Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science

J.K. Scholma

Lie-algebraic aspects of the classical nonrelativistic Calogero-Moser models

Department of Algebra, Analysis and Geometry Report AM-R8901 January

The Centre for Mathematics and Computer Science is a research institute of the Stichting Mathematisch Centrum, which was founded on February 11, 1946, as a nonprofit institution aiming at the promotion of mathematics, computer science, and their applications. It is sponsored by the Dutch Government through the Netherlands Organization for the Advancement of Research (N.W.O.).

Lie-Algebraic Aspects of the Classical Nonrelativistic Calogero-Moser Models

J.K. Scholma
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

In this paper we give a Lie-algebraic construction of the phase space and Poisson structure of the classical nonrelativistic Calogero-Moser models. The construction uses a real noncompact semisimple Lie algebra $\mathfrak{g}$ with an involutive automorphism σ and requires the existence of an element $\mu \in \mathfrak{g}$ with some special properties. We derive the Lax equation and give a Lie-algebraic proof that the *Ad*-invariant functions on $\mathfrak{g}$ are in involution. Two well-known examples are described from this point of view and a possible generalization is discussed. Finally we compare our construction with that of Kostant-Adler-Symes.

1980 Mathematics subject classification: 58F07, 34A34, 70F10

Key words and phrases: Calogero-Moser systems, Poisson manifolds, Lie algebras, coadjoint orbits, Lax equation, Kostant-Adler-Symes theorem, Yang-Baxter operator

1. Introduction

In this paper we consider the classical nonrelativistic Calogero-Moser models (for a review of the developments until 1980 we refer to [1]). These are finite-dimensional Hamiltonian systems with Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + V(q) \tag{1.1}$$

where $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ are the momentum and coordinate vectors in $\mathbb{R}^n$ and the potential $V(q)$ can be of the following type:

$$V(q) = \sum_{j<k} g^2\, v(q_j - q_k) \tag{1.2}$$

where the function $v(x)$ may be one of the following form:

$$\begin{aligned} v(x) &= x^{-2} \quad (I) \\ &= a^2 \sinh^{-2} ax \quad (II) \\ &= a^2 \sin^{-2} ax \quad (III) \\ &= a^2\, \wp(ax) \quad (IV) \\ &= x^{-2} + \omega^2 x^2 \quad (V) \end{aligned} \tag{1.3}$$

and where $\wp(x)$ is the Weierstrass function. These Hamiltonians describe one-dimensional n-particle systems with pairwise interaction.
$V(q)$ can also have the form:

$$V(q) = \sum_{j=1}^{n-1} g_j^2\, v(q_j - q_{j+1}) \quad (VI) \tag{1.4}$$

or

Report AM-R8901
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

$$V(q) = \sum_{j=1}^{n} g_j^2 \, v(q_j - q_{j+1}), \; q_{n+1} = q_1 \; (VI') \tag{1.5}$$

Where

$$v(x) = \exp 2x$$

These Hamiltonians describe n-particle systems with nearest-neighbour interaction and are known as the nonperiodic Toda lattice (VI) and the periodic Toda lattice (VI'). In this paper we will restrict ourselves to systems of type I. Now all these systems are related to real semisimple Lie algebras and the corresponding restricted root systems. To each root system (A_n especially) various integrable systems can be associated. It is still not quite clear which functions work. The construction of Olshanetski-Perelomov has little group-theoretical flavour. A much more Lie-algebraic / group-theoretical method of associating an integrable system to a Lie algebra with extra structure is the so-called Kostant-Adler-Symes-Reyman-Semenov-Tyanshansky (K.A.S.R.S.) construction ([2-10]). It is an open question, and indeed one of the central open problems of integrable system theory whether all integrable systems can be obtained in such a way. This is e.g. not known for the systems of type I-IV.
Here we describe a way to obtain the Calogero-Moser system (type I) which is reminiscent of K.A.S.R.S. and which seems to generalize that setting. In particular there seems to be no Yang-Baxter operator that generates the Hamiltonian operator of the C-M model.

In recent years there have been discovered many other interesting properties of the C-M models. For example, there exist corresponding quantum integrable systems (see [11] for a review), integrable relativistic generalizations [12-15], master integrals [16], relativistic generalizations with an external potential [17], generalizations to loop algebras [18] and generalizations with extra internal degrees of freedom [19].

The setup of this paper is as follows:
In section 2 we describe the C-M model which corresponds to the root system A_{n-1}. In section 3 we review some properties of Poisson manifolds. In section 4 we describe the K.A.S.R.S construction which requires the existence of a so-called Yang-Baxter operator and in which case the solution of the equations of motion is equivalent with a factorization problem in a (matrix) Lie group. In section 5 we describe the construction of the Calogero-Moser models. The equations of motion can be viewed as a Hamiltonian vector field on a bundle P_μ which is the phase space and requires the existence of an element μ (the moment) with some special properties. Sufficient for integrability is condition (5.40) and if μ satisfies (5.42) then the equations of motion can be written in Lax form. The main results are Theorem (5.18) and (5.23). In section 6 we describe the two known C-M models which correspond to the classical root systems A_l, B_l, C_l and D_l and give the explicit form of μ and the Lax pair. In section 7 a theorem is proved which tells us that in a certain sense the construction of the A_{n-1} model is unique. In section 8 we prove that a certain operator which generates the Hamiltonian vector field is not a Yang-Baxter operator.

Note: the symbol 'i' denotes $\sqrt{-1}$ unless it is used as an index.

2. Description of the A_{n-1} model [1]

Let $\{p_i, q_i, i = 1,\ldots,n\}$ denote canonical coordinates, with Poisson bracket

$$\{p_i, q_j\} = \delta_{ij} \tag{2.1}$$

and consider the Hamiltonian given by

$$H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \frac{1}{2} g^2 \sum_{i \neq j} (q_i - q_j)^{-2} \tag{2.2}$$

where $g \in \mathbb{R}$ is a coupling constant and $q_i \neq q_j$ if $i \neq j$. The Hamilton equations are:

$$\dot{q}_i = \{H, q_i\} = \frac{\partial H}{\partial p_i} = p_i \tag{2.3}$$

$$\dot{p}_i = \{H, p_i\} = -\frac{\partial H}{\partial q_i}$$

$$= 2g^2 \sum_{\substack{j \\ j \neq i}} (q_i - q_j)^{-3} \tag{2.4}$$

These equations define the Calogero-Moser model associated with rootsystem A_{n-1}. Define

$$\bar{q} = \sum_{i=1}^{n} q_i \quad \bar{p} = \sum_{i=1}^{n} p_i \tag{2.5}$$

then we derive from (2.3) and (2.4):

$$\dot{\bar{p}} = 0 \quad \dot{\bar{q}} = \bar{p} \tag{2.6}$$

so the system is translation invariant and in center-of-mass coordinates the phase space is

$$M = \{(q,p) | q = (q_1, \ldots, q_n), p \in (p_1, \ldots, p_n) \in \mathbb{R}^n,$$

$$q_i \neq q_j \text{ if } i \neq j, \ \bar{q} = \bar{p} = 0\} \subset \mathbb{R}^{2n} \tag{2.7}$$

Let $f,g \in C^\infty(M)$ then the Poisson bracket is defined by

$$\{f.g\} = \sum_{i=1}^{n} \left[\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right] \tag{2.8}$$

Now define the matrices L and M as follows:

$$L = \sum_{i=1}^{n} p_i e_{ii} + g \sum_{i<j} (q_i - q_j)^{-1} \sqrt{-1} (e_{ij} - e_{ji}) \tag{2.9}$$

$$M = g \sum_{i<j} (q_i - q_j)^{-2} \sqrt{-1} (e_{ij} + e_{ji} - e_{ii} - e_{jj} + \frac{2}{n} I) \tag{2.10}$$

Where $\{e_{ij}, i,j = 1,\ldots,n\}$ is the standard basis of $gl(n,\mathbb{C})$. A straightforward calculation shows that (2.3) and (2.4) are equivalent with the matrix equation

$$\dot{L} = [M,L] \tag{2.11}$$

and the Hamiltonian H is given by:

$$H = \frac{1}{2} \, tr \, L^2 \tag{2.12}$$

(2.11) is known as the Lax equation of the Calogero-Moser model and (L,M) is called the Lax pair. Observe that L is a traceless hermitian matrix and that M is traceless skew-hermitian, so the functions

$$H_k = \frac{1}{k} \, tr \, L^k, \quad 2 \leqslant k \leqslant n \tag{2.13}$$

are real-valued and from (2.11) it follows that the H_k are conserved quantities. It is also known [1] that the H_k are in involution, so

$$\{H_k, H_l\} = 0, \quad 2 \leqslant k,l \leqslant n \tag{2.14}$$

and that they are functionally independent, so it follows from the Liouville theorem [20] that the Calogero-Moser model is a completely integrable Hamiltonian system and that there exist coordinates, the so-called action-angle coordinates $\{J_i, \theta_i, \ i = 1,\ldots,n\}$ in which the equations (2.3) and (2.4) are given by

$$\dot{J}_i = 0 \quad \dot{\theta}_i = c_i, \quad c_i \text{ constants} \tag{2.15}$$

which describes a motion on tori. Now define

$$Q = \sum_{i=1}^{n} q_i \, e_{ii} = \mathrm{diag}(q_1, \ldots, q_n) \tag{2.16}$$

$$P = \sum_{i=1}^{n} p_i e_{ii} = \mathrm{diag}(p_1, \ldots, p_n) \tag{2.17}$$

$$\mu = g \sum_{i<j} \sqrt{-1}\,(e_{ij} + e_{ji}) \tag{2.18}$$

then $q_i - q_j = \alpha_{ij}(Q)$, $i \neq j$, where α_{ij} are the roots of the root system A_{n-1} and Q is a socalled regular diagonal matrix because $q_i \neq q_j$ if $i \neq j$, so $\alpha_{ij}(Q) \neq 0$ $\forall_i \neq j$. From (2.3), (2.4), (2.9) and (2.10) we can derive the following relations between the various matrices:

$$\dot{Q} = P = L_{\mathrm{diag}} \tag{2.19}$$

$$[Q, L] = \mu \tag{2.20}$$

$$[M, \mu] = 0 \tag{2.21}$$

$$[Q, M] = L_{\mathrm{off}} \tag{2.22}$$

where L_{diag} and L_{off} denote the projections on the diagonal component resp. the off-diagonal component of L. From these relations we derive:

$$\begin{aligned}
&[\dot{Q}, L] + [Q, \dot{L}] \\
&= [P, L] + [Q, [M,L]] \\
&= [P, L] + [[Q, M],L] + [M, [Q, L]] \\
&= [P, L] + [L_{\mathrm{off}}, L] + [M, \mu] \\
&= 0
\end{aligned} \tag{2.23}$$

so (2.19) and (2.11) are consistent with (2.20).

In fact, it will turn out that (2.20) characterizes an abstract phase space P_μ on which the Calogero-Moser model is defined. This space P_μ is a bundle over the regular diagonal elements. On this space we can define a new Poisson bracket and we can prove that the H_k are in involution w.r.t this bracket, where we interprete the H_k as the Ad-invariant functions on the underlying Lie algebra. To prove this we use the properties of Ad-invariant functions, some properties of μ and various commutation relations.

3. Poisson manifolds [21-24]

Let M be a n-dimensional smooth manifold.

Definition 3.1. A Poisson structure or Poisson bracket is a skew-symmetric bilinear map

$$\{\ ,\ \}: C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

which satisfies the following properties:

(i) $\{fg, h\} = f\{g, h\} + \{f, h\}g$ (Leibniz property) (3.1)

(ii) $\{\{f, g\}, h\} + \mathrm{cycl} = 0$ (Jacobi identity) (3.2)

So a Poisson structure turns the commutative algebra $C^\infty(M)$ into a Lie algebra such that $\{h,\ \}: C^\infty(M) \to C^\infty(M)$ is a derivation for all $h \in C^\infty(M)$.

DEFINITION 3.2. A Poisson manifold is a manifold M with a Poisson structure.

DEFINITION 3.3. A Poisson mapping $\phi: M \to N$, with M and N Poisson manifolds, is a smooth map which satisfies:

$$\{f \circ \phi, g \circ \phi\}_M = \{f, g\}_N \circ \phi \quad \forall f, g \in C^\infty(N) \tag{3.3}$$

or equivalently

$$\phi^* \{f, g\}_N = \{\phi^*(f), \phi^*(g)\}_M \tag{3.4}$$

where the pullback $\phi^*: C^\infty(N) \to C^\infty(M)$ is defined as:

$$\phi^*(f)(m) = f(\phi(m)) \quad (\forall m \in M) \tag{3.5}$$

A diffeomorphic Poisson mapping $\phi: M \to M$ is called an automorphism of the Poisson manifold M.

DEFINITION 3.4 An infinitesimal endomorphism A of a Poisson manifold M is a linear map $A: C^\infty(M) \to C^\infty(M)$ which satisfies:

$$A(\{f,g\}) = \{A(f),g\} + \{f,A(g)\} \tag{3.6}$$

so it is a derivation of the Lie algebra $C^\infty(M)$ and it is called an inner endomorphism if there exists a $h \in C^\infty(M)$ such that

$$A(f) = \{h,f\} \tag{3.7}$$

DEFINITION 3.5. A submanifold $N \subset M$ of a Poisson manifold M is a Poisson submanifold if the inclusion $i: N \to M$ is a Poisson mapping.

DEFINITION 3.6. A function $f \in C^\infty(M)$ is called a Casimir function (or distinguished function, cyclic function or invariant) if $f \in Z(C^\infty(M))$, so

$$\{f, h\} = 0 \quad \forall h \in C^\infty(M) \tag{3.8}$$

OBSERVATION: The constant functions are Casimir functions.

PROPOSITION 3.7. *Let M and N be Poisson manifolds. Identify $C^\infty(M \times N)$ with $C^\infty(M) \otimes C^\infty(N)$, then there exists a natural Poisson bracket on $C^\infty(M) \otimes C^\infty(N)$, defined as follows*

$$\{f_1 \otimes g_1, f_2 \otimes g_2\} = \{f_1, f_2\}_M \otimes g_1 g_2 + f_1 f_2 \otimes \{g_1, g_2\}_N \tag{3.9}$$

$$\forall f_1, f_2 \in C^\infty(M), \forall g_1, g_2 \in C^\infty(N)$$

where we have identified $C^\infty(M)$ with $C^\infty(M) \otimes 1$ and $C^\infty(N)$ with $1 \otimes C^\infty(N)$. Moreover, $C^\infty(M)$ and $C^\infty(N)$ are subalgebras and

$$\{C^\infty(M), C^\infty(N)\} = 0 \tag{3.10}$$

and the projections $\pi_1: M \times N \to M$, $\pi_2: M \times N \to N$ are Poisson mappings.
$M \times N$ with this Poisson structure is called the product of the Poisson manifolds M and N.

PROOF. The bilinearity and skew-symmetry of (3.9) are obvious. The Leibniz property follows from that of the brackets on M and N. Than (3.9) satisfies the Jacobi identity is a straightforward calculation. Also

$$\{f_1 \otimes 1, f_2 \otimes 1\} = \{f_1, f_2\}_M \otimes 1 \tag{3.11}$$

so $C^\infty(M)$ is a subalgebra and

$$\{f_1 \otimes 1, 1 \otimes f_2\} = 0$$

Now define the pullbacks $\pi_1^*(f) = f \otimes 1$, $\pi_2^*(g) = 1 \otimes g$ for $f \in C^\infty(M)$, $g \in C^\infty(N)$ then

$$\begin{aligned} \pi_1^*\{f,g\}_M &= \{f,g\}_M \otimes 1 \\ &= \{f \otimes 1, g \otimes 1\} = \{\pi_1^*(f), \pi_1^*(g)\} \end{aligned}$$

so π_1 is a Poisson mapping. □

DEFINITION 3.8. With each $h \in C^\infty(M)$ we associate the Hamiltonian vector field V_h as follows:

$$V_h(f) = \{h,f\} \quad \forall f \in C^\infty(M) \tag{3.12}$$

LEMMA 3.9. *V_h has the following properties*

(i) $V_h(fg) = V_h(f)g + fV_h(g)$ *so it is a vector field.*

(ii) $V_{\{f,g\}} = [V_f, V_g]$ (3.13)

(iii) $V_h(\{f,g\}) = \{V_h(f),g\} + \{f, V_h(g)\}$ (3.14)

(iv) $V_{fg} = f V_g + g V_f$ (3.15)

PROOF: this follows immediately from the properties of the Poisson bracket. □

So the map $V : C^\infty(M) \to V(M)$ defined by $V(h) = V_h$ is a Lie algebra homomorphism from $C^\infty(M)$ into the inner infinitesimal endomorphisms of M.

DEFINITION 3.10. Let V_h be a Hamiltonian vector field, $h \in C^\infty(M)$. Let $\phi : \mathbb{R} \to M$ be a smooth curve in M such that

$$\frac{d}{dt}\phi(t) = V_h(\phi(t)) \tag{3.16}$$

(3.16) are called Hamilton's equations with Hamiltonian h. The unique maximal integral curve passing through $x \in M$ is denoted by $\psi(t,x)$ and is called the flow of V_h and is often written as

$$\psi(t,x) = \exp(tV_h)x \tag{3.17}$$

LEMMA 3.11. *For each t, the flow $\exp(tV_h) : M \to M$ determines a (local) Poisson automorphism of M.*

From the properties of the Poisson bracket it follows that in each point $p \in M$, V_h depends only on dh and $\{f,g\}$ depends only on df and dg, so there is a bundle map

$$B : T^*M \to TM \tag{3.18}$$

such that $V_h = B(dh)$ and $V = B \circ d : C^\infty(M) \to V(M)$. B is sometimes called a Hamiltonian operator. We may also think of B as defining a contravariant skew-symmetric 2-tensor W in M, for which

$$\{f,g\} = W(df, dg) = dg(B(df)) \tag{3.19}$$

The tensor W is sometimes called a cosymplectic structure.

Now let $\{x^i, i = 1, \ldots, n\}$ be local coordinates on M. Then it follows from the properties of the

Poisson bracket that we can write

$$\{f,g\} = \sum_{i=1}^{n} \sum_{j=1}^{n} W^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} \tag{3.20}$$

where

$$W^{ij}(x) = \{x^i, x^j\}(x) \tag{3.21}$$

So a Poisson structure is determined by the functions W^{ij}. These are sometimes called the structure functions of M relative to the local coordinates x^i, and the matrix W is called the structure matrix of M. We can also view W^{ij} as the components of the cosymplectic structure. The Jacobi identity for $\{\ ,\ \}$ is equivalent with the Jacobi identity for the coordinate functions. One often writes (3.10) as follows

$$\{f,g\} = \nabla f.\, J \nabla g \tag{3.22}$$

where J is the matrix W and $\nabla f(x)=$ grad $f(x)$. In local coordinates we also have in each point $x \in M$

$$B(dx^i) = \sum_{j=1}^{n} W^{ij}(x) \frac{\partial}{\partial x^j} \tag{3.23}$$

Definition 3.12. The rank of a Poisson structure at a point $x \in M$ is defined as the rank of the linear map $B_x : T_x^* M \to T_x M$. In local coordinates it is also the rank of the matrix $W^{ij}(x)$.

Lemma 3.13. *The rank of a Poisson manifold at any point is always an even integer.*

Proof. This follows from the skew-symmetry of W.

Lemma 3.14. *The rank of a Poisson manifold is constant along flows of Hamiltonian vector fields.*

Definition 3.15. A Poisson manifold M is called symplectic if the rank is everywhere equal to the dimension n of M.

Corollary. *A symplectic manifold is even-dimensional*

Theorem 3.16 *Each Poisson manifold M naturally splits into a collection of even-dimensional symplectic manifolds, the leaves of the symplectic foliation. The dimension of any such leaf N equals the rank of the Poisson structure at any point* $y \in N$.

Theorem 3.17. *(Darboux). Let M be an n-dimensional Poisson manifold of constant rank* $2m \leq n$. *At each* $x_0 \in M$ *there exist local coordinates* $(p,q,z)=(p_1,\ldots,p_m,\ q_1,\ldots,q_m)$, $2m+l=n$, *in terms of which the Poisson bracket takes the form*

$$\{f,g\} = \sum_{i=1}^{m} \left[\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right] \tag{3.24}$$

so

$$\{p_i, q_j\} = \delta_{ij} \tag{3.25}$$

The leaves of the symplectic foliation intersect the coordinate chart in the slices $\{z_1=c_1,\ldots,z_l=c_l\}$ determined by the distinguished (also called cyclic) coordinates z.

So locally a Poisson structure always has the form (3.24).

Finally, a useful criterium to decide whether a submanifold is a Poisson submanifold is the following:

LEMMA 3.18. *A submanifold $N \subset M$ of a Poisson manifold M is a Poisson submanifold iff all Hamiltonian vector fields are tangent to N.*

COROLLARY. *If M is a vector space and N a subspace this simplifies to the condition*

$$V_f(x) \in N \quad (\forall x \in N)\ (\forall f \in C^\infty(M)) \tag{3.26}$$

4. POISSON STRUCTURES ON LIE ALGEBRAS AND THE K.A.S.R.S. THEOREM

Let $\mathfrak{g}$ be a real semisimple Lie algebra with Killing form $<,>$ and identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by the Killing form, i.e.

$$\phi : \mathfrak{g} \to \mathfrak{g}^*,\ \phi(x)(y) = <x,y>\ \forall x,y \in \mathfrak{g} \tag{4.1}$$

Define

$$\{f,g\}(x) = <x,\ [\mathrm{grad} f(x),\ \mathrm{grad}\, g(x)]> \tag{4.2}$$

where

$$<\mathrm{grad} f(x),y> = df(x)(y) = \frac{d}{dt}|_{t=0} f(x+ty) \tag{4.3}$$

so

$$\mathrm{grad} f(x) = \phi^{-1}(df(x)) \tag{4.4}$$

PROPOSITION 4.1. *(4.2) defines a Poisson structure on $\mathfrak{g}$, the socalled Lie-Poisson bracket or Kirillov-Kostant bracket.*

PROOF. see [25].

DEFINITION 4.2. A Poisson structure on a vector space V is called linear if $\{\lambda,\mu\} \in V^*$ for all $\lambda,\mu \in V^*$.

LEMMA 4.3. *The Lie-Poisson bracket (4.2) is linear.*

PROOF. If $\lambda \in \mathfrak{g}^*$ then $<\mathrm{grad}\,\lambda(x),y> = \lambda(y)$ so $\mathrm{grad}\,\lambda(x) = \phi^{-1}(\lambda)$ and so

$\{\lambda,\mu\}(x) = <x,\ [\phi^{-1}(\lambda),\phi^{-1}(\mu)]>$ which is linear in x so $\{\lambda,\mu\} \in \mathfrak{g}^*$. ☐

DEFINITION 4.4. A function $f \in C^\infty(\mathfrak{g})$ is called Ad-invariant if

$$f(Adg(x)) = f(x)\ (\forall g \in G)\ (\forall x \in \mathfrak{g}) \tag{4.5}$$

where G is the connected Lie group with Lie algebra $\mathfrak{g}$ and Ad is the adjoint representation of G on $\mathfrak{g}$.

LEMMA 4.5. *Let $f,g \in C^\infty(\mathfrak{g})$ be Ad-invariant functions, then they have the following properties:* $(\forall x \in \mathfrak{g})$

(i) $[\mathrm{grad} f(x),x] = 0$ (4.6)

(ii) $\mathrm{grad} f(Adg(x)) = Adg(\mathrm{grad} f(x)),\quad \forall g \in G$ (4.7)

(iii) $[\mathrm{grad} f(x),\ \mathrm{grad}\, g(x)] = 0$ (4.8)

(iv) $h(x) = <\mathrm{grad} f(x),\ \mathrm{grad}\, g(x)>$ is Ad-invariant (4.9)

PROOF. (i) Let exp ty, $y \in \mathfrak{g}$ denote a 1-parameter subgroup of G then

$$f(x) = f(Ad(\exp ty)(x))$$
$$= f(\exp t(ady)(x))$$
$$= f(x + t[y,x] + O(t^2)) \quad (\forall x, y \in \mathfrak{g}) \tag{4.10}$$

so

$$0 = \frac{d}{dt}|_{t=0} f(x + t[y,x])$$
$$= \langle \text{grad} f(x), [y,x] \rangle$$
$$= \langle y, [x, \text{grad} f(x)] \rangle \quad (\forall y \in \mathfrak{g})$$

and so $[x, \text{grad} f(x)] = 0$

(ii) $\langle \text{grad} f(Adg(x)), y \rangle$

$$= \frac{d}{dt}|_{t=0} f(Adg(x) + ty)$$
$$= \frac{d}{dt}|_{t=0} f(x + tAdg^{-1}(y))$$
$$= \langle \text{grad} f(x), Adg^{-1}(y) \rangle$$
$$= \langle Adg(\text{grad} f(x)), y \rangle \quad (\forall y \in \mathfrak{g})$$

and so $\text{grad} f(Adg(x)) = Adg(\text{grad} f(x))$

(ii) $\langle [\text{grad} f(x), \text{grad} g(x)], y \rangle$

$$= \langle \text{grad} f(x), [\text{grad} g(x), y] \rangle$$

but from (i) it follows that

$$[\text{grad} g(x + sy), x + sy] = 0 \quad \forall x, y \in \mathfrak{g}, \ \forall s \in \mathbb{R}$$

And so

$$[\frac{d}{ds}|_{s=0} \text{grad} g(x + sy), x] + [\text{grad} g(x), y] = 0 \tag{4.11}$$

Substituting this yields:

$\langle [\text{grad} f(x), \text{grad} g(x)], y \rangle$

$$= \langle \text{grad} f(x), [x, \frac{d}{ds}|_{s=0} \text{grad} g(x + sy)] \rangle$$
$$= \langle [\text{grad} f(x), x], \frac{d}{ds}|_{s=0} \text{grad} g(x + sy) \rangle$$

$= 0 \ \forall y \in \mathfrak{g}$ and so $[\text{grad} f(x), \text{grad} g(x)] = 0$

(iv) $h(Adg(x)) = \langle \text{grad} f(Adg(x)), \text{grad} g(Adg(x)) \rangle$

$$= \langle Adg(\text{grad} f(x)), Adg(\text{grad} g(x)) \rangle$$
$$= \langle \text{grad} f(x), \text{grad} g(x) \rangle = h(x) \quad \square$$

THEOREM 4.6. *[26] Let G be a connected Lie group with Lie algebra* $\mathfrak{g}$ *and adjoint representation Ad on* $\mathfrak{g}$. *Then the Lie-Poisson bracket on* $\mathfrak{g}$ *has the following properties:*

(i) *the Ad-invariant function are Casimir functions.*

(ii) *the symplectic leaves are the orbits of Ad G on* $\mathfrak{g}$ *and are the common level sets of the Ad-invariant functions.*

(iii) *for all* $g \in G$, *Adg is a linear Poisson automorphism, which preserves the leaves of the symplectic foliation.*

(iv) $V_f(x) = [x, \operatorname{grad} f(x)]\ \forall f \in C^\infty(\mathfrak{g}),\ \forall x \in \mathfrak{g}$.

(v) *if* $\lambda \in \mathfrak{g}^*$ *then* $V_\lambda = -ad\phi^{-1}(\lambda)$ *and*

$$\exp(tV_\lambda)\,(x) = Ad\exp(-t\phi^{-1}(\lambda)\,(x) \tag{4.12}$$

so the flow of V_λ *is the orbit of* $\exp(-t\,\phi^{-1}(\lambda))$ *through* x.

LEMMA 4.7. *Let* $f \in C^\infty(\mathfrak{g})$ *be an Ad-invariant function and let* $x(t)$ *be a smooth curve in* $\mathfrak{g}$ *with* $\dot{x} = [y,x]$ *and let* $F(t) = \operatorname{grad} f(x(t))$ *then*

$$\dot{F} = [y, F], \quad (y \in \mathfrak{g}) \tag{4.13}$$

PROOF.

$$<\dot{F}, z> = \frac{d}{dt} <\operatorname{grad} f(x(t)), z>$$

$$= \frac{d}{dt}\frac{d}{ds}|_{s=0}\, f(x(t)+sz)$$

$$= \frac{d}{ds}|_{s=0}\, \frac{d}{dt} f(x(t)+sz) = \frac{d}{ds}|_{s=0}\, <\operatorname{grad} f(x(t)+sz), \dot{x}>$$

$$= \frac{d}{ds}|_{s=0}\, <\operatorname{grad} f(x+sz), [y,x]>$$

$$= <y, [x, \frac{d}{ds}|_{s=0}\, \operatorname{grad} f(x+sz)]>$$

$$= <y, [\operatorname{grad} f(x), z]> \quad \text{(using 4.11)}$$

$$= <[y, F], z> \quad \forall z \in \mathfrak{g}$$

$$\text{so} \quad \dot{F} = [y, F] \qquad \square$$

PROPOSITION 4.8. *The Hamiltonian operator* $B \circ \phi : T\mathfrak{g} \to T\mathfrak{g}$ *is given by* $B(x) = adx$ *and so* $V_f(x) = B(x)\,(\operatorname{grad} f(x))$. *The rank of the Lie-Poisson bracket is equal to*

$$\begin{aligned} \operatorname{rank} B(x) &= \dim \operatorname{im} adx \\ &= \dim \mathfrak{g} - \dim \ker(adx) \end{aligned} \tag{4.14}$$

Let $\dim \mathfrak{g} = n$, $\operatorname{rank} \mathfrak{g} = l$ and $\operatorname{rank} B(x) = 2m$, then

$$\dim \ker(adx) \geqslant l \quad \forall x \in \mathfrak{g}$$

so $2m \leqslant n - l$

DEFINITION 4.9. An element $x \in \mathfrak{g}$ is called regular if $\dim \ker(adx) = l$ and the corresponding orbit through x is called a generic orbit. If x is not regular then it is called singular. So for a regular element x we have $2m = n - l$.

If f is an Ad-invariant function, then grad $f(x) \in \ker(adx)$ and from Lemma 4.5 it follows that $\mathfrak{a}_x$ = real span of $\{\text{grad} f(x), f\, Ad - \text{invariant}\}$ forms an abelian subalgebra of $\ker adx$ and dim $\mathfrak{a}_x \leqslant l$. If x is regular then dim $\mathfrak{a}_x = l$, so the H_k as defined in (2.13) are functionally independent. But in general $n - l > 2l$ so the Ad-invariant functions are not sufficient to make the corresponding Hamiltonian system integrable, and thus one has to go to a Poisson submanifold with dimension equal to $2l$ or one has to find more conserved quantities.

EXAMPLE 4.10. Take $\mathfrak{g} = sl(l+1,\mathbb{R})$, than we have $n = (l+1)^2 - 1$, so $n - l = l(l+1) > 2l$ if $l > 1$ and $n - l = 2l$ if $l = 1$. Now take $x = \text{diag}(1,1,-2) \in sl(3,\mathbb{R})$, so $l = 2$ and dim $\ker(adx) = 2+2 = 4$ and so dim $\text{im}(adx) = 8-4 = 4 = 2l$ but grad $H_3(x) = -$ grad $H_2(x) = -x$ so dim $\mathfrak{a}_x = 1$.

Now let $R \in$ End $\mathfrak{g}$ and define:

$$[x,y]_R = \frac{1}{2}[Rx,y] + \frac{1}{2}[x,Ry] \tag{4.16}$$

PROPOSITION 4.11. *If R satisfies the Yang-Baxter equation*

$$R[Rx,y] + R[x,Ry] - [Rx,Ry] = 0 \tag{4.17}$$

or the modified Yang-Baxter equation

$$R[Rx,y] + R[x,Ry] - [Rx,Ry] = [x,y] \tag{4.18}$$

then (4.16) defines a Lie bracket on $\mathfrak{g}$ *and R is called a Yang-Baxter operator.*

PROOF. We can rewrite the Jacobi identity for $[\ ,\ ]_R$ using the Jacobi identity for $[\ ,\ ]$ and we get:

$$[x,\ R[Ry,z] + R[y,Rz] - [Ry,Rz]] + cycl = 0 \tag{4.19}$$

now substitute (4.17) or (4.18) in (4.19) and use the Jacobi identity for $[\ ,\ ]$. □

DEFINITION 4.12. A pair $(\mathfrak{g},R)$ is called a double Lie algebra if R is a Yang-Baxter operator and we use the symbol $\mathfrak{g}_R$ to denote the vector space $\mathfrak{g}$ equipped with the bracket (4.16). Let R^* denote the adjoint of R w.r.t the Killing form then R is called unitary if $R^* = -R$.

THEOREM 4.13. *[8] Let* $\mathfrak{g}_R$ *be a double Lie algebra and define*

$$\{f,g\}_R(x) = \langle x, [\text{grad} f(x), \text{grad}\, g(x)]_R \rangle \tag{4.20}$$

then (4.20) defines a Poisson bracket on $\mathfrak{g}$ *and*

(i) *the* $Ad-$*invariant functions on* $\mathfrak{g}$ *are in involution w.r.t* (4.20)

(ii) $$V_f(x) = \frac{1}{2}[x, R\, \text{grad} f(x)] + \frac{1}{2}R^*[x, \text{grad} f(x)] \tag{4.21}$$

(iii) if f *is* $Ad-$*invariant then*

$$V_f(x) = \frac{1}{2}[x,\ R\, \text{grad} f(x)] \tag{4.22}$$

(iv) $$B(x) = \frac{1}{2}\, adx \circ R + \frac{1}{2} R^* \circ adx \tag{4.23}$$

$$= \frac{1}{2}[adx, R] \text{ if } R \text{ is unitary} \tag{4.24}$$

REMARK 4.14. From (4.18) it is clear that $R = Id$ is a Yang-Baxter operator and in this case $[\ ,\ ]_R = [\ ,\ ]$ so (4.20) reduces to (4.2)

An important example of a Yang-Baxter operator is the following:

PROPOSITION 4.15. *Let* $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ *(vector space direct sum) with* $\mathfrak{a}$ *and* $\mathfrak{b}$ *subalgebras, let* π_a *and* π_b *denote the corresponding projections and define*

$$R = \pi_a - \pi_b \tag{4.25}$$

then R satisfies the modified Yang-Baxter equation (4.18).

PROOF.

$$
\begin{aligned}
& R[Rx,y] + R[x,Ry] - [Rx,Ry] \\
&= R[\pi_a x - \pi_b x, \pi_a y + \pi_b y] + R[\pi_a x + \pi_b x, \pi_a y - \pi_b y] \\
&\quad - [\pi_a x - \pi_b x, \pi_a y - \pi_b y] \\
&= R[\pi_a x, \pi_a y] - R[\pi_b x, \pi_b y] + R[\pi_a x, \pi_b y] - R[\pi_b x, \pi_a y] \\
&+ R[\pi_a x, \pi_a y] - R[\pi_b x, \pi_b y] - R[\pi_a x, \pi_b y] + R[\pi_b x, \pi_a y] \\
&- [\pi_a x, \pi_a y] - [\pi_b x, \pi_b y] + [\pi_b x, \pi_a y] + [\pi_a x, \pi_b y] \\
&= [\pi_a x, \pi_a y] + [\pi_b x, \pi_b y] + [\pi_a x, \pi_a y] + [\pi_b x, \pi_b y] \\
&- [\pi_a x, \pi_a y] - [\pi_b x, \pi_b y] + [\pi_b x, \pi_a y] + [\pi_a x, \pi_b y] \\
&= [\pi_a x, \pi_a y] + [\pi_b x, \pi_b y] + [\pi_a x, \pi_b y] + [\pi_b x, \pi_a y] \\
&= [x,y] \qquad \square
\end{aligned}
$$

LEMMA 4.16. *Let* $\mathfrak{a}$ *be a subalgebra of* $\mathfrak{g}$ *and let* $\mathfrak{a}^\perp$ *denote the orthogonal complement of* $\mathfrak{a}$ *in* $\mathfrak{g}$ *w.r.t the Killing form, then* $\mathfrak{a}^\perp$ *is an* $ad\,\mathfrak{a}$*-invariant subspace, so*

$$[\mathfrak{a}, \mathfrak{a}^\perp] \subset \mathfrak{a}^\perp \tag{4.26}$$

PROOF. Let $x \in \mathfrak{a}$, $y \in \mathfrak{a}^\perp$ then for each $z \in \mathfrak{a}$

$<z, [x,y]> = <[z,x],y> = 0$ so $[x,y] \in \mathfrak{a}^\perp$ $\square$

PROPOSITION 4.17. [8] *Let* $R \in End\, \mathfrak{g}$ *be a solution of (4.17) then*

(i) $\frac{1}{2}R : \mathfrak{g}_R \to \mathfrak{g}$ *is a Lie algebra homomorphism*

(ii) *im R is a subalgebra of* $\mathfrak{g}$ *and ker R is an abelian ideal of* $\mathfrak{g}_R$

(iii) *[im R, ker R]* $\subset$ *ker R*

(iv) $(ker\, R)^\perp$ is a Poisson submanifold of $\mathfrak{g}$

PROOF. (i), (ii) and (iii) follow immediately from (4.16) and (4.17). Now, im $R^* \subset (ker\, R)^\perp$ and from (iii) it follows that

$$[(ker\, R)^\perp, im\, R] \subset (ker\, R)^\perp \tag{4.27}$$

and now (iv) follows from (4.21 and Lemma 3.18 $\square$

DEFINITION 4.18. A unitary Yang-Baxter operator R is called invariant w.r.t $x \in \mathfrak{g}$ if

$$[R, adx] = 0 \tag{4.28}$$

COROLLARY. *From (4.24) we see that this is equivalent with $B(x)=0$ and so dim im $B(x)=0$ which means that the orbit is a point.*

LEMMA 4.19. *If R is a Yang-Baxter operator and $\phi \in Aut\mathfrak{g}$ then $\overline{R}=\phi R\phi^{-1}$ is again a Yang-Baxter operator and R and $\overline{R}$ are called equivalent.*

PROOF: Straightforward. □

DEFINITION 4.20. R is called invariant w.r.t $\phi \in Aut\mathfrak{g}$ if $\overline{R} = R$

COROLLARY. *If R is invariant w.r.t ϕ then*

$$B(\phi(x)) = \phi\circ B(x)\circ\phi^{-1} \tag{4.29}$$

Now all the integrable systems, coming from the K.A.S.R.S. construction are related to a solution of the modified Yang-Baxter equation and not to the equation (4.17). Indeed:

PROPOSITION 4.21. [8] *Let $R \in$ End $\mathfrak{g}$ be a solution of (4.18) and define $R_\pm = \frac{1}{2}(R \pm I)$, $\mathfrak{g}_\pm = Im\, R_\pm$, $\mathfrak{k}_\pm = ker\, R_\mp$ then:*
(i) $R_\pm : \mathfrak{g}_R \to \mathfrak{g}$ *are Lie algebra homomorphisms.*
(ii) $\mathfrak{g}_\pm$ are subalgebras of $\mathfrak{g}$
(iii) $\mathfrak{k}_\pm$ *are ideals of* $\mathfrak{g}_R$
(iv) *ker R is abelian in* $\mathfrak{g}$
(v) $[\mathfrak{g}_\pm, \mathfrak{k}_\pm] \subset \mathfrak{k}_\pm$
(vi) $[\mathfrak{k}_+, \mathfrak{k}_-]_R = 0$
(vii) $\mathfrak{k}_-^\perp$ *and* $\mathfrak{k}_+^\perp$ *are Poisson submanifolds of* $\mathfrak{g}$
(viii) $\mathfrak{k}_\pm$ *are subalgebras of* $\mathfrak{g}$
(ix) $[\mathfrak{g}, ker\, R] \subset im\, R$

PROOF. (i), (ii) and (iii) are direct consequences of the fact that we can rewrite (4.18) as :

$$R_\pm\, [x,y]_R = [R_\pm x,\ R_\pm y] \tag{4.30}$$

(iv) follows directly from (4,18) and (v) and (vi) follow from the fact that

$$\begin{aligned}[x,y]_R &= [R_+x,y] + [x,R_-y] \\ &= [R_-x,y] + [x,R_+y]\end{aligned} \tag{4.31}$$

Now

$$\begin{aligned}V_f(x) &= [x,R_+ \operatorname{grad} f(x)] + R_-^*\,[x, \operatorname{grad} f(x)] \\ &= [x,\ R_- \operatorname{grad} f(x)] + R_+^*\,[x, \operatorname{grad} f(x)]\end{aligned} \tag{4.32}$$

and so

$$B(x) = adx\circ R_- + R_+^*\circ adx \tag{4.33}$$

Also

$$\begin{aligned} Im\, R_-^* &\subset (ker\, R_-)^\perp = \mathfrak{k}_+^\perp \\ Im\, R_+^* &\subset (ker\, R_+)^\perp = \mathfrak{k}_-^\perp \end{aligned} \tag{4.34}$$

and from (v) we get

$$[\mathfrak{k}_+^\perp, \mathfrak{g}_+] \subset \mathfrak{k}_+^\perp$$

$$[\mathfrak{k}_-^{\perp}, \mathfrak{g}_-] \subset \mathfrak{k}_-^{\perp} \tag{4.35}$$

and now (vii) follows from (4.32) and Lemma 3.18. Now suppose $x,y \in ker\, R_-$ so, $Rx = x$, $Ry = y$, using (4.18) we get

$$[x,y] = 2R[x,y] - [x,y]$$

so $R_- [x,y] = 0$ and thus $[x,y] \in ker\, R_-$

(ix) follows directly from (4.18) □

The most interesting examples of solutions of (4.18) are those defined in Prop. 4.15 and in fact all solutions of (4.18) are essentially of that kind (cf. [8, Prop. 9, 10]) For those solutions we have the following theorem:

THEOREM 4.22. *(Factorization theorem of Kostant-Adler-Symes-Reyman- Semenov Tyanshanski) Let* $\mathfrak{g}$ *be a real semisimple Lie algebra and* $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ *with* $\mathfrak{a}$ *and* $\mathfrak{b}$ *subalgebras. Let* $R = \pi_a - \pi_b$ *be the corresponding Yang-Baxter operator and* $\{\ ,\ \}_R$ *the corresponding Lie-Poisson bracket. Let* $h \in C^\infty(\mathfrak{g})$ *be an Ad-invariant function and* $H(x) =$ *grad* $h(x)$ *then we have*

$$[x,y]_R = [\pi_a x, \pi_a y] - [\pi_b x, \pi_b y] \tag{4.36}$$

$$R_+ = \pi_a,\ R_- = -\pi_b,\ \mathfrak{g}_+ = \mathfrak{k}_+ = \mathfrak{a},\ \mathfrak{g}_- = \mathfrak{k}_- = \mathfrak{b}$$

and

$$\begin{aligned} V_h(x) &= [x, \pi_a H(x)] \\ &= -[x, \pi_b H(x)] \end{aligned} \tag{4.37}$$

Now let A and B be the connected subgroups of G corresponding to the subalgebras $\mathfrak{a}$ *and* $\mathfrak{b}$ *and let* $g_a(t)$, $g_b(t)$ *be the solution (for small t) of the factorization problem*

$$\exp(-t\, H(0)) = g_b^{-1}(t)\, g_a(t) \tag{4.38}$$

where $H(0) =$ *grad* $h(x(o))$*, then the solution of the Lax equation (4.37) is given by:*

$$x(t) = g_b(t)\, x(0)\, g_b^{-1}(t) = g_a(t)\, x(0)\, g_a(t)^{-1} \tag{4.39}$$

PROOF. Differentiating (4.39) gives:

$$\begin{aligned} \dot{x} &= \dot{g}_b\, x(0)\, g_b^{-1} - g_b\, x(0)\, g_b^{-1}\, \dot{g}_b\, g_b^{-1} \\ &= [\dot{g}_b\, g_b^{-1}, x] = [\dot{g}_a\, g_a^{-1}, x] \end{aligned} \tag{4.40}$$

and

$$\begin{aligned} H(x) &= \text{grad}\, h(x) \\ &= \text{grad}\, h\, (g_b\, x(0)\, g_b^{-1}) \\ &= g_b\ \text{grad}\, h(x(0))\, g_b^{-1} (\text{Lemma 4.5}) \\ &= g_b\, H(0)\, g_b^{-1} \end{aligned} \tag{4.41}$$

Now differentiate (4.38):

$$\dot{g}_b\, g_b^{-1} = \frac{d}{dt}(g_a \exp t\, H(0))\, g_b^{-1}$$

$$= \dot{g}_a \exp t\, H(0)\, g_b^{-1} + g_a \exp(t\, H(0))\, H(0)\, g_b^{-1}$$

$$= \dot{g}_a\, g_a^{-1}\, g_b\, g_b^{-1} + g_b\, H(0)\, g_b^{-1}$$

$$= \dot{g}_a\, g_a^{-1} + H(x) \tag{4.42}$$

and so

$$H(x) = \dot{g}_b\, g_b^{-1} - \dot{g}_a\, g_a^{-1}$$

which implies $\pi_a\ H(x) = -\dot{g}_a\, g_a^{-1}$ and $\pi_b\, H(x) = \dot{g}_b\, g_b^{-1}$ because $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras, and substituting this in (4.40) gives $\dot{x} = [x, \pi_a\, H(x)]$ □

So the solution of the Lax equation (4.37) is reduced to a factorization problem in the Lie group G. This is the finite-dimensional group-theoretical analogue of the Riemann-Hilbert problem in the case of partial differential equations.

From Prop. 4.21 it follows that we may restrict ourselves to the Poisson submanifold $\mathfrak{k}_+^\perp = \mathfrak{a}^\perp$ or $\mathfrak{k}_-^\perp = \mathfrak{b}^\perp$. In the second case the Poisson structure reduces to

$$\{f,g\}_R\,(x) = <x,\ [\pi_a\ \text{grad} f(x),\ \pi_a\ \text{grad} f(x)]> \ \forall x \in \mathfrak{b}^\perp \tag{4.43}$$

and the Hamiltonian vector field for an Ad-invariant $f \in C^\infty(\mathfrak{g})$ becomes:

$$V_f(x) = [\pi_b\ \text{grad} f(x), x]\ \ \forall x \in \mathfrak{b}^\perp \tag{4.44}$$

5. Phase space and Poisson structure of the Calogero-Moser models

Now let $\mathfrak{g}$ be a real noncompact semisimple Lie algebra (see [27-30] for some facts about (real) Lie algebras), let $\theta \in$ Aut $\mathfrak{g}$ denote the Cartan involution and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition with commutation relations:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}\quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}\quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k} \tag{5.1}$$

Let $<\ ,\ >$ be the Killing form, which is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$ and define and inner product on $\mathfrak{g}$ by:

$$<x,y>_\theta = -<x, \theta(y)> \tag{5.2}$$

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$ and let $\mathfrak{m} \subset \mathfrak{k}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Let R be the root system corresponding to the pair $(\mathfrak{g}, \mathfrak{a})$, let Δ be the simple roots w.r.t some ordening, R_+ the positive roots, R_- the negative roots and g_α the root space corresponding to the root α. Then we have the root space decomposition:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha \tag{5.3}$$

where the decomposition is orthogonal w.r.t the Killing form and

$$\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a} \tag{5.4}$$

Also we have $\theta\, \mathfrak{g}_\alpha = \mathfrak{g}_{-\alpha}$ (5.5)

Now define

$$\mathfrak{n} = \sum_{\alpha \in R} g_\alpha$$

$$\mathfrak{n}_+ = \sum_{\alpha \in R_+} g_\alpha$$

$$\mathfrak{n}_- = \sum_{\alpha \in R_-} g_\alpha \tag{5.6}$$

And we have

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta} \text{ if } \alpha+\beta \in R \tag{5.7}$$

from which we get the commutation relations:

$$[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0$$
$$[\mathfrak{g}_0, \mathfrak{n}_+] \subset \mathfrak{n}_+$$
$$[\mathfrak{g}_0, \mathfrak{n}_-] \subset \mathfrak{n}_-$$
$$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{g}_0 \tag{5.8}$$

DEFINITION 5.1. The rank of $\mathfrak{g}$ is defined as $l = \dim \mathfrak{a}$ and $m_\alpha = \dim \mathfrak{g}_\alpha$ is called the multiplicity of the root α.

REMARK 5.2. In contrast to the case of a complex simple Lie algebra R can be a non-reduced rootsystem, m_α can be >1 and $\mathfrak{m}$ is in general a proper non-zero subspace of $\mathfrak{g}_0$ and is not necessarily abelian.

The Killing form restricted to $\mathfrak{a}$ is positive definite and nondegenerate, so $\mathfrak{a}$ is an Euclidian vector space and we can identify $\mathfrak{a}$ and $\mathfrak{a}^*$ by means of the Killing form, so for all $\lambda \in \mathfrak{a}^*$ define $h_\lambda \in \mathfrak{a}$ by:

$$<h, h_\lambda> = \lambda(h) \quad \forall h \in \mathfrak{a} \tag{5.9}$$

Using this we define an inner product (,) on $\mathfrak{a}^*$ by:

$$(\lambda, \mu) := <h_\lambda, h_\mu> \tag{5.10}$$

Now let $\alpha \in R_+$ and choose $0 \neq e_\alpha \in \mathfrak{g}_\alpha$ in such a way that

$$<e_\alpha, e_\alpha>_\theta = 1 \tag{5.11}$$

and define:

$$e_{-\alpha} := \theta(e_\alpha). \tag{5.12}$$

then we also have:

$$<e_{-\alpha}, e_{-\alpha}>_\theta = 1 \tag{5.13}$$

and $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$.
From (5.11) and (5.12) we derive the commutation relations:

$$[e_\alpha, e_{-\alpha}] = -h_\alpha$$
$$[h_\alpha. e_\alpha] = (\alpha,\alpha)e_\alpha$$
$$[h_\alpha, e_{-\alpha}] = -(\alpha,\alpha)e_{-\alpha} \tag{5.14}$$

Define for all $\alpha \in R_+$:

$$s_\alpha = e_\alpha + e_{-\alpha} \quad a_\alpha = e_\alpha - e_{-\alpha} \tag{5.15}$$

then $s_\alpha \in \mathfrak{k}$, $a_\alpha \in \mathfrak{p}$ and

$$[s_\alpha, a_\alpha] = 2h_\alpha \tag{5.16}$$

Moreover;

$$\mathfrak{k} = \mathfrak{k} \cap \mathfrak{n} \oplus \mathfrak{m}$$
$$\mathfrak{p} = \mathfrak{p} \cap \mathfrak{n} \oplus \mathfrak{a} \tag{5.17}$$

where the decomposition is orthogonal w.r.t the Killing form and $s_\alpha \in \mathfrak{k} \cap \mathfrak{n}$ and $a_\alpha \in \mathfrak{p} \cap \mathfrak{n}$.

From (5.15) it also follows that:

$$[h, s_\alpha] = \alpha(h)\, a_\alpha$$
$$[h, a_\alpha] = \alpha(h)\, s_\alpha \quad \forall h \in \mathfrak{a} \tag{5.18}$$

Now let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{g}$ which contains $\mathfrak{a}$. Such a subalgebra exists and it follows that $\theta(x) \in \mathfrak{h}$ for all $x \in \mathfrak{h}$, so we have the direct decomposition:

$$\mathfrak{h} = \mathfrak{h} \cap \mathfrak{k} \oplus \mathfrak{h} \cap \mathfrak{p} \tag{5.19}$$

and $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$ and we write $\mathfrak{h}_k = \mathfrak{h} \cap \mathfrak{k}$. Let $\mathfrak{h}_c$ be the complexification of $\mathfrak{h}$ and $\mathfrak{g}_c$ the complexification of $\mathfrak{g}$, then $\mathfrak{g}_c$ is a complex semisimple Lie algebra and $\mathfrak{h}_c$ is a Cartan subalgebra of $\mathfrak{g}_c$.
We also have:

$$\mathfrak{h}_k \subset \mathfrak{m} \tag{5.20}$$

so $\mathfrak{m}$ is abelian iff dim $\mathfrak{h}_k$ = dim $\mathfrak{m}$. A Lie algebra is called quasi-split if $\mathfrak{h} = \mathfrak{m} \oplus \mathfrak{a}$ and split if $\mathfrak{m} = \{0\}$.

DEFINITION 5.3. An element $q \in \mathfrak{a}$ is called regular if the centralizer of q in $\mathfrak{g}$ equals $\mathfrak{g}_0$.

LEMMA 5.4. *$q \in \mathfrak{a}$ is regular iff $\alpha(q) \neq 0 \;\; \forall \alpha \in R$*

PROOF. ($\Rightarrow$) suppose $\exists \alpha \in R$ such that $\alpha(q) = 0$ then there exists an $0 \neq x \in \mathfrak{g}_\alpha$ such that $[q,x] = 0$ and so $x \in \mathfrak{g}_0 \cap \mathfrak{g}_\alpha = \{0\}$ $\rightarrow\leftarrow$

($\Leftarrow$) suppose $\alpha(q) \neq 0 \;\; \forall \alpha \in R$ and $q \in \mathfrak{a}$ is not regular then there exists an $0 \neq x \in \mathfrak{n}$ such that $[q,x] = 0$ but $\mathfrak{n} = \sum_{\alpha \in R} \mathfrak{g}_\alpha$ so there must exist an $\alpha \in R$ such that $\alpha(q) = 0$ or $x = 0$ $\rightarrow\leftarrow$ □

Now $ad\, q \in$ End $\mathfrak{g}$ is a semisimple linear transformation and so

$$\mathfrak{g} = \ker(ad\, q) \oplus \operatorname{im}(ad\, q) \tag{5.21}$$

but because q is regular we have $\ker(ad\, q) = \mathfrak{g}_0$ and $\operatorname{im}(ad\, q) = \mathfrak{n}$ and

$$ad\, q\colon\; \mathfrak{n} \longrightarrow \mathfrak{n} \tag{5.22}$$

acts as an isomorphism. Let $(ad\, q)^{-1}\colon \mathfrak{n} \longrightarrow \mathfrak{n}$ denote the inverse and $(ad\, q)^{-1} \circ \pi_n$ the extension of this inverse to $\mathfrak{g}$, where π_n denotes the projection on $\mathfrak{n}$.

LEMMA 5.5. *Let $q \in \mathfrak{a}$ be a regular element then for all $x,y \in \mathfrak{g}$, $h \in \mathfrak{g}_0$:*
(i) $<(ad\, q)^{-1}\pi_n\, x, y> \; = -<x, (ad\, q)^{-1}\pi_n\, y>$
(ii) $(ad\, q)^{-1}\pi_n\, [h,x] = [h, (ad\, q)^{-1}\pi_n\, x]$
(iii) $[(ad\, q)^{-1}\pi_n x, \pi_n y] + [\pi_n x, (ad\, q)^{-1}\pi_n y] \in \mathfrak{n}$
(iv) $\pi_n\, [(ad\, q)^{-1}\pi_n x, (ad\, q)^{-1}\pi_n y]$
$= (ad\, q)^{-1} \{[(ad\, q)^{-1}\pi_n x, \pi_n y] + [\pi_n x, (ad\, q)^{-1}\pi_n y]\}$

PROOF. (i) follows from the skew-symmetry of $ad\, q$ w.r.t the Killing form and (ii), (iii) and (iv) follow from the fact that $ad\, q$ is a derivation and the commutation relations (5.8). □

DEFINITION 5.6. The positive Weyl chamber $\mathfrak{a}_+$ of $\mathfrak{a}$ is defined as:

$$\mathfrak{a}_+ = \{h \in \mathfrak{a} \mid \alpha(h) > 0, \;\; \forall \alpha \in R_+\} \tag{5.23}$$

So $\mathfrak{a}_+$ consists of regular elements and is a open connected subset of $\mathfrak{a}$.

DEFINITION 5.7. A bundle is a triple (P, B, π) where P and B are topological spaces and $\pi: P \to B$ is a continuous surjective map. B is called the base and P the total space. $F_x = \pi^{-1}(x) = \{y \in P | \pi(y) = x, x \in B\}$ is called the fibre at x. If F_x is homeomorphic to F for all $x \in B$, then F is called the typical fibre. A (global) section σ is a map $\sigma: B \to P$ such that $\pi \circ \sigma = id_B$.

An example of a bundle is the product bundle $P = \mathfrak{a}_+ \times \mathfrak{g}$, with $\pi(h,x) = h$ for all $h \in \mathfrak{a}_+$, $x \in \mathfrak{g}$, $B = \mathfrak{a}_+$ and $F = \mathfrak{g}$

DEFINITION 5.8. A bundle (P', B', π') is a subbundle of (P, B, π) provided P' is a subspace of P, B' is a subspace of B and $\pi' = \pi / p' : P' \to B'$.

Now let $\mu \in \mathfrak{n}$ be a fixed element and define

$$P_\mu = \{(q,x) \in P | [q,x] = \mu\} \tag{5.24}$$

PROPOSITION 5.9. *P_μ is a subbundle of the product bundle P and $\pi : P_\mu \to B$ is defined by*

$$\pi(q,x) = q \tag{5.25}$$

PROOF. Because $\mu \in \mathfrak{n}$ and $ad\, q : \mathfrak{n} \to \mathfrak{n}$ acts as an isomorphism for all $q \in \mathfrak{a}_+$ it is clear that π is surjective. □

Now define $\pi_2 : P \to \mathfrak{g}$ by $\pi_2(q,x) = x$. Then π_2 restricts to P_μ and we define $\hat{P}_\mu := im\, \pi_2 / P_\mu$.

LEMMA 5.9.1. *$\pi_2 : P_\mu \to \hat{P}_\mu$ is injective iff*

$$C_{\mathfrak{a}}(\mu) = \{0\}$$

PROOF. ($\Leftarrow$) Suppose that $C_{\mathfrak{g}}(\mu) \cap \mathfrak{a} = \{0\}$ and suppose there exists a $x \in \mathfrak{g}$ and $q_1, q_2 \in \mathfrak{a}_+$ such that $[q_1, x] = [q_2, x] = \mu$ then $[q_1 - q_2, x] = 0$ and so $[q_1 - q_2, \mu] = 0$ which implies $q_1 - q_2 \in C_{\mathfrak{g}}(\mu) \cap \mathfrak{a} = \{0\}$ so $q_1 = q_2$ and π_2 is injective.

($\Rightarrow$) Suppose π_2 is injective and $h \in C_{\mathfrak{g}}(\mu) \cap \mathfrak{a}$. Then there always exist $q_1, q_2 \in \mathfrak{a}_+$ such that $h = q_1 - q_2$ and because q_1 is regular there exists a $x \in \mathfrak{g}$ such that $[q_1, x] = \mu$ and so $0 = [h, \mu] = [h, [q_1, x]] = [q_1, [h, x]]$ but q_1 is regular so $[h, x] = 0$ which implies $[q_1, x] = [q_2, x]$ and so $q_1 = q_2$ but then $h = 0$ □

So we have constructed a bundle P_μ over the positive Weyl chamber $\mathfrak{a}_+$ and with typical fibre homeomorphic to $\mathfrak{g}_0$. P_μ will be our phase space and the next step is to define a Poisson bracket on P_μ.

Now define for each $q \in \mathfrak{a}_+$ and for all $x, y \in \mathfrak{g}$:

$$[x,y]_q = [\pi_a x, (ad\, q)^{-1} \pi_n y] + [(ad\, q)^{-1} \pi_n x, \pi_a y] \tag{5.26}$$

PROPOSITION 5.10. *(5.26) defines a Lie bracket on $\mathfrak{g}$ for all $q \in \mathfrak{a}_+$*

PROOF. Bilinearity and skew-symmetry are obvious, so we only have to check the Jacobi identity. Now observe that $[x,y]_q \in \mathfrak{n}$ for all $x, y \in \mathfrak{g}$, so

$$
\begin{aligned}
&[[x,y]_q, z]_q + cycl \\
&= [(ad\,q)^{-1}\pi_n [x,y]_q, \pi_a z] + cycl \\
&= [(ad\,q)^{-1}\pi_n [\pi_a x, (ad\,q)^{-1}\pi_n y], \pi_a z] \\
&\quad + [(ad\,q)^{-1}\pi_n [(ad\,q)^{-1}\pi_n x, \pi_a y], \pi_a z] + cycl \\
&= [[\pi_a x, (ad\,q)^{-2}\pi_n y], \pi_a z] \\
&\quad + [[(ad\,q)^{-2}\pi_n x, \pi_a y], \pi_a z] + cycl \text{ (Lemma 5.5)} \\
&= [[\pi_a x, (ad\,q)^{-2}\pi_n y], \pi_a z] \\
&\quad + [[(ad\,q)^{-2}\pi_n y, \pi_a z], \pi_a x] + cycl \\
&= [(ad\,q)^{-2}\pi_n y, [\pi_a z, \pi_a x]] + cycl = 0 \quad \square
\end{aligned}
$$

Now define the corresponding Lie-Poisson bracket on $\mathfrak{g}$:

$$\{f,g\}^q(x) = <x, [\mathrm{grad} f(x), \mathrm{grad}\, g(x)]_q> \tag{5.27}$$

Now we first define a Poisson bracket on P and then show that P_μ is a Poisson submanifold of P. Now $T_pP \cong \mathfrak{a}\oplus\mathfrak{g}$ and because the Killing form restricted to $\mathfrak{a}$ is nondegenerate we can define a non-degenerate bilinear form or $\mathfrak{a}\oplus\mathfrak{g}$ by defining $<(a_1,x_1),(a_2,x_2)> = <a_1,a_2> + <x_1,x_2>$.
Now let $f\in C^\infty(P)$ and define

$$
\begin{aligned}
&<\mathrm{grad} f(q,x), (\dot q,\dot x)> \\
&= <\pi_1 \nabla f(q,x), \dot q> + <\pi_2 \nabla f(q,x), \dot x>
\end{aligned}
$$

where $p=(q,x)\in P$ and $\dot p = (\dot q,\dot x)\in T_pP$.

Let $f,g\in C^\infty(P)$, $p=(q,x)\in P$ and write $F=\nabla f(P)$, $G=\nabla g(P)$ and define

$$\{f,g\}_1(p) = <x.[F_2, G_2]_q> \tag{5.28a}$$

$$\{f,g\}_2(p) = <\pi_a G_2, F_1> - <\pi_a F_2, G_1> \tag{5.28b}$$

$$\{f,g\} = \{f,g\}_1 + \{f,g\}_2 \tag{5.28c}$$

where F_1, F_2 denote $\pi_1 F$ resp. $\pi_2 F$.

PROPOSITION 5.11. *$\{,\}_1$, $\{,\}_2$ and $\{,\}$ define Poisson brackets on P. In other words: $\{,\}_1$ and $\{,\}_2$ form a Hamiltonian pair.*

PROOF. It is clear that all the brackets are skew-symmetric and bilinear and they only depend on df and dg, so it is sufficient to prove the Jacobi identity for coordinate functions.
Now observe that

$$\mathrm{grad}\,\{f,g\}_2(p) = 0 \tag{5.28d}$$

Also it follows from (5.28a) that

$$
\begin{aligned}
&\pi_2\, \mathrm{grad}\,\{f,g\}_1(p) \\
&= [F_2, G_2]_q
\end{aligned} \tag{5.28e}
$$

and

$$
\begin{aligned}
&\pi_1\, \mathrm{grad}\,\{f,g\}_1(p) \\
&= [[x,\pi_a F_2], (ad\,q)^{-2}\pi_n G_2]
\end{aligned}
$$

$$+ [[\pi_a G_2, x], (ad\, q)^{-2} \pi_n F_2] \qquad (5.28f)$$

We can rewrite the Jacobi identity for $\{,\}$ as:

$$\begin{aligned} 0 = &\{\{f,g\}_1, h\}_1 + cycl \\ &+ \{\{f,g\}_2, h\}_1 + cycl \\ &+ \{\{f,g\}_1, h\}_2 + cycl \\ &+ \{\{f,g\}_2, h\}_2 + cycl \end{aligned} \qquad (5.29g)$$

De first term is zero because this reduces to the Jacobi-identity for $\{,\}^q$, so $\{,\}_1$ is a Poisson bracket.

The fourth term is zero because of (5.28d), so $\{,\}_2$ is also a Poisson bracket.

The second term is also zero because of (5.28d). Now using (5.28e) and (5.28f) and the Jacobi identity on $\mathfrak{g}$ the third term can be rewritten as:

$$< x, [(ad\, q)^{-2} \pi_n G_2, [\pi_a F_2, \pi_a H_2]] > + cycl$$

which is zero. □

From (5.28a) and (5.28b) we derive that for $f \in C^\infty(P)$ and $p=(q,x) \in P$ the Hamiltonian vector field is given by:

$$V_f(p) = (\dot{q}, \dot{x}) = (-\pi_a F_2, \pi_a [x, (ad\, q)^{-1} \pi_n F_2] - (ad\, q)^{-1} \pi_n [x, \pi_a F_2] + \pi_a F_1) \qquad (5.29)$$

If $f \in C^\infty(\mathfrak{g})$ then f extends to a function on P by defining $f(q,x) = f(x)$ and so

$$\pi_1 \, \mathrm{grad} f(q,x) = 0 \text{ and } \pi_2 \, \mathrm{grad} f(q,x) = \mathrm{grad} f(x) \qquad (5.30)$$

PROPOSITION 5.12. *P_μ is a Poisson submanifold of P with Poisson bracket (5.28c).*

PROOF. Because $[\dot{q},x] + [q,\dot{x}] = 0$, $V_f(P)$ is tangent to P_μ for all $f \in C^\infty(P)$ and $p \in P_\mu$ and using Lemma 3.18 we conclude that P_μ is a Poisson submanifold. □

So if $i: P_\mu \to P$ denotes the imbedding of P_μ in P and $i^*: C^\infty(P) \to C^\infty(P_\mu)$ the pull-back, then it follows from Prop. 5.12 that i is a Poisson mapping and

$$\{i^*(f), i^*(g)\} = i^*(\{f,g\}) \qquad (5.31)$$

with $f,g \in C^\infty(P)$, defines the induced Poisson structure on P_μ.

For the Calogero-Moser models we choose as Hamiltonian an *Ad*-invariant $f \in C^\infty(\mathfrak{g})$ and view this as a function on P and on P_μ. Let $F = \mathrm{grad} f(x)$, then $[F,x] = 0$.

PROPOSITION 5.13. *Let $f \circ \pi_m \in C^\infty(\mathfrak{m})$ then f is a Casimir function w.r.t (5.27) and $\mathfrak{m}^\perp = \mathfrak{a} \oplus \mathfrak{n}$ is a Poisson submanifold of $\mathfrak{g}$.*

PROOF. It is sufficient to prove it for coordinate functions on $\mathfrak{m}$, but then grad $f(x) \in \mathfrak{m}$ so from (5.26) and (5.27) it follows that $\{f,g\}^q = 0$ for all $g \in C^\infty(\mathfrak{g})$ and so f is a Casimir function, so we may restrict ourselves to $\mathfrak{m}^\perp = \mathfrak{a} \oplus \mathfrak{n}$ □

Let $\pi_{\mathfrak{m}}^\perp : \mathfrak{g} \to \mathfrak{m}^\perp$ denote the projection on $\mathfrak{m}^\perp$ then $(\pi_{\mathfrak{m}}\perp)^* : C^\infty(\mathfrak{m}^\perp) \to C^\infty(\mathfrak{g})$ and the induced bracket on $\mathfrak{m}^\perp$ becomes:

$$\{f,g\}^q(x) = i^* \{\pi_{\mathfrak{m}}^* \perp(f), \pi_{\mathfrak{m}}^* \perp(g)\}^q(x)$$

$$= \{\pi^*_{\mathfrak{m}} \perp(f),\ \pi^*_{\mathfrak{m}} \perp(g)\}^q\ (i(x)) \tag{5.32}$$

It is also clear that $P_\mu \cap \mathfrak{m}^\perp$ is a Poisson submanifold of $P \cap \mathfrak{m}^\perp$.

To obtain the phase space of the Calogero-Moser systems we need more structure.
Suppose $\sigma \in Aut\ \mathfrak{g}$ is an involutive automorphism which commutes with the Cartan involution θ and let

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{s} \tag{5.33}$$

denote the decomposition in eigenspaces of σ with commutationrelations:

$$[\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{t} \quad [\mathfrak{t}, \mathfrak{s}] \subset \mathfrak{s} \quad [\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{t} \tag{5.34}$$

then we also have the decompositions:

$$\mathfrak{k} = \mathfrak{k} \cap \mathfrak{t} \oplus \mathfrak{k} \cap \mathfrak{s} \quad \mathfrak{p} = \mathfrak{p} \cap \mathfrak{t} \oplus \mathfrak{p} \cap \mathfrak{s} \tag{5.35}$$

and all the decompositions are orthogonal w.r.t the Killing form. Now it is always possible to choose $\mathfrak{a}$ in such a way that $\mathfrak{g}_0$ is a σ-invariant subspace and then we get:/

$$\mathfrak{a} = \mathfrak{a} \cap \mathfrak{t} \oplus \mathfrak{a} \cap \mathfrak{s} \tag{5.36}$$

and

$$\mathfrak{m} = \mathfrak{m} \cap \mathfrak{t} \oplus \mathfrak{m} \cap \mathfrak{s} \tag{5.37}$$

Now suppose that $\mathfrak{a} \cap \mathfrak{t} = \{0\}$ so $\mathfrak{a} \subset \mathfrak{s}$. This is true for the A_n and BC_n models. Because $\mathfrak{g}_0$ is σ-invariant $\mathfrak{n}$ is also σ-invariant and so we have the decomposition:

$$\mathfrak{n} = \mathfrak{n} \cap \mathfrak{k} \cap \mathfrak{t} \oplus \mathfrak{n} \cap \mathfrak{k} \cap \mathfrak{s} \oplus \mathfrak{n} \cap \mathfrak{p} \cap \mathfrak{t} \oplus \mathfrak{n} \cap \mathfrak{p} \cap \mathfrak{s} \tag{5.38}$$

Now take $\mu \in \mathfrak{n} \cap \mathfrak{k} \cap \mathfrak{s}$ then it immediately follows that

$$C_{\mathfrak{g}}(\mu) = C_{\mathfrak{g}}(\mu) \cap \mathfrak{k} \cap \mathfrak{t} \oplus C_{\mathfrak{g}}(\mu) \cap \mathfrak{k} \cap \mathfrak{s} \oplus C_{\mathfrak{g}}(\mu) \cap \mathfrak{p} \cap \mathfrak{t} \oplus C_{\mathfrak{g}}(\mu) \cap \mathfrak{p} \cap \mathfrak{s} \tag{5.39}$$

LEMMA 5.14. *Suppose* $\mu \in \mathfrak{n} \cap \mathfrak{k} \cap \mathfrak{s}$ *and* $p = (q,x) \in P_\mu \cap \mathfrak{m}^\perp$ *then* $x \in \mathfrak{p}$.

PROOF. From $[q,x] = \mu$ it follows that $[\pi_k x, q] = 0$ and so $\pi_k x \in \mathfrak{g}_0$ but $x \in \mathfrak{m}^\perp$ so $\pi_k x = 0$ □

LEMMA 5.15. *Let* $\mu \in \mathfrak{n} \cap \mathfrak{k} \cap \mathfrak{s}$ *and* $p = (q,x) \in P_\mu \cap \mathfrak{m}^\perp$ *and let* $f \in C^\infty(\mathfrak{g})$ *Ad-invariant. Let* $F =$ *grad* $f(x)$ *then*

(i) $[\pi_k F, x] = 0$, $[\pi_p F, x] = 0$
(ii) $[(ad\, q)^{-1} \pi_{n \cap p} F, \mu] \in \mathfrak{n} \cap \mathfrak{k}$
(iii) $[(ad\, q)^{-1} \pi_{n \cap k} F, \mu] \in \mathfrak{n} \cap \mathfrak{p}$

PROOF. (i) follows directly from $[F, x] = 0$ and the fact that $x \in \mathfrak{p}$.

(ii) $\pi_m\ [(ad\, q)^{-1} \pi_{n \cap p}\ F, \mu]$

$$\begin{aligned}
&= -\pi_m [\pi_{n \cap p}\ F,\ (ad\, q)^{-1} \mu] \text{ (Lemma 5.5)}\\
&= -\pi_m [\pi_{n \cap p}\ F,\ \pi_{n \cap p} x]\\
&= -\pi_m [\pi_{n \cap p}\ F,\ x]\\
&= \ \pi_m [\pi_a\ F,\ x] \text{ (Lemma 5.15)}\\
&= 0
\end{aligned}$$

(iii) $\pi_a[(ad\,q)^{-1}\pi_{n\cap k}\,F,\ \mu]$

$$= -\pi_a[\pi_{n\cap k}\,F,\ (ad\,q)^{-1}\mu] \quad \text{(Lemma 5.5)}$$
$$= -\pi_a[\pi_{n\cap k}\,F,\ \pi_{n\cap p}\,x]$$
$$= -\pi_a[\pi_{n\cap k}\,F,\ x]$$
$$= \pi_a[\pi_m\,F,\ x] \quad \text{(Lemma 5.15)}$$
$$= 0 \qquad \square$$

COROLLARY 5.16. $[(ad\,q)^{-1}\pi_{n\cap p\cap s}\,F,\ \mu] \in \mathfrak{n}\cap\mathfrak{k}\cap\mathfrak{s}$

PROOF. $\pi_m\,[(ad\,q)^{-1}\pi_{n\cap p}\,F,\ \mu]=0$ but $\mathfrak{m}=\mathfrak{m}\cap\mathfrak{t}\oplus\mathfrak{m}\cap\mathfrak{s}$ so also

$$\pi_{m\cap s}\,[(ad\,q)^{-1}\pi_{n\cap p}\,F,\ \mu]=0$$

and so $[(ad\,q)^{-1}\pi_{n\cap p\cap s}\,F,\ \mu] \in \mathfrak{n}\cap\mathfrak{k}\cap\mathfrak{s}$ $\square$

LEMMA 5.17. *Suppose* $\mu\in\mathfrak{n}\cap\mathfrak{k}\cap\mathfrak{s}$ *satisfies the following condition:*

$$x\in\mathfrak{n}\cap\mathfrak{k}\cap\mathfrak{t} \text{ and } [x,\mu]\in\mathfrak{n} \Rightarrow [x,\mu]=0 \tag{5.40}$$

then $[(ad\,q)^{-1}\pi_{n\cap p\cap s}\,F,\ \mu]=0$

PROOF. This follows from Corollary 5.16 and the fact that $(ad\,q)^{-1}:\mathfrak{n}\cap\mathfrak{p}\cap\mathfrak{s}\longrightarrow\mathfrak{n}\cap\mathfrak{k}\cap\mathfrak{t}$ $\square$

THEOREM 5.18. *Suppose* $\mu\in\mathfrak{n}\cap\mathfrak{k}\cap\mathfrak{s}$ *satisfies condition (5.40). Let* $f,g\in C^\infty(\mathfrak{g})$ *Ad-invariant. Then they are in involution on* $P_\mu\cap\mathfrak{m}^\perp$.

PROOF. Let $F=$ grad $f(x)$ and $G=$ grad $g(x)$, where $(q,x)\in P_\mu\cap\mathfrak{m}^\perp$ then we have:

$$\{f,g\}(q,x)$$
$$= <x,\ [\pi_a F,\ (ad\,q)^{-1}\pi_n G]+[(ad\,q)^{-1}\pi_n F,\ \pi_a G]>$$
$$= <x,\ [\pi_a F,\ (ad\,q)^{-1}\pi_{n\cap p} G]+[(ad\,q)^{-1}\pi_{n\cap p} F,\ \pi_a G]>$$
(because $x\in\mathfrak{p}$)
$$= <x,\ [(ad\,q)^{-1}\pi_{n\cap p} G,\ \pi_{n\cap p} F]+[\pi_{n\cap p} G,\ (ad\,q)^{-1}\pi_{n\cap p} F]>$$
(Lemma 5.15)
$$= <\pi_n x,\ [(ad\,q)^{-1}\pi_{n\cap p} G,\ \pi_{n\cap p} F]+[\pi_{n\cap p} G,\ (ad\,q)^{-1}\pi_{n\cap p} F]>$$
(Lemma 5.5)
$$= <(ad\,q)^{-1}\mu,\ [(ad\,q)^{-1}\pi_{n\cap p} G,\ \pi_{n\cap p} F]+[\pi_{n\cap p} G,\ (ad\,q)^{-1}\pi_{n\cap p} F]>$$
$$= -<\mu,\ (ad\,q)^{-1}\{[(ad\,q)^{-1}\pi_{n\cap p} G,\ \pi_{n\cap p} F]+[\pi_{n\cap p} G,\ (ad\,q)^{-1}\pi_{n\cap p} F]\}>$$
(Lemma 5.5)
$$= <\mu,\pi_n[(ad\,q)^{-1}\pi_{n\cap p} F,\ (ad\,q)^{-1}\pi_{n\cap p} G]>$$
(Lemma 5.5)
$$= <\mu,\ [(ad\,q)^{-1}\pi_{n\cap p} F,\ (ad\,q)^{-1}\pi_{n\cap p} G]>$$
$$= <\mu,\ [(ad\,q)^{-1}\pi_{n\cap p\cap t} F,\ (ad\,q)^{-1}\pi_{n\cap p\cap t} G]>$$

(Lemma 5.17)

$= 0$ because $(ad\, q)^{-1} : \mathfrak{n}\cap\mathfrak{p}\cap\mathfrak{t} \rightarrow \mathfrak{n}\cap\mathfrak{k}\cap\mathfrak{s}$

and $[\mathfrak{s}, \mathfrak{s}]\cap\mathfrak{t}$ but $\mu\in\mathfrak{s}$ □

REMARK 5.19. If μ does not satisfy condition (5.40) then we are left with the term:

$$<\mu, [(ad\, q)^{-1}\pi_{n\cap p\cap t}F, (ad\, q)^{-1}\pi_{n\cap p\cap s}G] + [(ad\, q)^{-1}\pi_{n\cap p\cap s}F, (ad\, q)^{-1}\pi_{n\cap p\cap t}G]> \tag{5.41}$$

REMARK 5.20. We will show in section 6 that the μ corresponding to the A_n and BC_n models satisfies condition (5.40) and so Theorem 5.18 is true for these models.

So we have proved that under a certain condition on μ the *Ad*-invariant functions restricted to the Poisson submanifold $P_\mu\cap\mathfrak{m}^\perp$ are in involution w.r.t the induced Poisson bracket. To prove that the Hamilton equations can be written in Lax form we need a somewhat stronger condition on μ, namely

$$\pi_n : C_{\mathfrak{g}}(\mu)\cap\mathfrak{k}\cap\mathfrak{s} \longrightarrow \mathfrak{n}\cap\mathfrak{k}\cap\mathfrak{s} \text{ is surjective} \tag{5.42}$$

LEMMA 5.21. *Suppose* $\mu\in\mathfrak{n}\cap\mathfrak{k}\cap\mathfrak{s}$ *satisfies condition (5.42) then it also satisfies condition (5.40).*

PROOF. Suppose $x\in\mathfrak{n}\cap\mathfrak{k}\cap\mathfrak{t}$ and $[x,\mu]\in\mathfrak{n}\cap\mathfrak{k}\cap\mathfrak{s}$ then for all $y\in\mathfrak{n}\cap\mathfrak{k}\cap\mathfrak{s}$

$$<[x,\mu], y> = <x, [\mu,y]>$$
$$= <x, [\mu, m(y)]> \text{ for some } m(y)\in\mathfrak{m}\cap\mathfrak{s}$$
$$= <[x, \mu], m(y)> = 0$$

but this implies $[x,\mu]=0$ because the Killing form is nondegenerate on $\mathfrak{n}\cap\mathfrak{k}\cap\mathfrak{s}$. □

Now suppose μ satisfies (5.42) and consider the map

$$ad\, q : C_{\mathfrak{g}}(\mu)\cap\mathfrak{k}\cap\mathfrak{s} \longrightarrow \mathfrak{n}\cap\mathfrak{p}\cap\mathfrak{t} \tag{5.43}$$

because of (5.42) this map is also surjective. Suppose that

$$C_{\mathfrak{g}}(\mu)\cap\mathfrak{m}\cap\mathfrak{s} = \{0\} \tag{5.44}$$

then this map is also injective, so we can define the inverse which we denote by S:

$$S : \mathfrak{n}\cap\mathfrak{p}\cap\mathfrak{t} \longrightarrow C_{\mathfrak{g}}(\mu)\cap\mathfrak{k}\cap\mathfrak{s} \tag{5.45}$$

LEMMA 5.22. *Suppose* $\mu\in\mathfrak{n}\cap\mathfrak{k}\cap\mathfrak{s}$ *satisfies (5.42) and (5.44) then for all* $x,y\in\mathfrak{g}$

$$\pi_{n\cap p\cap s}[(ad\, q)^{-1}\pi_{n\cap k\cap s}x, S\,\pi_{n\cap p\cap t}y] = (ad\, q)^{-1}\{[\pi_{n\cap k\cap s}x, S\pi_{n\cap p\cap t}y] + [(ad\, q)^{-1}\pi_{n\cap k\cap s}x, \pi_{n\cap p\cap t}y]\} \tag{5.46}$$

PROOF. This follows from the fact that *ad q* is a derivation and so

$$ad\, q\,[x,y] = [ad\, q\, x,y] + [x, ad\, q\, y] \tag{5.47}$$

Now take $x\in\mathfrak{n}\cap\mathfrak{p}\cap\mathfrak{t}$ and $y\in C_{\mathfrak{g}}(\mu)\cap\mathfrak{k}\cap\mathfrak{s}$ and invert (5.47) □

THEOREM 5.23. *Suppose* $\mu\in\mathfrak{n}\cap\mathfrak{k}\cap\mathfrak{s}$ *satisfies (5.42) and (5.44), let* $f\in C^\infty(\mathfrak{g})$ *Ad-invariant, then the* *restriction of the Hamiltonian vector field of f to $P_\mu \cap \mathfrak{m}^\perp$ can be written in Lax form:*

$$\dot{x} = V_f(x) = [x, M]$$

where

$$M = S\pi_{n\cap p\cap t} F + (ad\, q)^{-1}\pi_{n\cap p\cap s} F \tag{5.49}$$

and

$$F = \text{grad} f(x), \quad p = (q,x) \in P_\mu \cap \mathfrak{m}^\perp$$

PROOF. $V_f(x) = [x, S\pi_{n\cap p\cap t} F + (ad\, q)^{-1}\pi_{n\cap p\cap s} F]$

$$= [\pi_a x,\ S\pi_{n\cap p\cap t} F + (ad\, q)^{-1}\pi_{n\cap p\cap s} F]$$
$$+ [(ad\, q)^{-1}\mu,\ S\,\pi_{n\cap p\cap t} F] + [(ad\, q)^{-1}\mu, (ad\, q)^{-1}\pi_{n\cap p\cap s} F] \tag{5.50}$$

The first term of (5.50) becomes:

$$[\pi_a x,\ (ad\, q)^{-1}\, \pi_{n\cap p} F]$$
$$= (ad\, q)^{-1}\ [\pi_a x,\ \pi_{n\cap p} F] \quad \text{(Lemma 5.5.)}$$
$$= (ad\, q)^{-1}\{[x,\ \pi_{n\cap p} F] - [(ad\, q)^{-1}\mu,\ \pi_{n\cap p} F]\}$$
$$= (ad\, q)^{-1}\pi_n\, [x, \pi_{n\cap p} F] - (ad\, q)^{-1}\, \pi_n\, [(ad\, q)^{-1}\mu,\ \pi_{n\cap p} F]$$
$$= -(ad\, q)^{-1}\, \pi_n\, [x, \pi_a F] - (ad\, q)^{-1}\pi_n\, [(ad\, q)^{-1}\mu,\ \pi_{n\cap p} F]$$

(Lemma 5.15)

$$= -(ad\, q)^{-1}\, [x, \pi_a F] - (ad\, q)^{-1}\pi_n\, [(ad\, q)^{-1}\mu,\ \pi_{n\cap p} F] \tag{5.51}$$

The second term can be rewritten as:

$$[(ad\, q)^{-1}\mu,\ S\,\pi_{n\cap p\cap t} F]$$
$$= \pi_a\, [(ad\, q)^{-1}\mu,\ S\,\pi_{n\cap p\cap t} F] + \pi_{n\cap p\cap s}\, [(ad\, q)^{-1}\mu,\ S\,\pi_{n\cap p\cap t} F]$$
$$= \pi_a\, [(ad\, q)^{-1}\mu,\ (ad\, q)^{-1}\, \pi_{n\cap p\cap t} F] + (ad\, q)^{-1}\pi_n\, [\mu,\ S\,\pi_{n\cap p\cap t} F] + (ad\, q)^{-1}\, \pi_n\, [(ad\, q)^{-1}\mu,\ \pi_{n\cap p\cap t} F]$$

(Lemma 5.22)

$$= \pi_a\, [(ad\, q)^{-1}\mu,\ (ad\, q)^{-1}\, \pi_{n\cap p\cap t} F] + (ad\, q)^{-1}\pi_n\ [(ad\, q)^{-1}\mu,\ \pi_{n\cap p\cap t} F] \tag{5.52}$$

The third term becomes:

$$\pi_a\, [(ad\, q)^{-1}\mu, (ad\, q)^{-1}\pi_{n\cap p\cap s} F] + \pi_{n\cap p}[(ad\, q)^{-1}\mu, (ad\, q)^{-1}\pi_{n\cap p\cap s} F]$$
$$= (ad\, q)^{-1}\pi_n\, [\mu, (ad\, q)^{-1}\pi_{n\cap p\cap s} F] + (ad\, q)^{-1}\pi_n\, [(ad\, q)^{-1}\mu, \pi_{n\cap p\cap s} F]$$

(because the first term vanishes and using Lemma 5.5).

$$= (ad\, q)^{-1}\, \pi_n\, [(ad\, q)^{-1}\mu,\ \pi_{n\cap p\cap s} F] \quad \text{(Lemma 5.17)} \tag{5.53}$$

Combining (5.51), (5.52) and (5.53) we finally get:

$$V_f(x) = \pi_a\, [(ad\, q)^{-1}\mu, (ad\, q)^{-1}\, \pi_{n\cap p\cap t} F] - (ad\, q)^{-1}\ [x, \pi_a F]$$
$$= \pi_a\, [(ad\, q)^{-1}\mu,\ (ad\, q)^{-1}\, \pi_{n\cap p} F]\ - (ad\, q)^{-1}\ [x, \pi_a F]$$
$$= \pi_a\, [(ad\, q)^{-1}\mu,\ (ad\, q)^{-1}\, \pi_n F]\ - (ad\, q)^{-1}\, [x, \pi_a F]$$
$$= \pi_a\, [x, (ad\, q)^{-1}\pi_n F] - (ad\, q)^{-1}\, [x, \pi_a F]$$

which is exactly (5.29). □

REMARK 5.24. We will show in section 6 that the μ corresponding to the A_n and BC_n models satisfies (5.42) and (5.44) so Theorem 5.23 is true for these models.

THEOREM 5.25. *(Liouville) (see [20])*
Let M be a symplectic manifold with dim $M = 2n$ and consider Hamilton's equations on M with Hamiltonian h. If there are n functionally independent global integrals of motion which are in involution then the Hamiltonian system is completely integrable, which means that there exist global action-angle coordinates $\{I_i, \phi_i, i = 1,...,n\}$ on M in which the equations become:

$$\dot{I}_i = 0 \quad \ddot{\phi} = 0 \tag{5.54}$$

COROLLARY 5.26. *Consider Hamilton's equations on $P_\mu \cap \mathfrak{m}^\perp$, where $\mu \in \mathfrak{n} \cap \mathfrak{k} \cap \mathfrak{s}$ satisfies (5.40) with Hamiltonian $h \in C^\infty(P_\mu \cap \mathfrak{m}^\perp)$ where h is the restriction of an Ad-invariant function on $\mathfrak{g}$. Suppose $P_\mu \cap \mathfrak{m}^\perp$ is a symplectic manifold and the restrictions of the Ad-invariant functions to $P_\mu \cap \mathfrak{m}^\perp$ are functionally independent, then the Hamilton equations are completely integrable.*

PROOF. This follows from Theorem 5.18 and Theorem 5.25. □

REMARK 5.27. For the A_n and BC_n systems it has been proved [1] that $P_\mu \cap \mathfrak{m}^\perp$ is a symplectic manifold and that the *Ad*-invariant functions are functionally independent so these systems are completely integrable.

6. EXAMPLES: THE A_{n-1} AND BC_n MODELS [1]

In this section we want to apply the theory of the previous sections to construct two examples, namely the A_{n-1} model and the BC_n model. It turns out that for these models there exists a μ which satisfies (5.42) and (5.44), so the Hamilton equations can be written in Lax form.

6.1. The A_{n-1} model

Let $\mathfrak{g} = sl(n, \mathbb{C})$, viewed as a real Lie algebra and choose as the Killing form:

$$<x, y> = \text{Re } tr(xy) \tag{6.1}$$

Let $\mathfrak{k}$ be the compact real form of $\mathfrak{g}$ and θ the corresponding conjugation, then $\theta(x) = -x^\dagger$ and

$$\mathfrak{g} = \mathfrak{k} \oplus \sqrt{-1}\,\mathfrak{k} \tag{6.2}$$

is the corresponding Cartan decomposition. So $\mathfrak{k} = $ {traceless skew-hermitian matrices} and $\mathfrak{p} = $ { traceless hermitian matrices}. We can choose $\mathfrak{a} = $ { real diagonal matrices}, so

$$\mathfrak{a} = \{\text{diag}(q_1, \ldots, q_n), \; q_i \in \mathbb{R}, \; \sum_{i=1}^{n} q_i = 0\} \tag{6.3}$$

and then $\mathfrak{m} = \sqrt{-1}\,\mathfrak{a}$, so $\mathfrak{m}$ is abelian and the rank of $\mathfrak{g}$ is $n-1$. Let $\{e_{ij}, i,j = 1,...,n\}$ denote the standard basis of $gl(n, \mathbb{C})$, and let $d_i = e_{ii}$ then $\{d_i, i = 1,...,n\}$ forms a basis of $\mathfrak{a}$ and we write:

$$Q = \text{diag}(q_1, \ldots, q_n) = \sum_{i=1}^{n} q_i \, d_i \tag{6.4}$$

Now define $\lambda_i(Q) = q_i$ and $\alpha_{ij} = \lambda_i - \lambda_j$, then $\{\alpha_{ij}, i \neq j\}$ forms the root system A_{n-1}. Observe that $<d_i, d_j> = \delta_{ij}$ so $h_{\lambda_i} = d_i$ and $h_{\alpha_{ij}} := h_{ij} = d_i - d_j$. The simple roots are:

$$\Delta = \{\alpha_i := \alpha_{i,i+1}, \; 1 \leqslant i \leqslant n-1\} \tag{6.5}$$

and

$$R_+ = \{\alpha_{ij}, i < j\}, \; R_- = \{\alpha_{ij}, \; i > j\}.$$

Let $\mathfrak{g}_{ij}$ denote the root space of α_{ij} then

$$\mathfrak{g}_{ij} = \mathbb{R} < e_{ij},\ \sqrt{-1}\, e_{ij} > \tag{6.6}$$

so $m_{ij} = 2$.
Also e_{ij} and $\sqrt{-1}\, e_{ij}$ satisfy (5.11), so define

$$e^1_{\alpha_{ij}} = e_{ij} \quad e^1_{-\alpha_{ij}} = -e_{ji}$$
$$e^2_{\alpha_{ij}} = \sqrt{-1}\, e_{ij} \quad e^2_{-\alpha_{ij}} = \sqrt{-1}\, e_{ji} \tag{6.7}$$

then:

$$s^1_{ij} = e_{ij} - e_{ji} \quad a^1_{ij} = e_{ij} + e_{ji} \quad (i<j) \tag{6.8}$$
$$s^2_{ij} = \sqrt{-1}(e_{ij} + e_{ji}) \quad a^2_{ij} = \sqrt{-1}(e_{ij} - e_{ji}) \quad (i<j) \tag{6.9}$$

The regular elements of $\mathfrak{a}$ are those elements for which $q_i \neq q_j$ if $i \neq j$ and the positive Weyl chamber consists of the elements for which $q_1 > q_2, \ldots, > q_n$.

Now define $\sigma(X) = -X^T$ then σ is an involutive automorphism of $\mathfrak{g}$ which commutes with σ, $\mathfrak{g}_0$ is σ-invariant and $\mathfrak{t} =$ {skew-symmetric matrices with zero trace}, $\mathfrak{s} =$ {symmetric matrices with zero trace} and we see that $\mathfrak{g}_0 \subset \mathfrak{s}$, so $\mathfrak{m} \cap \mathfrak{t} = \{0\}$, and

$$\mathfrak{n} \cap \mathfrak{k} \cap \mathfrak{t} = \mathbb{R} < s^1_{ij}, i<j >$$
$$\mathfrak{n} \cap \mathfrak{k} \cap \mathfrak{s} = \mathbb{R} < s^2_{ij}, i<j >$$
$$\mathfrak{n} \cap \mathfrak{p} \cap \mathfrak{t} = \mathbb{R} < a^2_{ij}, i<j >$$
$$\mathfrak{n} \cap \mathfrak{p} \cap \mathfrak{s} = \mathbb{R} < a^1_{ij}, i<j > \tag{6.10}$$

Now define:

$$\mu = g \sum_{i<j} s^2_{ij} = g \sum_{\alpha \in R_+} s^2_\alpha \quad (0 \neq g \in \mathbb{R}) \tag{6.11}$$

which is the same element as defined in (2.18).

LEMMA 6.1.

$$C_{\mathfrak{g}}(\mu) \cap \mathfrak{g}_0 = \{0\}$$

PROOF: Because $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$ and $\mathfrak{m} = \sqrt{-1}\,\mathfrak{a}$ it is sufficient to prove it for elements of $\mathfrak{a}$. So suppose $Q \in C_{\mathfrak{g}}(\mu) \cap \mathfrak{a}$ then:

$$\begin{aligned} 0 &= [Q, \mu] \\ &= [Q, g \sum_{\alpha \in R_+} s^2_\alpha] = g \sum_{\alpha \in R_+} [Q, s^2_\alpha] \\ &= g \sum_{\alpha \in R_+} \alpha(Q)\, a^2_\alpha \Rightarrow (\forall \alpha \in R_+)\ \alpha(Q) = 0 \end{aligned}$$

which implies $Q \in Z(\mathfrak{g})$ and because $\mathfrak{g}$ is semisimple $Q = 0$ □

So μ satisfies (5.44) and the condition in Prop. 5.9.

LEMMA 6.2. *The projection $\pi_n : C_{\mathfrak{g}}(\mu) \cap \mathfrak{s} \rightarrow \mathfrak{n} \cap \mathfrak{s}$ is surjective.*

PROOF: Because $C_{\mathfrak{g}}(\mu) \cap \mathfrak{s} = C_{\mathfrak{g}}(\mu) \cap \mathfrak{s} \cap \mathfrak{k} \oplus C_{\mathfrak{g}}(\mu) \cap \mathfrak{s} \cap \mathfrak{p}$ and $\mathfrak{k} = \sqrt{-1}\,\mathfrak{p}$ it is sufficient to prove that

$\pi_n : C_{\mathfrak{g}}(\mu) \cap \mathfrak{p} \cap \mathfrak{s} \rightarrow \mathfrak{n} \cap \mathfrak{p} \cap \mathfrak{s}$ is surjective. Now one easily checks that the matrices

$$Z_{ij} = a_{ij}^{1} + \frac{2}{n} I - d_i - d_j \tag{6.12}$$

belong to $C_{\mathfrak{g}}(\mu) \cap \mathfrak{p} \cap \mathfrak{s}$ and $\pi_n Z_{ij} = a_{ij}^{1}$ □

So in this case μ satisfies an even stronger condition than (5.42).

COROLLARY 6.3. *The projection* $\pi_n : C_{\mathfrak{g}}(\mu) \cap \mathfrak{s} \rightarrow \mathfrak{n} \cap \mathfrak{s}$ *is a bijection.*

Now consider an element $(Q,L) \in P_\mu \cap \mathfrak{m}^\perp$ then L has the form:

$$L = P + g \sum_{\alpha \in R_+} \alpha^{-1}(Q)\, a_\alpha^2 \tag{6.13}$$

where $P = \operatorname{diag}(p_1, \ldots, p_n) \in \mathfrak{a}$ and $Q \in \mathfrak{a}_+$ and this is exactly the Lax matrix as defined in (2.9) and so $H = \frac{1}{2}\, tr\, L^2$. But one easily shows that grad$H(L) = L$, so /

$$\begin{aligned} M &= S\ \pi_{n \cap p \cap t} \operatorname{grad} H(L) \\ &= S\ (\pi_n L) \\ &= S((adQ)^{-1}\mu) \\ &= \pi_n^{-1}\ (adQ)^{-2}\mu \end{aligned} \tag{6.14}$$

where π_n^{-1} is the inverse of the map in Cor. 6.3 and using the construction of π_n^{-1} in (6.12) we see that M is the matrix defined in (2.10). So the Hamilton equations derived in Theorem 5.23 are indeed the same as the Lax equation in (2.11), and the conserved quanties H_k in (2.13) correspond with the restriction of Ad-invariant functions to the phase space $P_\mu \cap \mathfrak{m}^\perp$.

The element μ defined in (6.11) has some other interesting properties. Let $e = (1, \ldots, 1) \in \mathbb{R}^n$ then we can write μ in the following way:

$$\mu = ig\,(|e\rangle\langle e| - I) \tag{6.15}$$

where we have used the dyadic notation $|e\rangle\langle e|$ for the rank-1 matrix $e \otimes e$ (see [31] for some useful properties of dyads).

Now define:

$$A = \frac{1}{n} i(|e\rangle\langle e| - I) \tag{6.16}$$

so $\mu = ngA$ then

$$Ae = i(1 - \frac{1}{n})e \tag{6.17}$$

and

$$Av = -\frac{i}{n} v \quad \text{if} \quad \langle e,v\rangle = 0 \tag{6.18}$$

so A has eigenvalue $i(1 - \frac{1}{n})$ with multiplicity 1 and eigenvalue $-\frac{i}{n}$ with multiplicity $n-1$ and there exists an orthogonal transformation T such that

$$\mathfrak{A} = T^{-1}AT = i\operatorname{diag}(1 - \frac{1}{n}, -\frac{1}{n}, \ldots, -\frac{1}{n}) \tag{6.19}$$

Now $A \in \mathfrak{k} = \mathfrak{su}(n)$ and one easily calculates that:

$$ker(ad\,A) \cap \mathfrak{k} = \left\{ \begin{pmatrix} ia & 0.....0 \\ \cdot & \\ \cdot & \\ \cdot & u \\ \cdot & \\ 0 & \end{pmatrix}, \begin{matrix} a \in \mathbb{R} \\ u \in \mathfrak{u}(n-1) \\ ia + tr\,u = 0 \end{matrix} \right\} \tag{6.20}$$

$$im\,(ad\,A) \cap \mathfrak{k} = \left\{ \begin{pmatrix} 0 & -z^{\dagger} \\ & \\ z & 0 \end{pmatrix}, z \in \mathbb{C}^{n-1} \right\} \tag{6.21}$$

and

$$\mathfrak{k} = ker(ad\,A) \cap \mathfrak{k} \oplus im(ad\,A) \cap \mathfrak{k} \tag{6.22}$$

$ker(ad\,A) \cap \mathfrak{k}$ is a subalgebra of $\mathfrak{k}$ and we have furthermore the commutation relations in $\mathfrak{k}$:

$$[ker(ad\,A),\ im(ad\,A)] \subset im(ad\,A) \tag{6.23}$$

$$[im(ad\,A),\ im(ad\,A)] \subset ker(ad\,A) \tag{6.24}$$

Also

$$ker(ad\,A) \cap \mathfrak{k} \cong \mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(n-1)) \tag{6.25}$$

and

$$dim\ ker(ad\,A) \cap \mathfrak{k} = (n-1)^2 \tag{6.26}$$

$$dim\ im(ad\,A) \cap \mathfrak{k} = 2(n-1) \tag{6.27}$$

and $J = ad\,A : im(ad\,A) \cap \mathfrak{k} \rightarrow im(ad\,A) \cap \mathfrak{k}$ acts as an isomorphism with $J^2 = -Id$, so we conclude that J is the hermitian structure corresponding to the hermitian symmetric space $SU(n)/S(U(1) \times U(n-1))$ and $im(ad\,A) \cap \mathfrak{k} = im(ad\mu) \cap \mathfrak{k}$ is the tangent space in the identity and also:

$$ad\,\mu : im(ad\,\mu) \cap \mathfrak{k} \rightarrow im(ad\,\mu) \cap \mathfrak{k} \tag{6.28}$$

has the property: $(ad\,\mu)^2 = -n^2 g^2\,Id$.

Maybe these special properties of μ can be used to obtain more information about the solutions of the Lax equation.

6.2. The BC_n model

Let $\mathfrak{g} = su(n, n+1)$ which is a noncompact real form of $\mathfrak{g}_c = sl(2n+1, \mathbb{C})$ and which consists of the matrices

$$\begin{pmatrix} A & w & B \\ -v^{\dagger} & ia & -w^{\dagger} \\ C & v & -A^{\dagger} \end{pmatrix}, \quad \begin{matrix} v, w \in \mathbb{C}^n, a \in \mathbb{R} \\ B, C \in u(n) \\ A \in gl(n, \mathbb{C}) \end{matrix} \tag{6.29}$$

and which we denote by (A, B, C, w, v, ia).

The Cartan involution is given by $\theta(X) = -X^\dagger$ and:

$$\mathfrak{k} = \{(A, B, B, w, w, ia),\ A \in u(n)\} \tag{6.30}$$

$$\mathfrak{p} = \{(A, B, -B, w, -w, 0),\ A^\dagger = A\} \tag{6.31}$$

For $\mathfrak{a}$ we can choose:

$$\mathfrak{a} = \{(D, 0, 0, 0, 0, 0),\ D = \mathrm{diag}(q_1, \ldots, q_n)\} \tag{6.32}$$

and so the rank of $\mathfrak{g}$ is n. If follows that

$$\mathfrak{m} = \{(A, 0, 0, 0, 0, ia),\ A = i\,\mathrm{diag}(q_1, \ldots, q_n)\} \tag{6.33}$$

so $\mathfrak{m}$ is abelian.
As the Killing form we choose:

$$<x,y> = \frac{1}{2}\, tr\,(xy) \tag{6.34}$$

Define $d_i = (e_{ii}, 0, 0, 0, 0, 0)$ then $<d_i, d_j> = \delta_{ij}$ and if $Q \in \mathfrak{a}$ then/$Q = \sum_{i=1}^{n} q_i\, d_i$.

Define $\lambda_i(Q) = q_i$, then $h_{\lambda_i} = d_i$ and so $(\lambda_i, \lambda_i) = 1$
Now define $\alpha_{ij} = \lambda_i - \lambda_j$, $\beta_{ij} = \lambda_i + \lambda_j$ then the rootsystem BC_n is the set of roots:

$$R = \{\pm\lambda_i,\ \pm 2\lambda_i,\ \pm(\lambda_i + \lambda_j),\ \pm(\lambda_i - \lambda_j),\ i<j\} \tag{6.35}$$

and the corresponding rootspaces are:

$$\begin{aligned}
\mathfrak{g}_{\lambda_i} &= \{(0, 0, 0, \lambda e_i, 0, 0),\ \lambda \in \mathbb{C}\}\\
\mathfrak{g}_{-\lambda_i} &= \{(0, 0, 0, 0, \lambda e_i, 0),\ \lambda \in \mathbb{C}\}\\
\mathfrak{g}_{2\lambda_i} &= \{(0, a\sqrt{-1}\, e_{ii}, 0, 0, 0, 0),\ a' \in \mathbb{R}\}\\
\mathfrak{g}_{-2\lambda_i} &= \{(0, 0, a\sqrt{-1}\, e_{ii}, 0, 0, 0),\ a \in \mathbb{R}\}\\
\mathfrak{g}_{\alpha_{ij}} &= \{(\lambda\, e_{ij}, 0, 0, 0, 0, 0),\ \lambda \in \mathbb{C},\ i<j\}\\
\mathfrak{g}_{-\alpha_{ij}} &= \{(\lambda\, e_{ji}, 0, 0, 0, 0, 0),\ \lambda \in \mathbb{C},\ i<j\}\\
\mathfrak{g}_{\beta_{ij}} &= \{(0, a\sqrt{-1}\,(e_{ij} + e_{ji}) + b(e_{ij} - e_{ji}), 0, 0, 0, 0), a,\ b \in \mathbb{R},\ i<j\}\\
\mathfrak{g}_{-\beta_{ij}} &= \{(0, 0, a\sqrt{-1}\,(e_{ij} + e_{ji}) + b(e_{ij} - e_{ji}), 0, 0, 0),\ a,\ b \in \mathbb{R}, i<j\}
\end{aligned} \tag{6.36}$$

from which it follows that $m_{\lambda_i} = m_{\alpha_{ij}} = m_{\beta_{ij}} = 2$ and $m_{2\lambda_i} = 1$

The simple roots are:

$$\Delta = \{\lambda_i - \lambda_{i+1},\ 1 \leqslant i \leqslant n-1,\ \lambda_n\}$$

and:

$$\begin{aligned}
R_+ &= \{\lambda_i, 2\lambda_i,\ \alpha_{ij}, \beta_{ij},\ i<j\}\\
R_- &= \{-\lambda_i,\ -2\lambda_i, \alpha_{ji}, -\beta_{ij},\ i<j\}
\end{aligned} \tag{6.37}$$

Now define:

$$\begin{aligned}
&e^2_{\lambda_i} = (0, 0, 0, e_i, 0, 0),\ e^1_{\lambda_i} = (0, 0, 0, \sqrt{-1}\, e_i, 0,0)\\
&e^2_{-\lambda_i} = (0, 0, 0, 0, e_i, 0),\ e^1_{-\lambda_i} = (0, 0, 0, 0, \sqrt{-1}\, e_i, 0)\\
&e^1_{2\lambda_i} = (0,\ \sqrt{2}\ \sqrt{-1}\, e_{ii}, 0, 0, 0, 0),\ e^1_{-2\lambda_i} = (0, 0,\ \sqrt{2}\ \sqrt{-1}\, e_{ii}, 0, 0, 0)
\end{aligned}$$

$$e^2_{\alpha_{ij}} = (e_{ij}, 0, 0, 0, 0, 0),\ e^1_{\alpha_{ij}} = (\sqrt{-1}\, e_{ij}, 0, 0, 0, 0, 0)$$
$$e^2_{-\alpha_{ij}} = (-e_{ji}, 0, 0, 0, 0, 0),\ e^1_{-\alpha_{ij}} = (\sqrt{-1}\, e_{ji}, 0, 0, 0, 0, 0)$$
$$e^1_{\beta_{ij}} = (0, \sqrt{-1}(e_{ij}+e_{ji}), 0, 0, 0, 0),\ e^2_{\beta_{ij}} = (0, e_{ij}-e_{ji}, 0, 0, 0, 0)$$
$$e^1_{-\beta_{ij}} = (0, 0, \sqrt{-1}(e_{ij}+e_{ji}), 0, 0, 0),\ e^2_{-\beta_{ij}} = (0, 0, e_{ij}-e_{ji}, 0, 0, 0) \tag{6.38}$$

then these vectors span the corresponding root spaces and satisfy (5.11) and (5.12).

Now define $\sigma(X) = -X^T$ for all $X \in \mathfrak{g}$, then σ is an involutive automorphism of $\mathfrak{g}$ which commutes with θ, $\mathfrak{g}_0$ is σ-invariant and $\mathfrak{g}_0 \subset \mathfrak{s}$ and one easily verifies that:

$$\mathfrak{n} \cap \mathfrak{k} \cap \mathfrak{t} = \mathbb{R} < s^2_{\lambda_i}, s^2_{\alpha_{ij}}, s^2_{\beta_{ij}}, i<j >$$
$$\mathfrak{n} \cap \mathfrak{k} \cap \mathfrak{s} = \mathbb{R} < s^1_{\lambda_i}, s^1_{2\lambda_i}, s^1_{\alpha_{ij}}, s^1_{\beta_{ij}}, i<j >$$
$$\mathfrak{n} \cap \mathfrak{p} \cap \mathfrak{t} = \mathbb{R} < a^1_{\lambda_i}, a^1_{2\lambda_i}, a^1_{\alpha_{ij}}, a^1_{\beta_{ij}}, i<j >$$
$$\mathfrak{n} \cap \mathfrak{p} \cap \mathfrak{s} = \mathbb{R} < a^2_{\lambda_i}, a^2_{\alpha_{ij}}, a_{\beta_{ij}}, i<j > \tag{6.39}$$

Now define:

$$\mu = g \sum_{i<j} s^1_{\alpha_{ij}} + g \sum_{i<j} s^1_{\beta_{ij}} + g_1 \sum_i s^1_{\lambda_i} + g_2 \sum_i s^1_{2\lambda_i} \tag{6.40}$$

where $g, g_1, g_2 \in \mathbb{R}$ and $g \neq 0$

Using (6.38) we find:

$$\mu = (\hat{\mu}, \hat{\mu} + \sqrt{2} g_2 \sqrt{-1} I, \hat{\mu} + \sqrt{2} g_2 \sqrt{-1} I, \sqrt{-1} g_1 e, \sqrt{-1}\, g_1 e, 0) \tag{6.41}$$

where $\hat{\mu} = \sqrt{-1}\, g \sum_{i<j} (e_{ij}+e_{ji})$ is the μ of the A_{n-1} model and $e = (1, \ldots, 1) \in \mathbb{R}^n$.

We have to distinguish two cases: $g_1 = 0$ and $g_1 \neq 0$. If $g_1 = 0$ we can restrict ourselves to $\mathfrak{g} = su(n,n)$. and R reduces to the rootsystem C_n.

LEMMA 6.4. $C_{\mathfrak{g}}(\mu) \cap \mathfrak{g}_0 = \{0\}$

PROOF: a) Let $Q \in C_{\mathfrak{g}}(\mu) \cap \mathfrak{a}$ then

$$0 = [Q,\mu] = g \sum_{i<j} \alpha_{ij}(Q) a^1_{ij} + g \sum_{i<j} \beta_{ij}(Q) a^1_{\beta_{ij}} + g_1 \sum_i \lambda_i(Q) a^1_{\lambda_i} + g_2 \sum_i 2\lambda_i(Q) a^1_{2\lambda_i} \tag{6.42}$$

(i) $g \neq 0$, $g_1 \neq 0$, then (6.42) implies $Q \in Z(\mathfrak{g})$ so $Q = 0$ because $\mathfrak{g}$ is semisimple.
(ii) $g \neq 0$, $g_1 = 0$, now $\mathfrak{g} = su(n,n)$ and (6.42) implies $\alpha_{ij}(Q) = \beta_{ij}(Q) = 0$ so in particular $(\lambda_i - \lambda_{i+1})(Q) = (\lambda_i + \lambda_{i+1})(Q) = 0$ but then also $2\lambda_n(Q) = 0$ and therefore $\alpha(Q) = 0$, $\forall \alpha \in \Delta$ which implies $Q \in Z(\mathfrak{g})$ and so $Q = 0$

b) Now let $Q = (iD, 0, 0, 0, 0, ia) \in \mathfrak{m} \cap C_{\mathfrak{g}}(\mu)$ then we get the equations:

$$(1) \quad [D, \hat{\mu}] = 0$$
$$(2) \quad g_1\, De = g_1\, ae \tag{6.43}$$

Because $g \neq 0$ (1) implies $D = bI$, $b \in \mathbb{R}$ and from $a + 2tr\, D = 0$ it follows that $a + 2nb = 0$ and so (2) becomes

$$(2)' \quad g_1 \; b=0 \tag{6.44}$$

If $g_1 \neq 0$ then $b=0$ and so $a=0$ and if $g_1=0$ then $\mathfrak{g}=su(n,n)$ and so $a=0$ and so also $b=0$. So in both cases it follows that $Q=0$. □

So we have proved that μ satisfies (5.44) and the condition in Prop. 5.9. Now we shall prove that μ also satisfies (5.42) and therefore (5.40):

LEMMA 6.5. *The projection $\pi_n : C_{\mathfrak{g}}(\mu) \cap \mathfrak{k} \cap \mathfrak{s} \to \mathfrak{n} \cap \mathfrak{k} \cap \mathfrak{s}$ is surjective if one of the following conditions is satisfied: (i) $g \neq 0$, $g_1=0$ (ii) $g \neq 0$, $g_1 \neq 0$ and $g_1^2 - 2g^2 + \sqrt{2}\, g g_2 = 0$*

PROOF: Let $X=i(A, B, B, w, w, a) \in \mathfrak{k} \cap \mathfrak{s}$, where $w \in \mathbb{R}^n$, $a \in \mathbb{R}$, A, B real symmetric and $a+2trA=0$ and suppose $[X, \mu]=0$ then we get the following equations:

$$(1) \quad [A+B, \hat{\mu}] = i g_1 (|e\rangle\langle w| - |w\rangle\langle e|)$$
$$(2) \quad i g_1 (A+B)e - i g_1 a\, e = 2\hat{\mu} w + i\sqrt{2} g_2 w \tag{6.45}$$

Now

$$\begin{aligned}
& i(|e\rangle\langle w| - |w\rangle\langle e|) \\
&= \sum_{i,j} w_j\, i\,(|e_i\rangle\langle e_j| - |e_j\rangle\langle e_i|) \\
&= \sum_{i,j} w_j\, i\,(e_{ij} - e_{ji}) \\
&= \sum_{i\neq j} w_j\, i\,(e_{ij} - e_{ji}) \\
&= \sum_{i<j} (w_j - w_i)\, i(e_{ij} - e_{ji}) \\
&= -\sum_{i<j} \alpha_{ij}(W) i(e_{ij} - e_{ji}), \text{ where } W = \mathrm{diag}(w_1, \ldots, w_n) \\
&= -g^{-1} [W, \hat{\mu}]
\end{aligned} \tag{6.46}$$

Furthermore we have:

$$\begin{aligned}
2\hat{\mu} w &= 2ig(|e\rangle\langle e| - I)w \\
&= 2ig \langle e,w\rangle e - 2ig\, w
\end{aligned} \tag{6.47}$$

Substituting (6.46) and (6.47) in (6.45) gives:

$$(i)' \quad [A+B+g_1 g^{-1} W, \hat{\mu}] = 0$$
$$(2)' \quad g_1(A+B)e - g_1 a e$$
$$= 2g\langle e, w\rangle e - 2g\, w + \sqrt{2} g_2\, w \tag{6.48}$$

Now we consider first the case $g \neq 0$, $g_1=0$, then $\mathfrak{g}=su(n,n)$ and so $w=a=0$ and we are left with the condition:

$$(3) \quad [A+B, \hat{\mu}] = 0, \quad tr\, A=0 \tag{6.49}$$

We must distinguish the following cases:

(a) $B = e_{ii}, A = -e_{ii} + \frac{1}{n} I \Rightarrow \pi_n X = \frac{1}{\sqrt{2}}\, s^1_{2\lambda_i}$

(b) $B = e_{ij} + e_{ji}, i<j, A = -e_{ii} - e_{jj} + \frac{2}{n} I$

$$\Rightarrow \pi_n X = s^1_{\beta_{ij}}$$

(c) $B = 0, A = e_{ij} + e_{ji} - e_{ii} - e_{jj} + \frac{2}{n} I, i<j$

$$\Rightarrow \pi_n X = s^1_{\alpha_{ij}} \tag{6.50}$$

and comparing this with (6.39) we conclude that π_n is surjective if $g \neq 0$, $g_1 = 0$

Now consider the case $g \neq 0$, $g_1 \neq 0$ then we have to distinguish the following cases:

(a) $B = 0,\ w = 0,\ A = e_{ij} + e_{ji} - e_{ii} - e_{jj} + \frac{4}{2n+1} I,\ a = \frac{4}{2n+1}$

$$\Rightarrow \pi_n\ X = s^1_{\alpha_{ij}},\quad i<j$$

(b) $w = 0,\ B = e_{ii},\ a = \frac{2}{2n+1},\ A = \frac{2}{2n+1} I - e_{ii}$

$$\Rightarrow \pi_n\ X = \frac{1}{\sqrt{2}}\ s^1_{2\lambda_i}$$

(c) $w = 0,\ B = e_{ij} + e_{ji},\ (i<j),\ a = \frac{4}{2n+1},\ A = -e_{ii} - e_{jj} + \frac{4}{2n+1} I$

$$\Rightarrow \pi_n\ X = s^1_{\beta_{ij}}$$

(d) $B = 0$, A diagonal. Now the conditions (6.48) reduce to:

$$(4)\ [A + g_1 g^{-1}\ W,\ \hat{\mu}] = 0$$

$$(5)\ g_1\ Ae - g_1\ ae = 2g<e,w>e + (\sqrt{2} g_2 - 2g)w \tag{6.51}$$

Because A and W are diagonal, (4) implies:

$$A + g_1 g^{-1}\ W = b\ I,\quad b \in \mathbb{R} \tag{6.52}$$

substituting this in (5) yields:

$$(g_1^2 - 2g^2 + \sqrt{2} g_2 g)w = (gg_1 b - gg_1 a - 2g^2<e,w>)e \tag{6.53}$$

because $We = w$.

So we must require:

$$g_1^2 - 2g^2 + \sqrt{2}\ g_2\ g = 0 \tag{6.54}$$

otherwise the only solution is a scalar multiple of e, and also:

$$g_1 b - g_1\ a - 2g\ <e,w> = 0 \tag{6.55}$$

Furthermore we have the condition $a + 2\,tr\,A = 0$ and combining this with (6.52) gives:

$$a = -2\,tr\,A = -2(bn - g_1 g^{-1}<e,w>) \tag{6.56}$$

Substituting this in (6.55) gives:

$$b = \frac{2}{2n+1} \left(\frac{g}{g_1} + \frac{g_1}{g} \right) <e,w> \tag{6.57}$$

$$a = \frac{2}{2n+1} \left(\frac{g_1}{g} - 2n\frac{g}{g_1} \right) <e,w> \tag{6.58}$$

and so if we take $w=e_i$ then A and a are given by (6.52), (6.57) and (6.58) and $\pi_n X=(0,0,0,ie_i,0)=s^1_{\lambda_i}$ and comparing this with (6.39) we conclude that π_n is surjective. Summarizing we have proved that $\pi_n : C_{\mathfrak{g}}(\mu)\cap\mathfrak{k}\cap\mathfrak{s}\to\mathfrak{n}\cap\mathfrak{k}\cap\mathfrak{s}$ is surjective if one of the following conditions is satisfied:

(i) $g\neq 0,\ g_1=0$
(ii) $g\neq 0,\ g_1\neq 0,\ g_1^2-2g^2+\sqrt{2}g_2g=0$

and the inverse of π_n is explicitly given by:

$$\text{(i)}\quad \pi_n^{-1}(s^1_{2\lambda_i}+i\sqrt{2}(\frac{1}{n}I-e_{ii},0,0,0,0,0)$$
$$\pi_n^{-1}(s^1_{\alpha_{ij}})=s^1_{\alpha_{ij}}+i(\frac{2}{n}I,-e_{ii}-e_{jj},0,0,0,0,0)$$
$$\pi_n^{-1}(s^1_{\beta_{ij}})=s^1_{\beta_{ij}}+i(\frac{2}{n}I-e_{ii}-e_{jj},0,0,0,0,0) \tag{6.59}$$
$$\text{(ii)}\quad \pi_n^{-1}(s^1_{\alpha_{ij}})=s^1_{\alpha_{ij}}+i(\frac{4}{2n+1}I-e_{ii}-e_{jj},0,0,0,0,\frac{4}{2n+1})$$
$$\pi_n^{-1}(s^1_{\beta_{ij}})=s^1_{\beta_{ij}}+i(\frac{4}{2n+1}I-e_{ii}-e_{jj},0,0,0,0,\frac{4}{2n+1})$$
$$\pi_n^{-1}(s^1_{2\lambda_i})=s^1_{2\lambda_i}+i\sqrt{2}(\frac{2}{2n+1}I-e_{ii},0,0,0,0,\frac{2}{2n+1})$$
$$\pi_n^{-1}(s^1_{\lambda_i})=s^1_{\lambda_i}+i(b\,I-g_1g^{-1}e_{ii},0,0,0,0,a) \tag{6.60}$$

where a,b are given by (6.57) and (6.58) □

Now we are ready to construct the corresponding Calogero-Moser model. Let $L\in P_\mu\cap\mathfrak{m}^\perp$ then L has the form:

$$\begin{aligned} L &= P+(adQ)^{-1}\mu \\ &= P+g\sum_{i<j}\alpha_{ij}^{-1}(Q)\,a_{\alpha_{ij}}^{-1}+g\sum_{i<j}\beta_{ij}^{-1}(Q)\,a^1_{\beta_{ij}} \\ &\quad +g_1\sum_i\lambda_i^{-1}(Q)a^1_{\lambda_i}+g_2\sum_i(2\lambda_i)^{-1}(Q)\,a^1_{2\lambda_i} \end{aligned} \tag{6.61}$$

where $Q\in\mathfrak{a}_+$ and $P\in\mathfrak{a}$ From the normalization (5.11) and (5.12) follows that

$$<e_\alpha,e_{-\alpha}>=-1 \tag{6.62}$$

and so

$$<a_\alpha,a_\alpha>=2 \tag{6.63}$$

From this it follows that $H=\frac{1}{2}tr\,L^2$ is given by:

$$\begin{aligned} H &= \frac{1}{2}\sum_{i=1}^n p_i^2+g^2\sum_{i<j}\left[(q_i-q_j)^{-2}+(q_i+q_j)^{-1}\right] \\ &\quad +g_1^2\sum_i q_i^{-2}+g_2^2\sum_i(2q_i)^{-2} \end{aligned} \tag{6.64}$$

Because grad $H(L)=L$, M is given by:

$$M=S\,\pi_{n\cap p\cap t}\,L$$

$$= S\,(ad\,Q)^{-1}\,\mu$$
$$= (\pi_n)^{-1}\,(ad\,Q)^{-2}\,\mu$$
$$= \pi_n^{-1}\,\{g\sum_{i<j}\alpha_{ij}^{-2}(Q)\,s^1_{\alpha_{ij}} + g\sum_{i<j}\beta_{ij}^{-2}(Q)\,s^{-1}_{\beta_{ij}}$$
$$+ g_1\sum_i \lambda_i^{-2}(Q)\,s^1_{\lambda_i} + g_2\sum_i (2\lambda_i(Q))^{-1}\,s^{-1}_{2\lambda_i}\}$$
$$= g\sum_{i<j}\alpha_{ij}^{-2}(Q)\,\pi_n^{-1}(s^1_{\alpha_{ij}}) + g\sum_{i<j}\beta_{ij}^{-2}(Q)\,\pi_n^{-1}(s^1_{\beta_{ij}})$$
$$+ g_1\sum_i \lambda_i^{-2}(Q)\,\pi_n^{-1}(s^1_{\lambda_i}) + g_2\sum_i [2\lambda_i(Q)]^{-1}\,\pi_n^{-1}(s^1_{2\lambda_i}) \qquad (6.65)$$

where π_n^{-1} is defined by (6.59) or (6.60).

And we choosed $Q = \sum_i q_i\, d_i$, $P = \sum_i p_i d_i$. From section 5 we know that

$$\dot{Q} = -\pi_a\,\nabla H(L) = -\pi_a\,L = -P \qquad (6.65)$$

which is equivalent with

$$\dot{q}_i = -p_i \qquad (6.65)$$

and from (5.29) we derive that:

$$\dot{P} = \pi_a\,\dot{L} = \pi_a\,[L,\,(ad\,Q)^{-1}\pi_n\,\nabla H(L)]$$
$$= \pi_a\,[L,\,(ad\,Q)^{-1}\,\pi_n\,L]$$
$$= \pi_a\,[(ad\,Q)^{-1}\mu,\,(ad\,Q)^2\mu]$$
$$= -\pi_a\,[\mu,\,(ad\,Q)^{-3}\mu] \text{ (Lemma 5.5).} \qquad (6.66)$$

Now substituting μ and using the commutation relations (5.16) this reduces to the equations:

$$\dot{P} = -2g^2\sum_{i<j}(q_i - q_j)^{-3}\,h_{\alpha_{ij}} - 2g^2\sum_{i<j}(q_i + q_j)^{-3}\,h_{\beta_{ij}}$$
$$-2g_1^2\sum_i q_i^{-3}\,h_{\lambda_i} - 2g_2^2\sum_i (2q_i)^{-3}\,h_{2\lambda_i} \qquad (6.67)$$
$$= 2g^2\sum_{i<j}(q_i - q_j)^{-3}\,(d_i - d_j) - 2g^2\sum_{i<j}(q_i + q_j)^{-3}(d_i + d_j)$$
$$-2g_1^2\sum_i q_i^{-3}d_i - 2g_2^2\sum_i (2q_i)^{-3}\,2d_i$$
$$= -2g^2\sum_{i\neq j}(q_i - q_j)^{-3}d_i - 2g^2\sum_{i\neq j}(q_i + q_j)^{-3}d_i$$
$$-2g_1^2\sum_i q_i^{-3}d_i - 4g_2^2\sum_i (2q_i)^{-3}d_i \qquad (6.68)$$

And this is equivalent with:

$$\dot{p}_i = -2g^2\sum_{j\neq i}[(q_i - q_j)^{-3} + (q_i + q_j)^{-3}]$$
$$-2g_1^2\,q_i^{-3} - \frac{1}{2}g_2^2\,q_i^{-3} \qquad (6.69)$$

(6.65) and (6.69) are the equations that define the BC_n Calogero-Moser model.

From (6.64) it is clear that we can get the classical root systems B_n, C_n and D_n by choosing special values for g, g_1 and g_2. We can distinguish the following cases:

1) $g \neq 0$, $g_1 = 0$, $\mathfrak{g} = su(n,n)$, $\mathfrak{k} = s(u(n) \oplus u(n))$

   1a) $g_2 \neq 0 \Rightarrow C_n$ model

   1b) $g_2 = 0 \Rightarrow D_n$ model

   1c) $\sqrt{2} g_2 = g$: this can be viewed as a reduced A_{2n-1}-model by choosing

$$q_i + q_{2n+1-i} = 0 \quad (1 \leqslant i \leqslant n) \tag{6.70}$$

2) $g \neq 0$, $g_1 \neq 0$, $g_1^2 - 2g^2 + \sqrt{2} g_2 g = 0$ and

   $\mathfrak{g} = su(n,n+1)$, $\mathfrak{k} = s(u(n) \oplus u(n+1))$

   2a) $g_2 \neq 0 \Rightarrow BC_n$ model

   2b) $g_2 = 0 \Rightarrow g_1^2 = 2g^2$ B_n model

   2c) $g_2 \neq 0$, $g_1 = \sqrt{2} g_2 = g$: this can be viewed as a reduced A_{2n} model by imposing the conditions:

$$q_{n+1} = 0, \quad q_{2n+2+i} = -q_i \quad (1 \leqslant i \leqslant n) \tag{6.71}$$

7. Some ideas on the construction of Calogero-Moser models

In section 5 we described an abstract construction of Calogero-Moser models. This construction starts with a real noncompact semisimple Lie algebra and two commuting involutive automorphisms θ and σ and requires the existence of an element μ with some special properties. In section 6 we described two examples, the A_{n-1} and BC_n models, corresponding to the Lie algebras $sl(n,\mathbb{C})$ and $su(n,n+1)$. In these two cases μ is known and the corresponding equations of motion can be written in Lax form. Next we would like to construct C-M models for other Lie algebras, especially those where the restricted root system is of exceptional type. In order to apply the construction of section 5 we have to find other examples of elements μ which satisfy (5.40) or (5.42). One possibility would be to generalize the construction of the μ corresponding to the A_{n-1} model, but we shall see that for other Lie algebras this μ does not satisfy (5.42). [32]

So let $\mathfrak{g}_c$ be a complex simple Lie algebra with Cartan subalgebra $\mathfrak{h}_c$, root system R, simple roots Δ, positive roots R_+, negative roots R_- and Chevalley basis $\{h_\alpha, e_\alpha, e_{-\alpha}, \alpha \in R\}$ with commutation relations:

$$\begin{aligned}
&[h_\alpha, e_\alpha] = 2 e_\alpha \\
&[h_\alpha, e_{-\alpha}] = -2 e_{-\alpha} \\
&[e_\alpha, e_{-\alpha}] = h_\alpha \\
&[e_\alpha, e_\beta] = c_{\alpha,\beta}\, e_{\alpha+\beta} \ \text{ if } \alpha+\beta \in R \\
&\qquad = 0 \ \text{ if } 0 \neq \alpha+\beta \notin R.
\end{aligned} \tag{7.1}$$

and the structure constants satisfy the following conditions:

$$\begin{aligned}
c_{\alpha,\beta} &= -c_{-\alpha,-\beta} \\
c_{\alpha,\beta} &= -c_{\beta,\alpha}
\end{aligned} \tag{7.2}$$

which imply that the $c_{\alpha,\beta}$'s are real.

Define:

$$s_\alpha = e_\alpha + e_{-\alpha},\ a_\alpha = e_\alpha - e_{-\alpha},\ \alpha \in R_+ \tag{7.3}$$

and

$$\mathfrak{k} = \mathbb{R}< ih_\alpha, is_\alpha, a_\alpha, \alpha \in R_+ > \tag{7.4}$$

then $\mathfrak{k}$ is a compact real form of $\mathfrak{g}_c$ and

$$(\mathfrak{g}_c)_R = \mathfrak{k} \oplus i\,\mathfrak{k} \tag{7.5}$$

where $(\mathfrak{g}_c)_R$ denotes $\mathfrak{g}_c$ considered as a real Lie algebra. Now let θ denote the conjugation of $\mathfrak{g}_c$ w.r.t $\mathfrak{k}$.

Then θ is a Cartan involution of $(\mathfrak{g}_c)_R$ and (7.5) is the corresponding Cartan decomposition with $\mathfrak{p} = i\mathfrak{k}$
Define:

$$\mathfrak{h}_R = \mathbb{R}< h_\alpha, \alpha \in R > \tag{7.6}$$

then $\mathfrak{h}_R$ is a real form of $\mathfrak{h}_c$ and is a maximal abelian subspace of $\mathfrak{p}$. So $\mathfrak{a} = \mathfrak{h}_R$ and $\mathfrak{m} = i\mathfrak{a}$. The root system of $((\mathfrak{g}_c)_R, \mathfrak{h}_R)$ coincides with the root system R of $(\mathfrak{g}_c, \mathfrak{h}_c)$ and the root spaces $\mathfrak{g}_\alpha$ also coincide. However, g_α considered as a real vector space has dimension 2.

Now define:

$$\sigma(h_\alpha) = -h_\alpha, \quad \sigma(e_\alpha) = -e_{-\alpha} \tag{7.7}$$

then σ is an involutive automorphism of $(\mathfrak{g}_c)_R$ which commutes with θ and $\mathfrak{g}_0 \subset \mathfrak{s}$. Moreover we have:

$$\mathfrak{n} \cap \mathfrak{k} \cap \mathfrak{t} = \mathbb{R} < a_\alpha, \alpha \in R_+ >$$

$$\mathfrak{n} \cap \mathfrak{k} \cap \mathfrak{s} = \mathbb{R} < is_\alpha, \alpha \in R_+ >$$

$$\mathfrak{n} \cap \mathfrak{p} \cap \mathfrak{t} = \mathbb{R} < ia_\alpha, \alpha \in R_+ >$$

$$\mathfrak{n} \cap \mathfrak{p} \cap \mathfrak{s} = \mathbb{R} < s_\alpha, \alpha \in R_+ > \tag{7.8}$$

Instead of the normalization (5.11) we now have:

$$< e_\alpha, e_{-\alpha} > = \frac{2}{(\alpha,\alpha)} \tag{7.9}$$

$$< e_\alpha, e_\beta > = 0 \ \text{ if } \beta \neq -\alpha \tag{7.10}$$

Also σ restricts to an involutive automorphism of $\mathfrak{k}$ and the corresponding noncompact real form is:

$$\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{p}_1 \tag{7.11}$$

where

$$\mathfrak{k}_1 = \mathfrak{k} \cap \mathfrak{t} = \mathbb{R} < a_\alpha, \alpha \in R_+ >$$

$$\mathfrak{p}_1 = i(\mathfrak{k} \cap \mathfrak{s}) = \mathbb{R} < h_\alpha, s_\alpha, \alpha \in R_+ > \tag{7.12}$$

This is a normal real form of $\mathfrak{g}_c$ because $\mathfrak{h}_R$ is a maximal abelian subspace of $\mathfrak{p}_1$ and is also maximal abelian in $\mathfrak{g}$. The rootsystem of the pair $(\mathfrak{g}, \mathfrak{h}_R)$ coincides with the rootsystem R of $(\mathfrak{g}_c, \mathfrak{h}_c)$. If $\alpha \in R$ then $\mathfrak{g}_\alpha = \mathbb{R}e_\alpha$, so dim $\mathfrak{g}_\alpha = 1$ and the dimension of the centraliser of $\mathfrak{h}_R$ in $\mathfrak{k}_1$ is zero. Let $\mathfrak{g}$ denote the orthogonal complement of $\mathfrak{h}_R$ in $\mathfrak{p}_1$.
Now choose $\mu \in \mathfrak{n} \cap \mathfrak{k} \cap \mathfrak{s}$, then it has the form:

$$\mu = \sum_{\alpha \in R_+} g_\alpha \, is_\alpha, \quad g_\alpha \in \mathbb{R} \tag{7.13}$$

and suppose μ satisfies (5.42). Then $i\mu \in \mathfrak{g}$ and (5.42) is equivalent with the following condition:

$$\pi_q : C_{\mathfrak{p}_1}(i\mu) \to \mathfrak{q} \text{ is surjective} \tag{7.14}$$

or equivalently: can we find a $\mu \in \mathfrak{q}$ such that for each $\alpha \in R_+$ there exists an element $h(\alpha) \in \mathfrak{h}_R$ such that

$$[h(\alpha) + s_\alpha, \mu] = 0 \tag{7.15}$$

This would mean that the centralizer of μ in $\mathfrak{p}_1$ would contain at least $|R_+|$ linearly independent elements. So the question becomes: can a nonzero element in $\mathfrak{p}_1$ have a centralizer in $\mathfrak{p}_1$ of dimension $\geqslant |R_+|$?

Now let K_1 be the compact group with Lie algebra $\mathfrak{k}_1$ then we know that

$$\mathfrak{p}_1 = Ad(K)\,\mathfrak{h}_R \tag{7.16}$$

so the question is equivalent to: can $0 \neq x \in \mathfrak{h}_R$ have a centralizer in $\mathfrak{p}_1$ of dimension $\geqslant |R_+|$?

A general element of $\mathfrak{p}_1$ has the form

$$\sum_{\alpha \in R_+} c_\alpha \, s_\alpha + h \tag{7.17}$$

with $c_\alpha \in \mathbb{R}$ and $h \in \mathfrak{h}_R$. For $x \in \mathfrak{h}_R$ we have:

$$[x, \sum_{\alpha \in R_+} c_\alpha \, s_\alpha + h] = 0$$

$$\Leftrightarrow \sum_{\alpha \in R_+} c_\alpha \, \alpha(x) \, s_\alpha = 0$$

$$\Leftrightarrow \quad c_\alpha \, \alpha(x) = 0 \quad \forall \alpha \in R_+ \tag{7.18}$$

Hence the centralizer of x in $\mathfrak{p}_1$ has dimension

$$\dim \mathfrak{h}_R + |\{\alpha \in R_+ \,|\, \alpha(x) = 0\}| \tag{7.19}$$

So our question becomes equivalent: can it happen for $0 \neq x \in \mathfrak{h}_R$ that:

$$|\{\alpha \in R_+ \,|\, \alpha(x) \neq 0\}| \leqslant \dim \mathfrak{h}_R \tag{7.20}$$

Let $l = \dim \mathfrak{h}_R$ and let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ denote the simple roots. Define $\dot{\alpha}_i^\vee := \dfrac{2\alpha_i}{(\alpha_i, \alpha_i)}$ and define the fundamental weights $\{\lambda_1, \ldots, \lambda_l\}$ by the condition:

$$(\lambda_1, \alpha_j^\vee) = \delta_{ij} \tag{7.21}$$

So $\{\lambda_i\}$ is the dual basis of $\alpha_i^\vee$ w.r.t (,).

By identification of $\mathfrak{h}_R$ and $\mathfrak{h}_R^*$ and by Weyl group invariance of our problem the question becomes equivalently: can it happen for $0 \neq \lambda = c_1\lambda_1 + \cdots c_l\lambda_l$ with $c_1, \ldots, c_l \geqslant 0$ that:

$$|\{\alpha \in R_+ \,|(\alpha, \lambda) \neq 0\} \leqslant \dim \mathfrak{h}_R? \tag{7.22}$$

Clearly, if such a λ exists then we may choose λ as one of the fundamental weights so it is sufficient to find a fundamental weight λ_k such that

$$a_k = |\{\alpha \in R_+ \,|\, (\alpha, \lambda_k) \neq 0\}| \leqslant l \tag{7.23}$$

Now we can check Planches I-IX in BOURBAKI [33] and it follows that for all rootsystems except A_l $a_k > l$, in particular we have for B_l, C_l and D_l:

$$a_k = \frac{1}{2} k (4l - 3k + 1) \tag{7.24}$$

For A_l we get:

$$a_k = k(l+1-k) \geqslant l, \ 1 \leqslant k \leqslant l \tag{7.25}$$

so we must require $a_k = l$ which implies $k=1$ or $k=l$. Using the explicit form of the λ_i's in [27] we see that λ_1 corresponds with the μ defined in (6.11), (6.16) and (6.19) and λ_l gives the same μ.

So we conclude that the construction of the μ of the A_{n-1} model does not work for the other root systems and moreover that there is only one μ corresponding to root system A_{n-1} that has the desired property (5.42).

8. COMPARISON OF CONSTRUCTION C-M MODELS WITH K.A.S.R.S. THEOREM

In this section we want to compare the construction of the Calogero-Moser models in section 5 with the construction of integrable models in section 4. So suppose μ satisfies (5.42) and write the Lax equation (5.48) in the following form:

$$\dot{x} = [x, \frac{1}{2} R_q (F)] \tag{8.1}$$

where $R_q \in \operatorname{End} \mathfrak{g}$ is given by:

$$R_q = 2 S \circ \pi_{\mathfrak{n} \cap \mathfrak{p} \cap \mathfrak{t}} + 2 (ad\, q)^{-1} \circ \pi_{\mathfrak{n} \cap \mathfrak{p} \cap \mathfrak{s}} + k(q) Id \tag{8.2}$$

and $k(q) \in \mathbb{R}$ is a constant which may depend on q.

If we compare (8.1) with (4.22) we see that we could have applied the K.A.S.R.S construction if R_q is a Yang-Baxter operator for each $q \in \mathfrak{a}_+$. But this is not true, as we show in the following propositions:

PROPOSITION 8.1 *R_q does not satisfy (4.17)*

PROOF: If $x \in \mathfrak{k}$ then $R_q x = k(q)x$ so taking $x, y \in \mathfrak{k}$ in (4.17) we conclude that $k(q)=0$, but this implies $\ker R_q = \mathfrak{k} \oplus \mathfrak{a}$. Now from Prop. 4.17 it follows that $[\operatorname{im} R_q, \ker R_q] \subset \ker R_q$, but from (8.2) it is clear that $\mathfrak{n} \cap \mathfrak{k} \cap \mathfrak{t} \subset \operatorname{im} R_q$ and

$$[\mathfrak{n} \cap \mathfrak{k} \cap \mathfrak{t}, \mathfrak{a}] \subset \mathfrak{n} \cap \mathfrak{p} \cap \mathfrak{s} \not\subset \ker R_q \tag{8.3}$$

so (8.2) cannot be a solution of (4.17). □

PROPOSITION 8.2 *R_q does not satisfy (4.18).*

PROOF Again taking $x, y \in \mathfrak{k}$ in (4.18) we conclude that $k(q)^2 = 1$ so $k(q) = 1$ or $k(q) = -1$ but if $k(q)=1$ then $\ker (R_q)_- = \mathfrak{k} \oplus \mathfrak{a}$ and if $k(q) = -1$ then $\ker (R_q)_+ = \mathfrak{k} \oplus \mathfrak{a}$. Now from Prop. 4.21 it follows that $\ker (R_q)_-$ and $\ker R_q)_+$ must be subalgebras of $\mathfrak{g}$, but $\mathfrak{k} \oplus \mathfrak{a}$ is not a subalgebra because $[\mathfrak{a}, \mathfrak{k}] \subset \mathfrak{n} \cap \mathfrak{p}$, so (8.2) cannot be a solution of (4.18) □

So we conclude that there does not seem to exist a Yang-Baxter operator such that the Lax equation (5.48) can be derived as in section 4.

REFERENCES

1. M.A. OLSHANETSKY, A.M. PERELOMOV, Classical integrable finite-dimensional systems related to Lie algebras, Phys. Rep. **71**, 5 (1981), 313-400.
2. B. KOSTANT, The solution to a generalized Toda lattice and representation theory, Adv. Math. **34** (1979), 195-338.
3. M. ADLER, On a trace functional for formal pseudo-differential operators and the symplectic structure for the *KdV* type equations, Invent. Math. **50** (1979), 219-248.
4. W.W. SYMES, Systems of Toda type, inverse spectral problems and representation theory, Invent. Math. **59** (1980), 13-51.
5. W.W. SYMES, The *QR* algorithm and scattering for the finite nonperiodic Toda lattice, Physica **4 D** (1982), 275-280.
6. W.W. SYMES, Hamiltonian group actions and integrable systems, Physica **1 D** (1980), 339-374.
7. A.G. REYMAN, Integrable Hamiltonian systems connected with graded Lie algebras, J. Sov. Math. **19**, 5 (1982), 1507-1545.
8. M.A. SEMENOV-TYAN-SHANSKII, What is a classical *r*-matrix?, Funct. Anal. Appl. **17**, 4 (1983), 259-272.
9. M.A. SEMENOV-TYAN-SHANSKII, Classical *r*-matrices and the method of orbits, J. Sov. Math. **28**, 4 (1985), 513-524.
10 M.A. SEMENOV-TYAN-SHANSKII, Classical *r*-matrices, Lax equations, Poisson-Lie groups and dressing transformations, in: Field theory, quantum gravity and strings II (H.J. DE VEGA, N. SANCHEZ eds), LNP 280, 174-214, Springer-Verlag 1987.
11. M.A. OLSHANETSKY, A.M. PERELOMOV, Quantum integrable systems related to Lie algebras, Phys. Rep. **94**, 6 (1983), 313-404.
12. S.N.M. RUIJSENAARS, H. SCHNEIDER, A new class of integrable systems and its relation to solitons, Ann. Phys **170**, 2 (1986), 370-405.
13. S.N.M. RUIJSENAARS, Complete integrability of relativistic Calogero-Moser systems and elliptic function identities, Comm. Math. Phys. **110** (1987), 191-213.
14. S.N.M. RUIJSENAARS, Action-angle maps and scattering theory for some finite-dimensional integrable systems; I. The pure soliton case, Comm. Math. Phys. **115** (1988), 127-165.
15. M. BRUSCHI, F CALOGERO, The Lax representation for an integrable class of relativistic dynamical systems, Comm. Math. Phys. **109** (1987), 481-492.
16. W. OEVEL, M FALCK, Master symmetries for finite-dimensional integrable systems: the Calogero-Moser system, Progr. Theor. Phys. **75** (1986), 1328-1341.
17. H. SCHNEIDER, Integrable relativistic *N*-particle systems in an external potential, Physica 26D (1987), 203-209.
18. M.R. ADAMS, J. HARNAD, E. PREVIATO, Isospectral Hamiltonian flows in finite and infinite dimensions. I Generalized Moser systems and moment maps into loop algebras, Comm. Math. Phys. **117** (1988), 451-501.
19. J. GIBBONS, T. HERMSEN, A generalization of the Calogero-Moser system, Physica 11D (1984), 337-348.
20. V.I. ARNOLD, Mathematical methods of classical mechanics, Springer-Verlag 1984.
21. A. WEINSTEIN, The local structure of Poisson manifolds, J. Diff. Geom. **18** (1983), 523-557.
22. P.J. OLVER, Applications of Lie groups to differential equations, Springer-Verlag 1986.
23. R. HERMANN, Toda lattices, cosymplectic manifolds, Backlund transformations and kinks (Part A), Math. Sci. Press 1977.
24. R. HERMANN, Topics in the mathematics of quantum-mechanics, Math. Sci. Press. 1977.
25. A.A. KIRILLOV, Elements of the theory of representations, Springer-Verlag 1976.
26 A.T. FOMENKO, Differential geometry and topology, Consultants Bureau, New York 1987.
27. J.E. HUMPHREYS, Introduction to Lie algebras and representation theory, Springer-Verlag 1980.
28. S. HELGASON, Differential geometry, Lie groups and symmetric spaces, Academic Press 1978.
29. R. GILMORE, Lie groups, Lie algebras and some of their applications, Wiley 1974.

30. T.H. KOORNWINDER (ed.), The structure of real semisimple Lie groups, MC Syllabus 49, Mathematical Centre, Amsterdam 1982.
31. R. SHAW, Linear algebra and group representations, Vol I, Academic Press 1982.
32. T.H. KOORNWINDER, private communication.
33. N. BOURBAKI, Groupes et algèbres de Lie, Masson 1981.