Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science

N.M. Temme

A convolution integral equation solved by Laplace transformation

A CONVOLUTION INTEGRAL EQUATION SOLVED BY LAPLACE TRANSFORMATION

N.M. TEMME

Centre for Mathematics and Computer Science, Amsterdam

We consider the integral equation

$$p(t) = \int_0^t K(t-\tau)p(\tau)d\tau + r(t)$$

where both $K(t)$ and $r(t)$ behave as $\exp(\alpha t^3)$ as $t \to \infty$ ($\alpha > 0$). So straight-forward application of the Laplace transform technique is not possible. By introducing a complex parameter the equation is solved in the complex domain. Analytic continuation with respect to this parameter yields the desired solution. For a particular example (which arose in a statistical problem on estimating monotone densities) we describe the construction of the explicit solution of the equation.

1. Introduction

We are interested in the solution p_0 of the integral equation

$$p_0(t) = (2\pi)^{-\frac{1}{2}}\int_0^t p_0(y)e^{-\frac{1}{2}ty(t-y)}(t-y)^{\frac{1}{2}}dy + r_0(t) \tag{1}$$

where $t > 0$ and

$$r_0(t) = -(\pi/2)^{\frac{1}{2}} + t^{3/2} - (2\pi)^{-\frac{1}{2}}\int_0^t (t-y)^{\frac{1}{2}}y^{-3/2}[1-e^{-\frac{1}{2}ty(t-y)}]dy. \tag{2}$$

This equation arose in a statistical problem on the estimation of monotone densities considered in Groeneboom (1984).

First we give an expansion of $p_0(t)$ in powers of t, yielding a good approximation for small and intermediate values of t. Our main concern is an expansion at infinity. We obtain a convergent expansion which can be used for all positive values of t.

2. Expansion at $t = 0$ and at infinity

The function $r_0(t) - t^{3/2}$ is an entire function; this follows immediately from (2). Substituting the Taylor expansion of $r_0(t) - t^{3/2}$ into (1) we obtain

$$p_0(t) = -(\pi/2)^{\frac{1}{2}}\sum_{n=0}^{\infty}\alpha_n t^{3n} + t^{-3/2}\sum_{n=1}^{\infty}\beta_n t^{3n}, \tag{3}$$

with $\alpha_0 = 1$, $\alpha_1 = 7/48$, $\beta_1 = 2/3$, $\beta_2 = 4/189$. It follows that both series in (3) represent entire functions of t. Representation (3) was useful in obtaining numerical information on $p_0(t)$ for small and intermediate values of t. Higher coefficients α_n, β_n can be computed by recursion.

By substituting an asymptotic expansion of $r_0(t)$ into (1) we obtain for $p_0(t)$ the expansion

$$p_0(t) \sim -t^{-3/2}, \quad t \to \infty.$$

All remaining coefficients of t-powers seem to vanish. So we obtained the conjecture that $p_0(t) = -t^{-3/2} + q(t)$ with $q(t) = O(t^{-n})$ as $t \to \infty$, for any n. For the statistical problem this information on p_0 and q was not sufficient and it appeared that it was rather difficult to obtain the asymptotic behaviour of q directly from (1).

3. A convolution equation

When we write $-\frac{1}{2}ty(t-y) = \frac{1}{6}(t-y)^3 + \frac{1}{6}y^3 - \frac{1}{6}t^3$ we obtain for (1)

$$p_0(t)e^{\frac{1}{6}t^3} = (2\pi)^{-\frac{1}{2}}\int_0^t p_0(y)e^{\frac{1}{6}y^3}(t-y)^{\frac{1}{2}}e^{\frac{1}{6}(t-y)^3}dy + e^{\frac{1}{6}t^3}r_0(t). \tag{4}$$

Introducing the functions

$$p_1(t) = p_0(t)e^{\frac{1}{6}t^3}, \quad r_1(t) = r_0(t)e^{\frac{1}{6}t^3}, \quad K_1(t) = (2\pi)^{-\frac{1}{2}}t^{\frac{1}{2}}e^{\frac{1}{6}t^3} \tag{5}$$

we have the convolution type equation

$$p_1(t) = \int_0^t p_1(y)K_1(t-y)dy + r_1(t). \tag{6}$$

A usual approach for such linear equations is the Laplace transform technique. It is readily seen that the Laplace transform of $K_1(t)$ is not defined. The same is true for $p_1(t)$, $r_1(t)$.

To avoid this complication we consider the modified equation

$$p(t) = \int_0^t p(y)K(t-y)dy + r(t) \tag{7}$$

where

$$p(t) = p_1(\lambda t), \quad r(t) = r_1(\lambda t), \quad K(t) = \lambda K_1(\lambda t), \tag{8}$$

with $\lambda \in S = \{\lambda \in \mathbb{C} \mid \lambda \neq 0, -\pi < arg\lambda < \pi\}$. Then (7) has to be solved for $\lambda = 1$, $t > 0$.

Remark Equation (7) can be solved by constructing a series expansion as in (3) or by the Neumann-series of iterated kernels. For each $t > 0$, the terms in these expansions are analytic functions of λ, $\lambda \in S$. The series converge uniformly with respect to λ in compact sets of S. Hence, $p(t)$ is analtyic in any finite point $\lambda \in S$.

4. Laplace transform techniques

We continue the analysis for $\lambda \in \Lambda$, where

$$\Lambda = \{\lambda \in \mathbb{C} \mid \lambda \neq 0, \pi/6 < arg\lambda < \pi/2\}.$$

Then the Laplace transform of

$$K(t) = (2\pi)^{-\frac{1}{2}} \lambda (\lambda t)^{\frac{1}{2}} e^{\frac{1}{6}\lambda^3 t^3} \tag{9}$$

exists; i.e., we can define

$$\hat{K}(s) = \int_0^\infty e^{-st} K(t) dt, \quad \lambda \in \Lambda \tag{10}$$

and $\hat{K}(s)$ is an entire function of s. Some asymptotical analysis is needed to show that $r(t) = \mathcal{O}[t^\alpha \exp(1/24\lambda^3 t^3)]$ as $t \to \infty$, $\lambda \in \Lambda$, where α is some real number. Hence $\hat{r}(s)$ exists as well ($\lambda \in \Lambda$) and is an entire function of s.

Using the convolution rule for Laplace transformations we obtain for the solution of (7)

$$p(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \hat{p}(s) ds, \quad \lambda \in \Lambda \tag{11}$$

where

$$\hat{p}(s) = \frac{\hat{r}(s)}{1 - \hat{K}(s)} \tag{12}$$

and where c should be large enough to locate all poles of the meromorphic function $\hat{p}(s)$ to the left of the path of integration.

To compute $\hat{r}(s)$ it is convenient to introduce

$$\phi_\alpha(t) = (\lambda t)^\alpha e^{\frac{1}{6}\lambda^3 t^3}, \quad \hat{\phi}_\alpha(s) = \int_0^\infty e^{-st} \phi_\alpha(t) dt \tag{13}$$

where in the integral Re $\alpha > -1$, $\lambda \in \Lambda$. The function $r(t)$ can be written in the form

$$r(t) = \phi_{3/2}(t) + (2/\pi)^{\frac{1}{2}} \lambda^2 t [\phi_{\frac{1}{2}} * \phi_{\frac{1}{2}}](t) - \frac{1}{2}(2\pi)^{-\frac{1}{2}} \lambda (2 + \lambda^3 t^3)[\phi_{-\frac{1}{2}} * \phi_{-\frac{1}{2}}](t), \tag{14}$$

where $*$ denotes the convolution operator. Using

$$\begin{cases} \dfrac{d}{ds} \hat{\phi}_\alpha(s) = -\lambda^{-1} \hat{\phi}_{\alpha+1}(s), \\ (\alpha + 1)\hat{\phi}_\alpha(s) = s\lambda^{-1}\hat{\phi}_{\alpha+1}(s) - \frac{1}{2}\hat{\phi}_{\alpha+3}(s) = -s\hat{\phi}'_\alpha(s) + \frac{1}{2}\lambda^3 \hat{\phi}'''_\alpha(s) \end{cases} \tag{15}$$

where the primes denote derivatives with respects to s. After elementary calculations we obtain

$$\hat{r}(s) = \hat{\phi}_{3/2}(s) - (2\pi)^{-\frac{1}{2}} \hat{\phi}_{\frac{1}{2}}(s)[\lambda\hat{\phi}_{3/2}(s) - 2s\hat{\phi}_{-\frac{1}{2}}(s)].$$

Taking into account the relation $\hat{K}(s) = (2\pi)^{-\frac{1}{2}}\lambda\hat{\phi}_{\frac{1}{2}}(s)$, see (9) and (13), we can write (12) as

$$\hat{p}(s) = -\hat{\phi}_{-3/2}(s) + 2\frac{\hat{\phi}_{3/2}(s) - s\lambda^{-1}\hat{\phi}_{-\frac{1}{2}}(s)}{1-\hat{K}(s)}.$$

(16)

Remark $\hat{\phi}_{-3/2}(s)$ is defined by means of the recursion in (15) and not by the integral in (13). However it is easily verified that for any $c, t > 0$, $\lambda \in \Lambda$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\hat{\phi}_{-3/2}(s)ds = \phi_{-3/2}(t).$$

(17)

5. Airy functions

A remarkable simplification of (16) is possible by using Airy functions. For properties of these functions we refer to Olver (1974). Starting with the well-known differential equation $\ddot{y} - zy = 0$, we can easily show that $w^{(3)} - 4z\dot{w} - 2w = 0$ is solved by a product of Airy functions. For $\alpha = -\frac{1}{2}$ the third order differential equation in (15) can be reduced to the above w-equation. By comparing Taylor series of $\hat{\phi}_{-\frac{1}{2}}(s)$ and the Airy functions we obtain

$$\hat{\phi}_{-\frac{1}{2}}(s) = 2^{5/6}\pi^{3/2}\lambda^{-1}Ai(z)[Bi(z)+iAi(z)], \quad z = 2^{-1/3}\lambda^{-1}s.$$

By using the Wronskian $\pi^{-1} = Ai(z)\dot{Bi}(z) - \dot{Ai}(z)Bi(z)$ and by expressing $\hat{\phi}_{1/2}(s)$ in terms of Airy functions, it follows that

$$1 - \hat{K}(s) = 2\pi iAi(z)\{\dot{Ai}(z) - i\dot{Bi}(z)\}.$$

(18)

Furthermore it follows from the recursions for $\hat{\phi}_{\alpha}(s)$

$$\hat{\phi}_{3/2}(s) - s\lambda^{-1}\hat{\phi}_{-\frac{1}{2}}(s) = i2^{7/6}\pi^{3/2}\lambda^{-1}\dot{Ai}(z)\{\dot{Ai}(z) - i\dot{Bi}(z)\}.$$

After these preparations the function $\hat{p}(s)$ of (16) can be written as

$$\hat{p}(s) = -\hat{\phi}_{-3/2}(s) + 2^{7/6}\pi^{\frac{1}{2}}\lambda^{-1}\dot{Ai}(z)/Ai(z), \quad z = 2^{1/3}\lambda^{-1}s.$$

(19)

With (11) and (17) we obtain

$$p(t) = -\phi_{-3/2}(t) + \frac{2^{7/6}\pi^{1/2}\lambda^{-1}}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\frac{\dot{Ai}(z)}{Ai(z)}ds,$$

(20)

where $c > 0$, $\lambda \in \Lambda$, $t > 0$. The Airy function $Ai(z)$ has an infinite number of negative zeros $a_1, a_2, \ldots$. These zeros are simple and there are no other zeros (Olver (1974, p. 414)). The poles of the integrand in (19) occur at $s_n = 2^{1/3}\lambda a_n (n = 1,2,\ldots)$, which for $\lambda \in \Lambda$ are located in the half plane Re$s < 0$. When we shift the contour of integration of (20) to the left we obtain a series of residues. It easily follows from asymptotic properties of the Airy functions and of the zeros that the infinite series of residues converges and that the rest integral vanishes. So we obtain the representation

$$p(t) = -\phi_{-3/2}(t) + 2(2\pi)^{\frac{1}{2}}\sum_{n=1}^{\infty} e^{s_n t}, \quad t > 0.$$

(21)

We now drop the condition $\lambda \in \Lambda$. The above series represents an analytic function of λ for the domain Re$\lambda > 0$, while the solution $p(t)$ is obtained for $\lambda \in \Lambda$. Also, $\phi_{-3/2}(t)$ is an analytic function of λ in Re$\lambda > 0$. From the Remark in section 3 it follows that p itself is analytic in Re$\lambda > 0$. By using the principle of analytic continuation it now follows that in (21) we can take $\lambda = 1$. In this case we have $p(t) = p_1(t) = p_0(t)\exp(\frac{1}{6}t^3)$. Thus we have for the solution of (1)

$$p_0(t) = -t^{-3/2} + 2(2\pi)^{\frac{1}{2}} e^{-\frac{1}{6}t^3} \sum_{n=1}^{\infty} e^{2^{1/3}a_n t}, \quad t > 0. \tag{22}$$

6. Concluding remarks.

1. We have solved (1) for the special case (2). With this function just one series of poles and residues occurs in (20), (21). A general function r_0 will not divide out the term $Ai(z) - iBi(z)$ in (18). This function vanishes at the zeros of $Ai(z\exp(2\pi i /3))$ (Olver (1974, p. 414)). In the s- plane the zeros of this function occur at $s_n' = 2^{1/3}\lambda a_n'\exp(-2\pi i /3)$, where a_n' are the negative zeros of $Ai(z)$. So, $arg s_n' = \pi/3 + arg\lambda$. For $\lambda \in \Lambda$ we have $\mathrm{Re}s_n' < 0$. In that case two series of residues appear in the solution. However, when λ leaves the sector Λ at $arg\lambda = \pi/6$, the zeros s_n' enter the half plane $\mathrm{Re}s > 0$. So, when the poles s_n' are not cancelled by the zeros of $\hat{r}(s)$, a representation in the form of two residue series is only possible when $\lambda \in \Lambda$, but not when $arg\lambda < \pi/6$. Consequently, the analytic continuation of the results from $\lambda \in \Lambda$ to $\lambda = 1$ will be much more complicated in the general case than in the case of r_0 as given in (2).

2. In (7) and (8) we have introduced the complex parameter λ. A similar approach is possible by making t complex and solving (6) for $t \in \Lambda$. The Laplace transform technique has to be modified then. To avoid this we have chosen for the λ-approach.

Acknowledgements

References

P. Groeneboom, *Estimating a monotone density*, Report MS-R8403, CWI Amsterdam, 1984. To appear in Proc. of the Neyman-Kiefer Conference, Berkeley, June-July 1983, Editor: L. LeCam.

F.W.J. Olver, *Asymptotics and Special Functions*, (Academic Press, 1974).