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SQUARES OF GEGENBAUER POLYNOMIALS AND MILIN TYPE INEQUALITIES

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De Branges, in his proof of the Bieberbach conjecture, was led to a specific solution with monotonicity properties of a certain linear system of differential equations. We present other solutions with similar monotonicity properties, the derivatives of their coordinates being multiples of squares of Gegenbauer polynomials. De Branges' solution is a nonnegative linear combination of our solutions. As a corollary we obtain Milin type inequalities for logarithmic power series coefficients of univalent analytic functions on the unit disk which are sharper than the Milin conjecture.

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1. Introduction

In Louis de Branges' recent proof (cf. [4], [6]) of the Bieberbach conjecture the following key observation is made.

Let $\kappa \in C([0,\infty))$, $|\kappa(t)| = 1$ for $t \geq 0$ and $\gamma_1,\gamma_2,... \in C^1([0,\infty))$ such that

$$\gamma_k'(t) = k\gamma_k(t) + 2\sum_{j=1}^{k-1} j\gamma_j(t)(\kappa(t))^{k-j} + (\kappa(t))^k, k = 1,2,.... \tag{1.1}$$

Fix $n \in \mathbb{N}$. Let $\tau_1,\tau_2, \ldots, \tau_{n+1} \in C^1([0,\infty))$ such that $\tau_{n+1} \equiv 0$ and

$$\tau_k + k^{-1}\tau_k' = \tau_{k+1} - (k+1)^{-1}\tau_{k+1}', \quad k=1,...,n. \tag{1.2}$$

Let

$$\phi(t) := \sum_{k=1}^{n} (k\,|\gamma_k(t)|^2-k^{-1})\,\tau_k(t) = \tag{1.3}$$

$$= -\sum_{k=1}^{n} [(p^{-1}\tau_p'(t) + (p+1)^{-1}\tau_{p+1}'(t))\cdot$$

$$\cdot \sum_{k=1}^{p} (k\,|\gamma_k(t)|^2 - k^{-1})].$$

Lemma 1.1. *With $\kappa,\gamma_k,\tau_k,\phi$ as above:*

$$\phi'(t) = -\sum_{k=1}^{n} k^{-1}|\gamma_k'(t)|^2\tau_k'(t). \tag{1.4}$$

Let S denote the class of all functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.5}$$

which are analytic and univalent in the unit disk and write

$$\log(z^{-1}f(z)) = 2\sum_{n=1}^{\infty} \gamma_n z^n. \tag{1.6}$$

Then Lemma 1.1 implies by use of Loewner's partial differential equation:

Theorem 1.2. *Let $\tau_1, \ldots, \tau_n$ satisfy (1.2) and let $\tau_k'(t) \leq 0$ for $k = 1,...,n$ and $t \geq 0$. Then for each $f \in S$ with coefficients γ_k given by (1.6) we have*

$$\sum_{k=1}^{n} \tau_k(0)\,(k\,|\gamma_k|^2-k^{-1}) \leq 0. \tag{1.7}$$

De Branges showed that the functions τ_k with

$$\tau_k'(t) := -k\begin{bmatrix} n+k+1 \\ n-k \end{bmatrix} e^{-kt}\,{}_3F_2\begin{bmatrix} -n+k,\kappa+½,n+k+2 \\ k+\frac{3}{2},2k+1 \end{bmatrix}\Big| e^{-t}\Big] \tag{1.8}$$

and $\tau_k(t) \to 0$ as $t \to \infty$, satisfy the conditions of Theorem 1.2 and, moreover,

$$\tau_k(0) = n+1-k. \tag{1.9}$$

Inequality (1.7) together with (1.9) is precisely the Milin conjecture, which was thus settled. It was already known that the Milin conjecture implies the Bieberbach conjecture.

In (1.8) we used the notation

$$pF_q\begin{bmatrix} a_1,...,a_p \\ b_1,...,b_q \end{bmatrix}\Big| z\Big] := \sum_{\nu=0}^{\infty} \frac{(a_1)_\nu \cdots (a_p)_\nu}{(b_1)_\nu \cdots (b_q)_\nu \nu!} z^\nu \tag{1.10}$$

for the (generalized) hypergeometric power series, where

$$(a)_\nu := \begin{cases} a(a+1)...(a+\nu-1), & \nu \in \mathbb{N}, \\ 1, & \nu = 0. \end{cases} \tag{1.11}$$

The inequality $\tau_k'(t) \leq 0$ for τ_k' given by (1.8) turned out to be contained in Askey & Gasper [2]. Their proof wrote the ${}_3F_2$ in (1.8) as a linear combination with positive coefficients of

$${}_3F_2\left[\begin{matrix} -m+k, m+k+1, k+½ \\ k+1, 2k+1 \end{matrix} \Big| e^{-t}\right], \tag{1.12}$$

m running over $n, n-2, n-4, ..., k$ or $k+1$. Now the inequality is settled by using Clausen's formula (cf. [3, p. 86]), which sets (1.12) equal to

$$[{}_2F_1(-½m+½k, ½m+½k+½; k+1; e^{-t})]^2 \tag{1.13}$$

It is the aim of the present paper to show that the functions (1.12) are not just an aid, exterior to Bieberbach type problems, for settling the nonpositivity of (1.8), but that these functions themselves belong to solutions of (1.2) and that they lead to Milin type inequalities which contain the original Milin inequality in their convex hull.

2. Main part

If follows by straightforward computation that the general solution $(\tau_1, \ldots, \tau_n)$ of (1.2) has the form

$$\tau_k(t) = k \sum_{\ell=k}^{n} \frac{(-1)^{\ell-k}(2\ell)!}{\ell(\ell-k)!(\ell+k)!} b_\ell e^{-\ell t}, \tag{2.1}$$

where $b_1, ..., b_n$ are arbitrarily complex (see also [4]). Then

$$\tau_k'(t) = -ke^{-kt} \sum_{\nu=0}^{n-k} \frac{(-1)^\nu(2k+2\nu)!}{\nu!(2k+\nu)!} b_{k+\nu} e^{-\nu t}. \tag{2.2}$$

De Branges made the choice

$$b_\ell := \begin{pmatrix} n+\ell+1 \\ n-\ell \end{pmatrix}. \tag{2.3}$$

Then (2.2) becomes (1.8). Now, for each $m = 1, ..., n$ we choose $b_1^m, ..., b_n^m$ by

$$b_\ell^m := \begin{cases} \dfrac{2^{-2\ell}(m+\ell)!}{\ell!\ell!(m-\ell)!}, & \ell = 1, ..., m, \\ 0, & \ell = m+1, \ldots, n. \end{cases} \tag{2.4}$$

Then (2.2) becomes

$$(\tau_k^m)'(t) = \begin{cases} -k\dfrac{2^{-2k}(m+k)!}{k!k!(m-k)!} e^{-kt} \cdot {}_3F_2\left(\begin{matrix} -m+k, m+k+1, k+½ \\ k+1, 2k+1 \end{matrix} \Big| e^{-t}\right) & \text{if } k = 1, ..., m, \\ 0 & \text{if } k = m+1, ..., n. \end{cases} \tag{2.5}$$

The n solutions $(\tau_1^m, \ldots, \tau_n^m)$, $m = 1, ..., n$, of (1.2) form a basis of its solution space. Note that we met the ${}_3F_2$'s in (2.5) already in (1.12), in the expansion with positive coefficients of (1.8). Because of the equality of (1.12) and (1.13), the solution $(\tau_1^m, \ldots, \tau_n^m)$ of (1.2) satisfies the conditions of Theorem 1.2. Hence, in view of (1.7) and (1.3) we have

$$-\sum_{k=1}^{p} (p^{-1}(\tau_p^m)'(0) + (p+1)^{-1}(\tau_{p+1}^m)'(0)). \tag{2.6}$$

$$\sum_{k=1}^{p}(k\,|\gamma_k|^2-k^{-1})\leqslant 0,$$

where γ_k is given by (1.6), $f\in S$. In order to compute $(\tau_p^m)'(0)$, use that

$$C_n^\lambda(x)=\frac{(2\lambda)_n}{n!}\,{}_2F_1(-\tfrac{1}{2}n,\tfrac{1}{2}n+\lambda;\lambda+\tfrac{1}{2};1-x^2), \tag{2.7}$$

where

$$C_n^\lambda(x):=\frac{(2\lambda)_n}{n!}\,{}_2F_1(-n,n+2\lambda;\lambda+\tfrac{1}{2};\tfrac{1}{2}-\tfrac{1}{2}x) \tag{2.8}$$

is a Gegenbaner polynomial. (Formula (2.7) follows from [5, 2.11 (2)].) Hence, by (1.12) = (1.13), (2.5) and (2.7):

$$(\tau_k^m)'(t)=-k\,\frac{2^{-2k}(m-k)!(2k)!}{(m+k)!\,k!\,k!}\,e^{-kt}\cdot[C_{m-k}^{k+\frac{1}{2}}((1-e^{-t})^{\frac{1}{2}})]^2,\ m\geqslant k. \tag{2.9}$$

Hence, by [5, 10.9 (19)]:

$$(\tau_k^m)'(0)=\begin{cases}-k\dfrac{(\frac{1}{2})_{\frac{1}{2}m+\frac{1}{2}k}(\frac{1}{2})_{\frac{1}{2}m-\frac{1}{2}k}}{(\frac{1}{2}m+\frac{1}{2}k)!(\frac{1}{2}m-\frac{1}{2}k)!} & \text{if } m-k\geqslant 0 \text{ and even,}\\ 0\ , & \text{otherwise.}\end{cases} \tag{2.10}$$

Thus, by substitution of (2.10) in (2.6) we obtain the following Milin type inequalities:

Theorem 2.1. *Let $f\in S$ and let $\gamma_1,\gamma_2,\ldots$ be given by (1.6). Then*

$$\sum_{p=1}^{m}d_p^m\sum_{k=1}^{p}(k\,|\gamma_k|^2-k^{-1})\leqslant 0, \tag{2.11}$$

where

$$d_p^m=d_{p-1}^m=\frac{(\frac{1}{2})_{\frac{1}{2}m+\frac{1}{2}p}(\frac{1}{2})_{\frac{1}{2}m-\frac{1}{2}p}}{(\frac{1}{2}m+\frac{1}{2}p)!(\frac{1}{2}m-\frac{1}{2}p)!},\ m-p \text{ even.} \tag{2.12}$$

Since $(\tau_1^m,\ldots,\tau_n^m)$, $m=1,\ldots,n$ is a basis of the solution space of (1.2), there are coefficients $e_1,\ldots,e_n$ such that

$$\tau_k'=\sum_{m=k}^{n}e_m(\tau_k^m)',\ k=1,\ldots,n. \tag{2.13}$$

By (1.8) and (2.5) this can be rewritten as

$$\begin{bmatrix}n+k+1\\ n-k\end{bmatrix}{}_3F_2\begin{bmatrix}-n+k,k+\frac{1}{2},n+k+2\\ k+\frac{3}{2},2k+1\end{bmatrix}x\Bigg]=\sum_{m=k}^{n}e_m\frac{2^{-2k}(m+k)!}{k!k!(m-k)!}\,{}_3F_2\begin{bmatrix}-m+k,m+k+1,k+\frac{1}{2}\\ k+1,2k+1\end{bmatrix}x\Bigg]. \tag{2.14}$$

It follows by Askey & Gasper [2, (3.1), (3.7)] that

$$\begin{cases}e_{n-2j}=\dfrac{(n-2j+\frac{1}{2})(\frac{1}{2})_j(n-j)!}{(n-j+\frac{1}{2})j!(\frac{1}{2})_{n-j}},\ j=0,\ldots,[\frac{1}{2}(n-1],\\ e_m=0,\ \ n-m\ \text{ odd.}\end{cases} \tag{2.15}$$

Since the coefficients e_{n-2j} are positive, we can get a new inequality from (2.11) by multiplying both sides of (2.11) with e_m and summing up from 1 to n. This new inequality will necessarily be the Milin inequality.

Let us conclude with a proof of (2.15) which is slightly different from the proof given in [2]. Put $x = 0$ in (2.14), multiply both sides with $2^{k+1}k(y-1)^{k-1}$ and sum up from $k = 1$ to n. Then, by substitution of (2.8) in both sides of the identity, we obtain

$$C_{n-1}^{2}(y) = \frac{1}{2} \sum_{m=1}^{n} e_m C_{m-1}^{\frac{3}{2}}(y). \tag{2.16}$$

This is a special case of Gegenbauer's formula expanding C_n^{λ} in terms of C_m^{μ}, cf. for instance Askey [1, (7.34)], and we obtain again (2.15).

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References

1 R. ASKEY, *Orthogonal polynomials and special functions,* Regional Conference Series in Applied Math. 21, SIAM, 1975.

2 R. ASKEY & G. GASPER, *Positive Jacobi polynomial sums,* II, Amer. J. Math. 98 (1976), 709-737.

3 W.N. BAILEY, *Generalized hypergeometric series,* Cambridge University Press, 1935.

4 L. de BRANGES, *A proof of the Bieberbach conjecture,* preprint, 1984.

5 A. ERDÉLYI et al., *Higher Transcendental Functions,* I, II, McGraw-Hill, 1953.

6 C.H. FITZGERALD & C. POMMERENKE, *The de Branges theorem on univalent functions,* Informal communication, August 1984.