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Calculation of the Availability of a Two-Unit Parallel System with Cold Standby:

An Illustration of the Embedding Technique

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A two-unit parallel system with cold standby, Markovian degradation of the working unit and one repair facility is considered. The calculation of the availability of this system under a control limit repair policy is discussed. Iterative schemes for computing the availability in case of Erlangian repair-time distributions are presented along with numerical results.

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1. Introduction

KAWAI [5] considers a two unit parallel system with cold standby, Markovian degradation of the working unit and one repair facility. He shows that under certain conditions (to be specified below) on the degradation- and repair-process the optimal preventive repair policy is of control limit type i.e. a preventive repair on the working unit is carried out if and only if the repair facility is free and the condition of the working unit is less than or equal to a specified critical level. Moreover, KAWAI [5] provides an explicit expression for the availability of the system under a given control limit rule. Numerical computations are given for the case of exponential distributed repair times.

In this paper we show that the embedding technique from Markov decision theory can be succesfully applied to obtain another, more rigorous derivation of the availability under a control limit rule. Since the explicit formula of the availability contains expressions which depend on the transient behaviour of a continuous time Markov chain we propose an iterative computational scheme to numerically compute the availability in case of Erlangian distributed repair times. Finally we present some indications of the influence of a second repair facility on the availability of the system.

This paper is organized as follows. In section 2 we describe the model and present some preliminaries which are used in the sequel. Also we briefly review the results of KAWAI [5] on the existence of an optimal control limit rule. In section 3 we show how the embedding technique from Markov decision theory can be succesfully employed to rigorously derive an explicit formula for the long-run availability of the system under a control limit rule. In section 4 we present an iterative scheme to numerically compute the availability in case of Erlangian distributed repair times. Finally, we consider in section 5 the case of ample repair facilities.

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2. Model description and preliminaries

A (production) mechanism consists of two parallel units. At most one of the units is working at a time. The other unit is either in repair or in cold standby position. The working condition of the operating unit is described by a state variable taking on values from the state space

$$\mathbb{S} := \{0,1,...,n+1\}.$$

State 0 denotes a perfect condition, states 1 upto n are degraded working conditions while $n+1$ denotes a malfunction. Under the absence of repair the working condition deteriorates according to a time homogeneous continuous time Markov chain with conservative infinitesimal generator $Q = (q_{ij})$. We will assume that a working unit cannot improve on its own, i.e.

$$q_{ij} = 0 \text{ for } j<i, \ i \in \mathbb{S}. \tag{2.1}$$

When a working unit enters state $n+1$ (which acts as an absorbing state) an emergency (or: type 1) repair is required. As long as the working unit has not yet entered state $n+1$ the option of a preventive (or: type 2) repair exists provided the repair facility is free. The type 1 and type 2 repair times form two mutually independent sequences of i.i.d.random variables with distribution functions G_1 and G_2 with finite means ν_1 en ν_2 and $G_i(0) = 0$, $i = 1,2$ respectively. The repair times are also independent of the sojourn times of the working unit in the various states.

When a unit enters the repair facility the cold standby unit takes the working position in state 0. After repair completion the unit takes the cold-standby position. A unit in cold-standby neither fails nor degrades. A system downtime starts when the working unit enters state $n+1$ while the other unit is still under repair and it ends as soon as this repair is finished.

The goal now is to schedule preventive repairs based on the continuous observation of the state of the working component in such a way that the long-run unavailability (the long-run fraction of down time) is minimized.

As an example of the model described above we mention a system driven by n parallel pumps, with a multivariate exponential lifetime distribution.

An intuitively appealing control rule for our model is the so-called control limit rule. A control limit rule is determined by a critical state $m \in \mathbb{S}$ (the control limit), such that preventive repair is prescribed as soon as the repair facility is free and the state of the working unit is greater than or equal to m. We denote a control limit rule with control limit m by $CLR(m)$ and the unavailability under the rule $CLR(m)$ is denoted by $g(m)$.

Sufficient conditions for the optimal control rule to be of CLR-type are given by Kawai [5]. For

$$q_i := \sum_{j=i+1}^{n+1} q_{ij}, \ i \in \mathbb{S}$$

we make the following assumptions

Assumption 2.1. For $i = 0,...,n$

$$0 < q_i \leqslant q_{i+1} \tag{2.2}$$

$$\frac{\sum_{j=k}^{n+1} q_{ij}}{q_i} \leqslant \frac{\sum_{j=k}^{n+1} q_{i+1,j}}{q_{i+1}}, \text{ for all } k \in \mathbb{S}. \tag{2.3}$$

With respect to the repair-time distributions G_1 and G_2 we introduce the following assumptions $(\overline{G}_i(t) := 1 - G_i(t), t \geqslant 0)$.

Assumption 2.2.

$$\overline{G}_1(t) \leqslant \overline{G}_2(t), \text{ for all } t \geqslant 0. \tag{2.4}$$

(We denote $G_1 \overset{st}{\leqslant} G_2$: stochastic order)

ASSUMPTION 2.3.

$$\frac{\bar{G}_1(t+s)}{\bar{G}_1(s)} \leqslant \frac{\bar{G}_2(t+s)}{\bar{G}_2(s)}, \text{ for all } s,t \geqslant 0. \tag{2.5}$$

(we denote $G_1 \overset{ucso}{\leqslant} G_2$: uniform conditional stochastic order)
For the proof of the next two theorems we refer to KAWAI [5].

THEOREM 2.4. *Under the assumptions 2.1 and 2.2 there exists a CLR-policy which minimizes the long-run unavailability.* □

THEOREM 2.5. *Under the assumptions 2.1 and 2.3 the long-run unavailability $g(m)$ under CLR(m) is unimodal as a function of m.* □

REMARK 2.6. (i) The notion of uniform conditional stochastic order (ucso) has been introduced by WHITT [10]. In [10] he shows that for absolutely continuous lifetime distributions ucso is equivalent to the monotone likelihood ratio property (cf. FERGUSON [4]).
(ii) $G_1 \overset{ucso}{\leqslant} G_2 \Rightarrow G_1 \overset{st}{\leqslant} G_2$. The converse is not true. For example take

$$G_1(t) = t,\ 0 \leqslant t \leqslant 1 \quad \text{and} \quad G_2(t) = \begin{cases} \frac{t}{2}, & 0 \leqslant t \leqslant \frac{1}{2} \\ 2t - \frac{3}{4}, & \frac{1}{2} \leqslant t \leqslant \frac{3}{4} \\ t, & \frac{3}{4} \leqslant t \leqslant 1 \end{cases}$$

then $G_1 \overset{st}{\leqslant} G_2$, while

$$\frac{\bar{G}_1(\frac{17}{24})}{\bar{G}_1(\frac{7}{24})} = \frac{7}{17} > \frac{16}{41} = \frac{\bar{G}_2(\frac{17}{24})}{\bar{G}_2(\frac{7}{24})}.$$

An important class of distribution functions satisfying assumption 2.3 are the Erlangian distributions under appropriate choice of the parameters.

LEMMA 2.7. *Let $G_{\lambda,p}(t)$ be the Erlang (λ,p) distribution function with $\lambda>0$ and $p \in \mathbb{N}$. If $p_1 \leqslant p_2$ and $\lambda_1 \geqslant \lambda_2$ then*

$$G_{\lambda_1,p_1} \overset{ucso}{\leqslant} G_{\lambda_2,p_2}.$$

PROOF.
(i) First assume $p_1 = p_2 = p$. Since

$$\bar{G}_{\lambda,p}(t) = \sum_{k=0}^{p-1} e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

the statement follows from the fact that the function $f(.,.,.)$ defined by

$$f(\lambda,t,s) := \frac{\sum_{k=0}^{p-1} e^{-\lambda(t+s)} \frac{(\lambda(t+s))^k}{k!}}{\sum_{k=0}^{p-1} e^{-\lambda s} \frac{(\lambda s)^k}{k!}}$$

is non-increasing in λ for all $s,t \geq 0$ (see also FERGUSON [4], p. 208).

(ii) Next we consider the case $\lambda_1 \geq \lambda_2$ and $p_2 = p_1+1$. Direct verification yields:

$$\frac{\overline{G}_{\lambda_2,p_1}(t+s)}{\overline{G}_{\lambda_2,p_1}(s)} \leq \frac{\overline{G}_{\lambda_2,p_1+1}(t+s)}{\overline{G}_{\lambda_2,p_1+1}(s)}.$$

(iii) The general case follows by straight forward induction from (ii). □

Throughout the rest of this paper we will assume that assumption 2.1 and 2.2 hold, which implies according to theorem 2.4 the optimality of a *CLR*.

We conclude this section with some additional notation and a useful lemma concerning entrance times for continuous time Markov chains.

Let $\{X(t), t \geq 0\}$ be a continuous time Markov chain on the state space $\mathbb{S}$ with infinitesimal generator Q, satisfying (2.1).

Define for $j \in \mathbb{S}$ and $k = 1,2$:

$$H_j(t) := P(X(t)=j \mid X(0)=0)$$

$$A_{jk} := \int_0^\infty H_j(t)\overline{G}_k(t)dt \tag{2.6}$$

$$a_{jk} := \int_0^\infty H_j(t)dG_k(t) \tag{2.7}$$

$$B_{jk} := \int_0^\infty H_j(t)G_k(t)dt. \tag{2.8}$$

Using Kolmogorov's forward differential equations:

$$H'_j(t) = -q_jH_j(t) + \sum_{i=0}^{j-1} H_i(t)q_{ij} \tag{2.9}$$

we find by partial integration for $k = 1,2$:

$$a_{jk} = \sum_{i=0}^{j} A_{ik}q_{ij}, \quad 1 \leq j \leq n+1 \tag{2.10}$$

$$a_{0k} = 1-q_0A_{0k} \tag{2.11}$$

$$B_{jk} = q_j^{-1}(a_{jk} + \sum_{i=0}^{j-1} B_{ik}q_{ij}), \; 0 \leq j \leq n. \tag{2.12}$$

Let

$$\tau_j := inf\{t \geq 0 : X(t)=j\}, \quad 0 \leq j \leq n+1$$

and

$$S(i,j) := \{X(\tau_j^-)=i\}$$

that is, τ_j is the entrance time of the Markov process into state j and $S(i,j)$ is the event that the entrance into j takes place by a jump from state i.

LEMMA 2.8.

$$P(\tau_j \geq t; S(i,j)) = \int_t^\infty q_{ij}H_i(s)ds.$$

PROOF. According to (2.1) we have

$$H_i(s) = \int_0^s e^{-q_i(s-u)} dP_{\tau_i}(u).$$

Hence

$$\int_t^\infty q_{ij} H_i(s) ds = \int_{s=t}^\infty q_{ij} \int_{u=0}^s e^{-q_i(s-u)} dP_{\tau_i}(u) ds$$

$$= \int_{u=0}^t \int_{s=t}^\infty q_{ij} e^{-q_i(s-u)} ds dP_{\tau_i}(u) + \int_{u=t}^\infty \int_{s=u}^\infty q_{ij} e^{-q_i(s-u)} ds dP_{\tau_i}(u).$$

This yields

$$\int_t^\infty q_{ij} H_i(s) ds = \frac{q_{ij}}{q_i} \int_{u=0}^t e^{-q_i(t-u)} dP_{\tau_i}(u) + \frac{q_{ij}}{q_i} P(t \leq \tau_i < \infty) \tag{2.13}$$

The first term on the right hand side of (2.13) denotes the probability that entrance into j takes place by a jump from state i, which itself is entered before t, while the process remains in i at least until t. The second term is the probability that entrance into j takes place from i which is entered beyond t. Together both terms give the probability that entrance into j takes place from state i beyond t. □

3. AN EXPLICIT FORMULA FOR $g(m)$

In this section we focus on the computation of $g(m)$, the unavailability of the system under the control rule $CLR(m)$ for some $1 \leq m \leq n$. We will show that

$$g(m) = \frac{(1-D_{m2})A_{n+1,1} + D_{m1}A_{n+1,2}}{(1-D_{m2})H_{m1} + D_{m1}H_{m2}} \tag{3.1}$$

where

$$D_{mk} := 1 - \sum_{j=m}^{n} q_{j,n+1} B_{jk}, \ k = 1,2$$

$$H_{mk} := \nu_k + \sum_{j=0}^{m-1} B_{jk}, \ k = 1,2,$$

while A_{jk} and B_{jk} are defined by (2.6) and (2.8).

This formula has been derived by KAWAI [5] using heuristic probabilistic arguments. The proof of (3.1) that we present here is based on an embedding technique from Markov (decision) theory. For other successful applications of this approach in the area of production control we refer to DE KOK et al. [3] and TIJMS et al. [8].

Let $\{Z(t), t \geq 0\}$ be the stochastic process on state space

$$\mathbb{S}_0 = \mathbb{S} \times [0,\infty) \times \{0,1,2\}$$

where $Z(t) = (Z_1(t), Z_2(t), Z_3(t))$ with $Z_1(t)$, $Z_2(t)$ and $Z_3(t)$ denoting the state of the working component, the elapsed "repair" time of the other component and the current type of "repair" respectively (type 0 ≡ cold standby).

Let

$$U := \{(n+1,x,k): x \geq 0, 1 \leq k \leq 2\} \subset \mathbb{S}_0$$

be the unavailability set.

Since $\{Z(t), t \geq 0\}$ is a regenerative process with regeneration state (0,0,2) we know that

$$g(m) := \lim_{t\to\infty} \frac{1}{t} \int_0^t P(Z(t) \in U) dt$$

exists (see e.g. ROSS [6] and TIJMS [7]).

To derive a formula for $g(m)$ we will study a properly chosen embedded Markov chain of the process $\{Z(t),t\geqslant 0\}$.

Unless stated otherwise we assume that the system $\{Z(t),t\geqslant 0\}$ starts in state (0,0,2). Let $T_0\equiv 0$ and, for $n\geqslant 1$, let T_n be the nth epoch at which the process $\{Z(t),t\geqslant 0\}$ enters the set $\{(0,0,1),(0,0,2)\}$, i.e. T_n denotes the nth epoch at which one unit goes into operation and the other goes under repair (either emergency or preventive).

Consider

$$X_n := Z_3(T_n).$$

Then $\{X_n:n\geqslant 0\}$ is a positive recurrent Markov chain on the state space $\{1,2\}$.

Define for $k = 1,2$:

$$\pi(k) := \lim_{n\to\infty} P(X_n=k),$$

$$\tau(k) := E(T_{n+1}-T_n \mid X_n=k)$$

$$c(k) := E(\int_{T_n}^{T_{n+1}} 1\{Z(t)\in U\}dt \mid X_n=k).$$

Now, it follows from the proof of theorem 7.5 in Ross [6] that

$$g(m) = \sum_{k=1}^{2}\pi(k)c(k)/\sum_{k=1}^{2}\pi(k)\tau(k). \tag{3.2}$$

For the computation of the stationary distribution $\{\pi(k)\}$ of $\{X_n:n\geqslant 0\}$ we introduce the generic random variables R_k, denoting the length of an arbitrary emergency ($k=1$) and preventive ($k=2$) repair time, with distribution function G_k and mean ν_k, $k = 1,2$.

Let (p_{ij}) be the matrix of one-step transition probabilities of $\{X_n:n\geqslant 0\}$ and recall that $\{X(t),t\geqslant 0\}$ denotes a continuous time Markov process on $\mathbb{S} = \{0,...,n+1\}$ with infinitesimal matrix Q.

Let

$$\sigma_m := inf\{t\geqslant 0:m\leqslant X(t)\leqslant n\}$$

$$\tau_{n+1} := inf\{t\geqslant 0:X(t)=n+1\}.$$

Then

$$p_{k2} = 1-P(\tau_{n+1}\geqslant R_k;\tau_{n+1}\geqslant\sigma_m). \tag{3.3}$$

Using lemma 2.8 we find from (3.3)

$$\begin{aligned} p_{k2} &= 1-\sum_{j=m}^{n} q_{j,n+1}\int_{t=0}^{\infty}\int_{s=t}^{\infty} H_j(s)ds\, dG_k(t) \qquad (3.4)\\ &= 1-\sum_{j=m}^{n} q_{j,n+1}\int_{s=0}^{\infty} G_k(s)H_j(s)ds \\ &= 1-\sum_{j=m}^{n} q_{j,n+1}B_{jk} \\ &= D_{mk}. \end{aligned}$$

From the balance equations of the stationary distribution of $\{X_n:n\geqslant 0\}$ we find with (3.4)

$$\pi(1) = \frac{1-D_{m2}}{D_{m1}+1-D_{m2}};\ \pi(2) = \frac{D_{m1}}{D_{m1}+1-D_{m2}}. \tag{3.5}$$

Next we note that

$$\tau(k) = \int_0^\infty P(T_{n+1}-T_n>t \mid X_n=k)dt \tag{3.6}$$

$$= \int_0^\infty (1-P(T_{n+1}-T_n\leq t \mid X_n=k))dt$$

$$= \int_0^\infty \{1-P(R_k\leq t)P(\min(\sigma_m,\tau_{n+1})\leq t)\}dt$$

$$= \int_0^\infty \{1-G_k(t)(1-\sum_{j=0}^{m-1} H_j(t))\}dt$$

$$= \nu_k+\sum_{j=0}^{m-1}\int_0^\infty G_k(t)H_j(t)dt$$

$$= \nu_k+\sum_{j=0}^{m-1} B_{jk}$$

$$= H_{mk}$$

and

$$c(k) = \int_0^\infty P(\max(\tau_{n+1},R_k)>t)dt - \int_0^\infty P(\tau_{n+1}>t)dt \tag{3.7}$$

$$= \int_0^\infty P(\tau_{n+1}\leq t)P(R_k>t)dt$$

$$= \int_0^\infty H_{n+1}(t)\overline{G}_k(t)$$

$$= A_{n+1,k}.$$

Together (3.2), (3.5), (3.6) and (3.7) yield (3.1). □

4. Computation of $g(m)$ and a numerical example

Suppose that the numbers $A_{jk}, 0\leq j\leq n+1$, $k = 1,2$ are known. Then it follows from (2.10), (2.11), (2.12) and (3.1) that $g(m)$ is known. Hence the computation of $g(m)$ is completed once the numbers A_{jk} have been computed.
Note that

$$A_{0k} = \int_0^\infty e^{-q_0 t}\overline{G}_k(t)dt. \tag{4.1}$$

In general the A_{jk} have to be computed numerically from (2.6). In this section we show that recursive computation of A_{jk} is possible in case of Erlangian repair time distributions.

Lemma 4.1. *Let* $\overline{G}_k(t) = e^{-\lambda_k t}$, $t\geq 0$; $k = 1,2$. *Then*

$$A_{0k} = (q_0+\lambda_k)^{-1} \tag{4.2}$$

$$A_{jk} = \sum_{i=0}^{j-1}\frac{q_{ij}}{q_j+\lambda_k}A_{ik},\ 1\leq j\leq n+1. \tag{4.3}$$

PROOF. Formula (4.2) is an immediate consequence of (4.1). Using Kolmogorov's forward integral equations:

$$H_j(t) = e^{-q_j t} \int_{s=0}^{t} \sum_{i=0}^{j-1} H_i(s) q_{ij} e^{q_j s} ds \tag{4.4}$$

we find from (2.6)

$$\begin{aligned} A_{jk} &= \int_0^\infty H_j(t)\overline{G}_k(t)dt \\ &= \int_{t=0}^\infty e^{-q_j t} \int_{s=0}^{t} \sum_{i=0}^{j-1} H_i(s) q_{ij} e^{q_j s} ds \overline{G}_k(t)dt \\ &= \sum_{i=0}^{j-1} q_{ij} \int_{s=0}^\infty H_i(s) e^{q_j s} \int_{t=s}^\infty e^{-q_j t}\overline{G}_k(t)dtds. \end{aligned} \tag{4.5}$$

Hence

$$\begin{aligned} A_{jk} &= \sum_{i=0}^{j-1} \frac{q_{ij}}{q_j+\lambda_k} \int_{s=0}^\infty H_i(s) e^{-\lambda_k s} ds \\ &= \sum_{i=0}^{j-1} \frac{q_{ij}}{q_j+\lambda_k} A_{ik}. \quad \square \end{aligned}$$

Now consider the case $\overline{G}(t) = \sum_{l=0}^{p-1} e^{-\lambda t}\frac{(\lambda t)^l}{l!}$ (For ease of notation we drop the subscript k). i.e. $G(\cdot)$ is the Erlang (λ,p)-distribution. Denote (see (2.6))

$$A_j^{(p)} := \int_0^\infty H_j(t)\overline{G}(t)dt.$$

THEOREM 4.2.

$$A_0^{(p)} = \lambda^{-1} \sum_{l=1}^{p} (\lambda/\lambda+q_0)^l \tag{4.6}$$

$$A_j^{(p)} = \lambda^{-1} \sum_{i=0}^{j-1} q_{ij} \sum_{m=1}^{p} (\lambda/\lambda+q_j)^m A_i^{(p-m+1)}, \ 1 \leq j \leq n+1. \tag{4.7}$$

PROOF. Formula (4.6) follows from (4.1). For $p=1$ (4.7) reduces to lemma 4.1. For $p>1$ we conclude from (4.5) and

$$\int_s^\infty e^{-q_j t}\overline{G}(t)dt = \lambda^{-1} e^{-(\lambda+q_j)s} \sum_{k=0}^{p-1} (\lambda/\lambda+q_j)^{k+1} \sum_{m=0}^{k} \frac{(\lambda+q_j)^m . s^m}{m!}$$

that

$$A_j^{(p)} = \lambda^{-1} \sum_{i=0}^{j-1} q_{ij} \int_{s=0}^\infty H_i(s) e^{-\lambda s} \sum_{k=0}^{p-1} (\lambda/\lambda+q_j)^{k+1} \sum_{m=0}^{k} \frac{(\lambda+q_j)^m . s^m}{m!} ds$$

which yields (4.7) after some elementary calculations. $\square$

REMARK 4.3. For $j = n+1$ relation (4.7) implies

$$A_{n+1}^{(p)} = A_{n+1}^{(p-1)} + \lambda^{-1} \sum_{i=0}^{n} q_{i,n+1} A_i^{(p)}. \tag{4.8}$$

As an example we apply the results on the following numerical data: $n=7$ and the Q-matrix is given by

$$Q=\begin{array}{c} \\ 0\\ 1\\ 2\\ 4\\ 4\\ 5\\ 6\\ 7\\ 8 \end{array}\left(\begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\\ -1 & 0.98 & & & & & & & 0.02\\ & -2 & 1.95 & & & & & & 0.05\\ & -3 & 2.90 & & & & & & 0.10\\ & & -4 & 3.80 & & & & & 0.20\\ & & & -5 & 4.70 & & & & 0.30\\ & & & & -6 & 5.50 & & & 0.50\\ & & & & & -7 & 6.30 & & 0.70\\ & & & & & & -8 & & 8\\ & & & & & & & & 0 \end{array}\right)$$

The following repair-time distributions are considered.

- $G_1(\cdot)$ is exponential with parameter μ_1 and Erlang $(2,2\mu_1)$ respectively with $\mu_1 = 1.1, 1.3, 1.5, 2.0$ and 4.0.
- $G_2(\cdot)$ is exponential with parameter 1 and Erlang (2,2.0)

In table 1 below we give m^*, the optimal value of m, as well as $1-g(m^*)$.

	G_2	Exp(1.0)		Erl.(2,2.0)	
G_1	μ_1	m^*	$1-g(m^*)$	m^*	$1-g(m^*)$
Exp(μ_1)	1.1	7	0.9535	7	0.9589
	1.3	6	0.9632	7	0.9690
	1.5	5	0.9708	6	0.9755
	2.0	3	0.9825	4	0.9857
	4.0	1	0.9940	2	0.9956
Erl.(2,2μ_1)	1.1	6	0.9607	7	0.9675
	1.3	5	0.9705	6	0.9758
	1.5	4	0.9772	4	0.9817
	2.0	2	0.9865	3	0.9898
	4.0	1	0.9946	1	0.9963

Table 1: the optimal control limits and availability

The general conclusion from table 1 is that a decreasing coefficient of variation of the repair time distributions yields a higher value of the optimal availability and a higher preventive repair limit.

5. The case of ample repair facilities

Unlike the system with a single repair facility no tractable exact formula can in general be obtained for the case of ample repair facilities. This already holds for the case with only two working conditions "on" and "off" (cf. Barlow and Proschan [1] and Tijms [7] for the single repair facility and Van der Heyden [9] for approximate formulae for two repair facilities).

For our model with more than two possible working conditions a description as a semi-Markov decision process is possible when we assume Erlangian distributed repair times. However, the big expansion of the state space makes this approach prohibitive for moderate n. A derivation of approximations for the unavailability under *CLR*-type control rules as in Van der Heyden [9], would be worthwhile.

To give an idea about the influence on the availability of a second repair facility we present in table 2 below the unavailability for the numerical data of the example from the previous section. The availability in case of ample repair facilities has been computed by straight forward value iteration.

G_1	G_2	Exp(1.0)		Erl.(2,2.0)	
	#fac.	1	2	1	2
	μ_1				
Exp(μ_1)	1.1	0.9535	0.9757	0.9589	0.9799
	1.3	0.9632	0.9806	0.9690	0.9822
	1.5	0.9708	0.9848	0.9755	0.9857
	2.0	0.9825	0.9909	0.9857	0.9914
	4.0	0.9940	0.9972	0.9956	0.9974
Erl.(2,2μ_1)	1.1	0.9607	0.9808	0.9675	0.9820
	1.3	0.9705	0.9854	0.9758	0.9862
	1.5	0.9772	0.9887	0.9817	0.9893
	2.0	0.9865	0.9935	0.9898	0.9939
	4.0	0.9946	0.9978	0.9963	0.9979

Table 2: the optimal availability for single vs. ample repair facilities.

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