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Strongly Sequential Term Rewriting Systems

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ABSTRACT
For regular term rewriting systems, G. Huet and J.-J. Lévy have introduced the property of 'strong sequentiality'. A strongly sequential regular term rewriting system admits an efficiently computable normalizing one-step reduction strategy. As shown by Huet and Lévy, strong sequentiality is a decidable property. In this paper we present a structural analysis of strongly sequential term rewriting systems, leading to two new and simplified proofs of the decidability of this property.

Key words and phrases: regular term rewriting systems, normalizing reduction strategy, needed redex.
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Introduction

The analysis of term rewriting systems is of growing interest for a large number of applications having to do with computing with equations. Two main streams can be distinguished in the study of term rewriting systems: (1) theory and applications of Knuth-Bendix completion procedures - here the point of departure is a given set of equations for which one tries to generate a complete (i.e. confluent and terminating) term rewriting system, and (2) theory and applications of regular term rewriting systems; here the term rewriting system is fixed but subject to the restrictions of 'left-linearity' and 'non-ambiguity', for short 'regularity'. The restriction of regularity enables one to develop a quite sizeable amount of theory, for a large part due to the efforts of the 'french school' (Berry, Boudol, Huet, Lévy e.a.; see [BL79], [B85], [HL79]).

The present paper is exclusively concerned with regular term rewriting systems. In an admirable paper ([HL79]), Huet and Lévy investigated the issue of parallel versus sequential reduction in a regular term rewriting system. More specifically, they formulated a criterion 'strong sequentiality', guaranteeing the possibility of an effective sequential normalizing reduction strategy, that is a strategy Φ such that its iteration on a given term M leads to a reduction sequence

$$M \rightarrow \Phi(M) \rightarrow \Phi^2(M) \rightarrow \ldots$$

which ends in the (unique) normal form of M if it exists and is infinite otherwise. The sequentiality is in the fact that the strategy indicates in each step just one redex to be rewritten, rather than a set of redexes to be rewritten in parallel. Actually, Huet and Lévy prove that every regular term rewriting system possesses a sequential normalizing 'call-by-need' strategy: a deep theorem in [HL79] says that every term M in a regular term rewriting system contains a 'needed' redex, that is one which has to be rewritten in any reduction to normal form. A call-by-need strategy is then obtained by rewriting in each step such a needed redex, and it is proved in [HL79] that such a strategy is normalizing. Unfortunately, it is undecidable in general whether a redex is needed or not. However, Huet and Lévy go on to show that in 'strongly sequential' term rewriting systems, a needed redex can be found effectively. This does not mean that in a strongly sequential term sewriting system *all* needed redexes can be determined effectively. For instance Combinatory Logic

$$Ap(Ap(Ap(S,x),y),z) \to Ap(Ap(x,z),Ap(y,z))$$
$$Ap(Ap(K,x),y) \to x$$
$$Ap(I,x) \to x$$

is a strongly sequential term rewriting system where this is impossible; cfr. the analogous statement for λ-calculus in [BKKS86]. In fact, a needed redex is very easy to determine in the case of CL: the leftmost redex is always needed. By contrast, consider CL⊕B, that is CL extended with B ('Berry's TRS', also called 'Gustave's TRS' in [H86]):

$$F(x,A,B) \to C$$
$$F(B,x,A) \to C$$
$$F(A,B,x) \to C$$

In the term rewriting system CL⊕B it is not clear at all how to find a needed redex: in a term $F(M_1,M_2,M_3)$ the redexes in M_1 may be non-needed because M_2,M_3 reduce to the constants A,B respectively, and likewise for redexes in M_2 and M_3. Actually, we do not know whether there is an algorithm to determine a needed redex in a term of CL⊕B (cfr. the suprising fact in [K87] where it is shown that every regular term rewriting system, including CL⊕B, has a computable normalizing one-step reduction strategy), but it seems safe to conjecture that if such an algorithm exists, it will not be very 'feasible'.

But in strongly sequential term rewriting systems a needed redex can be found really effectively, as shown in [HL79]. Moreover, it is decidable whether a term rewriting system is strongly sequential. This brings us to the point dealt with in this paper: in [HL79] a proof of the decidability of strong sequentiality is given with great ingenuity; but it is also very complicated, and in the present paper our endeavour is to analyze the notion of a strongly sequential term rewriting system in order to arrive at a simplified proof of the decidability. We present two proofs of which the first is the most direct; but the corresponding decision procedure itself is only of mathematical relevance as its computational complexity forbids a practical application. We feel however that this proof is conceptually simple and gives a good insight in the structure of a strongly sequential term rewriting system. Some of the underlying notions in [HL79] are eliminated here; notably: the 'matching dag', 'directions', 'increasing indexes' and 'Δ-sets' (or: 'properties Q_1, Q_2'). Also our proof is direct in the sense that it does not take the form of a correctness proof of some algorithm. The second proof is of comparable computational complexity as the one in [HL79]; conceptually it is harder than the first, though still simpler than the one in [HL79]. This proof is essentially already in [HL79] and uses their notions of increasing indexes and Δ-sets (the latter with a slight simplification by us). In both proofs our concepts of an 'atomic pre-redex' and of a 'tower of atomic pre-redexes' play a crucial role. We conclude with the simple but useful observation that strong sequentiality is a 'modular' property, i.e. depends on the 'disjoint pieces' of a term rewriting system, and with the construction of a term rewriting system which is 'inherently difficult', w.r.t. deciding strong sequentiality. Especially in the first part of our paper we follow [HL79] quite closely; also some proofs there are repeated for the sake of completeness. Finally, we refer to [HL79] for a number of interesting applications of strongly sequential term rewriting systems as well as a lot of information about implementing the decision procedure for strong sequentiality.

Contents

1. Regular Term Rewriting Systems: Preliminaries

We start with a number of definitions. A *term rewriting system* is a pair $(\Sigma, \mathcal{R})$. The *alphabet* Σ consists of

- V, a countably infinite set of variables,
- F, a finite set of function symbols; associated to every $F \in F$ is its arity $n \geqq 0$.

T is the set of terms built from F and V, i.e.

- $V \subset T$,
- if $F \in F$ has arity n and $M_1, \ldots, M_n \in T$ then $F(M_1, \ldots, M_n) \in T$.

$\mathcal{R}$ is a finite set of *reduction rules*, each of the form $\alpha \to \beta$, where $\alpha, \beta \in T$ with the restriction that

(1) α is not a variable, and

(2) the variables which occur in β also occur in α.

We usually write $\mathcal{R}$ instead of $(\Sigma, \mathcal{R})$. A *substitution* σ is a function from V to T. Substitutions are extended to T in the obvious way; we denote by M^σ the term obtained from M by applying the substitution σ. We call M^σ an *instance of* M. An instance of a left-hand side of a reduction rule is called a *redex*. The *reduction relation* $\to_{\mathcal{R}} \subset T \times T$ ($\to$ for short) is defined by

> $M \to_{\mathcal{R}} N$ iff M has a subterm which is a redex, say α^σ with $\alpha \to \beta \in \mathcal{R}$, and N is obtained from M by replacing that subterm by β^σ, the corresponding right-hand side of the reduction rule $\alpha \to \beta$.

The transitive and reflexive closure of $\to_{\mathcal{R}}$ is written as $\twoheadrightarrow_{\mathcal{R}}$ (we will omit the subscript $\mathcal{R}$). If $M \twoheadrightarrow N$, we say M *reduces to* N.

Example 1.1. Let $\mathcal{R} = \{A(x,0) \to x,\ A(x,S(y)) \to S(A(x,y))\}$. Consider the term $A(A(0,0),A(S(0),0))$. To this term we can apply the following reduction sequence (at each step the rewritten redex is underlined):

$$A(\underline{A(0,0)},A(S(0),0)) \to A(0,\underline{A(S(0),0)}) \to \underline{A(0,S(0))} \to S(\underline{A(0,0)}) \to S(S(0)).$$

The term $S(S(0))$ is a *normal form*, i.e. a term which contains no redexes.

We denote the set of normal forms of $\mathcal{R}$ by $NF_{\mathcal{R}}$ (NF for short).

A precise formalism is obtained through the notion of *occurrences*. We adopt the notations in [HL79] and [H86]. For any term $M \in T$, the set $O(M)$ of its occurrences is inductively defined as follows

- $\lambda \in O(M)$ (the empty occurrence),
- if $u \in O(M_i)$ then $i.u \in O(F(M_1,...,M_n))$.

If we write terms as trees, an occurrence of M denotes a unique node in the tree of M. If $u \in O(M)$, the *subterm of M at u*, notation M/u, is defined by

- $M/\lambda = M$,
- $F(M_1,...,M_n)/i.u = M_i/u$.

The *symbol of M at occurrence u*, notation $M(u)$, is defined for $u \in O(M)$ by

- $x(\lambda) = x$,
- $F(M_1,...,M_n)(\lambda) = F$,
- $F(M_1,...,M_n)(i.u) = M_i(u)$.

Finally, if $u \in O(M)$, for every term N the *replacement in M of the subterm at u by N*, notation $M[u \leftarrow N]$, is defined by

- $M[\lambda \leftarrow N] = N$,
- $F(M_1,...,M_n)[i.u \leftarrow N] = F(M_1,...,M_{i-1},M_i[u \leftarrow N],M_{i+1},...,M_n)$.

Example 1.2. Consider again the term rewriting system of example 1.1. The occurrences of $M = S(A(S(0),0))$ are exhibited in the following picture.

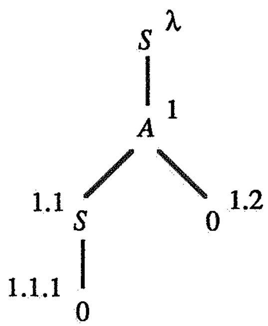


Figure 1.1.

We have $M/1 = A(S(0),0)$, $M(1.1.1) = 0$ and $M[1.1 \leftarrow M/1.2] = S(A(0,0))$.

The set of occurrences $O(M)$ is partially ordered by the *prefix ordering* $\leqq$, i.e. $u \leqq v$ iff there exists a w such that $uw = v$ (if such a w exists, it is unique). In this case we define $v/u = w$. We say that two occurrences u and v are *disjoint*, notation $u \mid v$, if neither $u \leqq v$ nor $v \leqq u$. If $u \leqq v$ and $u \neq v$, we write $u < v$. If $u_1, \ldots, u_n \in O(M)$ are pairwise disjoint, we write $M[u_i \leftarrow N_i \mid 1 \leqq i \leqq n]$ as an alternative for $M[u_1 \leftarrow N_1] \ldots [u_n \leftarrow N_n]$ (the order of the u_i's is irrelevant).

The size $|u|$ of an occurrence u is defined by

$$
\begin{aligned}
|\lambda| &= 0, \\
|i \cdot u| &= 1 + |u|.
\end{aligned}
$$

In this paper we restrict ourselves to the subclass of *regular* term rewriting systems. A term rewriting system is regular if it satisfies the following two constraints

(1) *left-linearity*: every left-hand side of the reduction rules does not have more than one occurrence of the same variable.

Example 1.3. $\mathcal{R} = \{IF(T,x,y) \rightarrow x,\ IF(F,x,y) \rightarrow y,\ IF(x,y,y) \rightarrow y\}$ is not left-linear: the left-hand side of the third reduction rule has two occurrences of the variable y.

(2) *non-ambiguity*: redex-patterns do not overlap. A *redex-pattern* is (informally) a left-hand side in which the variables are discarded.

Example 1.4. Consider again $\mathcal{R} = \{A(x,0) \rightarrow x,\ A(x,S(y)) \rightarrow S(A(x,y))\}$. In figure 1.2 the redex-patterns of the term $A(A(0,S(0)),S(A(0,0)))$ are indicated; they do not overlap. In fact it is not possible to find a term in which the redex-patterns do overlap: $\mathcal{R}$ is non-ambiguous. On the contrary, $\mathcal{R} = \{OR(T,x) \rightarrow T,\ OR(x,T) \rightarrow T\}$ is ambiguous, as shown in the same figure.

A formal definition of non-ambiguity is the following: $\mathcal{R}$ is non-ambiguous iff whenever α and β are left-hand sides and $u \in O(\alpha)$ such that $\alpha/u \notin V$, there are no substitutions σ and τ such that $\sigma(\alpha/u) = \tau(\beta)$, except in the case $\alpha = \beta$ and $u = \lambda$.

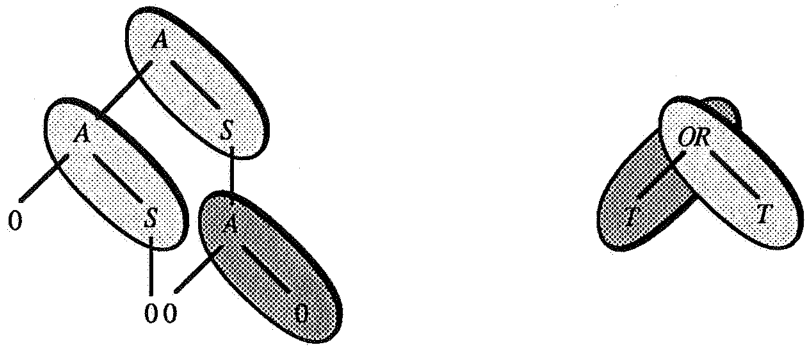


Figure 1.2.

(Note that there is a slight discrepancy between the formal definition of non-ambiguity and the intuitive one. For example, $\mathcal{R} = \{G(F(x,x)) \rightarrow x,\ F(A,B) \rightarrow A\}$ is non-ambiguous, according to the formal definition; but in the term $G(F(A,B))$ we have overlapping redex-patterns. The cause of the difference lies in the non-left-linearity of $\mathcal{R}$. Because we assume the term rewriting systems to be left-linear, the two notions of non-ambiguity coincide.)

Regular term rewriting systems have some very nice properties. Among these is the *unique normal form property*: if a term M has a normal form N (i.e. $M \twoheadrightarrow N$ with $N \in NF$), then N is the unique normal form of M (i.e. if $M \twoheadrightarrow N'$ and $N' \in NF$ then $N = N'$). In the next section we will see some more important properties of regular term rewriting systems.

2. Strongly Sequential Term Rewriting Systems

There are term rewriting systems in which some terms have a normal form, but also admit an infinite reduction sequence.

Example 2.1. (from [HL79]) Let $\mathcal{R} = \{F(x,B) \to B,\ A \to B,\ C \to C\}$. The term $F(C,A)$ has a normal form

$$F(C,\underline{A}) \to \underline{F(C,B)} \to B,$$

but always choosing the leftmost redex fails

$$F(\underline{C},A) \to F(\underline{C},A) \to F(\underline{C},A) \to \dots .$$

Therefore, it is important to have a "good" *reduction strategy*. Informally, a reduction strategy tells us, when presented a term, which redex(es) to rewrite. To be more precise, a *one-step reduction strategy* is a function $\Phi : T \to T$ such that

- $\Phi(M) = M$ if $M \in NF$,
- $M \to \Phi(M)$ otherwise.

A *many-step reduction strategy* is a function $\Phi : T \to T$ such that

- $\Phi(M) = M$ if $M \in NF$,
- $M \to^+ \Phi(M)$ otherwise ($\to^+$ is the transitive closure of $\to$).

A reduction strategy Φ is *normalizing* if for each term M having a normal form, the sequence

$$M,\ \Phi(M),\ \Phi(\Phi(M)), \dots,\ \Phi^n(M), \dots$$

contains a normal form. We are only interested in effective normalizing strategies. (A reduction strategy Φ is *effective* if $\Phi(M)$ can be computed from M.)

An important normalizing many-step reduction strategy for regular term rewriting systems is the *parallel-outermost* strategy: rewrite simultaneously all maximal (outermost) redexes. (In a term M a redex at occurrence u is *maximal* if for every v with $\lambda \leq v < u$, M/v is not a redex.) For a proof that the parallel-outermost strategy is normalizing for regular term rewriting systems, see [O'D77] or the Appendix of [BK84]. Alternatively, this fact can be obtained as a corollary of theorem 2.1 below. The following example shows that the parallel-outermost strategy not always gives the shortest reduction sequence to normal form.

Example 2.2. Let $\mathcal{R} = \{IF(T,x,y) \to x,\ IF(F,x,y) \to y,\ A \to B\}$. Consider the term $IF(IF(T,F,T),A,A)$. The parallel-outermost strategy rewrites a total of 4 redexes

$$IF(\underline{IF(T,F,T)},\underline{A},\underline{A}) \twoheadrightarrow \underline{IF(F,B,B)} \to B,$$

but the following sequence uses only 3 redexes

$$IF(\underline{IF(T,F,T)},A,A) \to \underline{IF(F,A,A)} \to \underline{A} \to B.$$

In the example above it is not necessary to rewrite the redex at occurrence 3 in the term $IF(IF(T,F,T),A,A)$ in order to get a normal form. Before we make this more precise, we introduce the notion of "descendants" in reductions.

Consider the reduction rule $F(x,y) \to G(F(x,x))$. When instantiated to, say, $F(M,N) \to G(F(M,M))$ it is clear that M in this step is doubled and that N has been erased. Obviously we have an intuition of the subterms in M as propagating to the right. We say that a subterm M' of M has (two) *descendants* in $G(F(M,M))$. We will not formalize this notion, as it is intuitively so clear.

A redex R in a term M is called *needed* if in every reduction of M to normal form, R or a descendant of R is rewritten. A needed redex must eventually be rewritten in order to get a normal form. In example 2.2 the underlined redex in $IF(IF(T,F,T),A,\underline{A})$ is not needed. Huet and Lévy proved the following very important

Theorem 2.1. *Let $\mathcal{R}$ be a regular term rewriting system. For all $M \in T$*

(1) *if M contains redexes then M contains a needed redex,*

(2) *if M has a normal form, repeated rewriting of needed redexes leads to that normal form.*

Unfortunately, in general needed redexes are not computable. Later on we will consider more restricted systems in which needed redexes are computable, but first we introduce some more formalism.

We add a new constant Ω to our alphabet. Ω will represent an unknown part of a term. Let T_Ω be the set of Ω-terms. The *prefix ordering* $\leq$ on T_Ω is defined by

- $\Omega \leq M$ for every $M \in T_\Omega$,

- if $M_i \leq N_i$ for $1 \leqq i \leqq n$ then $F(M_1, \ldots, M_n) \leq F(N_1, \ldots, N_n)$,
- $x \leq x$ for every $x \in V$.

If $M \leq N$ and $M \neq N$, we write $M < N$.

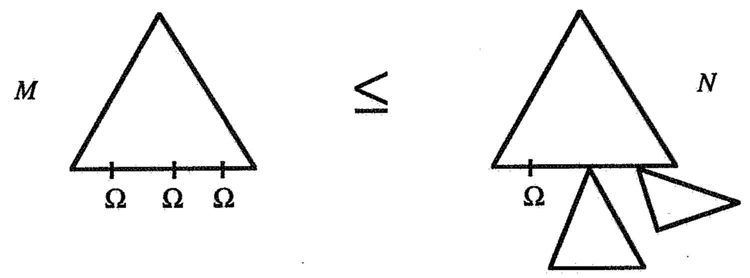


Figure 2.1.

If $M \in T_\Omega$ we write $O_\Omega(M)$ for the Ω-occurrences of M and $\overline{O}(M)$ for the other occurrences. The *greatest lower bound* of two Ω-terms M and N, notation $M \sqcap N$, is defined by

- $F(M_1, \ldots, M_n) \sqcap F(N_1, \ldots, N_n) = F(M_1 \sqcap N_1, \ldots, M_n \sqcap N_n)$,
- $x \sqcap x = x$,
- $M \sqcap N = \Omega$ in all other cases.

We write M_Ω for M^σ where σ maps all variables to Ω. $\omega'(M)$ is the term we get by replacing in M all (maximal) redexes by Ω. Finally, let NF_Ω consist of all Ω-*normal forms*, i.e. all $M \in T_\Omega$ such that $\omega'(M) = M$ and $O_\Omega(M) \neq \varnothing$.

Let $M \in T$ have redexes. Theorem 2.1 tells us that one of the redexes is needed. It is not difficult to see, using the non-ambiguity condition, that one of the maximal redexes is needed. Let $R_1, \ldots, R_n$ be the maximal redexes, at occurrences $u_1, \ldots, u_n$ respectively; thus $M = \omega'(M)[u_i \leftarrow R_i \mid 1 \leqq i \leqq n]$.

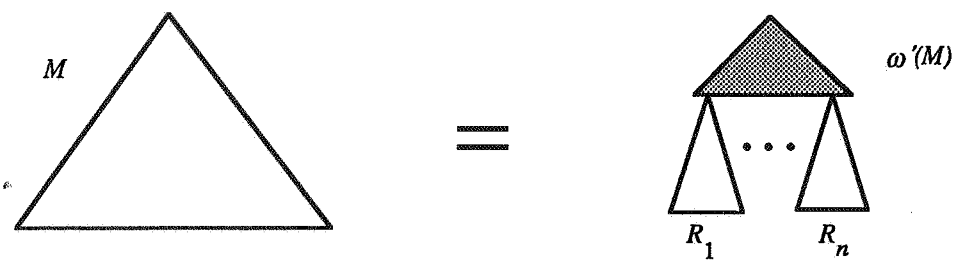


Figure 2.2.

We know that there is a $i \in \{1,...,n\}$ such that R_i is needed. The actual i depends on the substitution of the redexes $R_1,...,R_n$ for the Ω's in $\omega'(M)$.

In a *sequential* term rewriting system, such an i can be found uniformly: there is an i such that for all substitutions of redexes $R_1,...,R_n$ for the Ω's in $\omega'(M)$, the redex R_i is needed. The corresponding occurrence u_i in $\omega'(M)$ is called an *index*. However, sequentiality is undecidable and indexes are in general not computable, as the following example from [HL79] shows.

Example 2.3. $\mathcal{R}$ contains the following four rules

$$F(G(A,x),B) \to A$$
$$F(G(x,A),C) \to B$$
$$F(D,x) \to C$$
$$G(E,E) \to \ldots$$

Let $M = F(G(\Omega,\Omega),\Omega)$. Occurrence 1.1 is not an index: if we substitute redexes for the Ω's, it is possible to get a normal form without rewriting the redex at occurrence 1.1. E.g.

$$F(G(F(D,D),\underline{F(G(A,x),B)}),\underline{F(D,D)}) \twoheadrightarrow \underline{F(G(F(D,D),A),C)} \to B.$$

A similar argument shows that occurrence 1.2 is not an index. It is not difficult to see that occurrence is 2 an index iff $G(R_1,R_2)$ cannot reduce to D for all redexes R_1 and R_2. But this is not decidable, since $\mathcal{R}$ can have many other rules. (The reader familiar with [HL79] will remember a different definition of sequentiality. In fact, the two definitions do not completely agree. If we change our definition into "there is an i such that for all substitutions of *terms* $R_1,...,R_{i-1},R_{i+1},...,R_n$ and redexes R_i for the Ω's in $\omega'(M)$, R_i is needed", we get an equivalent to sequentiality of [HL79]. The reason for adopting our first definition is that it is intuitively more appealing. Furthermore, the forthcoming definition of strong sequentiality is equivalent to the one in [HL79].)

In a *strongly sequential* term rewriting system, there is an i such that for every substitution of redexes $R_1,...,R_n$ for the Ω's in $\omega'(M)$ and *independent of the right-hand sides of the reduction rules*, we always have R_i is needed. The phrase "independent of the right-hand sides of the reduction rules" is formalized by extending the reduction relation $\to_{\mathcal{R}}$ in such a way that a redex can be rewritten to any Ω-term. The new reduction relation is denoted by $\to'_{\mathcal{R}}$ ($\to'$ for short). The occurrence u_i in $\omega'(M)$ is again called an *index*. Notice that being an index depends on the underlying reduction relation: a term rewriting system is sequential iff every $M \in NF_\Omega$ has an index with respect to $\rightarrow_{\mathcal{R}}$; a term rewriting system is strongly sequential iff every $M \in NF_\Omega$ has an index with respect to $\rightarrow'_{\mathcal{R}}$. Because $\rightarrow_{\mathcal{R}} \subset \rightarrow'_{\mathcal{R}}$, we clearly have that a strongly sequential term rewriting system is also a sequential term rewriting system. The following example shows that the reverse does not hold.

Example 2.4. Let $\mathcal{R} = \{F(G(A,x),B) \rightarrow x,\ F(G(x,A),C) \rightarrow x,\ F(D,x) \rightarrow x,\ G(E,E) \rightarrow K\}$. Let $M \in NF_\Omega$. We leave it to the reader to show that every $u \in O_\Omega(M)$ is an index with respect to $\rightarrow$. Thus, $\mathcal{R}$ is sequential. But $\mathcal{R}$ is not strongly sequential. Consider the term $F(G(\Omega,\Omega),\Omega)$. Let R_1, R_2 and R_3 be any redexes. The following $\rightarrow'$-reductions show that none of the Ω-occurrences is an index with respect to $\rightarrow'$:

$$F(G(R_1,\underline{R_2}),\underline{R_3}) \twoheadrightarrow' \underline{F(G(R_1,A),C)} \rightarrow' K,$$
$$F(G(\underline{R_1},R_2),\underline{R_3}) \twoheadrightarrow' \underline{F(G(A,R_2),C)} \rightarrow' K,$$
$$F(G(\underline{R_1},\underline{R_2}),R_3) \twoheadrightarrow' F(\underline{G(E,E)},R_3) \rightarrow' F(D,R_3) \rightarrow' K.$$

Therefore, $\mathcal{R}$ is not strongly sequential.

In section 3 we will see that indexes w.r.t. $\rightarrow'$ are computable. Sections 4 and 5 contain two new proofs of the decidability of strong sequentiality. Therefore, a strongly sequential term rewriting system has a normalizing one-step reduction strategy which is effectively computable. In the rest of the paper when writing *index* we always mean *index w.r.t.* $\rightarrow'$. To conclude this section, if $M \in NF_\Omega$, $I(M)$ denotes the set of its indexes.

3. Indexes

In this section we will describe a procedure of Huet and Lévy to compute the indexes of a given term $M \in NF_\Omega$. First, we introduce some useful definitions.

We write $\mathcal{R}_\Omega = \{\alpha_\Omega \mid \alpha \to \beta \in \mathcal{R}\}$. So, by definition, for every redex R there is a $\alpha_\Omega \in \mathcal{R}_\Omega$ such that $\alpha_\Omega \leq R$. Two terms $M, N \in T_\Omega$ are said to be *compatible* if there exists a term $P \in T_\Omega$ such that $M \leq P$ and $N \leq P$, notation $M \uparrow N$. An Ω-term M is called *redex-compatible* if $\exists \alpha_\Omega \in \mathcal{R}_\Omega$ such that $M \uparrow \alpha_\Omega$, notation $M\uparrow$. A redex-compatible term M may become a redex by extending some of its Ω's. If M is not redex-compatible, we write $M\#$.

We now define the important function $\omega : T_\Omega \to T_\Omega$, which we will call the 'melting-procedure'.

$$\omega(\Omega) = \Omega,$$
$$\omega(x) = x,$$
$$\omega(F(M_1,\ldots,M_n)) = \begin{cases} \Omega & \text{if } F(\omega(M_1),\ldots,\omega(M_n))\uparrow, \\ F(\omega(M_1),\ldots,\omega(M_n)) & \text{otherwise.} \end{cases}$$

$\omega(M)$ can be seen as the result of repeated replacement of redex-compatible subterms by Ω. In the definition this happens in a bottom to top way, but one can prove (see [HL79]) that the order of replacing redex-compatible subterms by Ω does not matter for the final outcome. We will call $\omega(M)$ the *fixed part* of M, i.e. if $M' \geq M$ and $M' \twoheadrightarrow N$ then $\omega(M) \leq N$, whatever the right-hand sides of the rules may be. An example will clarify the melting-procedure.

Example 3.1. Let $\mathcal{R} = \{F(F(A,x),B) \to \ldots,\ G(A,B) \to \ldots\}$ and $M = F(A, F(\Omega, G(A,\Omega)))$. M contains one redex-compatible subterm, viz. $G(A,\Omega)$ at occurrence 2.2. When we replace this subterm by Ω we get $M' = F(A, F(\Omega,\Omega))$. Now the subterm $F(\Omega,\Omega)$ at occurrence 2 is redex-compatible, and replacing it by Ω gives us $M'' = F(A,\Omega)$. M'' contains no redex-compatible subterms, hence $\omega(M) = F(A,\Omega)$. If we substitute redexes for the Ω's in M, all terms N reachable through a sequence of $\to'$-reductions starting from this term have symbol F at occurrence λ and symbol A at occurrence 1.

The function $\bar{\omega}$ will be useful in inductive proofs. It takes one step less than the melting-procedure connected with ω does.

$$\bar{\omega}(\Omega) = \Omega,$$
$$\bar{\omega}(x) = x,$$
$$\bar{\omega}(F(M_1,...,M_n)) = F(\omega(M_1),...,\omega(M_n)).$$

The following properties of the melting-procedure will be used in the sequel. Their proofs are left to the reader as simple exercises.

- $\omega(M) \leq M$;
- $\omega(M) = \omega(M[u \leftarrow \omega(M/u)])$, for all $u \in O(M)$;
- if $M \leq N$ then $\omega(M) \leq \omega(N)$;
- $\omega(\omega(M)) = \omega(M)$;
- if $M \rightarrow' N$ then $\omega(M) \leq \omega(N)$;
- if M is redex-compatible then $\omega(M) = \Omega$.

Furthermore, we have the following useful

Lemma 3.1. ([HL79]) *If* $\alpha_\Omega \in \mathcal{R}_\Omega$ *and* $u \in O(\alpha_\Omega) - \{\lambda\}$ *then* $\omega(\alpha_\Omega/u) = \alpha_\Omega/u$.

Proof. We use induction on the complexity of α_Ω/u. If $\alpha_\Omega/u = \Omega$ it is trivial. If $\alpha_\Omega/u \neq \Omega$ we get by induction hypothesis $\bar{\omega}(\alpha_\Omega/u) = \alpha_\Omega/u$. Suppose there is a $\beta_\Omega \in \mathcal{R}_\Omega$ such that $\alpha_\Omega/u \uparrow \beta_\Omega$. Because $\mathcal{R}$ is left-linear we can find substitutions σ and τ such that $\sigma(\alpha/u) = \tau(\beta)$, but this a contradiction with the non-ambiguity condition. Thus $\omega(\alpha_\Omega/u) = \alpha_\Omega/u$. □

We are now ready to describe the procedure which determines what the indexes in a given term M are. Let ● be a fresh constant, the 'test symbol'. Let $M \in NF_\Omega$ and suppose $O_\Omega(M) = \{u_1,...,u_n\}$. If we want to know whether or not $u_i \in I(M)$, we perform the following steps:

(1) replace in M the Ω at occurrence u_i by ●, result $M' = M[u_i \leftarrow \bullet]$;

(2) apply the melting-procedure, result $\omega(M')$;

(3) u_i is an index of M iff the constant ● occurs in $\omega(M')$.

Graphically:

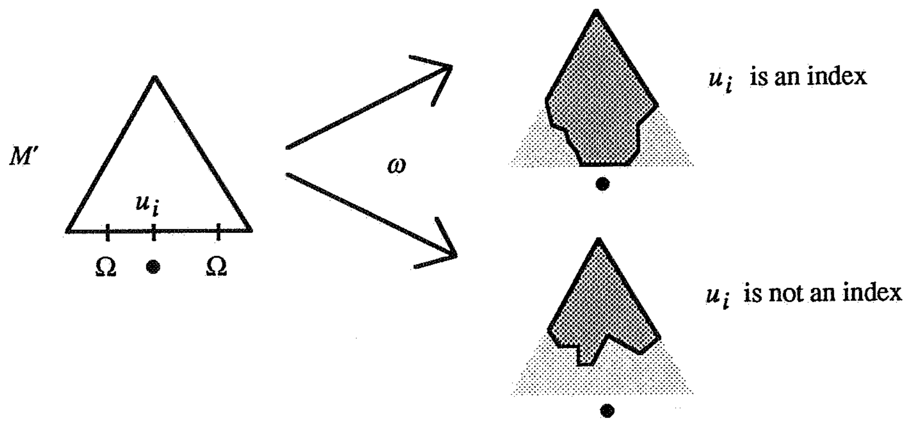


Figure 3.1.

This procedure is justified by the following

Lemma 3.2. ([HL79]) *Let* $M \in NF_\Omega$. *Suppose* $O_\Omega(M) = \{u_1, \ldots, u_n\}$. *The following three statements are equivalent.*

(1) $u_i \in I(M)$,

(2) $u_i \in O(\omega(M[u_i \leftarrow \bullet]))$,

(3) $\omega(M[u_i \leftarrow \bullet]) \neq \omega(M)$.

Proof. We first prove the equivalence of (1) and (2).

(1) $\Rightarrow$ (2) Suppose $u_i \notin O(\omega(M[u_i \leftarrow \bullet]))$. Let R be any redex. Consider the term $M' = M[u_i \leftarrow \bullet][u_j \leftarrow R \mid \text{for all } j \neq i]$. Because $\omega(R) = \Omega$ we have $\omega(M') = \omega(M[u_i \leftarrow \bullet])$, and thus $u_i \notin O(\omega(M'))$. We leave it to the reader to show that there is a term N in normal form such that $M' \twoheadrightarrow' N$ and $u_i \notin O(N)$ (Hint: the melting-procedure induces a $\rightarrow'$-reduction sequence). If we substitute in the reduction sequence $M' \twoheadrightarrow' N$ all occurrences of the constant $\bullet$ by a redex R', we get a reduction sequence $M'[u_i \leftarrow R'] \twoheadrightarrow' N$. In this sequence neither R' nor one of its descendants is contracted, thus $u_i \notin I(M)$.

(2) $\Rightarrow$ (1) Let $R_1, \ldots, R_n$ be redexes and consider the term $M' = M[u_j \leftarrow R_j \mid 1 \leqq j \leqq n]$. We have to prove that R_i is a needed redex. Suppose R_i is not needed, i.e. there is a reduction sequence to normal form in which neither R_i nor one of its descendants is contracted, say $A : M' \twoheadrightarrow' N$. Therefore it is possible to transform this sequence into

$A':M'[u_i \leftarrow \bullet] \twoheadrightarrow {}'N$. Because N is in normal form, N contains no descendants of R_i and thus no occurrences of $\bullet$. We have $\omega(M'[u_i \leftarrow \bullet]) \leq N$ and thus $\omega(M[u_i \leftarrow \bullet]) \leq N$ which, together with the assumption $u_i \in O(\omega(M[u_i \leftarrow \bullet]))$, implies $N/u_i = \bullet$. Contradiction. So R_i is needed and thus $u_i \in I(M)$.

The equivalence of (2) and (3) is almost trivial.

(2) $\Rightarrow$ (3) If $u_i \in O(\omega(M[u_i \leftarrow \bullet]))$ then $\omega(M[u_i \leftarrow \bullet])/u_i = \bullet$ and thus $\omega(M[u_i \leftarrow \bullet]) \neq \omega(M)$.

(3) $\Rightarrow$ (2) Suppose $u_i \notin O(\omega(M[u_i \leftarrow \bullet]))$. Then $\omega(M[u_i \leftarrow \bullet]) \leq M$ and thus (by one of the preceding 'simple exercises') $\omega(M[u_i \leftarrow \bullet]) \leq \omega(M)$. Because $M \leq M[u_i \leftarrow \bullet]$ we also have $\omega(M) \leq \omega(M[u_i \leftarrow \bullet])$. Therefore $\omega(M[u_i \leftarrow \bullet]) = \omega(M)$.

□

Example 3.2. ([HL79]) Let $\mathcal{R} = \{F(G(A,x),B) \rightarrow ..., F(G(x,A),C) \rightarrow ..., G(E,E) \rightarrow ...\}$. In the following terms, the Ω's at the index positions are underlined.

$F(\underline{\Omega},\underline{\Omega})$,
$G(\underline{\Omega},\underline{\Omega})$,
$F(G(\Omega,\Omega),\underline{\Omega})$.

The problem with indexes is that they are not 'transitive', as illustrated in the following

Example 3.3. Let $\mathcal{R}$ be the same as in example 3.2. We have $1 \in I(F(\Omega,\Omega))$ and $1 \in I(G(\Omega,\Omega))$, but $1.1 \notin I(F(G(\Omega,\Omega),\Omega))$.

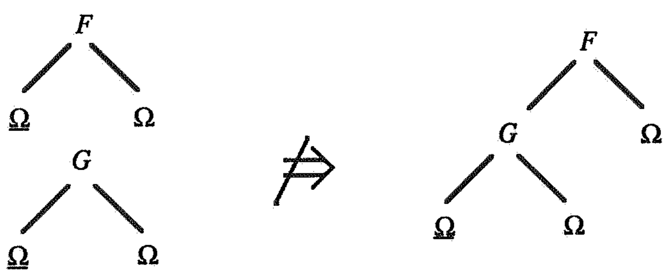


Figure 3.2.

The following properties of indexes which can all be found in [HL79] are heavily used in the sequel.

Lemma 3.3. *If* $u \in I(M)$, $M \leq M'$ *and* $u \in O_{\Omega}(M')$ *then* $u \in I(M')$.

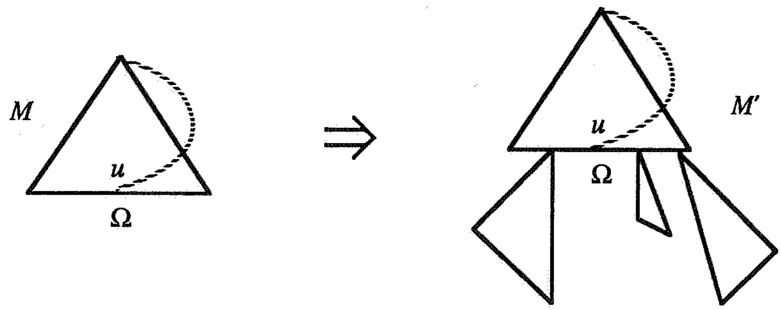


Figure 3.3.

Proof. Suppose $u \notin O(\omega(M'[u \leftarrow \bullet]))$. $M[u \leftarrow \bullet] \leq M'[u \leftarrow \bullet]$ implies $\omega(M[u \leftarrow \bullet]) \leq \omega(M'[u \leftarrow \bullet])$ and thus $u \notin O(\omega(M[u \leftarrow \bullet]))$. But $u \in I(M)$ and therefore $u \in O(\omega(M[u \leftarrow \bullet]))$ by lemma 3.2. Contradiction. So $u \in O(\omega(M'[u \leftarrow \bullet]))$ and, again by lemma 3.2, $u \in I(M')$. □

Lemma 3.4. *If* $uv \in I(M)$ *then* $v \in I(M/u)$.

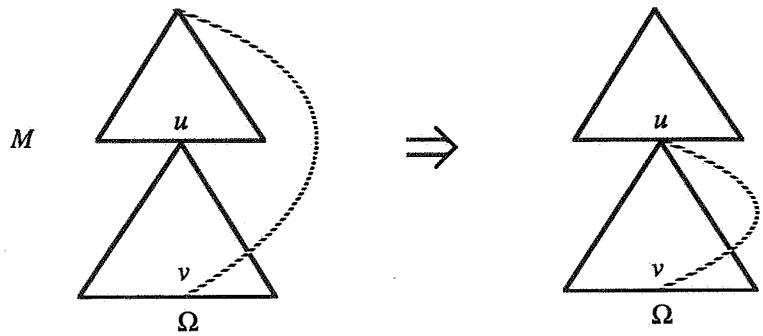


Figure 3.4.

Proof. Suppose $v \notin I(M/u)$. Then $\omega(M/u) = \omega(M/u[v \leftarrow \bullet])$ by lemma 3.2. Thus $\omega(M[uv \leftarrow \bullet]) = \omega(M[u \leftarrow \omega(M/u[v \leftarrow \bullet])]) = \omega(M[u \leftarrow \omega(M/u)]) = \omega(M)$, and therefore $uv \notin I(M)$. □

Lemma 3.5. *If* $\omega(M/u) = \Omega$, $v \in I(M)$ *and* $v \mid u$ *then* $v \in I(M[u \leftarrow \Omega])$.

Proof. If $v \notin I(M[u \leftarrow \Omega])$ we have by lemma 3.2 $\omega(M[u \leftarrow \Omega][v \leftarrow \bullet]) = \omega(M[u \leftarrow \Omega])$. But $\omega(M) = \omega(M[u \leftarrow \omega(M/u)]) = \omega(M[u \leftarrow \Omega])$ and likewise $\omega(M[v \leftarrow \bullet]) = \omega(M[v \leftarrow \bullet][u \leftarrow \Omega])$, and thus $\omega(M[v \leftarrow \bullet]) = \omega(M)$ which implies $v \notin I(M)$ by lemma 3.2. □

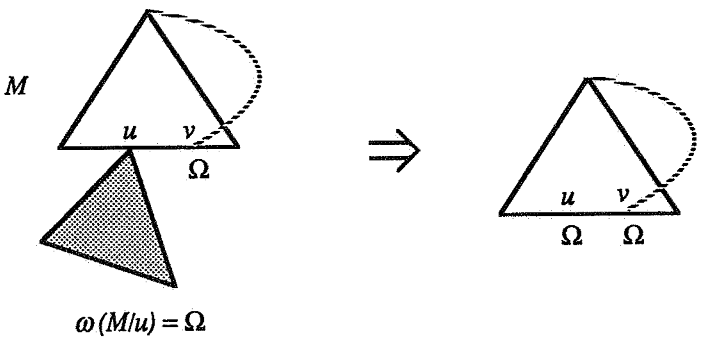


Figure 3.5.

The following example shows that the condition $\omega(M/u) = \Omega$ is necessary.

Example 3.4. Let $\mathcal{R}$ be the same as in example 3.2. Let $M = F(G(\Omega,\Omega),B)$. We have $1.1 \in I(M)$, $1.1 \mid 2$, $\omega(M/2) = B \neq \Omega$, but not $1.1 \in I(M[2 \leftarrow \Omega]) = I(F(G(\Omega,\Omega),\Omega))$.

Lemma 3.6. If $v \in I(M)$ and $\forall u \in \{u_1,\ldots,u_n\}$ we have $\omega(M/u) = \Omega$ and $v \mid u$ then $v \in I(M[u_i \leftarrow N_i \mid 1 \leqq i \leqq n])$ for all $N_1,\ldots,N_n \in T_\Omega$.

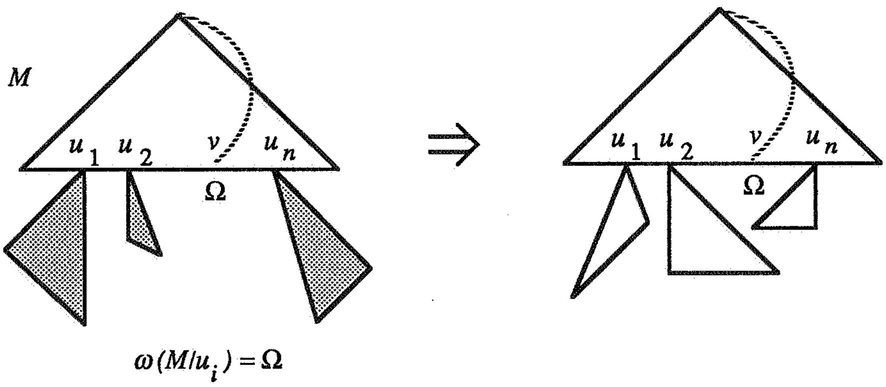


Figure 3.6.

Proof. Applications of lemma 3.5 gives $v \in I(M[u_i \leftarrow \Omega \mid 1 \leqq i \leqq n])$ and thus $v \in I(M[u_i \leftarrow N_i \mid 1 \leqq i \leqq n])$ by lemma 3.3. □

Lemma 3.7. If $uv \in I(M)$ and $\omega(M/u) = \Omega$ then $u \in I(M[u \leftarrow \Omega])$.

Proof. For every $w \leqq u$, define $M_w = M[u \leftarrow \bullet]/w$. We will show by induction on $|u/w|$ that $u/w \in O(\omega(M_w))$. The case $w = u$ is trivial. Assume the statement holds for all $w < w' \leqq u$. By induction hypothesis we have $u/w \in O(\bar{\omega}(M_w))$. Suppose $\bar{\omega}(M_w)\uparrow$. Because $\bar{\omega}(M_w)/(u/w) = \bullet$ this implies $\bar{\omega}(M_w)[u/w \leftarrow N]\uparrow$ for all Ω-terms N. In particular, letting

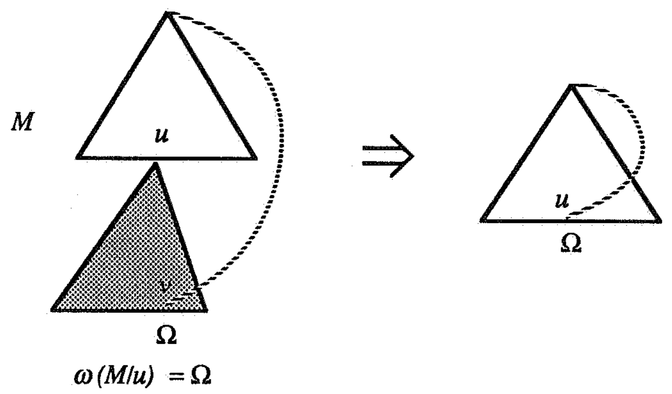


Figure 3.7.

$N = \omega(M/u[v \leftarrow \bullet])$, we get $\bar{\omega}(M[uv \leftarrow \bullet]/w)\uparrow$ and therefore $\omega(M[uv \leftarrow \bullet]/w) = \Omega$. But this implies that $uv \notin O(\omega(M[uv \leftarrow \bullet]))$, a contradiction with $uv \in I(M)$. So the case $\bar{\omega}(M_w)\uparrow$ is impossible, and thus $\omega(M_w) = \bar{\omega}(M_w)$ which implies $u/w \in O(\omega(M_w))$. Taking $w = \lambda$, we get $u \in O(\omega(M[u \leftarrow \bullet]))$ which implies $u \in I(M[u \leftarrow \Omega])$ by lemma 3.2. □

4. Decidability of Strong Sequentiality

A term $M \in NF_{\Omega}$ is called *parallel* if $I(M) = \varnothing$. Clearly, $\mathcal{R}$ is strongly sequential if and only if there are no parallel terms. In this section we will show that we can restrict the hunt for a parallel term to a finite set of Ω-terms, thus proving the decidability of strong sequentiality. We first prove that we can restrict ourselves to terms M with $\omega(M) = \Omega$.

Let $T_{\Diamond}$ consist of all terms $M \in NF_{\Omega}$ with $\omega(M) = \Omega$. Trivially, any term $M \in NF_{\Omega}$ can be written in the following format

$$M \;=\; \omega(M)[u_i \leftarrow N_i \mid 1 \leqq i \leqq n],$$

where $\{u_1, \ldots, u_n\} = O_{\Omega}(\omega(M))$ and $N_i = M/u_i$ for all $1 \leqq i \leqq n$. Note that all the N_i are members of $T_{\Diamond}$.

Figure 4.1.

We have the following important

Lemma 4.1. $v_i \in I(N_i) \;\Leftrightarrow\; u_i v_i \in I(M)$.

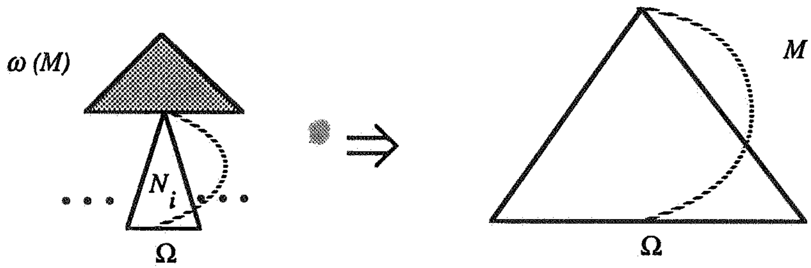


Figure 4.2.

Proof.

$\Rightarrow$ If $\omega(M) = \Omega$ then the statement is trivial. So assume $\omega(M) \neq \Omega$. Clearly $\omega(M[u_i v_i \leftarrow \bullet]) = \omega(M[u_i \leftarrow \omega(N_i[v_i \leftarrow \bullet])])$. For every $w \leqq u_i$, let

$M_w = M[u_i \leftarrow \omega(N_i[v_i \leftarrow \bullet])]/w$. Let us show by induction on $|u_i/w|$ that $\omega(M_w) = M_w$. The case $w = u_i$ is trivial. Otherwise, we have $\overline{\omega}(M_w) = M_w$ by induction hypothesis. If $M_w \uparrow$, we get $\omega(M[u_i \leftarrow \Omega]/w)\uparrow$ because $\omega(M[u_i \leftarrow \Omega]/w) \leq M[u_i \leftarrow \Omega]/w \leq M_w$. But $\omega(M[u_i \leftarrow \Omega]/w) = \omega(M)/w$. Contradiction. Therefore, $\omega(M_w) = M_w$ for every $w \leqq u_i$. In particular we have $\omega(M_\lambda) = M_\lambda$. Thus $\omega(M[u_i v_i \leftarrow \bullet]) = M[u_i \leftarrow \omega(N_i[v_i \leftarrow \bullet])]$, and because $v_i \in I(N_i)$ we have $\omega(N_i[v_i \leftarrow \bullet]) \neq \omega(N_i) = \Omega$ and thus $\omega(M[u_i v_i \leftarrow \bullet]) \neq \omega(M)$.

$\Leftarrow$ By lemma 3.4, $u_i v_i \in I(M)$ implies $u_i v_i / u_i \in I(M/u_i)$.

□

Corollary 4.1. If there exists a parallel term then there is also a parallel term in $T_\Diamond$. □

Let $\alpha_\Omega \in \mathcal{R}_\Omega$. A term $M \in T_\Omega$ is called a *pre-redex of* α_Ω if $M \prec \alpha_\Omega$. M is called a *half-redex with respect to* α_Ω if $M \uparrow \alpha_\Omega$ but neither $M \geq \alpha_\Omega$ nor $M \prec \alpha_\Omega$. M is called a *pre-redex* if $\exists \alpha_\Omega \in \mathcal{R}_\Omega$ such that M is a pre-redex of α_Ω. M is called a *half-redex* if $\exists \alpha_\Omega \in \mathcal{R}_\Omega$ such that M is a half-redex with respect to α_Ω.

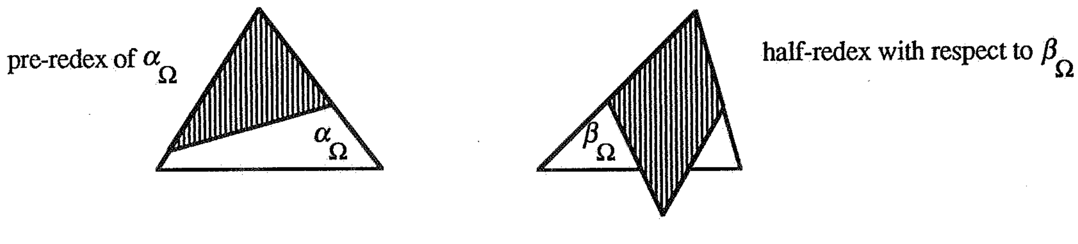


Figure 4.3.

A redex-compatible term is a redex, a pre-redex or a half-redex. The following example shows that it is possible that the same term M is a pre-redex and a half-redex.

Example 4.1. Let $\mathcal{R}_\Omega = \{\alpha_\Omega, \beta_\Omega\}$ where $\alpha_\Omega = F(A, F(A, A))$ and $\beta_\Omega = F(B, \Omega)$. $F(\Omega, F(\Omega, \Omega))$ is pre-redex of α_Ω and a half-redex with respect to β_Ω.

Let $M \in T_\Diamond$ be a half-redex. Like Procrustes, we cut off all parts of M that stick out:

$$\overline{M} = M \sqcap \alpha_\Omega^1 \sqcap \ldots \sqcap \alpha_\Omega^n,$$
$$\Gamma(M) = \overline{O}(M) \cap O_\Omega(\overline{M}),$$

where $\{\alpha_\Omega^1, \ldots, \alpha_\Omega^n\} = \{\alpha_\Omega \in \mathcal{R}_\Omega \mid M$ is a half-redex with respect to $\alpha_\Omega\}$. So $\Gamma(M)$ contains the occurrences of Ω that are created in cutting down M to $\overline{M}$. We have of course $\Gamma(M) \neq \emptyset$ if M is a half-redex. Also, if M is a half-redex then $\overline{M}$ is a "real" pre-redex, i.e. $\overline{M}$ is a pre-redex but not a half-redex.

Figure 4.4.

Lemma 4.2. Let M be a half-redex. If $u \in \Gamma(M)$ then $\exists\, \alpha_\Omega \in \mathcal{R}_\Omega$ with $M \uparrow \alpha_\Omega$ and $u \in O_\Omega(\alpha_\Omega)$.

Proof. Let $A = \{\alpha_\Omega \in \mathcal{R}_\Omega \mid M$ is a half-redex with respect to $\alpha_\Omega\}$. First of all, if $\alpha_\Omega \in A$ then $u \in O(\alpha_\Omega)$. Suppose that for all $\alpha_\Omega \in A$ we have $u \in \overline{O}(\alpha_\Omega)$. Then we have $\alpha_\Omega(u) = M(u)$ for all $\alpha_\Omega \in A$. But then $u \in \overline{O}(\overline{M})$. Contradiction with $u \in \Gamma(M)$. □

The next lemma says that the 'Procrustes procedure' does not create new indexes.

Lemma 4.3. Let M be a half-redex. If $u \in \Gamma(M)$ then $u \notin I(\overline{M})$.

Proof. Let $u \in \Gamma(M)$. By the previous lemma, $\exists\, \alpha_\Omega \in \mathcal{R}_\Omega$ with $M \uparrow \alpha_\Omega$ and $u \in O_\Omega(\alpha_\Omega)$. And because $\alpha_\Omega \geq \overline{M}$, we have $u \notin I(\overline{M})$ by lemma 3.3. □

Lemma 4.4. If M is a half-redex then $I(\overline{M}) \subset I(M)$.

Proof. If $u \in I(\overline{M})$ then we know from lemma 4.3 that $u \notin \Gamma(M)$. Thus $u \in O_\Omega(M)$ and lemma 3.3 gives us $u \in I(M)$. □

The following example shows that in general we have no equality between $I(M)$ and $I(\overline{M})$.

Example 4.2. Let $\mathcal{R}_\Omega = \{F(A, F(\Omega, A, A), A),\ F(B, \Omega, B)\}$ and $M = F(A, F(A, \Omega, \Omega), A)$. M is a half-redex with respect to the first element of $\mathcal{R}_\Omega$, $\overline{M} = F(A, F(\Omega, \Omega, \Omega), A)$, $\Gamma(M) = \{2.1\}$, $I(M) = \{2.2, 2.3\}$ and $I(\overline{M}) = \{2.3\}$.

We will now show that we can restrict ourselves to terms in $T_\Diamond$ which are built entirely from atomic pre-redexes. An Ω-term M is called an *atomic pre-redex* if M is a pre-redex and $\forall u \in \overline{O}(M) - \{\lambda\}$, M/u is not a half-redex nor a pre-redex. M is called an *atomic half-redex* if M is a half-redex and $\forall u \in \overline{O}(M) - \{\lambda\}$, M/u is not a half-redex nor a pre-redex.

Lemma 4.5. *If M is an atomic pre-redex or an atomic half-redex then $\overline{\omega}(M) = M$.*

Proof. Trivial. □

From the definition of the melting-procedure ω it follows that any term $M \in T_\Diamond$ is built from atomic pre- and half-redexes. This is formalized in the following simultaneous definition of $M_{(i)}$ and $\&_i$.

$$M_{(0)} = M,$$
$$\&_i(M) = \{u \in \overline{O}(M_{(i)}) \mid M_{(i)}/u \text{ is an atomic pre-redex or an atomic half-redex }\},$$
$$M_{(i+1)} = M_{(i)}[u \leftarrow \Omega \mid u \in \&_i(M)].$$

Let $\phi(M)$ be defined as the smallest n such that $M_{(n)} = \Omega$, or alternatively such that $\&_n(M) = \varnothing$.

$$\&(M) = \bigcup_{0 \leqq i < \phi(M)} \&_i(M).$$

Example 4.3. Let $\mathcal{R}_\Omega = \{F(A, G(\Omega, B)),\ G(B, A)\}$ and $M = F(F(\Omega, G(B, \Omega)), F(\Omega, G(A, B)))$. The following table reveals the structure of M (see figure 4.5).

i	$M_{(i)}$	$\&_i(M)$
0	$F(F(\Omega, G(B,\Omega)), F(\Omega, G(A,B)))$	$\{1.2, 2\}$
1	$F(F(\Omega,\Omega),\Omega)$	$\{1\}$
2	$F(\Omega,\Omega)$	$\{\lambda\}$
3	Ω	$\varnothing$

We have $\phi(M) = 3$ and $\&(M) = \{\lambda, 1, 1.2, 2\}$.

Let $n \geqq 1$. A term $M \in T_\Diamond$ is called a *level-n tower* if $\phi(M) = n$ and for all $0 \leqq i < n$ we have $|\&_i(M)| = 1$. We say that $M \in T_\Diamond$ *contains a level-n tower N at occurrence u* if $u \in \&(M)$,

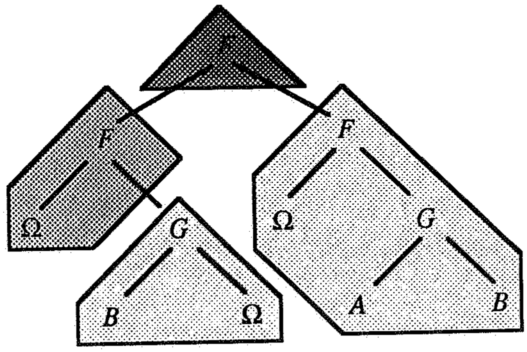


Figure 4.5.

$N \preceq M/u$ and $\forall v \in O_\Omega(N)$ we have $v \in O_\Omega(M/u)$ or $v \in \&(M/u)$. Let $M \in T_\Diamond$ and $1 \leqq n \leqq \phi(M)$. We define

$$\begin{aligned} Tow_n(M) &= \{[N,u] \mid M \text{ contains a level-}n\text{ tower } N \text{ at occurrence } u\}, \\ Tow_n^*(M) &= \{[N,u] \in Tow_n(M) \mid N \text{ is a half-redex}\}, \\ Tow(M) &= \bigcup_{1 \leqq n \leqq \phi(M)} Tow_n(M), \\ Tow^*(M) &= \bigcup_{1 \leqq n \leqq \phi(M)} Tow_n^*(M). \end{aligned}$$

Example 4.4. Let $\mathcal{R}_\Omega$ and M be the same as in example 4.3. The following table gives all towers contained in M. The underlined elements are member of $Tow^*(M)$.

n	$Tow_n(M)$
1	$\{\ [F(\Omega,\Omega),\lambda],\ [F(\Omega,\Omega),1],\ [G(B,\Omega),1.2],\ \underline{[F(\Omega,G(A,B)),2]}\ \}$
2	$\{\ [F(F(\Omega,\Omega),\Omega),\lambda],\ \underline{[F(\Omega,G(B,\Omega)),1]},\ [F(\Omega,F(\Omega,G(A,B))),\lambda]\ \}$
3	$\{\ [F(F(\Omega,G(B,\Omega)),\Omega),\lambda]\ \}$

We write $M \rhd M'$ if $M \in T_\Diamond$ and $\exists [N,u] \in Tow^*(M)$ such that $M' = M[uv \leftarrow \Omega \mid v \in \Gamma(N)]$, see figure 4.6. If M is a half-redex then $M \rhd \overline{M}$. The following lemma is a generalization of lemma 4.4 to terms in $T_\Diamond$.

Lemma 4.6. *If $M \rhd M'$ then $M' \prec M$ and $I(M') \subset I(M)$.*

Proof. The first part is obvious. Let $v \in I(M')$. If $v \in O_\Omega(M)$ then $v \in I(M)$ by lemma 3.3. So assume $v \notin O_\Omega(M)$. We know that $M' = M[uv' \leftarrow \Omega \mid v' \in \Gamma(N)]$ for some $[N,u] \in Tow^*(M)$. $v \notin O_\Omega(M)$, so $v/u \in \Gamma(N)$ and thus $v/u \notin I(\overline{N})$ by lemma 4.3. But $v \in I(M')$ implies $v/u \in I(M'/u)$, and because $\overline{N} \preceq M'/u$ and $v/u \in O_\Omega(\overline{N})$, we get $v/u \in I(\overline{N})$

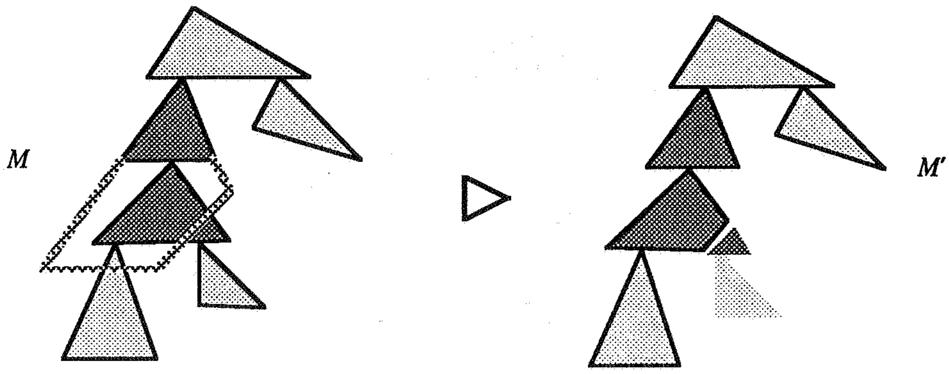


Figure 4.6.

by lemma 3.6. Contradiction. □

Let $\rhd\!\rhd$ be the transitive and reflexive closure of $\rhd$ ($\rhd$ is a binary relation on $T_\Diamond$). $M \in T_\Diamond$ is in $\rhd$*-normal form* if there exists no $M' \in T_\Diamond$ with $M \rhd M'$. So a $\rhd$-normal form does not contain a tower which is also a half-redex. (One can even proof that a $\rhd$-normal form does not contain a subterm which is a half-redex, but we will not need this.) Note that we are interested in eliminating half-redexes in favour of pre-redexes since there are only finitely many pre-redexes, but possibly infinitely many half-redexes.

Lemma 4.7. *If $M \rhd\!\rhd M'$ and M' is in $\rhd$-normal form, then $M' \leq M$, $Tow^*(M') = \varnothing$ and $I(M') \subset I(M)$.*

Proof. Immediate consequence of lemma 4.6 and definitions. □

Example 4.5. Consider again $\mathcal{R}_\Omega$ and M from examples 4.3 and 4.4. $M' = F(F(\Omega, G(\Omega, \Omega)), F(\Omega, G(\Omega, B)))$ is the unique $\rhd$-normal form of M. Although in this example we have $I(M) = I(M')$, in general equality does not hold. For instance, in example 4.2 we have $M \rhd \overline{M}$ and $\overline{M}$ is in $\rhd$-normal form.

Corollary 4.2. *If $\mathcal{R}$ is not strongly sequential then there is a parallel term in $\rhd$-normal form.* □

We will now show that we only need to consider terms in $\rhd$-normal form with a bounded depth.

The *depth* of $\mathcal{R}_\Omega$, ϱ, is defined by

$$depth(\Omega) = 0,$$
$$depth(F(M_1,\dots,M_n)) = 1 + \max\{depth(M_i) \mid 1 \leqq i \leqq n\},$$
$$\varrho = \max\{depth(\alpha_\Omega) \mid \alpha_\Omega \in \mathcal{R}_\Omega\}.$$

Lemma 4.8. *Let $M \in T_\Omega$ and $u \in O(M)$ such that $|u| \geqq \varrho$ and $M[u \leftarrow \bullet]\#$. Then $M\#$.*

Proof. Suppose $M \uparrow \alpha_\Omega$. If $u \in O(\alpha_\Omega)$ then $\alpha_\Omega/u = \Omega$, because $\alpha_\Omega/u \neq \Omega$ would imply

$$\varrho \geqq depth(\alpha_\Omega) \geqq |u| + 1 \geqq \varrho + 1$$

But if $\alpha_\Omega/u = \Omega$ then $M[u \leftarrow \bullet] \uparrow \alpha_\Omega$. Thus $u \notin O(\alpha_\Omega)$, but then again $M[u \leftarrow \bullet] \uparrow \alpha_\Omega$. So we conclude that $M\#$. □

The following lemma is a partial transivity result for indexes. It plays a crucial role in the first proof of the decidability of strong sequentiality, because it enables us to restrict the search for a parallel term to a finite set of Ω-terms which are entirely built from atomic pre-redexes.

Lemma 4.9 (u-v-w Lemma). *Let $M \in NF_\Omega$, $u, v, w \in O(M)$ such that $u < v < w$, $v \in I(M[v \leftarrow \Omega])$, $w/u \in I(M/u)$ and $|v/u| \geqq \varrho - 1$. Then $w \in I(M)$.*

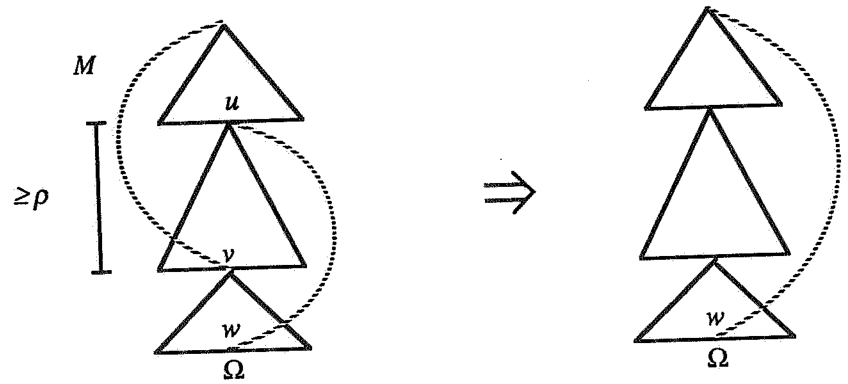


Figure 4.7.

Proof. Suppose $\omega(M[w \leftarrow \bullet]) = \omega(M)$. Then $\exists u' < w\ \overline{\omega}(M[w \leftarrow \bullet]/u')\uparrow$. Suppose $u' \geqq u$. Then $\overline{\omega}(M/u[w/u \leftarrow \bullet]/(u'/u))\uparrow$, which implies $\omega(M/u[w/u \leftarrow \bullet]) = \omega(M/u)$. But this is a contradiction with $w/u \in I(M/u)$. Thus $u' < u$. Take the maximum such u'. Clearly $|v/u'| \geqq |v/u| + 1 \geqq \varrho$. Because u' is maximum, we have $v/u' \in O(\overline{\omega}(M[w \leftarrow \bullet]/u'))$ and thus $\overline{\omega}(M[w \leftarrow \bullet]/u')[v/u' \leftarrow \bullet] = \overline{\omega}(M[v \leftarrow \bullet]/u')$, and $\overline{\omega}(M[v \leftarrow \bullet]/u')\#$ because

$v \in I(M[v \leftarrow \Omega])$. With the previous lemma we get $\overline{\omega}(M[w \leftarrow \bullet]/u')\#$. Contradiction, so $\omega(M[w \leftarrow \bullet]) \neq \omega(M)$ and thus $w \in I(M)$. □

The following example shows that the bound $\varrho - 1$ in the *u-v-w*-lemma cannot be relaxed.

Example 4.6. Let $\mathcal{R}_\Omega = \{F(G(H(\Omega),F(H(\Omega)))),\ G(B,F(H(\Omega)))\}$. Let $M = F(G(\Omega,F(H(\Omega))))$, $u = 1$, $v = 1.2.1$ and $w = 1.2.1.1$. We have $\varrho = 4$, $v \in I(M[v \leftarrow \Omega]) = \{1.1, 1.2.1\}$, $w/u \in I(M/u) = \{1, 2.1.1\}$ and $|v/u| = 2 = \varrho - 2$, but $w \notin I(M) = \{1.1\}$.

We call a parallel term M *minimal* if $|O(M)| \leqq |O(M')|$ whenever M' is a parallel term.

Let $M \in T_\Diamond$ and $[N,u]$, $[N',u'] \in Tow(M)$ with $u \neq u'$. We say that $[N,u]$ and $[N',u']$ *lie on the same branch of M* if $\exists [N'',\lambda] \in Tow(M)$ such that $[N,u]$, $[N',u'] \in Tow(N'')$.

A level-n tower N is called a *relevant tower* if $\&_0(N) = \{v\}$, $|v| \geqq \varrho - 1$ and $\forall u \in \&(N)$ $\lambda \neq u < v \Rightarrow |v/u| < \varrho - 1$.

Lemma 4.10. *Let M be a minimal parallel term and* $[N,u]$, $[N,u'] \in Tow(M)$ *relevant towers. Then* $[N,u]$ *and* $[N,u']$ *do not lie on the same branch of M.*

Proof. Suppose $[N,u]$ and $[N,u']$ do lie on the same branch of M. Then $\exists [N',\lambda] \in Tow(M)$ with $[N,u]$, $[N,u'] \in Tow(N')$. Without loss of generality $u < u'$. Define w by $w \in O_\Omega(N)$ and $uw \in \&(N')$. Notice that w is uniquely defined. Let $M' = M[uw \leftarrow M/u'w]$ (see figure 4.8). We will show that M' is a parallel term. Suppose $I(M') \neq \emptyset$. Let $v \in I(M')$. If $v \mid uw$ then $v \in I(M'[uw \leftarrow \Omega]) = I(M[uw \leftarrow \Omega])$ by lemma 3.5. And thus by lemma 3.3, $v \in I(M)$. But this is impossible because M is a parallel term. So, if $v \in I(M')$ then $v \geqq uw$. By lemma 3.4 we have $v/u \in I(M'/u)$. We get $v/u \in I(M/u')$ by lemma 3.6. Because M is a minimal parallel term, we have $u'v' \in I(M[u'v' \leftarrow \Omega])$, where $\{v'\} = \&_0(N)$. Because $|u'v'/u'| = |v'| \geqq \varrho - 1$, we can use the *u-v-w*-lemma to get $v \in I(M)$. Impossible, so $I(M') = \emptyset$. Thus M' is a parallel term.
But clearly $|O(M')| < |O(M)|$. Contradiction with the minimality of M. □

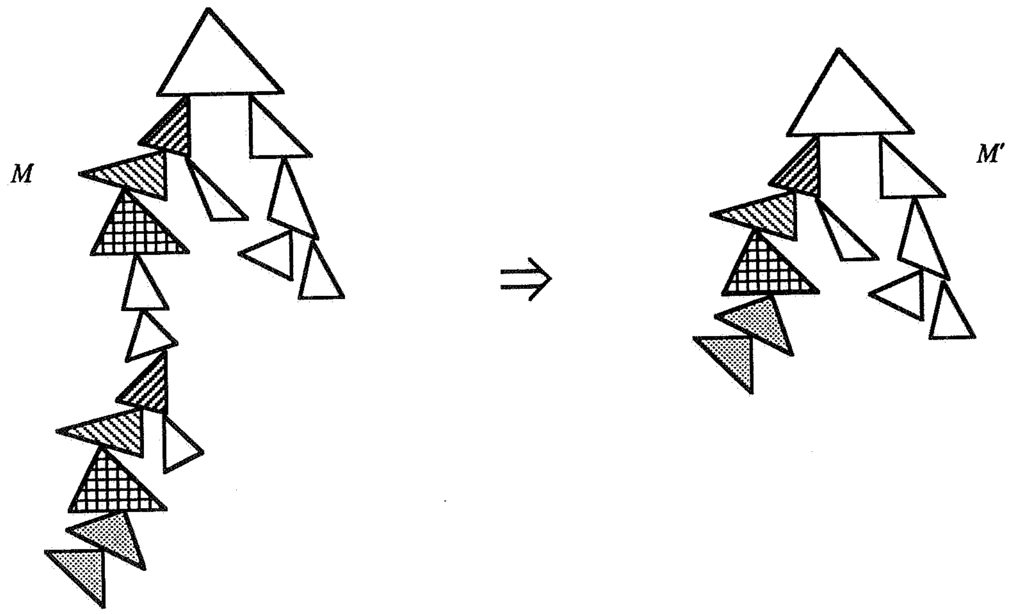


Figure 4.8.

Because the number of atomic pre-redexes is finite, there are only finitely many relevant towers of atomic pre-redexes. Thus a repetition of a relevant tower along a branch must occur within a computable bound, and inspection of all terms consisting of atomic pre-redexes with depth up to this bound reveals strong sequentiality or the absence of it.

The following simple procedure finds a parallel term if $\mathcal{R}$ is not strongly sequential.

step 0. $M_0 = \Omega$, $i = 0$

After *step* 0 we repeatedly perform *step* 1.

step 1. If $I(M_i) = \varnothing$ then we stop: M_i is a parallel term. Otherwise, choose $v_i \in I(M_i)$. Let $M_{i+1} = M_i[v_i \leftarrow P_i]$, where P_i is any atomic pre-redex. If $|v_i| \geqq \varrho - 1$ then M_{i+1} contains a relevant tower N_i at occurrence u_i such that $N_i/(v_i/u_i) = P_i$. If this particular tower has already occurred along the branch of M_{i+1} to P_i (i.e. $\exists u_i' < u_i$ with $[N_i, u_i'] \in Tow(M_{i+1})$ and $[N_i, u_i]$ and $[N_i, u_i']$ lie on the same branch of M_{i+1}), we stop unsuccesfully the construction of this particular term. This is justified by lemma 4.10: if M_{i+1} can be extended to a parallel term, then there is also a parallel term which contains only one occurrence of N_i along the branch of M_{i+1} to P_i.

$i = i + 1$, repeat *step* 1.

5. Δ-sets and Increasing Indexes

In [HL79] it is proved that strong sequentiality is equivalent to the existence of so-called Δ-sets. For every pre-redex M, $\Delta(M)$ is a non-empty subset of $I(M)$ subject to the following constraint: for all $u \in \Delta(M)$, for all pre-redexes N, if $M[u \leftarrow N]$ is a pre-redex then $\{v \mid uv \in \Delta(M[u \leftarrow N])\}$ is a non-empty subset of $\Delta(N)$. Assuming the existence of Δ-sets, Huet and Lévy constructed a 'matching dag', a special kind of graph on which they defined an efficient algorithm to find a needed redex in a given term. (In [HL79] it is proved that strong sequentiality is equivalent to the existence of a function Q satisfying two constraints (Q_1) and (Q_2). The equivalent notion of Δ-sets stems from [H86].) Actually, the notion of Δ-sets in [HL79] and [H86] is more complicated than the one we use, since in [HL79], [H86] it involves so-called 'directions', not introduced in the present paper.

The second part of the equivalence proof (existence Δ-sets ⇒ strong sequentiality) is in essence a correctness proof of their algorithm. In this section we will give a direct proof of this implication. For the other implication (strong sequentiality ⇒ existence Δ-sets) we use the increasing indexes of [HL79].

Let $M \in NF_\Omega$. The set of its *increasing indexes* $J(M)$ is defined by

$$J(M) = \{u \in I(M) \mid \forall N \in T_\diamond \; \exists v \geqq u \;\; v \in I(M[u \leftarrow N])\}.$$

The following lemma shows that if $\mathcal{R}$ is strongly sequential then every term $M \in NF_\Omega$ has at least one increasing index. (Lemma's 5.1-5.3 and theorem 5.1 are also in [HL79].)

Lemma 5.1. If $\mathcal{R}$ is strongly sequential then for any $M \in NF_\Omega$ we have $J(M) \neq \varnothing$.

Proof. Because $\mathcal{R}$ is strongly sequential $I(M) \neq \varnothing$, say $I(M) = \{u_1, \dots, u_n\}$. Assume $J(M) = \varnothing$. Then for every $i \in \{1, \dots, n\}$ there exists a $N_i \in T_\diamond$ such that $\{v \in I(M[u_i \leftarrow N_i]) \mid v \geqq u\} = \varnothing$. Consider the term $M' = M[u_i \leftarrow N_i \mid 1 \leqq i \leqq n]$. We leave it to the reader to show that M' contains no redexes (Hint: use lemma 3.1). Clearly $O_\Omega(M') \neq \varnothing$ and thus $I(M') \neq \varnothing$. Let $v \in I(M')$. If there exists a $k \in \{1, \dots, n\}$ with $u_k \leqq v$ then by lemma 3.6 we have $v \in I(M[u_k \leftarrow N_k])$. Contradiction, thus $v \mid u_i$ for all $i \in \{1, \dots, n\}$. Then $v \in I(M)$, again by lemma 3.6. But $v \notin \{u_1, \dots, u_n\}$. Contradiction. We conclude that $J(M) \neq \varnothing$. □

The suffix-property for indexes (lemma 3.4) also holds for increasing indexes.

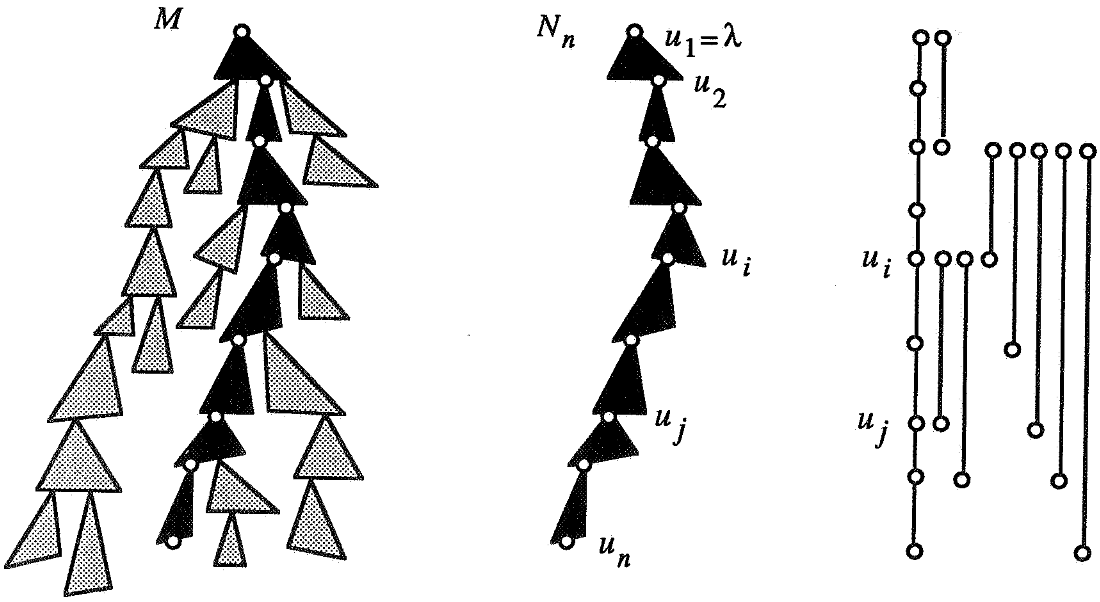
Lemma 5.2. *If $uv \in J(M)$ then $v \in J(M/u)$.*

Proof. Suppose $v \notin J(M/u)$. Thus there exists a term $N \in T_\Diamond$ such that $\{w \in I(M/u[v \leftarrow N]) \mid w \geqq v\} = \varnothing$. Let $M' = M[uv \leftarrow N]$. But $uv \in J(M)$ implies $\exists w \geqq uv$ with $w \in I(M')$ and thus $w/u \in I(M/u[v \leftarrow N])$ by lemma 3.4. Contradiction. □

Lemma 5.3. *Suppose $\mathcal{R}$ is strongly sequential. Let $M \in NF_\Omega$ and $N \in T_\Diamond$. For every $u \in J(M)$ there exists a $v \geqq u$ such that $v \in J(M[u \leftarrow N])$.*

Proof. Suppose $\{v \in J(M[u \leftarrow N]) \mid v \geqq u\} = \varnothing$. Let $\{u_1, \ldots, u_n\} = \{u_i \in I(M[u \leftarrow N]) \mid u_i \geqq u\}$. For every $i \in \{1, \ldots, n\}$ there exists a $N_i \in T_\Diamond$ such that $\{w \in I(M[u \leftarrow N][u_i \leftarrow N_i]) \mid w \geqq u_i\} = \varnothing$. Consider the term $M' = N[u_i/u \leftarrow N_i \mid 1 \leqq i \leqq n]$. We get a contradiction like in the proof of lemma 5.1. □

Now we are ready for the main theorem of this section. First we will give an intuitive description of the proof idea. As noted before, the problem with indexes is that they are not 'transitive'. However, 'partial transitivity' properties do hold; in our first proof of the decidability this was embodied by the u-v-w-lemma (4.9); in the following proof this is embodied by the Δ-sets. To show that the existence of Δ-sets guarantees the existence of an index in a term M, which may be supposed to be in $\rhd$-normal form (i.e. built from atomic pre-redexes and such that no tower in M is a half-redex), we select a maximal tower in M as in figure 5.1(a) which has the property that Δ-indexes are transmitted along the tower, in the following sense. The tower in figure 5.1(b) may contain next to the atomic pre-redexes, larger pre-redexes formed by some consecutive atomic pieces of the tower, e.g. as indicated in figure 5.1(c) where every line segment denotes a pre-redex between some u_i, u_j. Now for every such pre-redex between u_i, u_j we have that u_j/u_i is a Δ-index of that pre-redex. The result is that the tower under consideration leads indeed to a point u_n which is an index. This is seen as follows: if the test symbol $\bullet$ is inserted at u_n, then the tower is perfectly rigid: no chunk can be melted away. This is so first by our use of atomic pre-redexes (so no chunk away from the main path $u_1 - u_2 - \ldots - u_n$ of the tower can be melted away) and second of the arrangement that all pre-redexes in the tower 'looking at' the test symbol ● at u_n, have an index at that point. We will now give the precise proof.



(a) (b) (c)

Figure 5.1.

Theorem 5.1. $\mathcal{R}$ *is strongly sequential iff there exist* Δ*-sets for* $\mathcal{R}$.

Proof.

⇒ If $\mathcal{R}$ is strongly sequential then it follows from lemma's 5.1, 5.2 and 5.3 that the increasing indexes J satisfy the conditions for being Δ-sets.

⇐ We have to prove that every term $M \in NF_{\Omega}$ has an index. By previous results (lemma's 4.1 and 4.7) it is enough to prove that every term $M \in T_{\diamond}$ in $\rhd$-normal form has an index. Let M be any such term. We will define a sequence $N_1 \leq N_2 \leq \ldots \leq N_n \leq M$ such that for all $1 \leqq i \leqq n$ we have $[N_i, \lambda] \in \mathit{Tow}_i(M)$ and there exist $u_i, v_i \in O(N_i)$ such that for all $w \in \&(N_i)$ with $v_i \leqq w < u_i$ and $N_i/w \prec \mathcal{R}_{\Omega}$ we have $u_i/w \in \Delta(N_i/w)$ $(\star\star)$.

N_1 is defined by $[N_1, \lambda] \in \mathit{Tow}_1(M)$ $(N_1 = M_{(\phi(M)-1)})$, $v_1 = \lambda$ and $u_1 \in \Delta(N_1)$.

⋮

Suppose we have defined $N_1, \ldots, N_{i-1}$. Define N_i by $[N_i, \lambda] \in \mathit{Tow}_i(M)$, $N_{i-1} \leq N_i$ and $\&_1(N_i) = \{u_{i-1}\}$. Let $v_i \in \{\lambda, u_1, \ldots, u_{i-1}\}$ be minimal under the restriction that

$N_i/v_i \prec \Re_\Omega$. Clearly $v_{i-1} \leqq v_i$. If $v_i = u_{i-1}$ we choose $u_i \in \Delta(N_i/v_i)$. In this case the hypothesis (★★) is clearly satisfied. Otherwise $v_i < u_{i-1}$. We have $N_{i-1}/v_i < N_i/v_i \prec \Re_\Omega$. By induction hypothesis we know that $\forall w' \in \&(N_{i-1})$ with $v_{i-1} \leqq w' < u_{i-1}$ and $N_{i-1}/w' \prec \Re_\Omega$, we have $u_{i-1}/w' \in \Delta(N_{i-1}/w')$. In particular, letting $w' = v_i$, we get $u_{i-1}/v_i \in \Delta(N_{i-1}/v_i)$, and from the definition of Δ-sets it follows that $\exists u' \succ u_{i-1}/v_i$ with $u' \in \Delta(N_i/v_i)$ and $u'/u_{i-1} \in \Delta(N_i/u_{i-1})$. Define $u_i = v_i \cdot u'$. We have to prove that $\forall w \in \&(N_i)$ with $v_i \leqq w < u_i$ and $N_i/w \prec \Re_\Omega$, we have $u_i/w \in \Delta(N_i/w)$ (★★). The case $w = v_i$ has already been done. So assume $v_i < w$. Because $w \in \&(N_i)$, we know that $w = u_k$ for some $k < i$. We have $v_i \in \&(N_k)$, $v_k \leqq v_i < u_k$ and $N_k/v_i \prec \Re_\Omega$. By induction hypothesis we get $u_k/v_i \in \Delta(N_k/v_i)$. Thus we have $w/v_i \in \Delta(N_k/v_i) = \Delta(N_i/v_i[w/v_i \leftarrow \Omega])$. The definition of Δ-sets gives us $\{v \mid (w/v_i)v \in \Delta(N_i/v_i)\} \subset \Delta(N_i/w)$. Because $u_i/v_i \in \Delta(N_i/v_i)$ we finally get $u_i/w \in \Delta(N_i/w)$.

⋮

We stop with the triple (N_n, v_n, u_n) if there is no N_{n+1} subject to the constraints above.

We will now prove that $u_n \in I(N_n)$. Let $N^w = N_n[u_n \leftarrow \bullet]/w$ for every $w \in O(N_n)$. We will prove that $\omega(N^w) = N^w$. If $w \mid u_n$ then N^w is a subterm of a small pre-redex and thus $\omega(N^w) = N^w$ by lemma 4.5. Together with $u_n/v_n \in I(N_n/v_n)$ this implies that if $w \geqq v_n$ then $\omega(N^w) = N^w$. For the remaining case $w \leqq v_n$ we use induction on $|v_n/w|$. If $w = v_n$ we already know that $\omega(N^w) = N^w$. So assume $w < v_n$. By induction hypothesis we have $\bar{\omega}(N^w) = N^w$. If $w \notin \&(N_n) = \{\lambda, u_1, \ldots, u_{n-1}\}$ then clearly $N^w\#$ and thus $\omega(N^w) = N^w$. If $w \in \&(N_n)$ then $N_n/w\#$ because v_n was minimal under the restriction that $N_n/v_n\uparrow$. And $N_n/w\#$ implies $N^w\#$ and thus $\omega(N^w) = N^w$.

Therefore $u_n \in I(N_n)$ and by lemma 3.3 we finally get $u_n \in I(M)$.

□

Because it is straightforward to give an (inefficient) algorithm for finding Δ-sets, theorem 5.1 gives a decision procedure for strong sequentiality.

With theorem 5.1 we can give a very easy proof that strong sequentiality is a 'modular' property.

Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be regular term rewriting systems. The *direct sum* of $\mathcal{R}_1$ and $\mathcal{R}_2$, notation $\mathcal{R}_1 \oplus \mathcal{R}_2$, is the TRS obtained by taking the disjoint union of $\mathcal{R}_1$ and $\mathcal{R}_2$. That is, if the alphabets of $\mathcal{R}_1$ and $\mathcal{R}_2$ are disjoint ($F_{\mathcal{R}_1} \cap F_{\mathcal{R}_2} = \emptyset$), then $\mathcal{R}_1 \oplus \mathcal{R}_2 = \mathcal{R}_1 \cup \mathcal{R}_2$; otherwise we take renamed copies $\mathcal{R}_1'$ and $\mathcal{R}_2'$ of $\mathcal{R}_1$ and $\mathcal{R}_2$ such that these copies have disjoint alphabets and define $\mathcal{R}_1 \oplus \mathcal{R}_2 = \mathcal{R}_1' \cup \mathcal{R}_2'$.

Lemma 5.4. *$\mathcal{R}_1 \oplus \mathcal{R}_2$ is strongly sequential iff $\mathcal{R}_1$ and $\mathcal{R}_2$ are strongly sequential.*

Proof.

$\Rightarrow$ If $\mathcal{R}_1 \oplus \mathcal{R}_2$ is strongly sequential, we can find Δ-sets, say $\Delta_{\mathcal{R}_1 \oplus \mathcal{R}_2}$. The restriction of $\Delta_{\mathcal{R}_1 \oplus \mathcal{R}_2}$ to pre-redexes of $\mathcal{R}_1$ clearly satisfies the conditions for being Δ-sets (with respect to $\mathcal{R}_1$); likewise for $\mathcal{R}_2$.

$\Leftarrow$ Because $\mathcal{R}_1$ and $\mathcal{R}_2$ are strongly sequential we can find Δ-sets, say $\Delta_{\mathcal{R}_1}$ and $\Delta_{\mathcal{R}_2}$. Define $\Delta_{\mathcal{R}_1 \oplus \mathcal{R}_2}$ by

$$\Delta_{\mathcal{R}_1 \oplus \mathcal{R}_2}(M) = \begin{cases} \Delta_{\mathcal{R}_1}(M) & \text{if } M \text{ is a pre-redex in } \mathcal{R}_1, \\ \Delta_{\mathcal{R}_2(M)} & \text{if } M \text{ is a pre-redex in } \mathcal{R}_2. \end{cases}$$

It is not difficult to see that $\Delta_{\mathcal{R}_1 \oplus \mathcal{R}_2}$ satisfies the conditions for being Δ-sets. Therefore, $\mathcal{R}_1 \oplus \mathcal{R}_2$ is strongly sequential.

□

6. Term Rewriting Systems with Deep Parallel Terms

We conjectured for some time that, with help of the *u*-*v*-*w*-lemma, it should be possible to prove that the depth of a minimal parallel term is bounded by 2ϱ or perhaps 3ϱ (where ϱ is the maximum depth of the redex schemes as defined in section 4), which would imply a very simple decision procedure for strong sequentiality: just check all terms with depth up to 2ϱ (3ϱ). Unfortunately, this is not the case. Consider the following term rewriting systems $\mathcal{R}^n$ ($n \geqq 3$).

$$
\begin{aligned}
\mathcal{R}^n \;=\; \{\quad & F_0(A,B,x) \rightarrow \ldots && (\alpha_0),\\
& F_1(F_0(x,A,B),A) \rightarrow \ldots && (\alpha_1),\\
& F_2(F_1(F_0(B,x,A),B),A) \rightarrow \ldots && (\alpha_2),\\
& F_3(F_2(F_1(A,x),B),A) \rightarrow \ldots && (\alpha_3),\\
& \quad\vdots \\
& F_{n-1}(F_{n-2}(F_{n-3}(A,x),B),A) \rightarrow \ldots && (\alpha_{n-1}),\\
& F_n(F_{n-1}(F_{n-2}(A,x),y),z) \rightarrow \ldots && (\alpha_n) \qquad \}
\end{aligned}
$$

Lemma 6.1. $(\alpha_0)+(\alpha_1)+\ldots+(\alpha_{n-1})$ *is strongly sequential.*

Proof. We will inductively construct collections Δ_i of Δ-sets for $2 \leqq i \leqq n-1$, satisfying the conditions for being Δ-sets with respect to $(\alpha_0)+\ldots+(\alpha_i)$. The collection Δ_2 is defined by (the underlined Ω's denote the Δ-indexes for the relevant terms):

$F_1(\Omega,\underline{\Omega})$,
$F_2(\Omega,\underline{\Omega})$,
$F_2(F_1(\Omega,\underline{\Omega}),\underline{\Omega})$,
$F_2(F_1(\Omega,\underline{\Omega}),A)$.

For all other pre-redexes M of $(\alpha_0)+(\alpha_1)+(\alpha_2)$ we define

$$\Delta_2(M) = I(M).$$

It is not difficult to show that Δ_2 satisfies the conditions for being Δ-sets with respect to $(\alpha_0)+(\alpha_1)+(\alpha_2)$.

Suppose we have defined $\Delta_2,\ldots,\Delta_i$. For pre-redexes of $(\alpha_0)+(\alpha_1)+\ldots+(\alpha_i)$ we define Δ_{i+1} by

$F_{i-1}(\underline{\Omega},\underline{\Omega})$,

$$\Delta_{i+1}(M) = \Delta_i(M) \quad \text{if } M \text{ is different from } F_{i-1}(\Omega,\Omega).$$

For pre-redexes of (α_{i+1}) we define Δ_{i+1} by

$$F_{i+1}(\Omega,\underline{\Omega}),$$
$$F_{i+1}(F_i(\Omega,\underline{\Omega}),\underline{\Omega}),$$
$$F_{i+1}(F_i(\Omega,\underline{\Omega}),A),$$
$$F_{i+1}(F_i(F_{i-1}(\Omega,\Omega),\underline{\Omega}),\underline{\Omega}),$$
$$F_{i+1}(F_i(F_{i-1}(\Omega,\Omega),\underline{\Omega}),A),$$
$$\Delta_{i+1}(M) = I(M) \quad \text{if } M \text{ is different from the terms above.}$$

We leave it to the reader to verify that Δ_{i+1} indeed satisfies the conditions for being Δ-sets with respect to $(\alpha_0)+ \ldots +(\alpha_i)$.

Theorem 5.1 gives us the strong sequentiality of $(\alpha_0)+ \ldots +(\alpha_{n-1})$. □

Lemma 6.2. *If M is a minimal parallel term then for all $u \in \overline{O}(M)$ we have $I(M[u \leftarrow \Omega]) = \{u\}$.*

Proof. Suppose $\exists u \in \overline{O}(M)$ such that $I(M[u \leftarrow \Omega]) \neq \{u\}$. Because M is minimal and $|O(M[u \leftarrow \Omega])| < |O(M)|$, we have $I(M[u \leftarrow \Omega]) \neq \varnothing$. If $v \in I(M[u \leftarrow \Omega])$ and $v \,|\, u$ we have $v \in I(M)$ by lemma 3.3. Contradiction. Therefore, $I(M[u \leftarrow \Omega]) = \{u\}$. □

Lemma 6.3. *$\mathcal{R}^n$ is not strongly sequential; the minimal parallel term is $F_n(F_{n-1}(\ldots F_2(F_1(F_0(\Omega,\Omega,\Omega),\Omega)\ldots),\Omega)$.*

Proof. Because $I(F_n(F_{n-1}(\ldots F_2(F_1(F_0(\Omega,\Omega,\Omega),\Omega),\Omega)\ldots),\Omega) = \varnothing$, $\mathcal{R}^n$ is not strongly sequential. By lemma 6.1 we know that $(\alpha_0)+ \ldots +(\alpha_{n-1})$ is strongly sequential. This implies that every parallel term contains the symbol F_n. Let M be a minimal parallel term. Suppose $\exists u \neq \lambda$ with $M(u) = F_n$. Let $N = M/u$. It is not difficult to show that $N \in T_\diamond$, using the fact that if M is a minimal parallel term then $\omega(M) = \Omega$ (lemma 4.1). Because M is minimal we have $I(N) \neq \varnothing$. Let $v \in I(N)$. We leave it to the reader to show that $uv \in I(M)$ (Hint: prove by induction on $|uv/w|$ that $uv/w \in O(\omega(M[uv \leftarrow \bullet]/w))$ for all $w \leqq uv$). But this is impossible. We conclude that $M(\lambda) = F_n$ and $\forall u \in \overline{O}(M) - \{\lambda\}$ $M(u) \neq F_n$.

If we write $M = F_n(M_1,M_2)$, it is easy to prove that $M_2 = \Omega$. If $M_1(\lambda) \neq F_{n-1}$, a similar argument to the one above yields a contradiction with the minimality of M. Therefore $M = F_n(F_{n-1}(M_3,M_4),\Omega)$.

Repeated application of the argument above yields $M = F_n(F_{n-1}(\ldots F_2(F_1(F_0(N_1,N_2,N_3),\Omega),\Omega)\ldots),\Omega)$, and this implies that $F_n(F_{n-1}(\ldots F_2(F_1(F_0(\Omega,\Omega,\Omega),\Omega),\Omega)\ldots),\Omega)$ is the minimal parallel term. □

Corollary 6.1. *For every $n \geqq 1$ there is a term rewriting system which is not strongly sequential and whose minimum parallel term has depth $n \cdot \varrho$.*

The above gives evidence that deciding strong sequentiality is not a trivial matter. Indeed, there is no known efficient method for finding Δ-sets. Huet and Lévy pointed out that for the practically relevant case of systems with constructors, deciding strong sequentiality is easy. A *system with constructors* is a term rewriting system in which the set of function symbols F is partitioned into sets F_R, the set of (recursive) function symbols, and F_C, the set of constructors. Every left-hand side has the form $F(M_1,\ldots,M_n)$ with $F \in F_R$ and where the M_i's are terms over $F_C \cup V$.

Lemma 6.4. ([HL79]) *A system with constructors is strongly sequential iff the indexes I satisfy the conditions for being Δ-sets.*

Proof. Easy. □

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