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ASYMPTOTIC BEHAVIOUR OF MATRIX COEFFICIENTS RELATED TO REDUCTIVE SYMMETRIC SPACES

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We study the asymptotic behaviour of K-finite H-invariant matrix coefficients related to a reductive symmetric space G/H. In all directions to infinity this behaviour is described by absolutely converging series expansions similar to those in the group case. A generalization of Harish-Chandra's Schwartz space is introduced.

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0. Introduction.

Let G be a real reductive Lie group of the Harish-Chandra class, τ an involution of G, and H an open subgroup of $G^{\tau} = \{x \in G; \ \tau(x) = x\}$. Then G has a τ-stable maximal compact subgroup K (see Section 1).

In the harmonic analysis on the reductive symmetric space G/H, one often encounters K-finite functions annihilated by a cofinite ideal of the algebra $\mathbb{D}(G/H)$ of invariant differential operators on G/H. Such functions arise in a natural way as matrix coefficients of K-finite vectors and H-fixed distribution vectors of admissible representations (see e.g. [14], [15], [2]), or as K-finite functions in L^2-realizations of discrete series representations (cf. [6], [13], [2]).

It is to be expected that the asymptotic behaviour of such functions plays a fundamental role in the harmonic analysis on G/H. This is for instance true in the group case, where G is the direct product of a reductive group 'G of Harish-Chandra's class with itself, H the diagonal in 'G × 'G, and K = 'K × 'K with 'K a maximal compact subgroup of 'G. Then G/H ≃ 'G and $\mathbb{D}(G/H) \simeq {}'\mathfrak{Z}$, the centre of the universal enveloping algebra of 'G. In this case, a detailed knowledge of the asymptotic behaviour of $'\mathfrak{Z}$-finite and 'K × 'K-finite functions on 'G is one of the essentials in Harish-Chandra's derivation of the Plancherel formula for 'G. Harish-Chandra started the general study of such functions and the differential equations satisfied by them in two unpublished papers [7] and [8]. Later on this material was made more accessible by Casselman and Miličić [3] who used ideas of Deligne [4] on systems of differential equations with regular singularities. A different approach was followed by Wallach [19] .

In this paper we study the asymptotic behaviour of $\mathbb{D}(G/H)$-finite and K-finite functions on G/H (in fact we also deal with the more general case of $\mathfrak{Z}$ -finite and K-finite H-spherical functions on G). As it turns out, the methods of [8] and [3] apply very well to our situation.
In all directions to infinity the asymptotic behaviour can be described by converging series expansions similar to those in the group case.

There exists a simple relation between leading exponents and L^p-integrability properties (see Section 9). However, in the present case it is necessary to take leading exponents arising from expansions in different Weyl chambers into account (cf. Theorem 9.4), whereas in the group case all growth behaviour is governed by the leading exponents corresponding to a single chamber.

In Section 10 we introduce a space of rapidly decreasing functions on G/H which generalizes Harish-Chandra's Schwartz space. As an application of the preceding sections we then prove that every $\mathbb{D}(G/H)$-finite and K-finite function belongs to this Schwartz space if and only if it belongs to $L^2(G/H)$ (Theorem 10.3). This generalizes a well known result of Harish-Chandra [9] .

Having later applications involving parabolic induction in mind we present the material for a group G of the Harish-Chandra class. Sections 1 and 2 deal with the basic structure theory of the corresponding reductive symmetric spaces.

Often proofs would have been essentially identical to those of analogous results in [3] . Therefore our paper contains many references to proofs in that paper, and we have followed the presentation of [3] rather closely, using the same notations where possible. On the other hand, we have tried to keep this paper self-contained by not referring to [3] for notations or definitions.

1. Symmetric spaces of class $\mathcal{H}$.

If G is a group of class $\mathcal{H}$, τ an involution of G, H a closed subgroup with $(G^\tau)^\circ \subset H \subset G^\tau$, we call the homogeneous space G/H a symmetric space of the Harish-Chandra class (class $\mathcal{H}$). For the basic structure theory of groups of class $\mathcal{H}$, we refer the reader to [18, pp. 192-198].

Proposition 1.1. Let G be a group of class $\mathcal{H}$, τ,H as above. Then G carries a Cartan involution θ with $\theta\tau=\tau\theta$. Moreover, $[H:H^\circ]<\infty$ and $\theta(H) = H$.

Proof. Let X(G) denote the space of continuous multiplicative homomorphisms $G \to \mathbb{R}^* = \mathbb{R}\setminus\{0\}$, and put

$$^\circ G = \bigcap_{\chi\in X(G)} \ker |\chi| .$$

Let $\mathfrak{c}$ be the centre of the Lie algebra $\mathfrak{g}$ of G, $^\circ\mathfrak{g}$ the Lie algebra of $^\circ G$, and set $^\circ\mathfrak{c} = {}^\circ\mathfrak{g} \cap \mathfrak{c}$. Because τ is an automorphism, it leaves $^\circ G$ invariant. The associated infinitesimal involution, denoted by the same symbol τ, leaves $^\circ\mathfrak{g}$, $\mathfrak{c}$ and $^\circ\mathfrak{c}$ invariant. If we let $\mathfrak{h}$ and $\mathfrak{q}$ denote the +1 and -1 eigenspaces of τ in $\mathfrak{g}$ respectively, we have decompositions $\mathfrak{c} = \mathfrak{c}_h \oplus \mathfrak{c}_q$ and $^\circ\mathfrak{c} = {}^\circ\mathfrak{c}_h \oplus {}^\circ\mathfrak{c}_q$, where $\mathfrak{c}_h = \mathfrak{c}\cap\mathfrak{h}$, $\mathfrak{c}_q = \mathfrak{c}\cap\mathfrak{q}$, etc. Fix linear subspaces $\mathfrak{v}_+$ and $\mathfrak{v}_-$ of $\mathfrak{c}_h$ and $\mathfrak{c}_q$ such that

$$\mathfrak{c}_h = {}^\circ\mathfrak{c}_h \oplus \mathfrak{v}_+ , \qquad \mathfrak{c}_q = {}^\circ\mathfrak{c}_q \oplus \mathfrak{v}_- ,$$

and put $\mathfrak{v} = \mathfrak{v}_+ \oplus \mathfrak{v}_-$. Then $\mathfrak{c} = {}^\circ\mathfrak{c} \oplus \mathfrak{v}$, and so $V = \exp\mathfrak{v}$ is a τ-stable maximal closed vector subgroup of centre(G). On the other hand, since τ leaves the semisimple algebra

$\mathfrak{g}_1 = [\mathfrak{g},\mathfrak{g}]$ invariant, there exists a τ-stable maximal compact subalgebra $\mathfrak{k}_1$ of $\mathfrak{g}_1$ (cf. [1]). Moreover, there exists a unique maximal compact subgroup K of G, whose Lie algebra contains $\mathfrak{k}_1$ (cf. [18, p.197, Thm 12]). Finally, there exists a unique Cartan involution θ of G such that $G^\theta = K$ and $\theta(x) = x^{-1}$ for $x \in V$. We claim that $\tau\theta = \theta\tau$. In fact, $\tau(K)$ is a maximal compact subgroup of G, whose Lie algebra $\tau(\mathfrak{k})$ contains $\tau(\mathfrak{k}_1) = \mathfrak{k}_1$. Hence, by the uniqueness referred to above, $\tau(K) = K$. The infinitesimal Cartan involution θ leaves $\mathfrak{g}_1$ and $\mathfrak{c}$ invariant, so that $\mathfrak{p} = \mathfrak{g}^{-\theta} = \mathfrak{g}_1^{-\theta} \oplus \mathfrak{c}^{-\theta} = \mathfrak{g}_1^{-\theta} \oplus \mathfrak{v}$. Therefore $\mathfrak{p}$ is τ-stable, hence $\exp\mathfrak{p}$ is, whence the claim.

Finally, since τ and θ commute, G^τ and $(G^\tau)^o$ are θ-invariant, so that $G^\tau = (G^\tau \cap K)\exp(\mathfrak{p}\cap\mathfrak{h})$ and $(G^\tau)^o = [(G^\tau)^o \cap K]\exp(\mathfrak{p}\cap\mathfrak{h})$. It follows that $[H:H^o] \leqslant [G^\tau:(G^\tau)^o] \leqslant [G^\tau \cap K : (G^\tau)^o \cap K] < \infty$ (the latter by compactness of K). It also follows that $(G^\tau)^o \cap K = (G^\tau \cap K)^o$, hence $H^o \cap K = (H\cap K)^o$, and $H = (H\cap K)\exp(\mathfrak{p}\cap\mathfrak{h})$. In particular $H = \theta(H)$.

From now on, let G be a group of class $\mathcal{H}$, τ an involution of G, and θ a commuting Cartan involution. Without further comments we will use the notations of the above proof in the sequel. Moreover, we fix a bilinear form B on $\mathfrak{g}$ which is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, coincides with the Killing form on $\mathfrak{g}_1$ and for which $\mathfrak{g}_1$ and $\mathfrak{c}$ are orthogonal. Then B is non-degenerate and Ad(G)-invariant.

<u>Próposition 1.2.</u> The map $\varphi: K \times (\mathfrak{p}\cap\mathfrak{q}) \times (\mathfrak{p}\cap\mathfrak{h}) \longrightarrow G$, $(k,X,Y) \longmapsto k \exp X \exp Y$ is a diffeomorphism.

<u>Proof.</u> Since $G \simeq {}^{o}G \times V$ (cf. [18, p.196]), where $V \simeq \exp(\mathfrak{k} \cap \mathfrak{p} \cap \mathfrak{q}) \times \exp(\mathfrak{k} \cap \mathfrak{p} \cap \mathfrak{h})$, it suffices to prove the assertion when $G = {}^{o}G$. In this case $G = K\, G_1$, where G_1 is the connected analytic subgroup with Lie algebra $\mathfrak{g}_1$. G being of class $\mathcal{H}$, G_1 is a closed subgroup; moreover it is semisimple and has finite centre. Therefore the lemma is valid for G_1 (cf. [5, proof of Thm 4.1]), and it follows that φ is surjective. On the other hand, if $k, k' \in K$, $X, X' \in \mathfrak{p} \cap \mathfrak{q}$, $Y, Y' \in \mathfrak{p} \cap \mathfrak{h}$ and $\varphi(k,X,Y) = \varphi(k',X',Y')$, then $k^{-1}k' \in K \cap G_1$ and by the validity of the lemma for G_1 we deduce that $k = k'$, $X = X'$, $Y = Y'$. Hence φ is injective. Finally, by a standard calculation one finds that the differential of φ is everywhere injective, thereby completing the proof.

Let $\mathfrak{g}_+$ be the +1 eigenspace of $\tau\theta$ in $\mathfrak{g}$. Then

$$\mathfrak{g}_+ = (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{q}).$$

Select a maximal abelian subspace $\mathfrak{a}_{pq}$ of $\mathfrak{p} \cap \mathfrak{q}$. Then
$\mathfrak{a}_{pq} = ([\mathfrak{g}_+, \mathfrak{g}_+] \cap \mathfrak{a}_{pq}) \oplus (\text{centre}(\mathfrak{g}_+) \cap \mathfrak{p})$.
From the corresponding fact in the semisimple case it is immediate that the set $\Delta = \Delta(\mathfrak{g}, \mathfrak{a}_{pq})$ of restricted roots of $\mathfrak{a}_{pq}$ in $\mathfrak{g}$ is a (possibly non-reduced) root system (cf. [16]).

Since $\tau\theta = I$ on $\mathfrak{a}_{pq}$, every root space $\mathfrak{g}^{\alpha}$ ($\alpha \in \Delta$) is $\tau\theta$-invariant, so that we have a corresponding decomposition

$$\mathfrak{g}^{\alpha} = \mathfrak{g}^{\alpha}_{+} \oplus \mathfrak{g}^{\alpha}_{-}$$

into the +1 and -1 eigenspaces. Let

$$\Delta_+ = \{\alpha \in \Delta \,;\, \mathfrak{g}^{\alpha}_{+} \neq 0\} .$$

Then $\Delta_+ = \Delta(\mathfrak{g}_+, \mathfrak{a}_{pq})$, the restricted root system of $\mathfrak{a}_{pq}$ in $\mathfrak{g}_+$. Of course $\mathfrak{a}_{pq}$ may be central in $\mathfrak{g}_+$, so that $\Delta_+ = \phi$. We fix a choice Δ_+^+ of positive roots for Δ_+ (if $\Delta_+ = \phi$, then $\Delta_+^+ = \phi$), and put

$$\mathfrak{a}_{pq}^- = \{H \in \mathfrak{a}_{pq};\ \alpha(H) < 0 \text{ for all } \alpha \in \Delta_+^+\},$$

$$A = \exp(\mathfrak{a}_{pq}), \qquad A^- = \exp(\mathfrak{a}_{pq}^-)$$

(later on, it will be technically more convenient to work with negative Weyl chambers). Also, we let

$$\mathfrak{a}_{pq}' = \{H \in \mathfrak{a}_{pq};\ \alpha(H) \neq 0 \text{ for all } \alpha \in \Delta_+\},$$

and $A' = \exp(\mathfrak{a}_{pq}')$. If $\Delta_+ = \phi$ this is to be interpreted as $\mathfrak{a}_{pq}^- = \mathfrak{a}_{pq}' = \mathfrak{a}_{pq}$.

<u>Proposition 1.3.</u> For every $X \in \mathfrak{p} \cap \mathfrak{q}$ there exists a unique $Y \in \mathrm{cl}(\mathfrak{a}_{pq}^-)$ such that $X = \mathrm{Ad}(k)Y$ for some $k \in K \cap H^o$.

<u>Proof.</u> Without loss of generality we may assume that $G = {}^oG$, and then the same proof as in [5, p.118] applies.

<u>Corollary 1.4 (Cartan decomposition).</u> For every $x \in G$ there exists a unique $a \in \mathrm{cl}(A^-)$ such that $x \in KaH^o$.

<u>Proof.</u> This follows from a straightforward combination of Propositions 1.2 and 1.3.

Before stating the next result we introduce a few more notations. Let $\mathfrak{l}$ be the centralizer of $\mathfrak{a}_{pq}$ in $\mathfrak{g}$. Since

$\mathfrak{a}_{pq}$ is invariant under τ and θ, so is $\mathfrak{l}$, and we have the decomposition

$$\mathfrak{l} = \mathfrak{l}_{kq} \oplus \mathfrak{l}_{kh} \oplus \mathfrak{a}_{pq} \oplus \mathfrak{l}_{ph} , \tag{1}$$

where $\mathfrak{l}_{kq} = \mathfrak{l} \cap \mathfrak{k} \cap \mathfrak{q}$, etc. Put

$$\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}^{\alpha}, \qquad \bar{\mathfrak{n}} = \sum_{\alpha \in \Delta^+} \mathfrak{g}^{-\alpha} .$$

Then obviously

$$\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{l} \oplus \mathfrak{n} . \tag{2}$$

By the same proof as in [2 , Prop. 3.4] , we also have

$$\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{l}_{kq} \oplus \mathfrak{a}_{pq} \oplus \mathfrak{h} . \tag{3}$$

Moreover, the maps $\bar{\mathfrak{n}} \times \mathfrak{l}_k \to \mathfrak{k}$, $(X,U) \mapsto X + \theta X + U$ and $\bar{\mathfrak{n}} \times \mathfrak{l}_h \to \mathfrak{h}$, $(X,U) \mapsto X + \tau X + U$ are easily seen to be bijective. Using [2, Prop. 3.5] , we now obtain the following.

<u>Proposition 1.5</u> (<u>infinitesimal Cartan decomposition</u>). Let $\mathfrak{h}^c$ be the orthocomplement of $\mathfrak{l}_{kh}$ in $\mathfrak{h}$. Then for every $a \in A'$ we have the direct sum decomposition

$$\mathfrak{g} = \mathrm{Ad}(a^{-1})\,\mathfrak{k} \oplus \mathfrak{a}_{pq} \oplus \mathfrak{h}^c .$$

Let M be the centralizer of $\mathfrak{a}_{pq}$ in $K \cap H$, and put $d(M) = \{(m,m) \in K \times H^o ;\ m \in M\}$.

<u>Lemma 1.6.</u> The map $(K \times H^o)/d(M) \times A^- \to G$ given by

$$((k,h)d(M),\ a) \longmapsto kah^{-1} \tag{4}$$

is a diffeomorphism onto the open dense subset $G' = KA^-H^o$ of G.

Proof. From Propositions 1.2 and 1.3 it easily follows that G' equals the open dense subset $K\exp(\mathrm{Ad}(K\cap H^{o})\mathfrak{a}^{-}_{pq})\exp(\mathfrak{p}\cap\mathfrak{h})$ of G.

To see that (4) is injective, suppose that $k\in K$, $a,b\in A^{-}$, $h\in H^{o}$. Then it suffices to show that $kah^{-1}=b$ implies $k=h\in M\cap H^{o}$ and $a=b$. Now this is seen as follows. Write $h^{-1}=h_1h_2$, where $h_1\in K\cap H^{o}$, $h_2\in\exp(\mathfrak{p}\cap\mathfrak{h})$. Then by Proposition 1.2 we have $kh_1=1$, $h_2=1$, $h_1^{-1}ah_1=b$. Let Ad_+ be the adjoint representation of $G^{o}_{+}=(K\cap H^{o})\exp(\mathfrak{p}\cap\mathfrak{q})$ in $\mathfrak{g}_+$. Then it follows that $\mathrm{Ad}_+(k)$ maps $\log(a)\in\mathfrak{a}^{-}_{pq}$ into $\mathfrak{a}^{-}_{pq}$. But $\mathfrak{a}^{-}_{pq}$ is a Weyl chamber for $\Delta_+=\Delta(\mathfrak{g}_+,\mathfrak{a}_{pq})$ and so, by standard semisimple theory applied to $\mathrm{Ad}_+(G^{o}_+)$, it follows that k centralizes $\mathfrak{a}_{pq}$. Hence $k=h\in M\cap H^{o}$ and $a=b$.

Finally, fix $k\in K$, $h\in H^{o}$, $a\in A^{-}$, and consider the map

$$\psi:\mathfrak{k}\times\mathfrak{h}^{c}\times\mathfrak{a}_{pq}\longrightarrow G,\qquad (X,Y,Z)\mapsto k\exp(X)a\exp(Z)\exp(Y)h.$$

Then the differential $d\psi(0)$ of ψ at $(0,0,0)$ is given by

$$d\psi(0)(U,V,W) = d(\lambda_{ka}\rho_h)(e)\ (\mathrm{Ad}(a^{-1})U+V+W),$$

where λ_{ka} denotes left multiplication by ka, and ρ_h right multiplication by h on G. By Proposition 1.5 this differential is bijective. Consequently (4) has bijective differential everywhere.

2. Invariant differential operators.

In this section we have gathered a number of properties of the algebra $\mathbb{D}(G/H)$ of invariant differential operators on G/H. For G semisimple and connected these results are rather well known, but somewhat spread through the literature.

Given a real Lie algebra $\mathfrak{b}$, we let $U(\mathfrak{b})$ denote the universal enveloping algebra of its complexification $\mathfrak{b}_c$. Similarly, $S(\mathfrak{b})$ denotes the symmetric algebra of $\mathfrak{b}_c$.

If $u \in U(\mathfrak{g})$, let $R_u = R(u)$ denote the right regular action of u on $C^\infty(G)$. Then if X lies in the algebra $U(\mathfrak{g})^H$ of $Ad_G(H)$ - invariant elements in $U(\mathfrak{g})$, R_X leaves the space $C^\infty(G/H)$ of right H-invariant smooth functions on G invariant. Throughout this paper we shall identify $C^\infty(G/H)$ with the space of smooth functions $G/H \to \mathbb{C}$. This being said, R_X acts as an element of $\mathbb{D}(G/H)$ on $C^\infty(G/H)$. The induced algebra homomorphism $U(\mathfrak{g})^H \to \mathbb{D}(G/H)$ is again denoted by R.

Proposition 2.1. Let $\lambda: S(\mathfrak{g}) \to U(\mathfrak{g})$ be the symmetrization map. Then we have the direct sum decomposition

$$U(\mathfrak{g})^H = (U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h}) \oplus \lambda[S(\mathfrak{g})^H].$$

For connected reductive G, this result is proved in [10]. One easily checks that the proof goes through for non-connected G as well. The same is true for the following result.

Lemma 2.2. The map $R: U(\mathfrak{g})^H \to \mathbb{D}(G/H)$ maps $\lambda(S(\mathfrak{g})^H)$ bijectively onto $\mathbb{D}(G/H)$. Moreover,

$$\ker(R) = U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h} .$$

In particular, since R is a homomorphism of algebras, it follows that $U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h}$ is a two-sided ideal in $U(\mathfrak{g})^H$ and we have a natural isomorphism

$$\mathbb{D}(G/H) \simeq U(\mathfrak{g})^H / (U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h}). \qquad (5)$$

In a canonical way $U(\mathfrak{g})^H/(U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h})$ embeds onto a subalgebra of the algebra

$$\mathbb{D} = U(\mathfrak{g})^{\mathfrak{z}} / (U(\mathfrak{g})^{\mathfrak{z}} \cap U(\mathfrak{g})\mathfrak{h}).$$

The adjoint representation naturally induces an action of H on $\mathbb{D}$. Clearly H^o acts trivially, so that in fact we have an action of the finite group H/H^o on $\mathbb{D}$. Obviously $\mathbb{D}^H = \mathbb{D}^{H/H^o}$.

<u>Proposition 2.3.</u> $U(\mathfrak{g})^H/(U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h}) \simeq \mathbb{D}^H$.

<u>Proof.</u> Let j denote the canonical map $U(\mathfrak{g})^{\mathfrak{z}} \to \mathbb{D}$. Then $j(U(\mathfrak{g})^H) \subset \mathbb{D}^H$ and it suffices to show that j maps $U(\mathfrak{g})^H$ onto $\mathbb{D}^H = \mathbb{D}^{H/H^o}$. Fix $Y \in \mathbb{D}^H$. Then $Y = j(Z)$ for some $Z \in U(\mathfrak{g})^{\mathfrak{z}}$. Put

$$X = [H:H^o]^{-1} \sum_{h \in H/H^o} Z^h .$$

Then obviously $X \in U(\mathfrak{g})^H$ and $j(X) = Y$.

For the moment, let us focus on the algebra $\mathbb{D}$. The following argument, based on a duality used in [1], has been exploited by several authors (cf. [5], [13]).
We define the so called dual real form $\mathfrak{g}^d$ of $\mathfrak{g}$ in $\mathfrak{g}_c$ by:

$$\mathfrak{g}^d = i(\mathfrak{k}\cap\mathfrak{q}) \oplus (\mathfrak{k}\cap\mathfrak{h}) \oplus (\mathfrak{p}\cap\mathfrak{q}) \oplus i(\mathfrak{p}\cap\mathfrak{h}), \qquad (6)$$

and put $\mathfrak{k}^d = \mathfrak{h}_c \cap \mathfrak{g}^d$, $\mathfrak{p}^d = \mathfrak{q}_c \cap \mathfrak{g}^d$ (this may be read as: "the $\mathfrak{k}$ in the dual algebra", etc.). Then

$$\mathfrak{g}^d = \mathfrak{k}^d \oplus \mathfrak{p}^d$$

is a Cartan decomposition for the reductive algebra $\mathfrak{g}^d$. Since $\mathfrak{k}^d$ and $\mathfrak{h}$ are real forms of the same complexification $\mathfrak{h}_c$, we have

$$\mathbb{D} = U(\mathfrak{g})^{\mathfrak{k}^d} / (U(\mathfrak{g})^{\mathfrak{k}^d} \cap U(\mathfrak{g})\mathfrak{k}^d). \qquad (7)$$

In the semisimple case it is well known that the algebra on the right hand side of (7) is commutative and finitely generated as a module over the centre $\mathfrak{Z}$ of $U(\mathfrak{g})$. From this the following result easily follows.

<u>Proposition 2.4.</u> $\mathbb{D}$ is commutative and finitely generated as a $\mathfrak{Z}$-module.

In a natural way $\mathbb{D}$ embeds as a linear subspace of $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}$. The adjoint representation naturally induces a representation of $\mathfrak{h}$ in $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}$.

<u>Proposition 2.5.</u> $\mathbb{D} \simeq (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h})^{\mathfrak{h}}$. (8)

<u>Proof.</u> Obviously we have

$$(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h})^{\mathfrak{h}} = (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{k}^d)^{\mathfrak{k}^d}.$$

Now let K^d be the analytic subgroup of the adjoint group G_c

of $\mathfrak{g}_c$, generated by $\exp(\mathrm{ad}\,\mathfrak{k}^d)$. Then K^d is compact, and by a standard argument involving averaging over K^d (see also the proof of Proposition 2.3) it follows that the natural map

$$(U(\mathfrak{g})^{K^d}/(U(\mathfrak{g})^{K^d} \cap U(\mathfrak{g})\mathfrak{k}^d) \longrightarrow (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{k}^d)^{K^d} \qquad (9)$$

is an isomorphism onto. But obviously the left hand side of (9) equals (7), whereas the right hand side equals (8). This proves the assertion.

Let $\mathfrak{a}_{kq}$ be maximal abelian in $\mathfrak{l}_{kq}$. Then

$$\mathfrak{a}_q = \mathfrak{a}_{kq} \oplus \mathfrak{a}_{pq} \qquad (10)$$

is maximal abelian in $\mathfrak{q}$, and $\mathfrak{a}_p^d = \mathfrak{a}_{q,c} \cap \mathfrak{g}^d$ is maximal abelian in $\mathfrak{p}^d$. Obviously we may identify $\Delta_q = \Delta(\mathfrak{g}_c, \mathfrak{a}_{q,c})$ with $\Delta(\mathfrak{g}^d, \mathfrak{a}_p^d)$. Let $W(\Delta_q)$ be the Weyl group of Δ_q, and let $I(\mathfrak{a}_q) = I(\mathfrak{a}_p^d)$ denote the space of $W(\Delta_q)$-invariant elements in $S(\mathfrak{a}_q) = S(\mathfrak{a}_p^d)$. Then by (7), Harish-Chandra's canonical isomorphism associated with $\mathfrak{g}^d, \mathfrak{k}^d, \mathfrak{a}_p^d$ is actually an algebra isomorphism from $\mathbb{D}$ onto $I(\mathfrak{a}_q)$.

Fix a choice Δ_q^+ of positive roots for Δ_q, and put

$$\mathfrak{n}(\Delta_q^+) = \sum_{\alpha \in \Delta_q^+} \mathfrak{g}_c^\alpha .$$

Then by the Iwasawa decomposition in $\mathfrak{g}^d$ we have the direct sum

$$\mathfrak{g}_c = \mathfrak{h}_c \oplus \mathfrak{a}_{q,c} \oplus \mathfrak{n}(\Delta_q^+).$$

Thus, by the Poincaré-Birkhoff-Witt theorem, we obtain the direct sum

$$U(\mathfrak{g}) = (\mathfrak{n}(\Delta_q^+)U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{h}) \oplus S(\mathfrak{a}_q) .$$

The corresponding projection of $U(\mathfrak{g})$ into $S(\mathfrak{a}_q)$ induces an algebra homomorphism $\tilde{\gamma}$ from $\mathbb{D}$ into $S(\mathfrak{a}_q)$. Let $T(\Delta_q^+)$ be the automorphism of $S(\mathfrak{a}_q)$ determined by

$$T(\Delta_q^+)\ (H) \;=\; H \;+\; \tfrac{1}{2}\ \mathrm{tr}\ (\mathrm{ad}(H)|\mathfrak{n}(\Delta_q^+)\ ),$$

for $H \in \mathfrak{a}_{q,c}$. Then $T(\Delta_q^+)\circ\tilde{\gamma}$ maps $\mathbb{D}$ isomorphically onto $I(\mathfrak{a}_q)$ and is independent of the choice Δ_q^+ of positive roots; by definition we have $\gamma = T(\Delta_q^+)\circ\tilde{\gamma}$ (for proofs of these facts, see for instance [12, Chapter 10]).

The following result and its proof were privately communicated to me by Professor T. Oshima.

Lemma 2.6. $\mathbb{D} \simeq \mathbb{D}(G/H)$.

Proof. By (5) and Proposition 2.3, we must show that the elements of $\mathbb{D}$ are H-invariant.

Via the above isomorphism γ, we see that $\mathbb{D}$ is a commutative ring without zero divisors. Let $\mathcal{Z}$ be the canonical image of $\mathfrak{Z}$ in $\mathbb{D}$ and fix $P \in \mathbb{D}$. Then by the proposition below it follows that $PQ \in \mathcal{Z}$ for some $Q \in \mathcal{Z}$. Since G is of class $\mathcal{H}$, the elements of $\mathfrak{Z}$ are G-invariant. Hence the elements of $\mathcal{Z}$ are H-invariant. Therefore, if $h \in H$, then

$$PQ = (PQ)^h = P^hQ^h = P^hQ.$$

Because $\mathbb{D}$ has no zero divisors, this implies that $P = P^h$ for all $h \in H$.

Proposition 2.7. The quotient fields of $\mathbb{D}$ and the canonical image $\mathcal{Z}$ of $\mathfrak{Z}$ in $\mathbb{D}$ are the same.

<u>Proof.</u> Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}_q$, and let $\gamma_{\mathfrak{z}}$ be Harish-Chandra's canonical isomorphism of $\mathfrak{Z}$ onto the algebra $I(\mathfrak{a})$ of $W(\mathfrak{g}_c,\mathfrak{a}_c)$ -invariants in $S(\mathfrak{a})$. Then it is well known that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{Z} & \longrightarrow & \mathbb{D} \\ \gamma_{\mathfrak{z}}\downarrow & & \downarrow\gamma \\ I(\mathfrak{a}) & \xrightarrow{\ r\ } & I(\mathfrak{a}_q). \end{array}$$

Here r denotes restriction to $\mathfrak{a}_q$ via the form B. The assertion now follows from the fact that $I(\mathfrak{a}_q)$ and $r(I(\mathfrak{a}))$ have the same quotient fields (cf. [11, Theorem 7.6]).

<u>Corollary 2.8.</u> The algebra $\mathbb{D}(G/H)$ is commutative and finitely generated as a $\mathfrak{Z}$ -module.

After the above, it is straightforward to see how the results of [2] extend to symmetric spaces of the Harish-Chandra class. Therefore we will refer to [2] without further comments.

3. Spherical functions and the basic equations.

Let $\mu = (\mu_1, \mu_2)$ be a smooth representation of $K \times H$ in a finite dimensional complex linear space E. If $v \in E$, $h \in H$, we shall often write $v\mu_2(h^{-1})$ instead of $\mu_2(h)v$. A C^∞-function $F: G \to E$ such that for all $x \in G$, $k \in K$, $h \in H$ we have

$$F(kxh) = \mu_1(k)\, F(x)\, \mu_2(h)$$

is called μ-spherical. The space of all such functions is denoted by $C^\infty_\mu(G)$. The subspace of $\mathfrak{Z}$-finite elements in $C^\infty_\mu(G)$ is denoted by $A_\mu(G)$. As they are annihilated by an elliptic differential operator with real analytic coefficients (see for instance the argument in [18, p.310]), the elements of $A_\mu(G)$ are in fact real analytic functions.

A function $F \in C^\infty_\mu(G)$ belongs to $A_\mu(G)$ iff it is annihilated by a cofinite ideal I in $\mathfrak{Z}$. We write $A_\mu(G, I)$ for the space of $F \in A_\mu(G)$ satisfying

$$R_Z F = 0 \qquad (Z \in I).$$

Here we have used the notation $R_u = R(u)$ for the infinitesimal right regular action of an element $u \in U(\mathfrak{g})$.

For the sake of completeness we list the following proposition which is proved along the same lines as [2 , Cor. 3.10] , involving a finite basis of $\mathfrak{Z}/I$ over $\mathbb{C}$ (cf. [18, p.308, Thm. 8]). Let $\mathfrak{a} \subset \mathfrak{l}$ be a Cartan subalgebra containing $\mathfrak{a}_{pq}$, $\Phi = \Delta(\mathfrak{g}_c, \mathfrak{a}_c)$, $\Phi_0 = \Delta(\mathfrak{l}_c, \mathfrak{a}_c)$, and let $W(\Phi)$, $W(\Phi_0)$ be the Weyl groups of Φ and Φ_0 respectively.

<u>Proposition 3.1.</u> Let I be a cofinite ideal in $\mathfrak{Z}$. Then

$$\dim A_\mu(G,I) \leqslant \dim(\mathfrak{Z}/I)\dim(\mu)\ [W(\Phi) : W(\Phi_0)] .$$

Let π be an admissible representation of finite length of G in a Fréchet space V. Let the space V^∞ of C^∞-vectors in V be equipped with the topology induced by the collection of seminorms

$$N_{p,a} : \qquad v \mapsto p(\pi(a)v),$$

where p ranges over a complete set of seminorms for V, and $a \in U(\mathfrak{g})$. As a locally convex space V^∞ is isomorphic to the closed subspace $T = \{f \in C^\infty(G,V);\ f(x) = \pi(x)f(e)\}$ of the Fréchet space $C^\infty(G,V)$; this follows by a straightforward application of the Banach-Steinhaus theorem. Hence V^∞ is Fréchet. The topological dual $V^{-\infty}$ of V^∞ is a G-module in a natural way; we let $(V^{-\infty})^H$ denote the subspace of H-fixed elements in $V^{-\infty}$.

Given $\varphi \in (V^{-\infty})^H$, $u \in V_K$ (the K-finite vectors in V), we may form the (C^∞-) matrix coefficient

$$m(x) = m_{\varphi,u}(x) = \varphi(\pi(x^{-1})u). \tag{11}$$

If J is the annihilator of V^∞ in $\mathfrak{Z}$, then J is a cofinite ideal because π has finite length, and we have $R(Z^\vee)m = L(Z)m = 0$ for all $Z \in J$; here $L(Z) = L_Z$ denotes the left regular action of Z on $C^\infty(G)$, and $u \mapsto u^\vee$ denotes the unique anti-automorphism of $U(\mathfrak{g})$ determined by

$$X^\vee = -X \qquad\qquad (X \in \mathfrak{g}).$$

Consequently m is a $\mathfrak{Z}$-finite and K-finite real analytic function on G/H (cf. [18, p.308]).

In a natural way the function m (and more generally any $\mathfrak{Z}$-finite and K-finite function on G/H) gives rise to a spherical function $\nu(m)$. Let $\mathcal{V}$ be the set of K-types occurring in the linear span of $\{L_k m;\ k \in K\}$, and write E for the finite dimensional subspace $\bigoplus_{\delta \in \mathcal{V}} L^2(K)_\delta$ of $L^2(K)$; here $L^2(K)_\delta$ denotes the subspace of left K-finite elements of isotopy type $\delta \in \hat{K}$. Then $\nu(m)\colon G \to E$ is defined by

$$\nu(m)(x)\colon \quad k \longmapsto m(kx), \qquad K \longrightarrow \mathbb{C}, \tag{12}$$

for $x \in G$. Let $\mu = (\mu_1, \mu_2)$ where μ_1 is the right regular representation of K restricted to E, and where μ_2 is the trivial representation of H in E. Then obviously $\nu(m) \in A_\mu(G, I)$.

<u>Corollary 3.2.</u> Let π be an admissible representation of finite length of G in a Fréchet space V. Then $\dim(V^{-\infty})^H < \infty$.

<u>Proof.</u> Select a finite set of K-finite generators $u_1, \dots, u_s$ for the $(\mathfrak{g}, K)$-module V_K, and let $\mathcal{V}$ be the (finite) set of K-types occurring in the linear span of the $\pi(k)u_j$, $k \in K$, $1 \leq j \leq s$. Let $E = E(\mathcal{V})$ and μ be as above. Then the map $(V^{-\infty})^H \to A_\mu(G, J^\vee)\colon \varphi \mapsto (\nu(m_{\varphi, u_j});\ 1 \leq j \leq s)$ is injective. Now apply Proposition 3.1.

The matrix coefficient $m = m_{\varphi,u}$ can be retrieved from $\nu(m)$ as follows. Let $\alpha_{\mathcal{V}} = \sum_{\delta \in \mathcal{V}} \dim(\delta)\, \chi_\delta$, where χ_δ is the character of δ. Then $\alpha_{\mathcal{V}} \in E$ and

$$m_{\varphi,u}(x) = \langle \nu(m_{\varphi,u})(x), \alpha_{\mathcal{V}} \rangle,$$

where $\langle \cdot,\cdot \rangle$ denotes the restriction of the $L^2(K)$-inner product to E. Thus in a straightforward manner the asymptotic behaviour of $m_{\varphi,u}$ can be read off from the behaviour of the $(\mu_1, 1)$-spherical function $\nu(m_{\varphi,u})$.

From now on, let $\mu = (\mu_1, \mu_2)$ be a fixed smooth representation of $K \times H$ in a finite dimensional complex linear space E.

In view of the Cartan decomposition $G = K\mathrm{cl}(A^-)H$, a function $F \in C^\infty_\mu(G)$ is determined by its restriction $\mathrm{Res}(F)$ to A^-. Let M be the centralizer of A in $K \cap H$, and put

$$E^M = \{u \in E;\quad \mu_1(m)u = u\mu_2(m) \quad \text{for all } m \in M\} . \qquad (13)$$

Then obviously the restriction map Res maps $C^\infty_\mu(G)$ injectively into $C^\infty(A^-, E^M)$.

Let $\mathcal{P}$ be the collection of systems P of positive roots for Δ with $P \cap \Delta_+ = \Delta_+^+$. If $P \in \mathcal{P}$, we set

$$\mathfrak{a}^-_{pq}(P) = \{H \in \mathfrak{a}_{pq};\quad \alpha(H) < 0 \quad \text{for all} \quad \alpha \in P\} ,$$

and $A^-(P) = \exp \mathfrak{a}^-_{pq}(P)$. Then

$$\mathrm{cl}(A^-) = \bigcup_{P \in \mathcal{P}} \mathrm{cl}(\, A^-(P)\,).$$

Thus, without loss of generality, we may fix a system Δ^+ of positive roots in $\mathcal{P}$ and study the behaviour of $F \in A_\mu(G)$ on $\mathrm{cl}(\, A^-(\Delta^+)\,)$.

Let Σ be the collection of simple roots in Δ^+ (warning: in [3], Σ stands for the root system, Δ for the simple roots). Then Σ is a basis for $(\mathfrak{a}_{pq} \cap \mathfrak{g}_1)^*$ over $\mathbb{R}$. Select a basis $\Lambda_{\mathfrak{c}}$ for $(\mathfrak{a}_{pq} \cap \mathfrak{c})^*$ over $\mathbb{R}$. Identifying $(\mathfrak{a}_{pq} \cap \mathfrak{g}_1)^*$ and $(\mathfrak{a}_{pq} \cap \mathfrak{c})^*$ with subspaces of $\mathfrak{a}^*_{pq}$ via B, we put:

$$\Lambda = \Sigma \cup \Lambda_{\mathfrak{r}}$$

Let $\{H_\lambda ; \lambda \in \Lambda\}$ be the basis of $\mathfrak{a}_{pq}$ which is dual to the basis Λ of $\mathfrak{a}^*_{pq}$. Then $H_\alpha \in \mathfrak{a}_{pq} \cap \mathfrak{g}_1$ for $\alpha \in \Sigma$ and $H_\lambda \in \mathfrak{a}_{pq} \cap \mathfrak{r}$ for $\lambda \in \Lambda_{\mathfrak{r}}$.

As in [2], let $\mathcal{F}^+$ be the algebra of functions $A' \to \mathbb{R}$ generated by

$$f_+^\alpha(a) = (a^\alpha - a^{-\alpha})^{-1}, \qquad g_+^\alpha(a) = -a^{-\alpha} f_+^\alpha(a),$$

$$f_-^\beta(a) = (a^\beta + a^{-\beta})^{-1}, \qquad g_-^\beta(a) = -a^{-\beta} f_-^\beta(a)$$

($\alpha \in \Delta_+^+$; $\beta \in \Delta^+$, $\mathfrak{g}_-^\beta \neq 0$). Here we have used the notation

$$a^\gamma = e^{\gamma \log a},$$

for $\gamma \in \mathfrak{a}^*_{pq}$, $a \in A$. Moreover, let $\mathcal{F}$ be the ring generated by 1 and $\mathcal{F}^+$.

Let $\mathfrak{Z}(\mathfrak{k})$ denote the centre of $U(\mathfrak{k})$ and let $v_1 = 1$, $v_2, \ldots, v_r \in \mathfrak{Z}(\mathfrak{k})$ be as in [2 , Lemma 3.7] . Moreover, let I be a cofinite ideal in $\mathfrak{Z}$ and fix $D_1 = 1, D_2, \ldots, D_s \in \mathfrak{Z}$ such that their canonical images generate $\mathfrak{Z}/I$ over $\mathbb{C}$. Then by [2 , Lemma 3.8] there exist finitely many elements $f^{k\ell}_{\lambda ij} \in \mathcal{F}$, $\xi^{k\ell}_{\lambda ij} \in U(\mathfrak{k})$, $\eta^{k\ell}_{\lambda ij} \in U(\mathfrak{h})$ ($\lambda \in \Lambda$, $1 \leqslant i,k \leqslant s$, $1 \leqslant j,\ell \leqslant r$), such that

$$H_\lambda D_i v_j' \equiv \sum_{k,l} f^{kl}_{\lambda ij}(a)(\xi^{kl}_{\lambda ij})^a D_k v_\ell' \eta^{kl}_{\lambda ij} \quad \text{mod } I \qquad (14)$$

for all $a \in A'$. Here we have used the notation

$$Y^x = \mathrm{Ad}(x^{-1})\, Y \qquad (x \in G,\ Y \in U(\mathfrak{g})\ ), \qquad (15)$$

which is the technically more convenient notation of [3], but inconsistent with the notation in [2].

The centralizer L of $\mathfrak{a}_{pq}$ in G is of class $\mathcal{H}$, hence centralizes $\mathfrak{Z}(\mathfrak{k})$ (cf. [18, p.286, Theorem 13]). Therefore M centralizes $\mathfrak{Z}$ and $\mathfrak{Z}(\mathfrak{k})$. Consequently, if $F \in A_\mu(G,I)$ then the functions

$$\Phi_{ij} = \operatorname{Res}(R(D_i v_j')F) \tag{16}$$

($1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant r$) map A into E^M. By (14) it follows that

$$H_\lambda \Phi_{ij}(a) = \sum_{k,\ell} f^{k\ell}_{\lambda ij}(a)\, \mu_1(\xi^{k\ell}_{\lambda ij})\, \Phi_{kl}(a)\, \mu_2(\eta^{k\ell}_{\lambda ij}), \tag{17}$$

for all $a \in A'$. Now let $\Phi : A' \longrightarrow (E^M)^{sr}$ be the vector valued function with entries Φ_{ij} ($1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant r$). Then by (17) there exist elements

$$G_\lambda \in \mathcal{F} \otimes \operatorname{End}_{\mathbb{C}}\{(E^M)^{sr}\} \qquad (\lambda \in \Lambda)$$

such that the real analytic map $\Phi : A \rightarrow (E^M)^{sr}$ satisfies the differential equations

$$H_\lambda \Phi = G_\lambda \cdot \Phi \qquad (\lambda \in \Lambda)$$

on A'.

As in [3] we view A as embedded in $\mathbb{C}^\Lambda$ under the map

$$\underline{\lambda}(a) = (\, a^\lambda ;\quad \lambda \in \Lambda\,).$$

Under this map the differential operators H_λ ($\lambda \in \Lambda$) correspond to $z_\lambda \partial/\partial z_\lambda$ in $\mathbb{C}^\Lambda$. If $\gamma \in \mathbb{Z}\Sigma$, then the character $e^\gamma : a \mapsto a^\gamma$ corresponds to a rational function on $\mathbb{C}^\Lambda$. Identifying $\gamma \in \mathbb{Z}\Sigma$ with the element (γ_α ; $\alpha \in \Sigma$) of $\mathbb{Z}^\Sigma \subset \mathbb{Z}^\Lambda$ determined by $\gamma = \sum_{\alpha \in \Sigma} \gamma_\alpha \alpha$, and using the multi-index notation

$$z^t = \prod_{\lambda \in \Lambda} (z_\lambda)^{t_\lambda}$$

for $z \in \mathbb{C}^\Lambda$, $t \in \mathbb{Z}^\Lambda$, we have

$$a^\gamma = \underline{\lambda}(a)^\gamma .$$

Consequently the elements of $\mathcal{F}$ can be viewed as rational functions on $\mathbb{C}^\Lambda$. If $\alpha \in \Delta_+$, we put

$$Y_+^\alpha = \{ z \in \mathbb{C}^\Lambda ; \quad z^{2\alpha} = 1 \} , \tag{18}$$

and if $\beta \in \Delta$, $\mathfrak{g}_-^\beta \neq 0$, we put

$$Y_-^\beta = \{ z \in \mathbb{C}^\Lambda ; \quad z^{2\beta} = -1 \} .$$

Moreover, let $Y_+ = \bigcup \{ Y_+^\alpha ; \alpha \in \Delta_+^+ \}$, $\quad Y_- = \bigcup \{ Y_-^\beta ; \beta \in \Delta^+, \mathfrak{g}_-^\beta \neq 0 \}$ and

$$Y = Y_+ \cup Y_- . \tag{19}$$

Then the elements of $\mathcal{F}$ are regular on $\mathbb{C}^\Lambda \setminus Y$. Being real analytic on A, the map Φ extends to a holomorphic $(E^M)^{sr}$-valued map on an open neighbourhood Ω of $\underline{\lambda}(A)$ in $\mathbb{C}^\Lambda$. We conclude that it satisfies the system of differential equations

$$z_\lambda \frac{\partial}{\partial z_\lambda} \Phi = G_\lambda \cdot \Phi \qquad (\lambda \in \Lambda) \tag{20}$$

on $\Omega \setminus Y$.

The system (20) has simple singularities (in the sense of [3, Appendix]) along the coordinate hyperplanes $z_\lambda = 0$ ($\lambda \in \Lambda$), so that we may apply the theory described in [3 , Appendix]. Put

$$D = \{ z \in \mathbb{C} ; \quad |z| < 1 \} .$$

Then obviously $\underline{\lambda}(A^-(\Delta^+)) \subset D^\Sigma \times \mathbb{C}^{\Lambda \setminus \Sigma} \subset \mathbb{C}^\Lambda \setminus Y$, so that a result analogous to [3 , Lemma 5.1] holds. To formulate it, we need some definitions and notations. If $m \in \mathbb{N}^{\Lambda}$ ($\mathbb{N} = \{0,1,\dots\}$), $s \in \mathbb{C}^{\Lambda}$, we put

$$\log^{m}\underline{\lambda}(a) = \prod_{\lambda \in \Lambda} \{\lambda(\log a)\}^{m_\lambda},$$

$$\underline{\lambda}^{s}(a) = \prod_{\lambda \in \Lambda} \exp(s_\lambda \lambda(\log a)),$$

for $a \in A$. Two elements $s, t \in \mathbb{C}^{\Lambda}$ are called integrally equivalent iff $s - t \in \mathbb{Z}^{\Lambda}$.

<u>Lemma 3.3.</u> Let $F \in A_\mu(G,I)$. Then there exist

(i) a finite set S of mutually integrally inequivalent elements of $\mathbb{C}^{\Lambda}$, and

(ii) for each $s \in S$ a finite collection $F_{s,m}$ $(m \in \mathbb{N}^{\Lambda})$ of non-trivial holomorphic E^{M}-valued functions on $D^{\Sigma} \times \mathbb{C}^{\Lambda\setminus\Sigma}$ such that on each of the coordinate hyperplanes $z_\lambda = 0$ $(\lambda \in \Lambda)$ at least one of them is not identically zero, such that

$$F = \sum_{s,m} (F_{s,m} \circ \underline{\lambda})\, \underline{\lambda}^{s} \log^{m}\underline{\lambda}$$

on $A^{-}(\Delta^{+})$.

This S and the $F_{s,m}$ are unique.

Let $\sum_k c_{s+k,m}\, z^{k}$ (summation over $\mathbb{N}^{\Lambda}$) be the power series expansion of $F_{s,m}$. Then the series expansion

$$F = \sum_{t,m} c_{t,m}\, \underline{\lambda}^{t} \log^{m}\underline{\lambda} \qquad (21)$$

of F converges absolutely on $A^{-}(\Delta^{+})$. Any series expansion like (21) which converges absolutely to F on a non-empty open subset of $A^{-}(\Delta^{+})$ must be identical to (21). If $c_{t,m} \neq 0$

for some $m \in \mathbb{N}^{\Lambda}$, then t is called a Δ^+-exponent of F. On $\mathbb{C}^{\Lambda}$ we define the $\leq_{\Sigma}$-order by

$$s \leq_{\Sigma} t \quad \text{iff} \quad t-s \in \mathbb{N}^{\Sigma},$$

for $s,t \in \mathbb{C}^{\Lambda}$. The $\leq_{\Sigma}$-minimal elements in the set of Δ^+-exponents of F are called the Δ^+-leading exponents of F. Given a Δ^+-leading exponent $t \in \mathbb{C}^{\Lambda}$, the corresponding character $\underline{\lambda}^t : A \to \mathbb{C}^*$ is called a Δ^+-leading character, and

$$F_t = \sum_m c_{t,m} \underline{\lambda}^t \log^m \underline{\lambda}$$

is called the corresponding Δ^+-leading term of F.

In Section 4 we will develop the theory of radial components associated with the Cartan decomposition (Cor. 1.4) in order to limit the possible Δ^+-leading terms of F. Let σ be the injective algebra homomorphism of $\mathfrak{Z}$ into $\mathfrak{Z}(\mathfrak{l})$, determined by

$$Z - \sigma(Z) \in \mathfrak{n}U(\mathfrak{g}), \tag{22}$$

for $Z \in \mathfrak{Z}$ ($\theta \circ \sigma \circ \theta$ is the map denoted by $\tilde{\mu}$ in [2 , Prop. 3.6]). If I is a cofinite ideal in $\mathfrak{Z}$, then $\mathfrak{Z}(\mathfrak{l})\sigma(I)$ is a cofinite ideal in $\mathfrak{Z}(\mathfrak{l})$ (cf. [2 , Lemma 3.7]). Under left multiplication the space $U = \mathfrak{Z}(\mathfrak{l})/\mathfrak{Z}(\mathfrak{l})\sigma(I)$ is an $\mathfrak{a}_{pq}$-module; by exponentiation it becomes an A-module. Being finite dimensional, the A-module U splits into a finite direct sum of generalized A-weight spaces. A character $\omega: A \to \mathbb{C}^*$ is said to lie Δ^+-shifted over the cofinite ideal I in $\mathfrak{Z}$ if it is a generalized A-weight for the A-module $\mathfrak{Z}(\mathfrak{l})/\mathfrak{Z}(\mathfrak{l})\sigma(I)$.

__Remark.__ Here we do not follow the terminology of [3]. The reason is that we wish to make the dependence on the choice $\Delta^+ \in \mathcal{P}$ explicit. If I is a cofinite ideal in $\mathfrak{Z}$, then to each $P \in \mathcal{P}$ corresponds the set $X(P,I)$ of characters lying P-shifted over I. The sets $X(P,I)$, $P \in \mathcal{P}$, are mutually different, but related by certain " ρ-shifts". We discuss this in Section 6.

__Theorem 3.4.__ Let I be a cofinite ideal in $\mathfrak{Z}$, $F \in A_\mu(G,I)$. Then all Δ^+-leading characters lie Δ^+-shifted over I.

We postpone the proof of this theorem to the next section.

In particular, the set of Δ^+-leading characters is finite, so that with essentially the same proof we have the following analogue of [3 , Theorem 5.6]. Viewing $\mathbb{C}^\Sigma$ as a subspace of $\mathbb{C}^\Lambda$, we call two elements $s,t \in \mathbb{C}^\Lambda$ Σ-integrally equivalent if $s-t \in \mathbb{Z}^\Sigma$. Moreover, we define the map $\underline{\alpha} : A \to \mathbb{C}^\Sigma$ by

$$\underline{\alpha}(a) = (a^\alpha ; \quad \alpha \in \Sigma).$$

__Theorem 3.5.__ Let F be a $\mathfrak{Z}$-finite μ-spherical function on G. Then there exist

(i) a finite set S_Σ of mutually Σ-integrally inequivalent elements of $\mathbb{C}^\Lambda$, and

(ii) for each $s \in S_\Sigma$ a finite set of non-trivial holomorphic functions $F^\Sigma_{s,m} : D^\Sigma \longrightarrow E^M$ $(m \in \mathbb{N}^\Lambda)$ such that on each coordinate hyperplane $z_\alpha = 0$ $(\alpha \in \Sigma)$ at least one of them is not identically zero, such that on $A^-(\Delta^+)$ we have

$$F = \sum_{s,m} (F^\Sigma_{s,m} \circ \underline{\alpha}) \, \underline{\lambda}^s \log^m \underline{\lambda}$$

on $A^-(\Delta^+)$.

This S_Σ and the $F^\Sigma_{s,m}$ are unique.

<u>Remark 3.6.</u> As in [3] the set S_Σ can be characterized as follows. For each class Ω of Σ-integrally equivalent Δ^+-leading exponents we define the element $s(\Omega) \in \mathbb{C}^\Lambda$ by

$$s(\Omega)_\lambda = \min \{ t_\lambda ; \quad t \in \Omega \} .$$

Then S_Σ is the set of all $s(\Omega)$.

4. Radial components and leading characters.

In this section we develop the theory of the μ-radial component of a differential operator in order to prove Theorem 3.4. We start with a result related to the infinitesimal Cartan decomposition (see Proposition 1.5). Let $\mathcal{R}$ be the ring of functions $A' \to \mathbb{R}$ generated by 1, a^{α} $(\alpha \in \Sigma)$ and

$$(1 - a^{2\alpha})^{-1} \qquad (\alpha \in \Delta_+^+),$$

$$(1 + a^{2\beta})^{-1} \qquad (\beta \in \Delta^+, \quad \mathfrak{g}_-^{\beta} \neq 0).$$

Moreover, let $\mathcal{R}^+$ be the ideal in $\mathcal{R}$ generated by the functions a^{α}, $\alpha \in \Sigma$.

<u>Proposition 4.1.</u> Let $X_\alpha \in \mathfrak{g}_+^{\alpha}$ or $X_\alpha \in \mathfrak{g}_-^{\alpha}$ $(\alpha \in \Delta^+)$. Then there exist $f_1, f_2 \in \mathcal{R}^+$, such that for all $a \in A'$ we have

$$X_\alpha = f_1(a)\,(X_\alpha + \theta X_\alpha)^a + f_2(a)\,(X_\alpha + \tau X_\alpha). \tag{23}$$

<u>Proof.</u> First recall that we use the notation (15). If $X_\alpha \in \mathfrak{g}_+^{\alpha}$, then $\theta X_\alpha = \tau X_\alpha$ so that (23) holds with $f_1 = a^{\alpha}(1 - a^{2\alpha})^{-1}$, $f_2 = -a^{2\alpha}(1 - a^{2\alpha})^{-1}$. On the other hand, if $X_\alpha \in \mathfrak{g}_-^{\alpha}$, then $\theta X_\alpha = -\tau X_\alpha$ and (23) holds with $f_1 = a^{\alpha}(1 + a^{2\alpha})^{-1}$, $f_2 = a^{2\alpha}(1 + a^{2\alpha})^{-1}$. In both cases it is clear that $f_1, f_2 \in \mathcal{R}^+$.

After this, we are prepared for the radial decomposition of differential operators. As in [3], we define trilinear maps

$$B_a : \; U(\mathfrak{a}_{pq}) \times U(\mathfrak{k}) \times U(\mathfrak{h}) \longrightarrow U(\mathfrak{g})$$

$(a \in A)$ by $B_a(H,X,Y) = X^a H Y$. Now let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}_{pq}$ in $\mathfrak{k} \cap \mathfrak{h}$. Then $\mathfrak{m}$ is the Lie algebra of M. If $U \in U(\mathfrak{m})$ then obviously $B_a(H,XU,Y) = B_a(H,X,UY)$, so that B_a induces the linear map

$$\Gamma_a : U(\mathfrak{a}_{pq}) \otimes U(\mathfrak{k}) \otimes_{U(\mathfrak{m})} U(\mathfrak{h}) \longrightarrow U(\mathfrak{g})$$

determined by $\Gamma_a(H \otimes X \otimes Y) = X^a H Y$ for $a \in A$, $H \in \mathfrak{a}_{pq}$, $X \in U(\mathfrak{k})$, $Y \in U(\mathfrak{h})$. Let $\mathcal{A}$ denote $U(\mathfrak{a}_{pq}) \otimes U(\mathfrak{k}) \otimes_{U(\mathfrak{m})} U(\mathfrak{h})$, viewed as a linear space.

<u>Lemma 4.2.</u> If $a \in A'$ then the map $\Gamma_a : \mathcal{A} \rightarrow U(\mathfrak{g})$ is a linear isomorphism. For each $D \in U(\mathfrak{g})$ there exists a unique $\Pi(D) \in \mathcal{R} \otimes \mathcal{A}$ such that, for all $a \in A'$:

$$\Gamma_a(\Pi(D)) = D. \tag{24}$$

<u>Proof.</u> Since $\mathfrak{m} = \mathfrak{l}_{kh}$, we have $\mathfrak{h} = \mathfrak{h}^c \oplus \mathfrak{m}$, and the first assertion follows from the infinitesimal Cartan decomposition (see Proposition 1.5) and the Poincaré-Birkhoff-Witt theorem.

The uniqueness part of the last assertion will follow from (24) and the first assertion. Therefore it suffices to prove the existence part. We proceed by induction on the degree deg(D) of D. If deg(D) = 0 the assertion is trivial, so let $m > 0$ and assume that the assertion has been proved already for $\deg(D) < m$. Let $D \in U(\mathfrak{g})_m$ (the subalgebra of elements of degree $\leq m$). By the direct sum decomposition

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{l}_{kq} \oplus \mathfrak{a}_{pq} \oplus \mathfrak{h} \tag{25}$$

(cf. also (3)) and the Poincaré-Birkhoff-Witt theorem, there exists a $D_0 \in U(\mathfrak{l}_k)U(\mathfrak{a}_{pq})U(\mathfrak{h})$ such that

$$D - D_0 \quad \in \quad \mathfrak{n}\, U(\mathfrak{g})_{m-1}.$$

Since A centralizes $U(\mathfrak{l}_k)$, the assertion is true for D_0, so that we may restrict ourselves to the case $D_0 = 0$. Without loss of generality we may even assume that $D = X_\alpha V$, where $X_\alpha \in \mathfrak{g}_+^\alpha$ or $X_\alpha \in \mathfrak{g}_-^\alpha$ $(\alpha \in \Delta^+)$, and $V \in U(\mathfrak{g})_{m-1}$. By Proposition 4.1 there exist $f_1, f_2 \in \mathcal{R}^+$ such that

$$X_\alpha = f_1(a)\ (X_\alpha + \theta X_\alpha)^a + f_2(a)\ (X_\alpha + \tau X_\alpha),$$

for all $a \in A'$. Hence

$$D = f_1(a)\ (X_\alpha + \theta X_\alpha)^a V + f_2(a) \left\{ V(X_\alpha + \tau X_\alpha) + \tilde{V} \right\},$$

where $\tilde{V} = [X_\alpha + \tau X_\alpha, V] \in U(\mathfrak{g})_{m-1}$, so that the assertion follows if we apply the induction hypothesis to V and $\tilde{V}$.

In a natural way $\mathcal{R} \otimes \mathfrak{a}$ may be viewed as a M-module, the multiplication being given by

$$m\ (f \otimes H \otimes X \otimes_{U(\mathfrak{m})} Y) = f \otimes H \otimes Ad(m)X \otimes_{U(\mathfrak{m})} Ad(m)Y,$$

if $m \in M$, $f \in \mathcal{R}$, $H \in U(\mathfrak{a}_{pq})$, $X \in U(\mathfrak{k})$, $Y \in U(\mathfrak{h})$. Viewing $U(\mathfrak{g})$ as a M-module for the adjoint action, we now have the analogue of [3, Proposition 2.5]. We omit the proof, which is essentially the same.

<u>Proposition 4.3.</u> The linear map $\Pi : U(\mathfrak{g}) \longrightarrow \mathcal{R} \otimes \mathfrak{a}$ is a M-module homomorphism.

The filtration by degree on $U(\mathfrak{a}_{pq})$ naturally induces a filtration on $\mathcal{R} \otimes \mathfrak{a}$, which we call the $\mathfrak{a}_{pq}$-filtration. The corresponding degree is called the $\mathfrak{a}_{pq}$-degree.

Proposition 4.4. If $X \in \mathfrak{n}U(\mathfrak{g})_m$ $(m \in \mathbb{N})$, then $\Pi(X) \in \mathcal{R}^+ \otimes \mathcal{A}$ and $\Pi(X)$ has $\mathfrak{a}_{pq}$-degree $\leq m$.

Proof. This is easily verified in the course of the proof of Lemma 4.2.

By the definition (13) of E^M, it follows that for $Z \in U(\mathfrak{m})$ and $u \in E^M$ we have

$$\mu_1(Z)\, u = u\, \mu_2(Z) = \mu_2(Z^\vee)\, u .$$

Hence, if in addition $X \in U(\mathfrak{k})$ and $Y \in U(\mathfrak{h})$, then

$$\mu_1(X\, Z)\, \mu_2(Y^\vee) u = \mu_1(X)\, \mu_2((Z\, Y)^\vee) u .$$

Therefore the bilinear map $U(\mathfrak{k}) \times U(\mathfrak{h}) \to \mathrm{Hom}_{\mathbb{C}}(E^M, E)$ given by $(X,Y) \mapsto \mu_1(X)\, \mu_2(Y^\vee)$ naturally induces a linear map $\xi_\mu : U(\mathfrak{k}) \otimes_{U(\mathfrak{m})} U(\mathfrak{h}) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(E^M, E)$, determined by

$$\xi_\mu(X \otimes Y) = \mu_1(X)\, \mu_2(Y^\vee),$$

for $X \in U(\mathfrak{k})$, $Y \in U(\mathfrak{h})$. We now define the linear map $\Pi_\mu : U(\mathfrak{g}) \longrightarrow \mathcal{R} \otimes U(\mathfrak{a}_{pq}) \otimes \mathrm{Hom}_{\mathbb{C}}(E^M, E)$ by

$$\Pi_\mu = (1 \otimes 1 \otimes \xi_\mu) \circ \Pi .$$

The elements of $\mathcal{R} \otimes U(\mathfrak{a}_{pq}) \otimes \mathrm{Hom}_{\mathbb{C}}(E^M, E)$ may be viewed as differential operators on A', mapping $C^\infty(A', E^M)$ into $C^\infty(A', E)$, in the following way. If $f \in \mathcal{R}$, $H \in U(\mathfrak{a}_{pq})$, $T \in \mathrm{Hom}_{\mathbb{C}}(E^M, E)$, then for $F \in C^\infty(A', E^M)$ we have

$$(f \otimes H \otimes T)\, F = f\, R_H(\, T \circ F\,).$$

Thus, if $X \in U(\mathfrak{g})$, then $\Pi_\mu(X)$ may be viewed as a differential operator on A', called the μ-radial component of X. We now have the following analogue of [3 , Theorem 3.1] , the proof being essentially identical.

Lemma 4.5. If $F \in C^{\infty}_{\mu}(G)$ and $X \in U(\mathfrak{g})$, then

$$\mathrm{Res}(\, R_X F \,) \;=\; \pi_{\mu}(X)\, \mathrm{Res}(F).$$

We also have the following analogue of [3 , Proposition 3.2] , which is an immediate consequence of Proposition 4.3 and the definition of E^M.

Proposition 4.6. The map π_{μ} maps $U(\mathfrak{g})^M$ into $\mathcal{R} \otimes U(\mathfrak{a}_{pq}) \otimes \mathrm{End}_{\mathbb{C}}(E^M)$.

Let $\mathcal{D}$ denote $\mathcal{R} \otimes U(\mathfrak{a}_{pq}) \otimes \mathrm{End}_{\mathbb{C}}(E^M)$, viewed as a subalgebra of the algebra of differential operators mapping $C^{\infty}(A', E^M)$ into itself.

Lemma 4.7. The map $\pi_{\mu} : U(\mathfrak{g})^H \to \mathcal{D}$ is an algebra homomorphism.

Proof. If $X, Y \in U(\mathfrak{g})^H$, $F \in C^{\infty}_{\mu}(G)$, then $R_Y F \in C^{\infty}_{\mu}(G)$, so that by Lemma 4.5 we have

$$\mathrm{Res}(R_X R_Y F) \;=\; \pi_{\mu}(X)\mathrm{Res}(R_Y F) \;=\; \pi_{\mu}(X)\, \pi_{\mu}(Y)\mathrm{Res}(F).$$

Hence $\pi_{\mu}(X\,Y) = \pi_{\mu}(X)\pi_{\mu}(Y)$ on $\mathrm{Res}(C^{\infty}_{\mu}(G))$. Using Lemma 1.6, we may now complete the proof just as in [3, Theorem 3.3] .

Since L is of class $\mathcal{H}$, (cf. [18, p.286, Thm. 13]), its subgroup M centralizes $\mathfrak{Z}(\mathfrak{l})$, so that $\mathfrak{Z}(\mathfrak{l}) \subset U(\mathfrak{g})^M$.

By Proposition 4.6 it follows that π_μ maps $\mathfrak{Z}(\mathfrak{l})$ into $\mathcal{D}$. Moreover, we have the following.

<u>Proposition 4.8.</u> The map $\pi_\mu : \mathfrak{Z}(\mathfrak{l}) \to \mathcal{D}$ is an algebra homomorphism.

<u>Proof.</u> Let $X,Y \in \mathfrak{Z}(\mathfrak{l})$. Then by (1) and the Poincaré-Birkhoff-Witt theorem X can be written as a sum $\sum_i U_iH_iV_i$, and Y as a sum $\sum_j \tilde{U}_j\tilde{H}_j\tilde{V}_j$, where $U_i,\tilde{U}_j \in U(\mathfrak{l}_k)$, $H_i,\tilde{H}_j \in U(\mathfrak{a}_{pq})$ and $V_i,\tilde{V}_j \in U(\mathfrak{l}_h)$. Hence

$$\pi_\mu(X)\pi_\mu(Y) = \sum_{i,j} (1\otimes H_i \otimes \mu_1(U_i)\mu_2(V_i^\vee))(1\otimes \tilde{H}_j \otimes \mu_1(\tilde{U}_j)\mu_2(\tilde{V}_j^\vee))$$

$$= \sum_{i,j} 1\otimes H_i\tilde{H}_j \otimes \mu_1(U_i\tilde{U}_j)\,\mu_2(V_i^\vee\tilde{V}_j^\vee)$$

$$= \sum_{i,j} 1\otimes H_i\tilde{H}_j \otimes \mu_1(U_i\tilde{U}_j)\,\mu_2(\{\tilde{V}_jV_i\}^\vee) \quad . \tag{26}$$

On the other hand, since $Y \in \mathfrak{Z}(\mathfrak{l})$, we have $XY = \sum_i U_iH_iV_iY = \sum_i U_iH_iYV_i = \sum_{i,j} U_iH_i\tilde{U}_j\tilde{H}_j\tilde{V}_jV_i = \sum_{i,j} (U_i\tilde{U}_j)(H_i\tilde{H}_j)(\tilde{V}_jV_i)$, from which we infer that $\pi_\mu(X\,Y)$ equals (26). Hence the proposition.

<u>Proposition 4.9.</u> If $X \in {}^*U(\mathfrak{g})_m$ $(m \in \mathbb{N})$, then $\pi_\mu(X)$ lies in $\mathcal{R}^+ \otimes U(\mathfrak{a}_{pq}) \otimes \mathrm{Hom}_{\mathbb{C}}(E^M,E)$ and its degree as a differential operator is $\leq m$.

<u>Proof.</u> The degree of the differential operator $\pi_\mu(X)$ is less than or equal to the $\mathfrak{a}_{pq}$-degree of $\pi(X)$. Hence the assertion is an immediate consequence of Proposition 4.4.

Corollary 4.10. If $Z \in \mathfrak{Z}$, then $\pi_\mu(Z) - \pi_\mu(\sigma(Z))$ lies in $\mathcal{R}^+ \otimes U(\mathfrak{a}_{pq}) \otimes \mathrm{End}_{\mathbb{C}}(E^M)$.

Proof. This follows immediately from Propositions 4.7-9 and definition (22) of σ.

Theorem 4.11. Let $F \in A_\mu(G,I)$, $t \in \mathbb{C}^\Lambda$ a Δ^+-leading exponent of F, and F_t the corresponding leading term. Then for $Z \in I$, we have:

$$\pi_\mu(\sigma(Z))\, F_t = 0.$$

For a proof the reader is referred to the proof of the analogous [3 , Theorem 5.2], or to the end of the next section, where we will prove an analogous theorem for μ-spherical functions with $\mu_2 = 1$, annihilated by a cofinite ideal of the algebra $\mathbb{D}(G/H)$ (Theorem 5.7).

Proof of Theorem 3.4. The proof is essentially identical to the proof of [3 , Proposition 5.4]. Proposition 4.7 and Theorem 4.11 have to be used instead of [3, Proposition 3.6 and Theorem 5.2] . See also the proof of Theorem 5.8 at the end of Section 5.

5. Leading characters of right H-invariant spherical functions.

The matrix coefficient $m_{\varphi,u}$ defined by (11) gives rise to a (μ_1, μ_2)-spherical function F with $\mu_2 = 1$, annihilated by a cofinite ideal of $\mathfrak{Z}$. Therefore the situation with $\mu_2 = 1$ is of special interest to us. In this section we study this case in more detail; we put $\mu = \mu_1$ and write $C^\infty_\mu(G/H)$ instead of $C^\infty_{(\mu,1)}(G)$.

Proposition 5.1. Let $F: G/H \to E$ be a smooth function with values in a finite dimensional complex linear space E. Then the following statements are equivalent.

(i) F is $\mathfrak{Z}$-finite,

(ii) F is $\mathbb{D}(G/H)$-finite.

Proof. (ii) $\Rightarrow$ (i) is trivial, whereas (i) $\Rightarrow$ (ii) follows immediately from Corollary 2.8.

Let $A_\mu(G/H)$ denote the space of $\mathfrak{Z}$-finite functions in $C^\infty_\mu(G/H)$. Then by the above every element of $A_\mu(G/H)$ is annihilated by a cofinite ideal in $\mathbb{D}(G/H)$. If J is a cofinite ideal in $\mathbb{D}(G/H)$, we let $A_\mu(G/H,J)$ denote the space of $F \in A_\mu(G/H)$ annihilated by J. The purpose of this section is to relate the leading terms of an element F of $A_\mu(G/H,J)$ to the ideal J (see Theorem 5.8).

By Lemma 4.7 the map $\Pi_\mu: U(\mathfrak{g})^H \to \mathcal{D}$ is an algebra homomorphism. Moreover, since $\mu_2 = 1$, one easily checks that $\Pi_\mu = 0$ on $U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h}$, so that Π_μ induces an algebra homomorphism $\mathbb{D}(G/H) \to \mathcal{D}$ which we denote by $\overline{\Pi}_\mu$.

By Proposition 4.6 the map Π_μ maps $U(\mathfrak{l})^{L\cap H}$ into $\mathcal{D}$.

<u>Lemma 5.2.</u> Let $\mu_2 = 1$. Then the map $\pi_\mu : U(\mathfrak{l})^{L\cap H} \rightarrow \mathcal{D}$ is an algebra homomorphism.

<u>Proof.</u> One easily checks that $\pi_\mu = 0$ on $U(\mathfrak{l})\mathfrak{l}_h$. Moreover, $\mathfrak{l} = \mathfrak{l}_{kq} \oplus \mathfrak{a}_{pq} \oplus \mathfrak{l}_h$, so that by the Poincaré-Birkhoff-Witt theorem:

$$U(\mathfrak{l}) = U(\mathfrak{l}_k)U(\mathfrak{a}_{pq}) + U(\mathfrak{l})\mathfrak{l}_h.$$

Now fix $X,Y \in U(\mathfrak{l})^{L\cap H}$, and set $X = X_0 + X_1$, $Y = Y_0 + Y_1$, where $X_0,Y_0 \in U(\mathfrak{l}_k)U(\mathfrak{a}_{pq})$ and $X_1,Y_1 \in U(\mathfrak{l})\mathfrak{l}_h$. Then $\pi_\mu(X) = \pi_\mu(X_0)$ and $\pi_\mu(Y) = \pi_\mu(Y_0)$. Moreover, since Y commutes with $\mathfrak{l}_h$, it follows that $XY = X_0Y \mod U(\mathfrak{l})\mathfrak{l}_h$, so that $\pi_\mu(XY) = \pi_\mu(X_0Y_0)$.

If $U \in U(\mathfrak{l}_k)$, $H \in U(\mathfrak{a}_{pq})$, then $\pi_\mu(HU) = \pi_\mu(UH) = 1 \otimes H \otimes \mu_1(U)$. If we express both X_0 and Y_0 as sums of such terms, it follows easily that $\pi_\mu(X_0Y_0) = \pi_\mu(X_0)\pi_\mu(Y_0)$ as differential operators operating on $C^\infty(A',E)$. Since these differential operators leave $C^\infty(A',E^M)$ invariant, it follows that $\pi_\mu(XY) = \pi_\mu(X)\pi_\mu(Y)$ as elements of $\mathcal{D}$.

By the above, the map $\pi_\mu : U(\mathfrak{l})^{L\cap H} \rightarrow \mathcal{D}$ naturally induces an algebra homomorphism $\mathbb{D}(L/L\cap H) \rightarrow \mathcal{D}$, which we denote by $\overline{\pi}_\mu$.

By (25), (2) and the Poincaré-Birkhoff-Witt theorem we have a decomposition

$$U(\mathfrak{g}) = (\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{h}) + U(\mathfrak{l}),$$

where

$$(\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{h}) \cap U(\mathfrak{l}) = U(\mathfrak{l})\mathfrak{l}_h.$$

Consequently, if $X \in U(\mathfrak{g})$, then there exists a $X_0 \in U(\mathfrak{l})$ with

$X - X_0 \in \mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{h}$. Its canonical image $\bar{X}_0$ in $U(\mathfrak{l})/U(\mathfrak{l})\mathfrak{l}_h$ is uniquely determined by X. Let ω_0 denote the map $X \to \bar{X}_0$, $U(\mathfrak{g})^H \to U(\mathfrak{l})/U(\mathfrak{l})\mathfrak{l}_h$.

__Proposition 5.3.__ The map ω_0 maps $U(\mathfrak{g})^H$ homomorphically into $U(\mathfrak{l})^{L\cap H}/(U(\mathfrak{l})^{L\cap H} \cap U(\mathfrak{l})\mathfrak{l}_h)$.

__Proof.__ Since Ad(L∩H) leaves $\mathfrak{l}_h$ invariant, the adjoint action induces an action of L ∩ H on the linear space $U(\mathfrak{l})/U(\mathfrak{l})\mathfrak{l}_h$. Since L ∩ H normalizes $\mathfrak{n}$ and $\mathfrak{h}$, ω_0 maps $U(\mathfrak{g})^H$ into $(U(\mathfrak{l})/U(\mathfrak{l})\mathfrak{l}_h)^{L\cap H}$. Because L is of class $\mathcal{H}$, Propositions 2.3 and 2.5 imply that we have a natural isomorphism

$$(U(\mathfrak{l})/U(\mathfrak{l})\mathfrak{l}_h)^{L\cap H} \simeq U(\mathfrak{l})^{L\cap H}/(U(\mathfrak{l})^{L\cap H} \cap U(\mathfrak{l})\mathfrak{l}_h),$$

so that ω_0 maps into the space on the right hand side.

To see that ω_0 is a homomorphism, let $X, Y \in U(\mathfrak{g})^H$ and put $X = X_0 + X_1$, $Y = Y_0 + Y_1$ with $X_0, Y_0 \in U(\mathfrak{l})$ and $X_1, Y_1 \in \mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{h}$. Since Y commutes with $\mathfrak{h}$ it follows that $X_1 Y \in \mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{h}$. Moreover, since $\mathfrak{l}$ normalizes $\mathfrak{n}$, we have $U(\mathfrak{l})\mathfrak{n} \subseteq \mathfrak{n}U(\mathfrak{l})$, so that $X_0 Y_1 \in \mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{h}$. Consequently $XY - X_0Y_0 \in \mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{h}$, so that $\omega_0(XY) = \omega_0(X_0Y_0) = \omega_0(X_0)\,\omega_0(Y_0) = \omega_0(X)\,\omega_0(Y)$.

Since obviously ω_0 maps the ideal $U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h}$ into zero, we have an induced algebra homomorphism $\omega : \mathbb{D}(G/H) \to \mathbb{D}(L/L\cap H)$.

<u>Proposition 5.4.</u> The following diagram commutes:

$$\begin{array}{ccc} \mathfrak{Z} & \xrightarrow{\ \sigma\ } & \mathfrak{Z}(\mathfrak{l}) \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R} \\ \mathbb{D}(G/H) & \xrightarrow{\ \omega\ } & \mathbb{D}(L/L\cap H) \end{array} \qquad (27)$$

<u>Proof.</u> This follows immediately from the definitions of the maps involved.

<u>Corollary 5.5.</u> The algebra $\mathbb{D}(L/L\cap H)$ is finitely generated as a $\omega(\mathbb{D}(G/H))$ -module.

<u>Proof.</u> By [2 , Lemma 3.7] $\mathfrak{Z}(\mathfrak{l})$ is a finite $\sigma(\mathfrak{Z})$-module. Hence $R(\mathfrak{Z}(\mathfrak{l}))$ is a finite $\omega(R(\mathfrak{Z}))$-module. In addition, by Corollary 2.8 $\mathbb{D}(L/L\cap H)$ is a finite $R(\mathfrak{Z}(\mathfrak{l}))$-module, and therefore $\mathbb{D}(L/L\cap H)$ is a finite $\omega(\mathbb{D}(G/H))$-module.

In view of Proposition 5.4, the following may be seen as a stronger version of Corollary 4.10 for the case $\mu_2 = 1$.

<u>Proposition 5.6.</u> Let $\mu_2 = 1$, $D \in \mathbb{D}(G/H)$. Then

$$\widetilde{\Pi}_\mu(D) - \widetilde{\Pi}_\mu(\omega(D)) \quad \in \quad \mathcal{R}^+ \otimes U(\mathfrak{a}_{pq}) \otimes \mathrm{End}_{\mathbb{C}}(E^M).$$

<u>Proof.</u> Let $X \in U(\mathfrak{g})^H$ and put $X = X_0 + X_1 + X_2$, where $X_0 \in U(\mathfrak{l})$, $X_1 \in \mathfrak{n}\,U(\mathfrak{g})$, and $X_2 \in U(\mathfrak{g})\mathfrak{h}$. Then $\Pi_\mu(R_X) - \Pi_\mu(\omega(R_X)) = \Pi_\mu(X_1) + \Pi_\mu(X_2)$. Now $\mu_2 = 1$ implies $\Pi_\mu(X_2) = 0$, and so application of Proposition 4.9 completes the proof.

<u>Theorem 5.7.</u> Let $\mu_2 = 1$, J a cofinite ideal in $\mathbb{D}(G/H)$, and $F \in A_\mu(G/H,J)$. If $t \in \mathbb{C}^\Lambda$ is a Δ^+-leading exponent of F and F_t the corresponding leading term, then

$$\overline{\pi}_\mu(\omega(D))\, F_t = 0,$$

for every $D \in J$.

<u>Proof.</u> If $\gamma \in \Lambda$, we let $e_\gamma \in \mathbb{N}^\Lambda$ denote the element whose γ-th coordinate equals one, wheras the other coordinates equal zero. One easily checks that for $\ell \in \mathbb{C}^\Lambda$, $m \in \mathbb{N}^\Lambda$:

$$H_\gamma\, \underline{\lambda}^\ell \log^m \underline{\lambda} = \ell_\gamma\, \underline{\lambda}^\ell \log^m \underline{\lambda} + m_\gamma\, \underline{\lambda}^\ell \log^{m-e_\gamma} \underline{\lambda}. \tag{28}$$

Every $f \in \mathcal{R}^+$ has a power series expansion $\sum_k c_k \underline{\lambda}^k$, where $c_k \in \mathbb{C}$ and where the summation is taken over $k \in \mathbb{N}^\Sigma \setminus \{0\}$. Hence by Proposition 5.6 it follows that for $X \in U(\mathfrak{g})^H$ we have

$$\overline{\pi}_\mu(\omega(R_X))\, F_t \equiv \overline{\pi}_\mu(R_X)\, F_t$$

modulo terms involving $\underline{\lambda}^{t+k} \log^m \underline{\lambda}$, where $k \in \mathbb{N}^\Sigma \setminus \{0\}$, $m \in \mathbb{N}^\Lambda$. Also, since t is a leading exponent, (28) implies that

$$\overline{\pi}_\mu(R_X)\, F_t \equiv \overline{\pi}_\mu(R_X)\, F,$$

hence

$$\overline{\pi}_\mu(\omega(R_X))\, F_t \equiv \overline{\pi}_\mu(R_X)\, F, \tag{29}$$

modulo terms involving $\underline{\lambda}^{t+k} \log^m \underline{\lambda}$, where $k \in \mathbb{N}^\Sigma \setminus \{0\}$, $m \in \mathbb{N}^\Lambda$. Now $\omega(R_X)$ is the image of some $W \in U(\mathfrak{l})^{L \cap H}$ in $\mathbb{D}(L/L\cap H)$, and so $\overline{\pi}_\mu(\omega(R_X)) = \pi_\mu(W)$. But clearly $\pi_\mu(W)$ belongs to $1 \otimes U(\mathfrak{a}_{pq}) \otimes \mathrm{End}_{\mathbb{C}}(E^M)$, so that by (28) $\pi_\mu(W) F_t$ is a finite sum of terms involving $\underline{\lambda}^t \log^m \underline{\lambda}$, $m \in \mathbb{N}^\Lambda$. From this and (29) it follows that $\overline{\pi}_\mu(R_X) F = 0$ implies

$\overline{\Pi}_\mu(\omega(R_X))F_t = 0$, whence the theorem.

If J is a cofinite ideal in $\mathbb{D}(G/H)$, then by Corollary 5.5 $\mathbb{D}(L/L\cap H)\,\omega(J)$ is a cofinite ideal in $\mathbb{D}(L/L\cap H)$. Since obviously $\mathfrak{a}_{pq} \subset U(\mathfrak{l})^{L\cap H}$, the finite dimensional space $W = \mathbb{D}(L/L\cap H)/\mathbb{D}(L/L\cap H)\omega(J)$ is a $\mathfrak{a}_{pq}$-module under left multiplication. Thus, by exponentiation W becomes a A-module. A character $\chi : A \to \mathbb{C}^*$ is said to lie Δ^+-shifted over the ideal J, if it is a generalized A-weight for W. The set of such characters is denoted by $X(\Delta^+, J)$.

Theorem 5.8. Let $\mu_2 = 1$, and let F be a μ-spherical function $G/H \to E$ annihilated by a cofinite ideal J of $\mathbb{D}(G/H)$. Then all Δ^+-leading characters of F lie Δ^+-shifted over J.

Proof. Let $t \in \mathbb{C}^\Lambda$ be a Δ^+-leading exponent of F and let J(t) be the set of $D \in \mathbb{D}(L/L\cap H)$ such that $\overline{\Pi}_\mu(D)F_t = 0$. By Lemma 5.2 J(t) is an ideal in $\mathbb{D}(L/L\cap H)$, and by Theorem 5.7 $\mathbb{D}(L/L\cap H)/J(t)$ is a quotient of the finite dimensional A-module $\mathbb{D}(L/L\cap H)/\mathbb{D}(L/L\cap H)\omega(J)$. Hence every generalized A-weight of $\mathbb{D}(L/L\cap H)/J(t)$ lies Δ^+-shifted over J.

Let $d\underline{\lambda}^t$ be the differential of the character $\underline{\lambda}^t : A \to \mathbb{C}^*$. If $H \in \mathfrak{a}_{pq}$, then

$$d\underline{\lambda}^t(H) = \sum_{\gamma\in\Lambda} t_\gamma\, \gamma(H).$$

By (28) the differential operators $H - d\underline{\lambda}^t(H)$, $H \in \mathfrak{a}_{pq}$, act nilpotently on the linear space generated by the functions $\underline{\lambda}^t \log^m \underline{\lambda}$ $(m \in \mathbb{N}^\Lambda)$. The leading term F_t being contained in this space, it follows that for sufficiently large $n \in \mathbb{N}$,

$$(H - d\underline{\lambda}^t(H))^n \quad \in \quad J(t).$$

Therefore $d\underline{\lambda}^t$ is a generalized $\mathfrak{n}_{pq}$-weight for $\mathbb{D}(L/L\cap H)/J(t)$, so that $\underline{\lambda}^t$ lies Δ^+-shifted over J.

6. Relations between the P-shifted characters.

Let I be a cofinite ideal in $\mathfrak{Z}$. If $P \in \mathfrak{P}$, we let X(P,I) denote the set of A-characters lying P-shifted over I (for the definition see the remark preceding Theorem 3.4). In this section we discuss the relations between the sets X(P,I), for different $P \in \mathfrak{P}$.

Let σ_P be the homomorphism $\mathfrak{Z} \longrightarrow \mathfrak{Z}(\mathfrak{l})$ defined as in (22), with Δ^+ replaced by P. Thus, writing

$$\mathfrak{n}(P) = \sum_{\alpha \in P} \mathfrak{g}^{\alpha}$$

we have

$$Z - \sigma_P(Z) \in \mathfrak{n}(P)U(\mathfrak{g})$$

for $Z \in \mathfrak{Z}$. Let T_P be the automorphism of $U(\mathfrak{l})$ determined by

$$T_P(X) = X + \tfrac{1}{2} \operatorname{tr}(\operatorname{ad}(X) \mid \mathfrak{n}(P)), \quad X \in \mathfrak{l}.$$

Being an automorphism, T_P leaves $\mathfrak{Z}(\mathfrak{l})$ invariant and maps the ideal $\bar{I}_P = \mathfrak{Z}(\mathfrak{l})\sigma_P(I)$ of $\mathfrak{Z}(\mathfrak{l})$ onto the ideal $\bar{I} = \mathfrak{Z}(\mathfrak{l})T_P\sigma_P(I)$. Now the map

$$\mu = T_P \circ \sigma_P$$

is Harish-Chandra's isomorphism of $\mathfrak{Z}$ into $\mathfrak{Z}(\mathfrak{l})$, hence independent of P (cf. [18, p. 228] , see also [2, Section 3]). Therefore the ideal $\bar{I}$ is independent of the choice of $P \in \mathfrak{P}$. We denote the set of generalized A-weights of $\mathfrak{Z}(\mathfrak{l})/\bar{I}$ by X(I).

Define the element ρ_P of $\mathfrak{a}^*_{pq}$ by

$$\rho_P(X) = \tfrac{1}{2}\,\mathrm{tr}(\,\mathrm{ad}(X)\,|\,\mathfrak{n}(P)\,),$$

and let e^{ρ_P} denote the positive character of A given by

$$a \mapsto a^{\rho_P} = \exp(\,\rho_P \log a\,).$$

Lemma 6.1. Let I be a cofinite ideal in $\mathfrak{Z}$, $P \in \mathcal{P}$. Then the set X(P,I) of characters lying P-shifted over I is given by

$$X(P,I) = e^{\rho_P}\cdot X(I).$$

Proof. If $H \in \mathfrak{a}_{pq}$, then it easily follows from the definition of T_P that for $Z \in U(\mathfrak{l})$ we have

$$T_P(H\,Z) = (H + \rho_P(H))\;T_P(Z).$$

Hence ν is a generalized $\mathfrak{a}_{pq}$-weight of $\mathfrak{Z}(\mathfrak{l})/\bar{I}_P$ iff $\nu - \rho_P$ is a generalized $\mathfrak{a}_{pq}$-weight of $\mathfrak{Z}(\mathfrak{l})/\bar{I}$. The assertion now follows by exponentiation.

The commutativity of the diagram (27) suggests that we have analogous relations between the sets X(P,J) of characters lying P-shifted over an ideal J of finite codimension in $\mathbb{D}(G/H)$. We will sketch these relations below.

Let ω_{OP} be the homomorphism of $U(\mathfrak{g})^{\mathfrak{h}}$ into

$$\mathbb{D}_L = U(\mathfrak{l})^{\mathfrak{l}\cap\mathfrak{h}}/(U(\mathfrak{l})^{\mathfrak{l}\cap\mathfrak{h}} \cap U(\mathfrak{l})\,\mathfrak{l}_h),$$

defined as in Section 5, with Δ^+ replaced by P, $\mathfrak{n}$ replaced by $\mathfrak{n}(P)$. Then ω_{OP} induces a homomorphism ω_P of $\mathbb{D} = U(\mathfrak{g})^{\mathfrak{h}}/(U(\mathfrak{g})^{\mathfrak{h}} \cap U(\mathfrak{g})\mathfrak{h})$ into $\mathbb{D}_L$.

Proposition 6.2. The automorphism T_P of $U(\mathfrak{l})$ leaves the subspaces $U(\mathfrak{l})^{\mathfrak{l}\cap\mathfrak{h}}$, $U(\mathfrak{l})^{L\cap H}$ and $U(\mathfrak{l})(\mathfrak{l}\cap\mathfrak{h})$ invariant.

Proof. For simplicity of notation we put $\mathfrak{n} = \mathfrak{n}(P)$. Since L normalizes $\mathfrak{n}$, T_P commutes with the adjoint action of L on $U(\mathfrak{l})$. This implies the invariance of the first two spaces.

For the last assertion it suffices to show that $\mathrm{ad}(\mathfrak{l}\cap\mathfrak{h})$ acts with trace zero on $\mathfrak{n}$. This will follow if we can show that $\mathrm{ad}(\mathfrak{l}_c\cap\mathfrak{h}_c)$ acts with (complex) trace zero on $\mathfrak{n}_c$. Now recall the definition (6) of $\mathfrak{g}^d$ and $\mathfrak{k}^d$, and let $\mathfrak{z}$ be the centralizer of $\mathfrak{a}_{pq}$ in $\mathfrak{k}^d$. Then $\mathfrak{z}_c = \mathfrak{l}_c\cap\mathfrak{h}_c$. Also, $\mathfrak{n}_c = (\mathfrak{n}_c\cap\mathfrak{g}^d)_c$. The subgroup Z of the complex adjoint group G_c of $\mathfrak{g}_c$ generated by $\exp(\mathrm{ad}\,\mathfrak{z})$ normalizes $\mathfrak{n}_c\cap\mathfrak{g}^d$. Since it is connected and compact it acts with (real) determinant 1 on $\mathfrak{n}_c\cap\mathfrak{g}^d$. By differentiation this implies that $\mathrm{ad}(\mathfrak{z})$ and hence $\mathrm{ad}(\mathfrak{z}_c) = \mathrm{ad}(\mathfrak{l}_c\cap\mathfrak{h}_c)$ acts with complex trace zero on $\mathfrak{n}_c$.

By the above proposition, T_P naturally induces an automorphism $\bar{T}_P$ of the algebra $\mathbb{D}_L$. Let $\mathfrak{a}_q$ be as in (10), and let γ denote the canonical isomorphism of $\mathbb{D}$ onto $I(\mathfrak{a}_q)$. Moreover, let $I_L(\mathfrak{a}_q)$ be the subspace of $W(\mathfrak{l}_c, \mathfrak{a}_{q,c})$-invariants in $S(\mathfrak{a}_q)$, and let γ_L denote the canonical isomorphism of $\mathbb{D}_L$ onto $I_L(\mathfrak{a}_q)$. Then $I(\mathfrak{a}_q) \subset I_L(\mathfrak{a}_q)$, and by Lemma 2.6 we have natural isomorphisms $\mathbb{D}(G/H) \simeq \mathbb{D}$ and $\mathbb{D}(L/L\cap H) \simeq \mathbb{D}_L$. Put

$$\mu = \bar{T}_P \circ \omega_P.$$

Then from the definitions of ω_P, $\bar{T}_P$, γ and γ_L it follows that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{D}(G/H) & \xrightarrow{\ \mu\ } & \mathbb{D}(L/L\cap H) \\ \Big\downarrow \gamma & & \Big\downarrow \gamma_L \\ I(\mathfrak{a}_q) & \longrightarrow & I_L(\mathfrak{a}_q) \end{array}$$

In particular, μ is independent of the choice of P.

If J is an ideal of finite codimension in $\mathbb{D}(G/H)$, then $\mathbb{D}(L/L\cap H)/\mathbb{D}(L/L\cap H)\mu(J)$ is a finite dimensional $\mathfrak{a}_{pq}$-module under left multiplication. By exponentiation it becomes a A-module; the set of generalized A-weights is denoted by X(J). Proceeding now exactly as in the first part of this section, we obtain

<u>Lemma 6.3.</u> Let J be a cofinite ideal in $\mathbb{D}(G/H)$, $P \in \mathcal{P}$. Then the set X(P,J) of characters lying P-shifted over J is given by

$$X(P,J) \quad = \quad e^{\rho_P} \cdot X(J).$$

7. Extensions of $F^{\Sigma}_{s,m}$ over walls of $A^-(\Delta^+)$.

Let F be a $\mathfrak{Z}$-finite μ-spherical function $G \to E$. Here we do not require that μ_2 be trivial. Although in general the power series expansions for the functions $F^{\Sigma}_{s,m}$ of Theorem 3.5 will fail to converge on the boundary of D^{Σ}, so that the series expansion (21) for F will not converge for regions larger than $A^-(\Delta^+)$, the functions $F^{\Sigma}_{s,m} \circ \underline{\alpha}$ do extend as real analytic functions over walls corresponding to roots $\gamma \in \Delta \backslash \Delta_+$. More precisely, we have:

<u>Lemma 7.1.</u> Let F be a $\mathfrak{Z}$-finite μ-spherical function on G. Then with the notations of Theorem 3.5, there exists an open neighbourhood Ω of $\underline{\alpha}(A^-) \cup D^{\Sigma}$ in $\mathbb{C}^{\Sigma}$ such that the functions $F^{\Sigma}_{s,m}$ extend holomorphically to Ω.

Moreover, we have

$$F = \sum_{s,m} (F_{s,m} \circ \underline{\alpha}) \, \underline{\lambda}^s \log^m \underline{\lambda}$$

on the whole of A^-.

<u>Proof.</u> By the proof of Theorem 3.5 (see also [3, Theorem 5.6]), it suffices to prove the existence of a neighbourhood Ω such that the functions $F_{s,m}$ of Lemma 3.3 extend holomorphically to $\Omega \times \mathbb{C}^{\Lambda \backslash \Sigma}$.

If $\varepsilon > 0$, we put

$$S(\varepsilon) = \{ z \in \mathbb{C}; \quad |\mathrm{Im}\, z| < \varepsilon \,\mathrm{Re}\, z \}$$

$$V(\varepsilon) = \{ z \in S(\varepsilon)^{\Sigma}; \quad |z^{\beta}| < 1 \text{ for } \beta \in \Delta^+_+ \}.$$

Then clearly $V(\varepsilon)$ is a simply connected open neighbourhood of $\underline{\alpha}(A^-)$ in $\mathbb{C}^{\Sigma}$. Fixing ε sufficiently small we obviously have

$$V(\varepsilon) \times \mathbb{C}^{\Lambda \setminus \Sigma} \cap Y = \phi \qquad (30)$$

(recall the definition (19) of Y). We claim that $\Omega = D^{\Sigma} \cup V(\varepsilon)$ satisfies our requirements. Put $D^* = D \cap \mathbb{C}^*$, $\Omega^* = \Omega \cap (\mathbb{C}^*)^{\Sigma}$. Then clearly

$$\Omega^* = (D^*)^{\Sigma} \cup V(\varepsilon).$$

Moreover, since $V(\varepsilon)$ and $(D^*)^{\Sigma} \cap V(\varepsilon) = (D \cap S(\varepsilon))^{\Sigma}$ are simply connected, it follows by the Van Kampen theorem (cf. [17]), that the homomorphism of homotopy groups

$$\pi_1((D^*)^{\Lambda}) \longrightarrow \pi_1(\Omega^* \times (\mathbb{C}^*)^{\Lambda \setminus \Sigma})$$

induced by the inclusion is an isomorphism.

By (30), we have $\Omega^* \times (\mathbb{C}^*)^{\Lambda \setminus \Sigma} \cap Y = \phi$, so that the map $\Phi : \underline{\lambda}(A^-(\Delta^+)) \longrightarrow (E^M)^{sr}$ defined by (16) extends to a multivalued holomorphic solution of (20) on $\Omega^* \times (\mathbb{C}^*)^{\Lambda \setminus \Sigma}$ (cf. [3 , Theorem A.1.2]). Hence Φ_{11} extends to a multivalued holomorphic map $\Omega^* \times (\mathbb{C}^*)^{\Lambda \setminus \Sigma} \longrightarrow E^M$ satisfying

$$\Phi_{11} = \sum_{s,m} F_{s,m} \, z^s \log^m z$$

on $(D^*)^{\Lambda}$ (for the definition of the multivalued holomorphic maps z^s and $\log^m z$, the reader is referred to the appendix). The $F_{s,m}$ are holomorphic on D^{Λ}, and by the above Lemma A.1 is applicable. Therefore the $F_{s,m}$ extend holomorphically to $\Omega \times \mathbb{C}^{\Lambda \setminus \Sigma}$.

To prove the other main result of this section, we need a few technical propositions. If $\Psi \subset \Sigma$, we define the wall

$$A^-_\Psi = \{a \in A;\quad a^\alpha = 1 \ (\alpha \in \Psi),\quad a^\alpha < 1 \quad (\alpha \in \Sigma \setminus \Psi)\}.$$

Obviously $A^-_\Psi \subset cl(A^-(\Delta^+)) \subset cl(A^-)$.

<u>Proposition 7.2.</u> Let Ψ be a subset of the set Σ of simple roots for Δ^+. Then the following two conditions are equivalent.

(i) $A^-_\Psi \subset A^-$,

(ii) $\mathbb{N}\Psi \cap \Delta_+ = \phi$.

<u>Remark.</u> In particular, if $\alpha \in \Sigma$ and $\alpha, 2\alpha \notin \Delta_+$, then the wall $A^-_{\{\alpha\}}$ of codimension one is contained in A^-.

<u>Proof of Proposition 7.2.</u> Suppose (i). If $a \in A^-_\Psi$, then $a^\alpha = 1$ for $\alpha \in \mathbb{N}\Psi$. On the other hand $a \in A^-$, so that $a^\gamma < 1$ for $\gamma \in \Delta^+_+$. Hence (ii). Conversely, suppose (ii) and let $a \in A^-_\Psi$. If $\gamma \in \Delta^+_+$, then $\gamma = \sum_{\alpha\in\Sigma} k_\alpha \alpha$ for certain $k_\alpha \in \mathbb{N}$ $(\alpha \in \Sigma)$, so that

$$a^\gamma = \prod_{\alpha \in \Sigma\setminus\Psi} (a^\alpha)^{k_\alpha}.$$

Also, because of the assumption, $k_\alpha \neq 0$ for some $\alpha \in \Sigma\setminus\Psi$. But $a^\alpha < 1$ for $\alpha \in \Sigma\setminus\Psi$, so that $a^\gamma < 1$. Hence (i).

If $\delta, \varepsilon > 0$, we put

$$R(1+\delta, \varepsilon) = \{z \in \mathbb{C};\quad |\mathrm{Re}\, z| < 1+\delta,\quad |\mathrm{Im}\, z| < \varepsilon\}, \tag{31}$$

$$D(\varepsilon) = \{z \in \mathbb{C};\quad |z| < \varepsilon\}.$$

View $\mathbb{C}^{\Sigma}$ as a subspace of $\mathbb{C}^{\Lambda}$ and let $Y_{\Sigma} = Y \cap \mathbb{C}^{\Sigma}$. Then obviously $Y = Y_{\Sigma} \times \mathbb{C}^{\Lambda\setminus\Sigma}$.

<u>Proposition 7.3.</u> Let Ψ be a subset of Σ such that $A^-_{\Psi} \subset A^-$. Then:

(i) if $0 < \varepsilon < 1$, there exists a $\delta > 0$ such that

$$R(1+\delta,\delta)^{\Psi} \times D(\varepsilon)^{\Sigma\setminus\Psi} \cap Y_{\Sigma} = \phi \;, \qquad (32)$$

(ii) if $\delta > 0$, there exists a $\varepsilon > 0$ such that

$$R(1+\delta,\varepsilon)^{\Psi} \times D(\varepsilon)^{\Sigma\setminus\Psi} \cap Y_{\Sigma} = \phi \;. \qquad (33)$$

<u>Proof.</u> If $\gamma \in \Delta^+$, then $\gamma = \sum_{\alpha\in\Sigma} k_{\alpha}\alpha$ for certain $k_{\alpha} \in \mathbb{N}$ $(\alpha \in \Sigma)$, and hence for $z \in \mathbb{C}^{\Sigma}$ we have

$$z^{\gamma} = \prod_{\alpha\in\Psi} (z_{\alpha})^{k_{\alpha}} \prod_{\alpha\in\Sigma\setminus\Psi} (z_{\alpha})^{k_{\alpha}} \qquad (34)$$

If $\gamma \in \Delta^+_+$, then by Proposition 7.2 $k_{\alpha} > 0$ for at least one $\alpha \in \Sigma\setminus\Psi$. So, if $0 < \varepsilon < 1$, then from (18) and (34) it follows that

$$[-1, 1]^{\Psi} \times \text{cl}(D(\varepsilon)^{\Sigma\setminus\Psi}) \qquad (35)$$

does not intersect Y^{γ}_{+}. Since obviously (35) does not intersect Y^{β}_{-} for $\beta \in \Delta^+$, $\mathfrak{g}^{\beta}_{-} \neq 0$, it follows that (35) and Y_{Σ} have empty intersection. Therefore (32) holds if δ is sufficiently small.

To see that (ii) holds, fix $\delta > 0$. If $\gamma \in \Delta^+_+$, then as in the first part of the proof we infer that Y^{γ}_{+} does not intersect

$$[-1-\delta\;,\; 1+\delta]^{\Psi} \times \{0\}^{\Sigma\setminus\Psi} . \qquad (36)$$

Also, if $\beta \in \Delta^+$, $\mathfrak{g}^{\beta}_{-} \neq 0$, it is obvious that (36) does not intersect Y^{β}_{-}. Hence (36) does not intersect Y_{Σ}, so that (33) holds for ε sufficiently small.

Lemma 7.4. Let F be a $\mathfrak{z}$-finite μ-spherical function on G, and let $F^{\Sigma}_{s,m}$ be as in Theorem 3.5. If Ψ is any subset of Σ with $A^-_{\Psi} \subset A^-$, then:

(i) there exists an open subset Ω of $\mathbb{C}^{\Psi} \times D^{\Sigma \setminus \Psi}$ containing $[-1, 1]^{\Psi} \times D^{\Sigma \setminus \Psi}$ such that the functions $F^{\Sigma}_{s,m}$ extend holomorphically to Ω,

(ii) if $\delta > 0$, then there exists a $0 < \varepsilon < 1$ such that each function $F^{\Sigma}_{s,m}$ extends holomorphically to $R(1+\delta,\varepsilon)^{\Psi} \times D(\varepsilon)^{\Sigma \setminus \Psi}$.

Proof. If $\delta, \varepsilon, \eta > 0$, put

$$X(\delta,\varepsilon,\eta) = R(1+\delta,\eta)^{\Psi} \times D(\varepsilon)^{\Sigma \setminus \Psi}.$$

Then by Proposition 7.3 it suffices to prove that the functions $F^{\Sigma}_{s,m}$ extend holomorphically to $X(\delta,\varepsilon,\eta)$ under the condition that $X(\delta,\varepsilon,\eta) \cap Y_{\Sigma} = \phi$. Therefore let us assume this condition to hold and put $X = X(\delta,\varepsilon,\eta) \times \mathbb{C}^{\Lambda \setminus \Sigma}$. Then $X \cap Y = \phi$ and hence the function Φ defined by (16) extends as a multivalued holomorphic function to $X^* = X \cap (\mathbb{C}^*)^{\Lambda}$. Therefore the same holds for $F \circ \underline{\lambda} = \Phi_{11}$, and applying Lemma A.1 we infer that the functions $F_{s,m}$ of Lemma 3.3 extend holomorphically to X. By the proof of Theorem 3.5 this implies that the functions $F^{\Sigma}_{s,m}$ extend to holomorphic functions $X(\delta,\varepsilon,\eta) \rightarrow E^M$.

8. Asymptotic behaviour along the walls.

In this section we study the asymptotic behaviour of a $\mathfrak{Z}$-finite μ-spherical function $F: G \to E$ along the walls of $A^-(\Delta^+)$, following the methods of [3].

Recall that Σ is the system of simple roots for Δ^+. To a subset $\Theta \subset \Sigma$ we associate the wall

$$A^-_\Theta = \{a \in A; \quad a^\alpha = 1 \ (\alpha \in \Theta), \quad a^\alpha < 1 \ (\alpha \in \Sigma \setminus \Theta)\} .$$

Thus $A^-_\phi = A^-(\Delta^+)$ and we have the disjoint union

$$\mathrm{cl}(\ A^-(\Delta^+)\ ) = \bigcup_{\Theta \subset \Sigma} A^-_\Theta .$$

Also, if $\Theta \subset \Sigma$, we write

$$A^-(\Theta) = \{ a \in A; \ a^\alpha \leq 1 \ (\alpha \in \Theta), \ a^\alpha < 1 \ (\alpha \in \Sigma \setminus \Theta)\} .$$

So $A^-(\Sigma) = \mathrm{cl}(\ A^-(\Delta^+)\ )$, and we have the disjoint union

$$A^-(\Theta) = \bigcup_{\Psi \subset \Theta} A^-_\Psi .$$

First we will formulate the analogue of [3, Theorem 6.2] for our situation, using the same grouping of terms procedure as in [3]. At the end of this section we will study the behaviour along a wall A^-_Ψ entirely contained in A^- in more detail. As we will see, for such a wall, no grouping of terms is needed to get expansions along it. This is basically a consequence of Lemma 7.4.

At present, let Θ be any subset of Σ. We use the notations of Theorem 3.5 freely. Following [3], we view $\mathbb{C}^{\Lambda \setminus \Theta}$ as embedded in $\mathbb{C}^\Lambda$, and let

$$\mathrm{pr}_{\Lambda \setminus \Theta} : \ \mathbb{C}^\Lambda \longrightarrow \mathbb{C}^{\Lambda \setminus \Theta}$$

denote the projection map. A notion of $(\Sigma\setminus\Theta)$-integral equivalence in $\mathbb{C}^{\Lambda\setminus\Theta}$ is defined by

$$s \sim_{\Sigma\setminus\Theta} t \qquad \text{iff} \qquad t - s \in \mathbb{Z}^{\Sigma\setminus\Theta}$$

and the $(\Sigma\setminus\Theta)$-order on $\mathbb{C}^{\Lambda\setminus\Theta}$ is defined by

$$s \leqslant_{\Sigma\setminus\Theta} t \qquad \text{iff} \qquad t - s \in \mathbb{N}^{\Sigma\setminus\Theta}.$$

The set $pr_{\Lambda\setminus\Theta}(S_\Sigma)$ splits into a finite number of $\sim_{\Sigma\setminus\Theta}$-equivalence classes. To each such a class Ω we associate the element $\sigma(\Omega)$ of $\mathbb{C}^{\Lambda\setminus\Theta}$ defined by

$$\sigma(\Omega)_\gamma = \min\{t_\gamma ; \quad t\in\Omega\} \qquad (\gamma\in\Lambda\setminus\Theta).$$

Obviously $\sigma(\Omega) \leqslant_{\Sigma\setminus\Theta} t$ for all $t\in\Omega$. Let $S_{\Sigma\setminus\Theta}$ be the set of all $\sigma(\Omega)$, Ω as above. Then the elements of $S_{\Sigma\setminus\Theta}$ are mutually $(\Sigma\setminus\Theta)$-integrally inequivalent.

If $\lambda\in\Lambda$, we view $\log z_\lambda$ as a multivalued holomorphic function on $(\mathbb{C}^*)^\Lambda$ (see also the appendix). Moreover, for $m\in\mathbb{N}^\Lambda$, $s\in\mathbb{C}^\Lambda$ we define

$$\log^m z = \prod_{\lambda\in\Lambda} (\log z_\lambda)^{m_\lambda} ,$$

$$z^s = \quad = \prod_{\lambda\in\Lambda} \exp(s_\lambda \log z_\lambda).$$

For $s\in S_{\Sigma\setminus\Theta}$, $m\in\mathbb{N}^{\Lambda\setminus\Theta}$, we define

$$F^{\Sigma\setminus\Theta}_{s,m} = \sum_{t,n} F^{\Sigma}_{t,m+n} \, z^{t-s} \log^n z,$$

the sum being taken over $n\in\mathbb{N}^\Theta$ and over all $t\in S_\Sigma$ with $pr_{\Lambda\setminus\Theta}(t)$ $(\Sigma\setminus\Theta)$-integrally equivalent to s. Obviously $t - s \in \mathbb{C}^\Theta \times \mathbb{N}^{\Sigma\setminus\Theta}$, so that $F^{\Sigma\setminus\Theta}_{s,m}$ is well defined on $(0,1)^\Theta \times D^{\Sigma\setminus\Theta}$ and extends holomorphically to any simply connected open subset of $(D^*)^{\Theta} \times D^{\Sigma\setminus\Theta}$ containing $(0,1)^{\Theta} \times D^{\Sigma\setminus\Theta}$. By the above and Theorem 3.5 (ii) it is now straightforward to check the following.

<u>Proposition 8.1.</u> There exist a finite set $S_{\Sigma\setminus\Theta}$ of mutually $(\Sigma\setminus\Theta)$-integrally inequivalent elements of $\mathbb{C}^{\Lambda\setminus\Theta}$ and for each $s \in S_{\Sigma\setminus\Theta}$ a finite set $F^{\Sigma\setminus\Theta}_{s,m}$ $(m \in \mathbb{N}^{\Lambda\setminus\Theta})$ of non-trivial holomorphic functions defined on a neighbourhood of $(0,1)^{\Theta} \times D^{\Sigma\setminus\Theta}$ in $(D^*)^{\Theta} \times D^{\Sigma\setminus\Theta}$, such that the following conditions are fulfilled.

(i) If $s \in S_{\Sigma\setminus\Theta}$, $\gamma \in \Sigma\setminus\Theta$, then there exists a $m \in \mathbb{N}^{\Lambda\setminus\Theta}$ such that $F^{\Sigma\setminus\Theta}_{s,m}$ does not vanish identically on the coordinate hyperplane $z_\gamma = 0$.

(ii) On $A^-(\Delta^+)$ we have:

$$F = \sum_{s,m} (F^{\Sigma\setminus\Theta}_{s,m} \circ \underline{\alpha})\, \underline{\lambda}^s \log^m \underline{\lambda}.$$

We also have the following analogues of [3, Lemma 6.1, Theorem 6.2]. We omit the proofs, since they are essentially the same.

<u>Lemma 8.2.</u> There exists an open subset $C(\Theta)$ of $(\mathbb{C}^*)^{\Theta} \times D^{\Sigma\setminus\Theta}$ containing $(0,1]^{\Theta} \times D^{\Sigma\setminus\Theta}$, such that the functions $F^{\Sigma\setminus\Theta}_{s,m}$ extend to holomorphic functions $C(\Theta) \longrightarrow E^M$.

<u>Theorem 8.3.</u> Let $F: G \longrightarrow E$ be a $\mathfrak{Z}$-finite μ-spherical function. Then for any set $\Theta \subset \Sigma$, we have

$$F = \sum_{s,m} (F^{\Sigma\setminus\Theta}_{s,m} \circ \underline{\alpha})\, \underline{\lambda}^s \log^m \underline{\lambda}$$

on $A^-(\Theta)$. Here the summation extends over $s \in S_{\Sigma\setminus\Theta}$ and finitely many $m \in \mathbb{N}^{\Lambda\setminus\Theta}$.

Because of Lemma 8.2, each function $F^{\Sigma\setminus\Theta}_{s,m}$ has a converging power series expansion on $(0,1]^{\Theta} \times D^{\Sigma\setminus\Theta}$ in the second variable:

$$F^{\Sigma\setminus\Theta}_{s,m}\ (x,w) = \sum_{\ell} c^{\Sigma\setminus\Theta}_{s,m,\ell}(x)\ w^{\ell}\ , \tag{37}$$

for $x \in (0,1]^{\Theta}$, $w \in D^{\Sigma\setminus\Theta}$. Here the sum is taken over $\ell \in \mathbb{N}^{\Sigma\setminus\Theta}$, and the $c^{\Sigma\setminus\Theta}_{s,m,\ell}$ are real analytic functions $(0,1]^{\Theta} \longrightarrow E^{M}$.
More precisely, we have the following result. Recall that for $\varepsilon > 0$, $D(\varepsilon) = \{ z \in \mathbb{C};\ \ |z| < \varepsilon \}$.

<u>Lemma 8.4.</u> The functions $c^{\Sigma\setminus\Theta}_{s,m,\ell}$ extend to real analytic functions $(0,\infty)^{\Theta} \longrightarrow E^{M}$. Given any $\delta > 0$ there exists a $0 < \varepsilon < 1$ such that the series (37) converges absolutely on $(0,1+\delta)^{\Theta} \times D(\varepsilon)^{\Sigma\setminus\Theta}$, locally uniformly in all variables.

<u>Proof.</u> Let $\delta > 0$, $0 < \eta < 1$, and put

$$X_2 = X_2(\delta,\eta) = \{ z \in \mathbb{C}^{\Theta};\ \ 0 < \mathrm{Re}\ z_{\alpha} < 1+\delta,\ \ |\mathrm{Im}\ z_{\alpha}| < \eta\ \ (\alpha \in \Theta) \}.$$

If $0 < \varepsilon < 1$, we define

$$X_1 = X_1(\varepsilon) = D(\varepsilon)^{\Sigma\setminus\Theta} \times \mathbb{C}^{\Lambda\setminus\Sigma}\ .$$

We claim that for ε sufficiently small the set

$$X = X(\delta,\eta,\varepsilon) = X_1 \times X_2$$

satisfies the following condition:

$$\text{if } \gamma \in \Delta^{+},\qquad Y^{\gamma}_{\pm} \cap X \neq \emptyset \qquad \text{then} \quad \gamma \in \mathbb{N}\Theta\ . \tag{38}$$

In fact, let $\gamma \in \Delta^{+}$. Then $\gamma = \sum_{\alpha\in\Sigma} k_{\alpha}\alpha$, with $k_{\alpha} \in \mathbb{N}$, and so

$$z^{\gamma} = \prod_{\alpha\in\Theta} (z_{\alpha})^{k_{\alpha}} \prod_{\alpha\in\Sigma\setminus\Theta} (z_{\alpha})^{k_{\alpha}}\ ,$$

for $z \in \mathbb{C}^{\Lambda}$. It is obvious from this expression that for ε sufficiently small, $z \in X$ and $|z^{\gamma}| = 1$ imply that $k_{\alpha} = 0$ for $\alpha \in \Sigma \setminus \Theta$, so that $\gamma \in \mathbb{N}\Theta$. Hence (38).

Because X fulfills (38), the proof of [3 , Theorem 6.2] applies to the present situation and we may conclude that the functions $F^{\Sigma \setminus \Theta}_{s,m}$ extend holomorphically to X. The assertions of the lemma are now obvious.

We now turn our attention to the case of a wall A^{-}_{Ψ} entirely contained in A^{-}.

Given $t \in \mathbb{C}^{\Lambda}$, we write

$$t = (t',t''), \qquad t' \in \mathbb{C}^{\Psi}, \quad t'' \in \mathbb{C}^{\Lambda \setminus \Psi}.$$

If $s \in S_{\Sigma}$, $m \in \mathbb{N}^{\Lambda}$, we define

$$\tilde{F}^{\Sigma \setminus \Psi}_{s,m}(z) = F^{\Sigma}_{s,m}(z)\, z^{s'} \log^{m'} z ,$$

for $z \in (0,1)^{\Psi} \times D^{\Sigma \setminus \Psi}$. Obviously, the following conditions are fulfilled:

(i) the functions $\tilde{F}^{\Sigma \setminus \Psi}_{s,m}$ extend holomorphically to any simply connected open neighbourhood of $(0,1)^{\Psi} \times D^{\Sigma \setminus \Psi}$ in $(D^{*})^{\Psi} \times D^{\Sigma \setminus \Psi}$;

(ii) for every $s \in S_{\Sigma}$, $\gamma \in \Sigma \setminus \Psi$, there exists a $m \in \mathbb{N}^{\Lambda}$ such that the function $\tilde{F}^{\Sigma \setminus \Psi}_{s,m}$ does not vanish identically on the coordinate hyperplane $z_{\gamma} = 0$;

(iii) on $A^{-}(\Delta^{+})$ we have

$$F = \sum_{s,m} (\tilde{F}^{\Sigma \setminus \Psi}_{s,m} \circ \underline{\alpha})\, \underline{\lambda}^{s''} \log^{m''} \underline{\lambda} .$$

Having the notation (31) in mind, we put

$$R_{+}(1+\delta,\varepsilon) = \{ z \in \mathbb{C};\ 0 < \operatorname{Re} z < 1+\delta,\ |\operatorname{Im} z| < \varepsilon \}$$

for $\delta, \varepsilon > 0$. With this notation we have the following analogue of Lemmas 8.2 and 8.4.

<u>Lemma 8.5.</u> There exists an open subset $\tilde{C}(\Psi)$ of $(\mathbb{C}^*)^{\Psi} \times D^{\Sigma\setminus\Psi}$ containing $(0,1]^{\Psi} \times D^{\Sigma\setminus\Psi}$ to which each $\tilde{F}_{s,m}^{\Sigma\setminus\Psi}$ extends holomorphically.

Also, for every $\delta > 0$ there exists a $0 < \varepsilon < 1$ such that each function $\tilde{F}_{s,m}^{\Sigma\setminus\Psi}$ extends holomorphically to the set $R_+(1+\delta, \varepsilon)^{\Psi} \times D(\varepsilon)^{\Sigma\setminus\Psi}$.

<u>Proof.</u> Both assertions follow straightforwardly from Lemma 7.4.

The following result is now obvious.

<u>Theorem 8.6.</u> Let F be a $\mathfrak{z}$-finite μ-spherical function on G. Then for any subset Ψ of Σ with $A_{\Psi}^{-} \subset A^{-}$, we have

$$F = \sum_{s,m} (\tilde{F}_{s,m}^{\Sigma\setminus\Psi} \circ \underline{\alpha}) \, \underline{\lambda}^{s''} \log^{m''} \underline{\lambda}$$

on $A^{-}(\Psi)$. Here the summation is taken over $s \in S_{\Sigma}$ and finitely many $m \in \mathbb{N}^{\Lambda}$.

<u>Remark.</u> Because of Lemma 8.5 we have the obvious analogue of Lemma 8.4 for the power series expansion of $\tilde{F}_{s,m}^{\Sigma\setminus\Psi}(x,w)$ in the variable w (here $x \in (0,1]^{\Psi}$, $w \in D^{\Sigma\setminus\Psi}$).

9. Leading characters and global estimates.

Using the results of the preceding sections we are now able to describe the connections between leading characters and the global growth of $\mathfrak{Z}$-finite μ-spherical functions on G/H. Our results will be analogous to those of [3]. In fact they can be considered as more general, since every group of class $\mathcal{H}$ can be viewed as a symmetric space of class $\mathcal{H}$ (see also the introduction).

From now on, we will restrict ourselves to right H-invariant μ-spherical functions. Here μ is a smooth representation of K in a finite dimensional complex linear space E. We equip E with an inner product such that μ is unitary, and let $\|.\|$ denote the corresponding norm. If $F \in A_\mu(G/H)$, then

$$\| F(kah) \| = \| F(a) \| ,$$

for $h \in H$, $k \in K$, $a \in A$. Thus by the Cartan decomposition (Corollary 1.4), we see that $\| F \|$ can be estimated once its behaviour on $cl(A^-)$ is known. As we saw in the preceding sections, we cannot associate leading characters to F on the whole of A^-. However, for each $P \in \mathcal{P}$ we defined a finite set of P-leading characters, connected with the asymptotic behaviour of F on $A^-(P)$. As we will see, these govern the behaviour of F on the closed Weyl chamber $cl(A^-(P))$.
In view of the union

$$cl(A^-) = \bigcup_{P \in \mathcal{P}} cl(A^-(P)),$$

this enables us to connect global estimates for F with estimates of the P-leading characters for every $P \in \mathcal{P}$.

We start with some notations. If $P \in \mathcal{P}$, we define the ordering $\leqslant_P$ on positive characters of A by

$$\chi_1 \leqslant_P \chi_2 \quad \text{iff} \quad \chi_1(a) \leqslant \chi_2(a) \text{ for all } a \in A^-(P).$$

Put:

$$A_\Delta = \{a \in A;\ a^\alpha = 1 \text{ for all } \alpha \in \Delta\}.$$

With the notations of Section 1, we have that

$$G/H \simeq A_\Delta \times {}^oG/({}^oG \cap H). \tag{39}$$

Also, $A_\Delta \subset \mathrm{cl}(A^-(P))$, so that $\chi_1 \leqslant_P \chi_2$ implies that $\chi_1 = \chi_2$ on A_Δ. We put

$$\chi_1 <_P \chi_2 \quad \text{iff} \quad \chi_1(a) < \chi_2(a) \text{ for all } a \in \mathrm{cl}(A^-(P)) \setminus A_\Delta.$$

<u>Theorem 9.1.</u> Let F be a $\mathfrak{Z}$-finite μ-spherical function on G/H, let $P \in \mathcal{P}$, and let ω be a positive character of A. Then the following conditions are equivalent.

(i) for every P-leading character ν of F, we have

$$|\nu| \leqslant_P \omega ;$$

(ii) there exist $M \geqslant 0$ and $m \geqslant 0$ such that

$$\| F(a) \| \leqslant M\, \omega(a)\, (1 + \|\log a\|)^m$$

for all $a \in \mathrm{cl}(A^-(P))$.

<u>Proof.</u> We may restrict ourselves to $P = \Delta^+$ and use the notations and results of Sections 3,4,8. Then $\omega = \underline{\lambda}^t$ for some $t \in \mathbb{R}^\Lambda$, and the condition (i) is equivalent to

$$\mathrm{Re}\ s_\alpha \geqslant t_\alpha \qquad (\alpha \in \Sigma),$$
$$\mathrm{Re}\ s_\lambda = t_\lambda \qquad (\lambda \in \Lambda \setminus \Sigma),$$

for all Δ^+-leading exponents s of F. By the characterization of the set S_Σ in Remark 3.6, (i) can be reformulated as

$$\mathrm{Re}(\ S_\Sigma\ ) \subset t + \mathbb{R}_+^\Sigma .$$

Here $\mathbb{R}_+ = [0,\infty)$. Thus, if (i) holds, then for $\Theta\subset\Sigma$ we have

$$\mathrm{Re}(\ S_{\Sigma\setminus\Theta}\ ) \subset \mathrm{pr}_{\Lambda\setminus\Theta}(t) + \mathbb{R}_+^{\Sigma\setminus\Theta} .$$

Now fix $0<\varepsilon<1$, and put

$$A_\varepsilon^-(\Theta) = \{a\in A;\ \ \varepsilon\leqslant a^\alpha\leqslant 1\ \ (\alpha\in\Theta),\ \ a^\alpha<\varepsilon\ \ (\alpha\in\Sigma\setminus\Theta)\}, \tag{40}$$

for $\Theta\subset\Sigma$. If $a\in \mathrm{cl}(A^-(\Delta^+))$, then $0<a^\alpha\leqslant 1$ for all $\alpha\in\Sigma$. Thus we see that $a\in A_\varepsilon^-(\Theta_a)$, with $\Theta_a = \{\alpha\in\Sigma\ ;\ \varepsilon\leqslant a^\alpha\leqslant 1\}$. Hence we have the disjoint union

$$\mathrm{cl}(A^-(\Delta^+)) = \bigcup_{\Theta\subset\Sigma} A_\varepsilon^-(\Theta) .$$

If $\Theta\subset\Sigma$, then by Theorem 8.3 we have

$$F = \sum (F_{s,m}^{\Sigma\setminus\Theta}\circ\underline{\alpha})\ \underline{\lambda}^s \log^m\underline{\lambda}$$

on $A_\varepsilon^-(\Theta)$, and the functions $F_{s,m}^{\Sigma\setminus\Theta}\circ\underline{\alpha}$ are bounded on $A_\varepsilon^-(\Theta)$. It follows that there exist M_Θ and $m_\Theta\geqslant 0$ such that

$$\| F(a)\| \leqslant M_\Theta\ \omega(a)\ (1 + \|\log a\|\ )^{m_\Theta}$$

for all $a\in A_\varepsilon^-(\Theta)$. Hence (ii).

Of course the above proof of the implication (i)$\Rightarrow$(ii) is essentially the same as the proof of the analogous implication in [3 , Theorem 7.1] . We leave it to the reader to check that the same holds for the implication (ii)$\Rightarrow$(i).

A character ζ of A_Δ is called the A_Δ-character of the μ-spherical function F: G/H $\rightarrow$ E if

$$F(ax) = \zeta(a)\, F(x) \qquad (x \in G,\ a \in A_\Delta).$$

From the uniqueness statement in Theorem 3.5 we immediately obtain:

<u>Proposition 9.2.</u> Let F be a $\mathfrak{Z}$-finite μ-spherical function on G/H with the A_Δ-character ζ. Then the expansion of F in $A^-(\Delta^+)$ has the form

$$F = \sum (F^{\Sigma}_{s,m} \circ \underline{\alpha})\, \underline{\lambda}^s \log^m \underline{\lambda} ,$$

where the restrictions of $\underline{\lambda}^s$, $s \in S_\Sigma$, to A_Δ are equal to ζ, and where $m \in \mathbb{N}^\Lambda$.

We now come to results concerning L^p-integrability. We could set up the theory for μ-spherical functions with a unitary A_Δ-character (see also [3]). But because of the decomposition (39) and the above proposition, we can as well assume that

$$G = {}^{\circ}G.$$

So let this be assumed from now on.

Given $P \in \mathcal{P}$, we define the positive character δ_P of A by

$$\delta_P(a) = \det(\, Ad(a) \mid \mathfrak{n}(P)\,) \qquad (a \in A).$$

Thus, writing $m(\alpha) = \dim \mathfrak{g}^\alpha$ for $\alpha \in \Delta$, we have

$$\delta_P(a) = \prod_{\alpha \in P} (a^\alpha)^{m(\alpha)} \qquad (a \in A).$$

A function f on A with values in a normed linear space is said to vanish at infinity in $A^-(P)$ if for every $\eta > 0$ there exists a $0 < \varepsilon < 1$ such that $\|f(a)\| < \eta$ for all $a \in A^-(P)$ with

$\delta_P(a) < \varepsilon$.

Theorem 9.3. Let F be a $\mathfrak{Z}$-finite μ-spherical function on G/H, let $P \in \mathcal{P}$ and let ω be a positive character of A. Then the following conditions are equivalent:

(i) for every P-leading character ν of F we have

$$|\nu| <_P \omega \qquad ;$$

(ii) the function $\omega^{-1}F$ vanishes at infinity in $A^-(P)$.

Proof. Without loss of generality, we may assume that $P = \Delta^+$, and use the notations and results of Sections 3,4,8. It is then easy to see how to transfer the proof of [3 , Theorem 7.4] to the present case, using δ_{Δ^+} instead of the function δ defined there.

Recalling Proposition 1.2, we define the function $\sigma = \sigma_{G/H}$ from G into $[0,\infty)$ by

$$\sigma(k \exp X \exp Y) = \|X\| = [-B(X,\theta X)]^{\frac{1}{2}}$$

for $k \in K$, $X \in \mathfrak{p}\cap\mathfrak{q}$, $Y \in \mathfrak{p}\cap\mathfrak{h}$. Then σ is left K- and right H-invariant, and

$$\sigma(k\,a\,h) = \|\log a\| ,$$

for $k \in K$, $a \in A$, $h \in H$ (see also [2]).

Theorem 9.4. Let F be a $\mathfrak{Z}$-finite μ-spherical function on G/H and let $1 \leq p < \infty$. Then the following conditions are equivalent:

(i) for each $P \in \mathcal{P}$ and every P-leading character ν of

F, we have

$$|\nu| <_P \delta_P^{1/p} \quad ; \tag{41}$$

(ii) for every $\ell \geqslant 0$ the function $(1 + \sigma)^{\ell} F$ is L^p-integrable;

(iii) F is L^p-integrable.

<u>Proof.</u> If $\alpha \in \Delta$, we let

$$m_+(\alpha) = \dim(\mathfrak{g}_+^{\alpha}), \qquad m_-(\alpha) = \dim(\mathfrak{g}_-^{\alpha}).$$

Thus $m(\alpha) = m_+(\alpha) + m_-(\alpha)$. Now let

$$D(a) = \prod_{\alpha \in \Delta^+} |a^{-\alpha} - a^{\alpha}|^{m_+(\alpha)} \; |a^{-\alpha} + a^{\alpha}|^{m_-(\alpha)}. \tag{42}$$

Then by [6, Theorem 2.6] we can fix normalizations of Haar measures dx on G/H and da on A, such that for $f \in L^1(G/H)$ we have

$$\int_{G/H} f(x)\, dx = \int_{K \times cl(A^-)} f(kaH)\, D(a)\, dk\, da.$$

Therefore $(1 + \sigma)^{\ell} F$ is L^p-integrable on G iff for each $P \in \mathcal{P}$ we have

$$\int_{cl(A^-(P))} (1 + \|\log a\|)^{\ell p} \|F(a)\|^p D(a)\, da < \infty \quad . \tag{43}$$

Consequently it suffices to prove for a fixed $P \in \mathcal{P}$ the equivalence of the following statements:

(i)' every P-leading character ν of F satisfies (41),

(ii)' the estimate (43) holds for all $\ell \geqslant 0$,

(iii)' the estimate (43) holds for $\ell = 0$.

Moreover, it is immediate that (42) remains valid if we replace Δ^+ by P, so that we may restrict ourselves to proving the equivalence of (i)'-(iii)' for $P = \Delta^+$.

Put $\delta = \delta_{\Delta^+}$, suppose (i)' and fix $\ell \geq 0$. In the notations of Section 3 we have $\Lambda = \Sigma$. We define the element $t \in \mathbb{R}^\Sigma$ by

$$t_\alpha = \min \left\{ \mathrm{Re}(s_\alpha); \quad s \text{ a } \Delta^+\text{-leading character} \right\} ,$$

and the positive character $\omega : A \to \mathbb{R}_+$ by $\omega = \underline{\lambda}^t$. From the remarks at the beginning of the proof of Theorem 9.1 it follows that for each Δ^+-leading character s we have

$$|s| \leqslant_{\Delta^+} \omega \qquad \text{and} \qquad \omega <_{\Delta^+} \delta^{1/p} .$$

One easily checks that there exists a $M_1 > 0$ such that for all $a \in \mathrm{cl}(A^-(\Delta^+))$ we have

$$1 + \|\log a\| \quad \leqslant \quad M_1 \ (1 + \ |\log \delta(a)| \ ) .$$

Therefore, applying Theorem 9.1, we infer the existence of $M_2 \geq 0$ and $m \geq 0$ such that for $a \in \mathrm{cl}(A^-(\Delta^+))$ we have

$$(1 + \|\log a\|)^\ell \|F(a)\| \leqslant M_2 \ \omega(a) \ (1 + |\log \delta(a)| \ )^m.$$

Also, there exists a constant $M_3 \geq 0$ such that

$$D(a) \leqslant M_3 \ \delta(a)^{-1}, \qquad (a \in \mathrm{cl}(A^-(\Delta^+)).$$

It follows that for some $M \geq 0$ we have

$$\int_{\mathrm{cl}(A^-(\Delta^+))} (1 + \|\log a\|)^{\ell p} \ \|F(a)\|^p \ D(a) \ da$$

$$\leqslant \quad M \int_{\mathrm{cl}(A^-(\Delta^+))} \omega(a)^p \ (1 + |\log \delta(a)| \ )^{mp} \delta(a)^{-1} da.$$

The latter integral is finite because $\omega^p \delta^{-1} <_{\Delta^+} 1$.

The implication (ii)'$\Rightarrow$(iii)' is obvious. For the remaining implication, suppose that (iii)' holds. Fix $0 < \varepsilon < 1$, and recall the definition (40) of $A_\varepsilon^-(\phi)$. There exists a constant $0 < C < 1$ such that

$$D(a) \geqslant C\, \delta(a)^{-1}, \qquad a \in A_\varepsilon^-(\phi).$$

Combined with the estimate in (iii)' this implies that

$$\int_{A_\varepsilon^-(\phi)} \|F(a)\|^p\, \delta(a)^{-1}\, da < \infty .$$

It is now straightforward to check that the proof of the implication (ii)$\Rightarrow$(i) in [3 , Theorem 7.5] applies here too, and gives us (i)'.

10. Schwartz functions on G/H.

In this section we assume that $G = {}^{o}G$, so that $A_{\Delta} = \{1\}$. Given $1 \leq p < \infty$ we define the space $\mathcal{C}^{p}(G/H)$ as the space of functions $f \in C^{\infty}(G/H)$ for which all the seminorms

$$N_{r,u}(f) = (1 + \sigma)^{r} \| L_{u} f \|_{L^{p}(G/H)}$$

($r \geq 0$, $u \in U(\mathfrak{g})$) are bounded. By the classical Sobolev inequalities the space $\mathcal{C}^{p}(G/H)$, equipped with the topology induced by the above seminorms, is a Fréchet space. We call $\mathcal{C}(G/H) = \mathcal{C}^{2}(G/H)$ the space of rapidly decreasing, or Schwartz functions on G/H. In the group case our definition coincides with Harish-Chandra's definition of Schwartz space (cf. [18, p.348]). We leave it to the reader to check that by slightly modified proofs, we have the following analogues of [2, Lemmas 1.1, 1.2].

<u>Lemma 10.1.</u> Let $1 \leq p < \infty$. Then $C_{c}^{\infty}(G/H)$ is dense in $\mathcal{C}^{p}(G/H)$.

<u>Lemma 10.2.</u> Let $1 \leq p < \infty$. Then the algebra $\mathbb{D}(G/H)$ maps $\mathcal{C}^{p}(G/H)$ continuously into itself.

The main result of this section is the following generalization of a well known result of Harish-Chandra (cf. [9, Lemma 43]).

<u>Theorem 10.3.</u> Let G be a group of class $\mathcal{H}$ with $G = {}^{o}G$, and let f be a $\mathfrak{Z}$-finite and K-finite function on G/H. Fix $1 \leq p < \infty$. Then f belongs to $L^{p}(G/H)$ if and only if it belongs to $\mathcal{C}^{p}(G/H)$.

For the proof of this theorem we need some results which are of interest of their own. Let $\mathcal{R}$ and $\mathcal{R}^+$ be as in Proposition 4.1. One easily verifies that the following result can be proved in the same fashion as Proposition 4.1. Recall that we use the notation (15).

Proposition 10.4. Let $X_\alpha \in \mathfrak{g}_+^\alpha$ or $X_\alpha \in \mathfrak{g}_-^\alpha$ $(\alpha \in \Delta^+)$. Then there exist $f_1, f_2 \in \mathcal{R}^+$ such that

$$\theta X_\alpha = f_1(a)(X_\alpha + \theta X_\alpha) + f_2(a)(X_\alpha + \tau X_\alpha)^{a^{-1}}.$$

Proposition 10.5. Let $D \in U(\mathfrak{g})$. Then there exist finitely many $f_i \in \mathcal{R}$, $X_i \in U(\mathfrak{k})$, $H_i \in U(\mathfrak{a}_{pq})$, $Y_i \in U(\mathfrak{h})$ $(1 \leqslant i \leqslant I)$, such that for all $a \in A^-$ we have

$$D = \sum_{1 \leqslant i \leqslant I} f_i(a)\, Y_i^{a^{-1}} H_i X_i .$$

The proof goes by induction on $\deg(D)$, in the same fashion as the proof of Lemma 4.2. Here one has to use the decomposition

$$\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{a}_{pq} \oplus \mathfrak{l}_{ph} \oplus \mathfrak{k}$$

instead of (25), and Proposition 10.4 instead of Proposition 4.1.

Lemma 10.6. Let F be a $\mathfrak{Z}$-finite μ-spherical function $G/H \to E$, and let $u \in U(\mathfrak{g})$. Then there exists a $\mathfrak{Z}$-finite spherical function $\tilde{F}$ on G/H with values in a finite dimensional vector space $\tilde{E}$ such that the following conditions are fulfilled.

(i) There exists a $\xi \in \mathrm{Hom}_{\mathbb{C}}(\tilde{E},E)$ such that $L_u F = \xi \circ \tilde{F}$.

(ii) For each $P \in \mathcal{P}$ and every P-leading exponent $\tilde{t}$ of $\tilde{F}$ there exists a P-leading exponent t of F with $\tilde{t} \in t + \mathbb{N}P$.

Remark. Observe that $\tilde{t} \in t + \mathbb{N}P$ implies $\underline{\lambda}^{\tilde{t}} \leqslant_P \underline{\lambda}^t$

Proof. Let U be the finite dimensional linear subspace of $U(\mathfrak{g})$ spanned by the elements $Ad(k)u$, $k \in K$. Let τ denote the adjoint representation of K restricted to U, and let τ^* be the contragredient representation of K in U^*. Fix a basis $\{u_j;\ 1 \leqslant j \leqslant J\}$ of U and let $\{u_j^*\}$ be the dual basis of U^*. Finally, put $\tilde{E} = U^* \otimes E$ and define $\tilde{F}: G/H \to \tilde{E}$ by:

$$\tilde{F}(x) = \sum_{1 \leqslant j \leqslant J} u_j^* \otimes L_{u_j} F(x). \tag{44}$$

Then the annihilator of F in $\mathfrak{Z}$ annihilates $\tilde{F}$ too, so that $\tilde{F}$ is $\mathfrak{Z}$-finite. Moreover, one easily checks that $\tilde{F}$ is $\tau^* \otimes \mu$-spherical. Since (i) is evident, it remains to prove (ii).

Without loss of generality we may assume that $P = \Delta^+$. By the results of Section 3, F has a unique series expansion

$$F = \sum_{s,m} c_{s,m} \underline{\lambda}^s \log^m \underline{\lambda} \tag{45}$$

which converges absolutely on $A^-(\Delta^+)$. Here $c_{s,m} \in E$, $s \in \mathbb{C}^\Sigma$, $m \in \mathbb{N}^\Sigma$ (recall that $\Lambda = \Sigma$). We call $s \in \mathbb{C}^\Sigma$ an exponent of F if $c_{s,m} \neq 0$ for some $m \in \mathbb{N}^\Sigma$, and denote the set of exponents by $\mathcal{E}(F)$. Being $\mathfrak{Z}$-finite and spherical, $\tilde{F}$ also has a unique absolutely converging series expansion

$$\tilde{F} = \sum_{j,s,m} (u_j^* \otimes d^j_{s,m}) \underline{\lambda}^s \log^m \underline{\lambda} \tag{46}$$

on $A^-(\Delta^+)$. Here $d^j_{s,m} \in E$, $s \in \mathbb{C}^\Sigma$, $m \in \mathbb{N}^\Sigma$, $1 \leqslant j \leqslant J$. Clearly

$$\mathcal{E}(\tilde{F}) = \bigcup_{1 \leqslant j \leqslant J} \mathcal{E}_j(\tilde{F}),$$

where $\mathcal{E}_j(\tilde{F})$ is the set of $s \in \mathbb{C}^\Sigma$ such that $d^j_{s,m} \neq 0$ for some $m \in \mathbb{N}^\Sigma$. From (44) and (46) it is immediate that

$$L_{u_j} F = \sum_{s,m} d^j_{s,m} \underline{\lambda}^s \log^m \underline{\lambda} \tag{47}$$

on $A^-(\Delta^+)$, for each $1 \leqslant j \leqslant J$. Now fix j. Then by Proposition 10.5 there exist $f_i \in \mathcal{R}$, $X_i \in U(\mathfrak{k})$, $H_i \in U(\mathfrak{a}_{pq})$ $(1 \leqslant i \leqslant I)$, such that

$$u_j - \sum_{1 \leqslant i \leqslant I} f_i(a)\, H_i X_i \in \mathfrak{h}^{a^{-1}} U(\mathfrak{g})$$

for all $a \in A^-(\Delta^+)$. Since F is right H-invariant it now follows that

$$L_{u_j} F(a) = \sum_i f_i(a)\, \mu(X_i^\vee)\, L(H_i) F(a)$$

for all $a \in A^-(\Delta^+)$. From the definition of $\mathcal{R}$ it easily follows that there exist holomorphic functions $\varphi_i : D^\Sigma \to \mathbb{C}$ such that $f = \varphi_i \circ \underline{\lambda}$ on $A^-(\Delta^+)$. Moreover, via $\underline{\lambda}$, the differential operators H_i correspond to polynomials in the differential operators $z_\alpha \partial / \partial z_\alpha$ $(\alpha \in \Sigma)$ on $\mathbb{C}^\Sigma$. Since the expansion (45) arises from power series expansions in (z_α), it now follows that we may find an absolutely converging series expansion for $L(u_j)F$ on $A^-(\Delta^+)$ by formally applying the expansion for the differential operator $\sum_i \varphi_i \circ \underline{\lambda}\, \mu(X_i^\vee)\, L(H_i)$ to the expansion (45) for F. By uniqueness this must give the expansion (47). Finally, taking the formula

$$z_\alpha \frac{\partial}{\partial z_\alpha} (z^s \log^m z) = s_\alpha z^s \log^m z + m_\alpha z^s \log^{m-e_\alpha} z$$

($\alpha \in \Sigma$) into account (see also (28)), we infer that $\mathcal{E}_j(\tilde{F}) \subset \mathcal{E}(F) + \mathbb{N}\Sigma$. Hence $\mathcal{E}(\tilde{F}) \subset \mathcal{E}(F) + \mathbb{N}\Delta^+$ and (ii) follows from the definition of leading exponent.

<u>Proof of Theorem 10.3.</u> Fix $r \in \mathbb{N}$, $u \in U(\mathfrak{g})$. We must show that

$$(1 + \sigma)^r L_u f \in L^p(G/H).$$

Let F be the μ-spherical function $\gamma(f): G/H \to E$ defined as in (12). Then obviously $F \in L^p(G/H, E)$. There exists a $\xi \in E^*$ such that $f = \xi \circ F$. Hence $L_u f = \xi \circ L_u F$ and it suffices to show that

$$(1 + \sigma)^r L_u F \in L^p(G/H, E).$$

Now select $\tilde{F}: G \to U^* \otimes E$ as in Proposition 10.6. Then for some $M > 0$ we have $\| L_u F(x) \| \leq M \| \tilde{F}(x) \|$ $(x \in G/H)$. Therefore it suffices to show that

$$(1 + \sigma)^r \tilde{F} \in L^p(G/H, E).$$

Now this follows immediately from Theorem 9.4 and Lemma 10.6.

Appendix.

In this appendix we prove a refinement of [3, Lemma A.1.7], which is crucial for the extension of the functions $F^{\Sigma}_{s,m}$ over certain walls (see Section 7).

Let k, n be fixed integers with $1 \leq k \leq n$, and let X be some connected open subset of $\mathbb{C}^n$, containing 0. Thus, for $\varepsilon > 0$ sufficiently small, X contains

$$X_\varepsilon = D(\varepsilon)^n,$$

where $D(\varepsilon) = \{ z \in \mathbb{C};\ |z| < \varepsilon \}$. Let $D^*(\varepsilon) = D(\varepsilon) \setminus \{0\}$, and put

$$X^* = X \cap [(\mathbb{C}^*)^k \times \mathbb{C}^{n-k}],$$

$$X^*_\varepsilon = D^*(\varepsilon)^k \times D(\varepsilon)^{n-k};$$

then X^*_ε and X^* are connected. Fix a point $x_0 \in (0,\varepsilon)^k \times D(\varepsilon)^{n-k}$. The inclusion $X^*_\varepsilon \subset X^*$ naturally induces a homomorphism between the homotopy groups

$$\pi_1(X^*_\varepsilon, x_0) \longrightarrow \pi_1(X^*, x_0). \tag{48}$$

Let $p: \widetilde{X}^* \to \widetilde{X}$ be a universal covering and fix $\widetilde{x}_0 \in \widetilde{X}^*$ such that $p(\widetilde{x}_0) = x_0$. Then (48) is an isomorphism if and only if the set $p^{-1}(X^*_\varepsilon)$ is simply connected. From now on we assume this to be the case, and so

$$\widetilde{X}^*_\varepsilon = p^{-1}(X^*_\varepsilon)$$

is a universal covering space of X^*_ε.

If Z is a connected complex analytic manifold and W a finite dimensional complex linear space, we let $\mathcal{O}(Z,W)$ denote the space of holomorphic functions $Z \to W$. Let $\pi : \widetilde{Z} \to Z$ be a universal covering of Z. Then elements of $\mathcal{O}(\widetilde{Z},W)$ will also be called multivalued W-valued holomorphic functions on Z. If T is a covering transformation of Z, we define the endomorphism T^* of $\mathcal{O}(\widetilde{Z},W)$ by

$$T^*f = f \circ T^{-1}.$$

This being said, let us return to the set X.

If $1 \leq j \leq k$, then the multivalued logarithmic function $\log z_j$ on X_ε^* is defined to be the holomorphic function $L_j : \widetilde{X}_\varepsilon^* \to \mathbb{C}$ with

$$e^{L_j(w)} = p(w)_j \qquad (w \in \widetilde{X}_\varepsilon^*),$$

$$L_j(\widetilde{x}_0) \in \mathbb{R}.$$

The functions $\log z_j$ extend holomorphically to the simply connected manifold $\widetilde{X}^*$. If $s \in \mathbb{C}^k$, $m \in \mathbb{N}^k$, we define the multivalued holomorphic functions z^s and $\log^m z$ on X^* by:

$$z^s = \prod_{1 \leq j \leq k} \exp(s_j \log z_j), \tag{49}$$

$$\log^m z = \prod_{1 \leq j \leq k} (\log z_j)^{m_j}. \tag{50}$$

After these introductory remarks we can formulate the result of this appendix. Two elements $s,t \in \mathbb{C}^k$ are called integrally equivalent if $t-s \in \mathbb{Z}^k$.

<u>Lemma A.1.</u> Let X be a connected open subset of $\mathbb{C}^n$, containing $X_\varepsilon = D(\varepsilon)^n$, and assume that the natural homomorphism from $\pi_1(X_\varepsilon^*)$ into $\pi_1(X^*)$ is an isomorphism.

Moreover, let U be a connected open subset of X intersecting $\{0\} \times \mathbb{C}^{n-k}$. Suppose that Φ is a multivalued holomorphic function on X^* with values in a finite dimensional complex linear space W, and assume that there exist

(i) a finite set S of integrally inequivalent elements of $\mathbb{C}^k$,

(ii) for each $s \in S$ a finite set of holomorphic functions $\Phi_{s,m}: U \to W$ $(m \in \mathbb{N}^k)$, such that

$$\Phi = \sum_{s,m} (\Phi_{s,m} \circ p)\, z^s \log^m z \qquad \text{on } p^{-1}(U).$$

Then:

(a) the functions $\Phi_{s,m}$ extend holomorphically to X,

(b) the above formula holds on the whole of X^*, and it determines the $\Phi_{s,m}$ uniquely.

Before giving the proof, we make some preliminary observations.

Let $x_0 \in X^*_\varepsilon$ be a base point as above, and let γ_j be the element of $\pi_1(X^*_\varepsilon, x_0)$ corresponding to the loop in the complex line $\{z \in \mathbb{C}^n;\ z_r = (x_0)_r \text{ if } r \neq j\}$ going once in the counter clockwise direction around $z_j = 0$ $(1 \leqslant j \leqslant k)$. Then

$$\pi_1(\tilde{X}^*_\varepsilon, x_0) \simeq \mathbb{Z}\gamma_1 \oplus \dots \oplus \mathbb{Z}\gamma_k.$$

We let $T_j: \tilde{X}^*_\varepsilon \to \tilde{X}^*_\varepsilon$ denote the covering transformation corresponding to γ_j. Because of the assumption on X, T_j extends holomorphically to a covering transformation of $\tilde{X}^*$. Now on $\tilde{X}^*_\varepsilon$ we have

$$T_j^*(\log z_r) = \log z_r + \delta_{jr}\, 2\pi i, \qquad (51)$$

for $1 \leqslant j, r \leqslant k$. By analytic continuation, the same holds on $\tilde{X}^*$.

In the following we shall use the multi-index notations

$$|m| = m_1 + \dots + m_k ,$$
$$m! = m_1! \dots m_k! ,$$
$$r \leqslant m \quad \text{iff} \quad m-r \in \mathbb{N}^k,$$
$$st = s_1t_1 + \dots + s_kt_k ,$$

for $m,r \in \mathbb{N}^k$, $s,t \in \mathbb{C}^k$. Also if $1 \leqslant j \leqslant k$, we put

$$e_j = (\delta_{1j}, \dots , \delta_{kj}).$$

From (51) and the definitions (49) and (50) we infer that on $\tilde{X}^*$ we have:

$$(T_j^* - e^{-2\pi i s_j})\, z^s \log^m z \equiv -2\pi i m_j e^{-2\pi i s_j}\, z^s \log^{m-e_j} z \qquad (52)$$

modulo terms involving $z^s \log^r z$, $r \leqslant m-2e_j$. Consequently

$$\prod_{1\leqslant j\leqslant k} (T_j^* - e^{-2\pi i s_j})^{m_j}\, z^s \log^r z = \delta_{rm}(-2\pi i)^{|m|} m!\, e^{-2\pi i m s}\, z^s \qquad (53)$$

if $r,m \in \mathbb{N}^k$, $r \leqslant m$.

<u>Proof of Lemma A.1.</u> Following the proof of [3, Lemma A.1.7] , we use induction on the cardinality $|S|$ of S, which we call the first level induction. To make the step from $|S| = p-1$ to $|S| = p$ we use induction on the number of elements of the set

$$M(S) = \{ (s,m) \in S \times \mathbb{N}^k ; \quad \Phi_{s,m} \neq 0 \} .$$

This second level induction starts from $|M(S)| = 0$ in which case the lemma is trivial. We thus avoid the argument in [3], where the case $|S| = 1$, $|M(S)| = 1$ is dealt with by a division by $z^s \log^m z$; in our case $\log^m z$ may have zeros on $\tilde{X}^*$.

Let $|S| = 1$, $|M(S)| = q \geqslant 1$ and suppose that the lemma has been proved already for $|S| = 1$, $0 \leqslant M(S) < q$. Put $S = \{s\}$, let

$$M(s) = \{m \in \mathbb{N}^k ;\ \Phi_{s,m} \neq 0\},$$

and fix an element m of $M(s)$ which is maximal for $\leqslant$. Then by (53) we have

$$\prod_{j=1}^{k} (T_j^* - e^{-2\pi i s_j})^{m_j} \Phi = (-2\pi i)^{|m|} m!\ e^{-2\pi i m s} (\Phi_{m,s} \circ p)\ z^s$$

on $p^{-1}(U)$. Hence $\Phi_{s,m}$ is uniquely determined on U, and $\Phi_{s,m} \circ p$ extends holomorphically to $\tilde{X}^*$. By the assumptions on X and U, the natural map $\pi_1(U \cap X^*) \longrightarrow \pi_1(X^*)$ is surjective, so that $p^{-1}(U)$ is connected. Therefore, $\Phi_{s,m}$ extends as a single valued holomorphic map from U to $U \cup X^*$. By [3, Lemma A.1.8], the map $\Phi_{s,m}$ extends holomorphically to X. Applying the second level induction hypothesis to

$$\Phi - (\Phi_{s,m} \circ p)\ z^s \log^m z$$

we obtain the assertion for $|S| = 1$, $|M(S)| = q$.

Next, let $|S| = p \geqslant 2$, $|M(S)| = q \geqslant 1$ and suppose the assertion to be proved for $|S| < p$, and for $|S| = p$, $|M(S)| < q$. Fix distinct elements $\sigma, \tau \in S$. Then $\sigma_j - \tau_j \notin \mathbb{Z}$, for some $1 \leqslant j \leqslant k$. By (51) there exists a $r \in \mathbb{N}$ such that

$$(T_j^* - e^{-2\pi i \tau_j})^r\ \Phi = \sum_{s \neq \tau,\ m} (\Psi_{s,m} \circ p)\ z^s \log^m z,$$

where $\Psi_{s,m}$ is a linear combination of the $\Phi_{s,n}$, $n \in \mathbb{N}^k$. We may assume that $M(\sigma) \neq \phi$ since otherwise we could reduce to the case $|S| = p-1$ at once. Let m_0 be maximal with respect to $\leqslant$ in $M(\sigma)$. Then from (52) it is immediate that

$$(e^{-2\pi i \sigma_j} - e^{-2\pi i \tau_j})^{-r} \Psi_{\sigma, m_0} = \Phi_{\sigma, m_0} .$$

By the first level induction hypothesis Ψ_{σ, m_0} is unique and extends holomorphically to X, and a forteriori the same holds for Φ_{σ, m_0}. Application of the second level induction hypothesis to the function

$$\Phi - (\Phi_{\sigma, m_0} \circ p) \, z^{\sigma} \log^{m_0} z$$

now completes the proof.

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