.1) the first term has the desired weak limit and therefore it suffices to show that the second term converges to zero in probability. This is achieved by substituting E_n for n in

$$\frac{n^{\frac{1}{2}}}{\log n}(T^{(k_n)}(Y_1,...,Y_n)-S_y(1))\iota(n-k_n)\xrightarrow{w} 0$$

which is a consequence Székely's (1982) theorem on weak convergence.
Part c) can be treated similarly and part d) is analogous to part d) of theorem 3.3.

A.3 Móri & Székely's (1982) part iv) of theorem 3 corrected.

The proof of this part of theorem 3 contains an error, which is seen by taking $k_n=[n/2]$. The correct version of part iv) should read:

If $(2\pi)^{-1}\arcsin\sqrt{c}$ is a rational number of the form p/q where p and q are relative prime numbers, q is divisible by 8, $n^{\frac{1}{2}}|k_n/n-c|\to b$ and $0<c<1$, then the subsequences of the even and odd n converge to different weak limits:

$$(2n)^{1/4} S^{(k_{2n})}(X_1,\ldots,X_{2n}) \Big/ \begin{bmatrix} 2n \\ k_{2n} \end{bmatrix}^{\frac{1}{2}}$$

converges in distribution to

$$\left[\frac{2}{\pi c(1-c)}\right]^{1/4} \exp(N^2/4) \cos\left[2\pi V_q + \tfrac{1}{2} b (c(1-c))^{-\frac{1}{2}} N\right]$$

and

$$(2n+1)^{1/4} S^{(k_{2n+1})}(X_1,\ldots,X_{2n+1}) \Big/ \begin{bmatrix} 2n+1 \\ k_{2n+1} \end{bmatrix}^{\frac{1}{2}}$$

converges in distribution to

$$\left[\frac{2}{\pi c(1-c)}\right]^{1/4} \exp(N^2/4) \cos\left[2\pi W_q + \tfrac{1}{2} b (c(1-c))^{-\frac{1}{2}} N\right],$$

where V_q has the uniform distribution on the set $\{0, 2/q, 4/q, \ldots, (q-2)/q\}$, W_q has the uniform distribution on the set $\{1/q, 3/q, \ldots, (q-1)/q\}$, N is standard normally distributed and V_q, W_q and N are independent.