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THE LINEARLY IMPLICIT EULER METHOD FOR QUASI-LINEAR PARABOLIC DIFFERENTIAL EQUATIONS

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Following the method of lines approach parabolic problems discretized in space by any usual method are discretized in time by the linearly implicit Euler method. To avoid order reduction occuring for problems with time dependent boundary conditions a modification of the method is proposed. This modified method is proved to be of uniform (i.e. independently of the space discretization) order 1 of convergence for wide classes of semi-linear and quasi-linear problems.

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1. Introduction

Consider a real abstract Cauchy problem

$$u_t = \mathcal{F}(x,t,u) \qquad 0<t\le T \qquad u(x,0) = u_0(x) \tag{1.1}$$

where $\mathcal{F}$ represents a partial differential operator differentiating the unknown (scalar or vector) function $u(x,t)$ w.r.t. its space variable x in the space domain in $\mathbb{R}$, $\mathbb{R}^2$ or $\mathbb{R}^3$. The operator $\mathcal{F}$ will contain the boundary conditions too.

Following the method of lines approach discretization in space (e.g. by finite differences or finite elements) yields an (only formally explicit) initial value problem of ODE's

$$y_h'(t) = F(t,y_h(t);h) \qquad 0<t\le T \qquad y_h(0)=y_0 \ , \tag{1.2}$$

where h denotes a grid parameter (e.g. the vector of grid distances). For the time integration of this semi-discrete problem ODE-solvers are used. Our analysis of the full (both space and time) discretization error is

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centered around the semi-discrete problem (see VERWER/SANZ-SERNA [1984] and SANZ-SERNA/VERWER [1989]). The classical concepts of consistency and convergence known from ODE-theory are not applicable here because the Lipschitz constant of F is used to derive bounds for the local and global error. For semi-discrete problems with small grid distance h, however, the classical Lipschitz constant becomes arbitrary large, in general.

This paper deals with the analysis of the full local and global discretization error of linearly implicit one step methods applied to the semi-discrete problem. We derive bounds for the local and global error that hold uniformly for $h \to 0$ (i.e. the grid is refined arbitrary far in a suitable manner).

In Chapter 2 semi-linear problems with semi-discretizations

$$y_h'(t)=A_h y_h(t)+Q_h(t,y_h(t))+r_h(t) \qquad y_h(0)=y_0 \tag{1.3}$$

(described in more detail below) are considered. Linearly implicit one step methods for such problems with constant linear operator A_h were analysed in STREHMEL/WEINER [1987]. The obtained bounds for local and global error which hold uniformly for $h \to 0$ are in general of a lower order than the classical ones - order reduction occurs, a phenomenon well known from stiff ODE-theory (see VERWER [1986] e.g.). The linearly implicit Euler method has (uniform) order 1 of consistency only if $r_h(t) \equiv$ const (which refers to continuous problems with time independent boundary conditions), in general its order of consistency is reduced to 0. A rather simple modification of the method has, however, order 1 again.

The analysis of numerical methods for quasi-linear parabolic problems is more complicated than for semi-linear ones. Semi-discretization yields systems similar to (1.3) but with a linear operator A_h (and function r_h) depending on the solution vector y_h. Chapter 3 contains the analysis of the linearly implicit Euler method for quasi-linear problems. It is divided into two parts. First a general convergence theorem for a wide class of problems (e.g. $u_t=d(u)u_{xx}$ and Burger's equation; but not the quasi-linear heat flow equation) is given. After that we prove convergence of the linearly implicit Euler method applied to the one dimensional quasi-linear heat flow equation semi-discretized by central differences.

The numerical examples given in Chapter 4 demonstrate the theoretical results in a nice way.

2. Linearly implicit one step methods for semi-discretizations of semi-linear parabolic equations

Let the parabolic equation be given by

$$\frac{\partial u}{\partial t} = \sum_{i,k=1}^{\alpha} \frac{\partial}{\partial x_i}\left(b_{ik}(x)\frac{\partial u}{\partial x_k}\right) + \sum_{i=1}^{\alpha} \frac{\partial}{\partial x_i}\left(b_i(x)u\right) + q(t,x,u) \qquad (x\in G,\ t\in(0,T]) \tag{2.1}$$

$$u(x,t) = \varphi(x,t) \qquad (x\in\partial G,\ t\in[0,T]),$$

$$u(x,0) = u_0(x) \qquad (x\in G)$$

where the matrix $((b_{ik}))$ is uniformly positive definite and G is a spatial domain in $\mathbb{R}^{\alpha}$ (α=1,2 or 3) with boundary ∂G. It is assumed that (2.1) has a unique "smooth" solution $u(x,t)$, the source term $q(t,x,u)$ is Lipschitzian w.r.t. u .

We suppose that semi-discretization in space results in

$$\begin{aligned} y_h'(t) &= A_h y_h(t) + Q_h(t,y_h(t)) + r_h(t) \\ y_h(0) &= y_0 \end{aligned} \tag{2.2}$$

with

(i) $\mu[A_h] \le \mu_0 \le 0$,

(ii) $\|Q_h(t,v_1)-Q_h(t,v_2)\| \le L\|v_1-v_2\|$ for all $t\in[0,T]$, $v_1,v_2\in\mathbb{R}^N$,

(iii) $\|\frac{d^i}{dt^i}Q_h(t,u_h(t))\| \le M$ for all $t\in[0,T]$, $i=1(1)p_0$

where $\|\cdot\|$ is some norm, $\mu[\cdot]$ the corresponding logarithmic matrix norm (cf. DEKKER/VERWER [1984]) and $u_h(t)$ denotes the natural restriction of the exact solution $u(x,t)$ of (2.1) on the mesh.

The constant linear operator A_h is the discrete analogue of the differential operator in (2.1), Q_h represents the source term, $r_h(t)$ arises from inhomogeneous boundary conditions (note that, in general, $\|r_h(t)\| \longrightarrow \infty$ if $h\to 0$). The assumption $\mu_0\le 0$ allows to simplify the proofs. Analogous results, however, will hold for $\mu_0>0$ too. $p_0\ge 1$ denotes an integer specified below.

Remark 2.1.: An essential tool in our further investigations are bounds for rational matrix functions which hold uniformly in h (DEKKER/VERWER [1984], pp. 43, 46–47):

(i) The matrix $I-\tau A$ is invertible and $\|(I-\tau A)^{-1}\| \le \frac{1}{1-\tau\mu[A]}$ for all norms $\|\cdot\|$ (with corresponding logarithmic norms $\mu[\cdot]$), all matrices A and all τ with $\tau\mu[A]<1$.

(ii) Let R(z) be a rational function, the norm $\|\cdot\|$ be induced by an inner product and $\langle Aw,w\rangle \le \mu\|w\|^2$ $(w\in\mathbb{R}^N)$ (this one-sided Lipschitz condition is known to be equivalent to $\mu[A]\le\mu$). Then we get
$\|R(\tau A)\| \le \sup_{\mathrm{Re}\, z\le\tau\mu} |R(z)|$.

The assumption on the norm in (ii) to be induced by an inner product is essential. For the Chebyshev norm $\|\cdot\|_\infty$, e.g., a bound similar to (ii) does not hold in general. ■

Definition 2.1.: The semi-discretization (1.2) is called consistent, if the local space discretization error

$$\alpha_h(t) := u_h'(t) - F(t,u_h(t);h) \qquad \text{tends to zero if } h\to 0. \ ■$$

Remark 2.2.: We denote

$$\alpha_{h,0} := \max_{t\in[0,T]} \|\alpha_h(t)\|$$

and

$$\alpha_{h,p_0} := \alpha_{h,0} + \max_{t\in[0,T]} \sum_{i=1}^{p_0} \left\|\frac{d^i}{dt^i}\alpha_h(t)\right\| \qquad (p_0\ge 1). \ ■$$

Definition 2.2.: A one step method applied to (1.2) defines a sequence $y_0,y_1,\ldots$ approximating $u_h(0),u_h(t_1),\ldots$.
Let $\hat{y}_{n+1}$ be the result of one time step with $\hat{y}_n := u_h(t_n)$.

$$\beta_{n+1} := u_h(t_{n+1}) - \hat{y}_{n+1}$$

is called the (full) local discretization error.

$$\varepsilon_{n+1} := u_h(t_{n+1}) - y_{n+1}$$

is called the (full) global discretization error. ■

We want to derive bounds of the form (see VERWER [1986] and SANZ-SERNA, VERWER and HUNDSDORFER [1986])

$$C_0\left(\tau^q + \alpha_{h,p_0}\right) \qquad \text{for all } h\in(0,h_0] \text{ and } \tau\in(0,\tau_0]$$

for the global discretization error of linearly implicit one step methods applied to semi-discrete systems. Here τ denotes the time step, the constants C_0, h_0, τ_0 and q do not depend on the space discretization parameter h. The term α_{h,p_0} represents the error caused by discretization in space, the "τ^q-term" results from the discretization in time. q is called the uniform order of convergence (i.e. independent of the space discretization) of the one step method.

The global discretization error is studied via the local discretization error (uniform consistency). Under certain mild assumptions uniform convergence follows.

Example 2.1.: Linearly implicit Euler method with arbitrary stability function (classical order 1).

The linearly implicit Euler method applied to (2.2) is given by

$$y_{n+1} := R_0(\tau A_h)y_n + \tau R_1(\tau A_h)(Q_h(t_n,y_n) + r_h(t_n)) \; .$$

Here $R_0(z)$ denotes a rational function (an approximation of exp(z) for $z\to 0$ of order 1 at least). $R_1(z):=(R_0(z)-1)/z$, $R_2(z):=(R_1(z)-1)/z$ (see below) are rational functions with the same denominator as $R_0(z)$. $R_0(z)$ is proposed to be A-acceptable, i.e. Re $z < 0 \Rightarrow |R_0(z)|<1$.

We get

$$\hat{y}_{n+1}=R_0(\tau A_h)u_h(t_n)+\tau R_1(\tau A_h)(Q_h(t_n,u_h(t_n))+r_h(t_n))$$
$$=u_h(t_n)+\tau R_1(\tau A_h)(u_h'(t_n)-\alpha_h(t_n))$$

$$\beta_{n+1}=u_h(t_{n+1})-\hat{y}_{n+1}$$
$$=\tau(I-R_1(\tau A_h))u_h'(t_n)+\tau R_1(\tau A_h)\alpha_h(t_n)+\frac{\tau^2}{2}u_h''(t_n+\vartheta\tau)$$

$$\beta_{n+1}=-\tau^2R_2(\tau A_h)A_hu_h'(t_n)+\tau R_1(\tau A_h)\alpha_h(t_n)+\frac{\tau^2}{2}u_h''(t_n+\vartheta\tau) \qquad (2.3)$$

with $0<\vartheta<1$.

Let us now consider problems (2.1) with time independent boundary conditions, i.e. $r_h(t)\equiv$ const, and a norm $\|\cdot\|$ induced by an inner product. Then it holds (choose $p_0=1$ in (2.2) (iii))

$$A_h u_h'(t)=u_h''(t)-\frac{d}{dt}Q_h(t,u_h(t))-\alpha_h'(t)$$

$$\|A_h u_h'(t)\| \le C_1 + \alpha_{h,1}$$

Due to Remark 2.1 (ii) and $\mu[A_h]\le 0$ it follows that $\|R_0(\tau A_h)\|\le 1$ and $\|R_1(\tau A_h)\|$, $\|R_2(\tau A_h)\|$ are bounded uniformly in h.

Hence

$$\|\beta_{n+1}\| \le \mathcal{O}(\tau^2) + C_2\tau\alpha_{h,0} .$$

For the global discretization error we obtain

$$\varepsilon_{n+1}=u_h(t_{n+1})-y_{n+1}=u_h(t_{n+1})-\hat{y}_{n+1}+\hat{y}_{n+1}-y_{n+1}$$
$$=\beta_{n+1}+R_0(\tau A_h)\varepsilon_n+\tau R_1(\tau A_h)(Q_h(t_n,u_h(t_n))-Q_h(t_n,y_n))$$

Because of (2.2) (ii) and Remark 2.1 (ii) we get

$$\|\varepsilon_{n+1}\| \le (1+C_3\tau)\ \|\varepsilon_n\| + \|\beta_{n+1}\|$$

and by induction (provided that $\varepsilon_0=0$)

$$\|\varepsilon_{n+1}\| \le \frac{C_4}{\tau}\ \|\beta_{n+1}\| \le \mathcal{O}(\tau) + C_0\alpha_{h,0} .$$

I.e. for problems with time independent boundary conditions the linearly implicit Euler method is of uniform order 1 of convergence.

For time dependent boundary conditions of (2.1), however, $r_h(t)\not\equiv$ const , $\|A_h' u_h(t)\|$ is not bounded for $h\to 0$ and thus

$$\|\beta_{n+1}\| \le \mathcal{O}(\tau) + C_5\tau\alpha_{h,0}$$

only (cf. (2.3)).

Following the lines of the proof above we now only get

$$\|\varepsilon_{n+1}\| \le \mathcal{O}(1) + C_0^*\alpha_{h,1} .$$

Under the additional assumption

$$R_0(z) \ne 1 \quad \text{for all } z \in \mathbb{C}\cup\{\infty\}\ ,\mathrm{Re}\ z \le 0,\ z\ne 0$$

uniform convergence with order 1 can still be proved by taking into account cancellation and damping effects (cf. HUNDSDORFER [1986], Corollary 3.2), but the proof becomes more complicated. Below we will discuss some simple modifications on the method for which a direct proof can be given. ∎

In STREHMEL/WEINER [1987] for problems (2.2) fulfilling (i)-(iii) with a norm induced by an inner product a principle is given that allows to construct linearly implicit one step methods of arbitrary high uniform order of convergence. Because of occuring order reduction the number of stages increases rapidly with increasing uniform order. For time independent boundary conditions the order reduction is often decreased (cf. Example 2.1).

But (2.3) shows a natural way to modify the methods to get better results for $r_h'(t) \neq 0$ too.

Example 2.2.: Modified linearly implicit Euler method I.

$$y_{n+1} := R_0(\tau A_h)y_n + \tau R_1(\tau A_h)(Q_h(t_n,y_n)+r_h(t_n)) + \tau^2 R_2(\tau A_h) r_h'(t_n)$$

Instead of (2.3) we get

$$\beta_{n+1} = -\tau^2 R_2(\tau A_h)(A_h u_h'(t_n) + r_h'(t_n)) + \dots$$

and following the proof of Example 2.1 for time independent boundary conditions we now get the estimate for ε_{n+1} :

$$\|\varepsilon_{n+1}\| \leq C_0(\tau + \alpha_{h,1})$$

also for time dependent boundary conditions. Thus no order reduction occurs.

The additional numerical effort of the modification is small, in general, because almost all components of $r_h(t)$ are zero (for one dimensional problems only the first and the last ones are not zero). ∎

Similar modifications are useful for methods with more than one stage too. They involve the computation of higher derivatives of $r_h(t)$. For implementation it is possible to substitute the derivatives by a discrete approximation using values of $r_h(t)$ at different points t_i:

Example 2.3.: Modified linearly implicit Euler method II.

$$y_{n+1}:=R_0(\tau A_h)y_n+\tau R_1(\tau A_h)(Q_h(t_n,y_n)+r_h(t_n))+\tau R_2(\tau A_h)(r_h(t_{n+1})-r_h(t_n))$$

Because of

$$r_h(t_{n+1})-r_h(t_n)=\tau r_h'(t_n)+\frac{\tau^2}{2}r_h''(t_n+\vartheta\tau)$$

and

$$r_h''(t_n+\vartheta\tau)=u_h'''(t_n+\vartheta\tau)-A_h u_h''(t_n+\vartheta\tau)-\frac{d^2}{dt^2}Q_h(t_n+\vartheta\tau,u_h(t_n+\vartheta\tau))-\frac{d^2}{dt^2}\alpha_h(t_n+\vartheta\tau)$$

and the uniform bounds of $\|R_2(\tau A_h)\|$ and $\|\tau A_h R_2(\tau A_h)\|$, the estimate

$$\|\varepsilon_{n+1}\| \le C_0(\tau + \alpha_{h,2})$$

is proved as above (choose $p_o=2$ in (2.2) (iii)).

For the special choice $R_0(z)=1/(1-z)$ ((1,0)-Padé-approximation to exp(z))

$$R_1(z)=R_2(z)=1/(1-z)$$

holds and we get

$$y_{n+1}:=(I-\tau A_h)^{-1}(y_n+\tau(Q_h(t_n,y_n)+r_h(t_{n+1}))) \ . \tag{2.4}$$

We will consider this method for solving quasi-linear problems in Chapter 3. ■

3. The linearly implicit Euler method for quasi-linear-problems

As proved in STREHMEL/WEINER [1987] linearly implicit one step methods are suitable solvers for semi-linear parabolic problems. Because of the lack of B-stability we cannot expect positive uniform convergence results for general nonlinear problems. Even the analysis of the application of very simple linearly implicit methods to semi-discretizations of quasi-linear parabolic problems is much more complicated than for the semi-linear problems considered in Chapter 2.

LE ROUX [1980] considers linearly implicit multistep methods (including the linearly implicit Euler method) for the quasi-linear abstract Cauchy problem (1.1) with

$$\mathcal{F}(x,t,u) = -A(t,u(t))u(t)+f(t).$$

The linearly implicit Euler method is shown to be of order 1 of convergence for this problem. However, assumptions on the differential operator $A(\cdot,\cdot)$ are made which are in general difficult to verify and require homogeneous boundary conditions together with an inner product norm. In contrast to these results we investigate convergence of the fully discretized scheme under easily verifiable conditions on the differential operator and the space discretization (see (3.2) (i)-(iv)).

We give two theorems characterizing the behaviour of the (modified) linearly implicit Euler method. Following the basic ideas of Chapter 2 Theorem 3.1 is a (uniform) convergence result for a wide class of quasi-linear problems. The semi-discretized system is assumed to fulfil certain conditions but no special interest is given to the space discretization itself, it may be done by each usual method.

Among the examples of wide-spread interest not meeting the assumptions of Theorem 3.1 the one dimensional quasi-linear heat flow equation semi-discretized by central differences is considered in more detail in Theorem 3.2 . Again uniform convergence of order 1 is proved.

3.1. A special class of quasi-linear problems

Look at quasi-linear problems of the form (cf. (2.1))

$$\frac{\partial u}{\partial t} = \sum_{i,k=1}^{\alpha} b_{ik}(x,u)\frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{\alpha} b_i(x,u)\frac{\partial u}{\partial x_i} + q(t,x,u) \quad (x\in G, t\in[0,T]) \tag{3.1}$$

$$u(x,t) = \varphi(x,t) \quad (x\in\partial G,\ t\in[0,T])$$
$$u(x,0) = u_0(x) \quad (x\in G),$$

As before we assume the existence of a unique "smooth" solution of (3.1). The matrix $((b_{ik}))$ shall be uniformly positive definite, the source term $q(t,x,u)$ Lipschitzian.

We consider semi-discretizations of the form

$$y_h'(t)=A_h(y_h(t))y_h(t)+Q_h(t,y_h(t))+r_h(t,y_h(t)) \tag{3.2}$$

with

(i) $\mu[A_h(w)]\le\mu_0$ for all $w\in\mathbb{R}^N$,

(ii) $\|Q_h(t,v_1)-Q_h(t,v_2)\| \le L\ \|v_1-v_2\|$ for all $t\in[0,T]$, $v_1,v_2\in\mathbb{R}^N$,

(iii) $\|A_h(v_1)u_h(t)+r_h(t,v_1)-(A_h(v_2)u_h(t)+r_h(t,v_2))\| \le C\|v_1-v_2\|$

for all $t\in[0,T]$, $v_1,v_2\in\mathbb{R}^N$,

(iv) $\|A_h(u_h(t))u_h(t_1)+r_h(t_1,u_h(t))-$

$-(A_h(u_h(t))u_h(t_2)+r_h(t_2,u_h(t)))\| \le C\ |t_1-t_2|$

for all $t,t_1,t_2\in[0,T]$.

The constants μ_0, L and C are independent of h.

Notice that conditions (iii) and (iv) are fulfilled automatically for semi-linear problems (2.2). These conditions seem to be very technical but in fact they are often easy to verify:

Example 3.1.: Semi-discretization of

$$\frac{\partial u}{\partial t} = d(u)\frac{\partial^2 u}{\partial x^2} \qquad x\in(0,1)\ ,\ t\in(0,T]\ ,\ d(u)\ge\alpha>0 \quad (u\in\mathbb{R})$$

$$u(0,t)=u(1,t)=0 \qquad t\in[0,T]$$

$$u(x,0)=u_0(x) \qquad x\in[0,1]$$

by central differences on an equidistant grid yields the ODE-system

$$y_h'(t) = D_h(y_h(t))B_hy_h(t) \qquad \text{with} \quad y_h : [0,T] \longrightarrow \mathbb{R}^N$$

$y_h^{(i)}(t)$ approximating $u(x_i,t)$,$x_i=i\cdot h$ $(i=1(1)N)$, $h=1/(N+1)$.

$$D_h(w):=\mathrm{diag}(d(w_1),\dots,d(w_N))$$

is a diagonal matrix and

$$B_h := \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & & \ddots & \\ & & 1 & -2 \end{bmatrix}$$

the (constant) standard approximation of $\frac{\partial^2}{\partial x^2}$. Because $u(x,t)$ is assumed to be smooth $B_hu_h(t)$ (an approximation of $u_{xx}(x,t)$ on the grid) and $(B_hu_h(t_1)-B_hu_h(t_2))/(t_1-t_2)$ $(t_1\ne t_2)$ (an approximation of $(u_{xx}(t_1)-u_{xx}(t_2))/(t_1-t_2)=u_{xxt}(x,t_1+\vartheta(t_2-t_1))$ with $\vartheta\in(0,1)$) are bounded uniformly in the Chebyshev-norm $\|\cdot\|_\infty$.

Hence we get

$$A_h(v_1)u_h(t)-A_h(v_2)u_h(t)=(D_h(v_1)-D_h(v_2))B_hu_h(t)$$

and (iii) holds if

$$\|D_h(v_1)-D_h(v_2)\|_\infty = \max_{i=1(1)N} |d(v_1^{(i)})-d(v_2^{(i)})| \le L_d\|v_1-v_2\|_\infty ,$$

i.e. if $d(\cdot)$ is Lipschitzian with constant L_d, $(r_h(t)\equiv 0)$.

Analogously (iv) holds if

$$A_h(u_h(t))(u_h(t_1)-u_h(t_2))=D_h(u_h(t))B_h\cdot(u_h(t_1)-u_h(t_2)) ,$$

i.e. if $\|D_h(u_h(t))\|_\infty = \max\limits_{i=1(1)N} |d(u(x_i,t))|$ is bounded – this is a natural assumption to the parabolic problem.

This foregoing may obviously be generalized to more complicated problems (3.1) and its semi-discretizations. ■

Remark 3.1.: Until now we did not specify the norm that should be used to guarantee conditions (2.2) (i)-(iii) or (3.2) (i)-(iv) to be fulfilled. Norms induced by inner products are more favourable for our investigations of linearly implicit one step methods because Remark 2.1 (ii) gives bounds for such norms of rational matrix functions which allow to construct methods of arbitrary high uniform order for semi-linear problems (see Chapter 2).

The $\mathcal{L}^2$-norm is known to be an effective tool to analyse the semi-linear heat flow equation of form (2.1). Therefore the discrete analogue of this norm (e.g. on an equidistant grid with meshwidth h on $[0,1]\subset\mathbb{R}$ induced by the inner product $\langle x,y\rangle_2=h\cdot x^T y$) seems to be a natural choice for the norm $\|\cdot\|$ in (2.2). Indeed for the one dimensional semi-linear heat flow equation discretized in space similar to Example 3.1 (with $d=d(x)$, now) by central differences (on a not necessary equidistant grid) standard approach shows that conditions (2.2) (i)-(iii) hold for $\|\cdot\|_2$.

The use of the $\|\cdot\|_2$-norm for the "mild" quasi-linear generalization given by Example 3.1 to prove condition (3.2) (i), however, is already impossible (see ARNOLD [1988]):

Let $d(u)$ be given by $d(u)=1+2u$ and $y=y(\varepsilon)=(0,\varepsilon,0,\varepsilon,\ldots,0)^T$ for arbitrary $\varepsilon>0$ (the grid is assumed to have an odd number of grid points for convenience of presentation). Because of

$$\mu_2[D_h(y(\varepsilon))B_h] = \max_{\xi\neq 0} \frac{\langle\xi, D_h(y(\varepsilon))B_h\xi\rangle_2}{\langle\xi,\xi\rangle_2}$$

(see DEKKER/VERWER [1984] p. 31) we get with

$$\xi=\xi(\varepsilon)=(1+\varepsilon,1,1+\varepsilon,1,\dots,1+\varepsilon)^T$$

$$\langle\xi,\xi\rangle_2 = h\cdot\xi^T\xi \le (1+\varepsilon)^2$$

and

$$\langle\xi,D_h(y(\varepsilon))B_h\xi\rangle_2 = h\cdot\xi^T D_h(y(\varepsilon))B_h\xi = \frac{1}{h^2}(\varepsilon^2+2h(\varepsilon^2+\varepsilon-1))$$

$$\mu_2[D_h(y(\varepsilon))B_h] = \mathcal{O}(h^{-2}) \qquad (h\to 0) \qquad \text{for fixed } \varepsilon.$$

Obviously no constant μ_{max} independent of h with (3.2) (i) can be found. (Notice, however, that because of

$$\mu_2[D_h(u_h(t)B_h] \le \frac{1}{2}\,(\mu_1[D_h(u_h(t)B_h] + \mu_\infty[D_h(u_h(t)B_h])$$

and

$$\mu_\infty[D_h(u_h(t)B_h] = 0$$

we get by Taylor expansion

$$\mu_2[D_h(u_h(t))B_h] \le \frac{1}{2}\max\Big(\sup_{u\in\mathbb{R}}|d'(u)|\cdot \sup_{\substack{x\in[0,1]\\ t\in[0,T]}} |u_{xx}(x,t)|\;, \quad \sup_{u\in\mathbb{R}}|d''(u)|\cdot \sup_{\substack{x\in[0,1]\\ t\in[0,T]}} |u_x(x,t)|^2\Big)$$

for sufficiently "smooth" functions d(u), u(x,t) with $d(u)\ge\alpha>0$).

Because condition (3.2) (i) plays an essential role in the proof of Theorem 3.1 we have to look for other norms to prove convergence for Example 3.1 . Until now we found no norm induced by an inner product, for which conditions (3.2) (i)-(iv) hold, the Chebyshev-norm $\|\cdot\|_\infty$ seems to be the only alternative. The corresponding logarithmic norm is given by (see DEKKER/VERWER [1984], p. 31)

$$\mu_\infty[T] = \max_{i=1(1)N} \Big(t_{ii}+\sum_{\substack{j=1\\ j\neq i}}^{N}|t_{ij}|\Big) \quad \text{for} \quad T = ((t_{ij})) .$$

We get

$$\mu_\infty[D_h(y)B_h] = 0 \quad ,$$

i.e. (3.2) (i), and (3.2) (ii) follows immediately, (iii) and (iv) have been shown to be fulfilled in Example 3.1 .

As a result of this choice of the norm we have to restrict our analysis to the rather simple (first order) modified linearly implicit Euler method (see (3.3) below). Bounds of rational matrix functions needed for higher order methods do not hold for the $\|\cdot\|_{\gamma}$-norm (cf. Remark 2.1). ■

The modified linearly implicit Euler method for quasi-linear problems (3.1) with semi-discretization (3.2) is given by

$$y_{n+1}:=(I-\tau A_h(y_n))^{-1}(y_n+\tau Q_h(t_n,y_n)+\tau r_h(t_{n+1},y_n)) \tag{3.3}$$

Remark 3.2.:

i) Because of assumption (3.2) (i) a constant $\tau_0>0$ (independent of h) exists so that $(I-\tau A_h(y_n))$ is regular for all $\tau\in(0,\tau_0]$ (see Remark 2.1 (i)).

ii) For time independent source terms Q_h the vector y_{n+1} in (3.3) is the result of one simplified Newton step (with initial guess y_n and matrix $A_h(y_n)$) applied to the system of nonlinear equations arising in the solution of (3.2) by implicit Euler method:

$$y_{n+1}:= y_n+\tau(A_h(y_{n+1})y_{n+1}+Q_h(t_{n+1},y_{n+1})+r_h(t_{n+1},y_{n+1})) .$$

In practice this method is often implemented similar to (3.3), i.e. only one step with a simplified Newton method is computed and iteration is broken off (not looking at convergence of the Newton method). ■

It holds

Theorem 3.1.: (Uniform convergence for quasi-linear problems).
The modified linearly implicit Euler method (3.3) applied to semi-discrete systems (3.2) has uniform order 1 of convergence.

Proof: We get

$$\begin{aligned}
&(I-\tau A_h(u_h(t_n)))\beta_{n+1} = \\
&\quad =(I-\tau A_h(u_h(t_n)))u_h(t_{n+1})-u_h(t_n)-\tau(Q_h(t_n,u_h(t_n))-r_h(t_n,u_h(t_{n+1})))\\
&\quad =u_h(t_{n+1})-(u_h(t_n)+\tau(A_h(u_h(t_n))u_h(t_n)+Q_h(t_n,u_h(t_n))+r_h(t_n,u_h(t_n))))-\\
&\qquad -\tau(A_h(u_h(t_n))u_h(t_{n+1})+r_h(t_{n+1},u_h(t_n))-\\
&\qquad\qquad -(A_h(u_h(t_n))u_h(t_n)+r_h(t_n,u_h(t_n))))
\end{aligned}$$

and by (3.2) (iv) and Taylor expansion it follows that

$$\|(I-\tau A_h(u_h(t_n)))\beta_{n+1}\| \le C_6(\tau^2 + \tau\alpha_{h,0}) \text{ for } \tau\in(0,\tau_0]. \tag{3.4}$$

For the global discretization error the following recursion holds:

$$(I-\tau A_h(y_n))(u_h(t_{n+1})-y_{n+1})=$$

$$=(I-\tau A_h(u_h(t_n)))\beta_{n+1}+u_h(t_n)+\tau(Q_h(t_n,u_h(t_n))+r_h(t_n,u_h(t_n)))+$$

$$+\tau(A_h(u_h(t_n))-A_h(y_n))u_h(t_{n+1})-(y_n+\tau(Q_h(t_n,y_n)+r_h(t_n,y_n)))$$

Because of (3.2) (i) and Remark 2.1 (i) $\|(I-\tau A_h(y_n))^{-1}\| \le \frac{1}{1-\tau\mu_0}$ holds for $\tau\mu_0<1$. Using (3.2) (ii),(iii) we get

$$\|\varepsilon_{n+1}\| \le \frac{1}{1-\tau\mu_0}((1+\tau(C+L))\|\varepsilon_n\|+\|(I-\tau A_h(u_h(t_n)))\beta_{n+1}\|)$$

By induction and (3.4)

$$\|\varepsilon_{n+1}\| = \mathcal{O}(\tau) + \mathcal{O}(\max_{t\in[0,T]}\|\alpha_h(t)\|) \qquad \text{for } \tau\in(0,\tau_0) \text{ is}$$

proved, i.e. uniform convergence of order 1. ■

3.2. The one dimensional quasi-linear heat flow equation

Theorem 3.1 shows uniform convergence of the modified linearly implicit Euler method for quasi-linear problems. In contrast to the semi-linear case where conditions (2.2) (i)-(iii) are met by a very wide variety of problems occuring in practice, conditions (3.2) (i)-(iv) are fulfilled for a wide but not at all complete class of quasi-linear problems.

From a practical point of view the quasi-linear heat flow equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\Big(d(u)\cdot\operatorname{grad}(u)\Big) + q(t,x,u) \quad (x\in G, t\in(0,T]) \tag{3.5}$$

$$u(x,0)=u_0(x) \quad (x\in G) \qquad u(x,t)=\varphi(x,t) \quad (x\in\partial G, t\in[0,T])$$

$$d(u) \ge \alpha > 0 \quad (u\in\mathbb{R})$$

occuring in many applications is of special interest.

Theorem 3.2 shows uniform convergence of order 1 (in the discrete $\mathcal{L}^2$-norm) for the modified linearly implicit Euler method applied to the one dimensional heat flow equation (3.5) discretized in space by central differences. Notice that this discretization does not fulfil condition (3.2) (iii) - so Theorem 3.1 is not applicable. On the other hand the proof of Theorem 3.2 is based essentially on special properties of the used space discretization.

Let $G=(0,1)\subset\mathbb{R}$, the source term q be Lipschitzian. Furthermore equation (3.5) is again supposed to possess a unique "smooth" solution $u(x,t)$.
On the equidistant grid $\{ x_i=i\cdot h \;,\; i=1(1)N \}$ $h=1/(N+1)$ the differential operator is discretized by central differences

$$\frac{d}{dx}\left(d(u)\frac{du}{dx}\right)\Bigg|_{x_i} \approx \frac{1}{h^2}\Big(d_{i-1}(t,y_h(t))(y_h^{(i-1)}(t)-y_h^{(i)}(t)) +$$
$$+ d_i(t,y_h(t))(y_h^{(i+1)}(t)-y_h^{(i)}(t))\Big)$$

$$\left(y_h^{(0)}(t)=u_h(0,t)=\varphi(0,t) \qquad y_h^{(N+1)}(t)=u_h(1,t)=\varphi(1,t)\right)$$

with

$$d_i(s,y) := d\left(\frac{y^{(i)}+y^{(i+1)}}{2}\right) \qquad (i=1(1)N-1,\; y\in\mathbb{R}^N)$$

and

$$d_0(s,y) := d\left(\frac{\varphi(0,s)+y^{(1)}}{2}\right) \qquad d_N(s,y) := d\left(\frac{\varphi(1,s)+y^{(N-1)}}{2}\right)$$

$$y_h:[0,T] \longrightarrow \mathbb{R}^N \quad \text{with} \quad y_h^{(i)}(t) \approx u(x_i,t) \qquad (i=1(1)N).$$

We get the semi-discretized system

$$y_h'(t)=A_h(t,y_h(t))y_h(t)+Q_h(t,y_h(t))+r_h(t,t,y_h(t)) \tag{3.6}$$

with

$$A_h(s,y) = \frac{1}{h^2}\begin{bmatrix} -d_0-d_1 & d_1 & 0 & \dots & 0 & 0 \\ d_1 & -d_1-d_2 & d_2 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & d_{N-1} & -d_{N-1}-d_N \end{bmatrix}$$

$$r_h(s,t,y)=\left(\frac{d_0(s,y)}{h^2}\varphi(0,t),0,0,\dots,0,\frac{d_N(s,y)}{h^2}\varphi(1,t)\right)^T \in \mathbb{R}^N$$

$$Q_h(t,y)=\left(q(t,x_0,y^{(1)}),q(t,x_1,y^{(2)}),\;\dots\;,q(t,x_N,y^{(N)})\right)^T.$$

In contrast to (3.2), the discretized differential operator in (3.6) depends on the time t. The modified linearly implicit Euler method is therefore given by

$$y_{n+1}:=(I-\tau A_h(t_n,y_n))^{-1}(y_n+\tau Q_h(t_n,y_n)+\tau r_h(t_n,t_{n+1},y_n)) \tag{3.7}$$

Remark 3.4.: The matrix $(I-\tau A_h(s,y))$ is regular for all $\tau>0$ because of $\mu_\infty[A_h(s,y)]=0$ $(y\in\mathbb{R}^N)$ (see Remark 2.1 (i)). ■

Theorem 3.2.: The linearly implicit Euler method (3.7) applied to the semi-discretization (3.6) of the one dimensional quasi-linear heat flow equation has uniform order 1 of convergence.

Proof:

i) Write (3.7) in the form

$$y_{n+1}=y_n+\tau A_h(t_n,y_n)y_{n+1}+\tau Q_h(t_n,y_n)+\tau r_h(t_n,t_{n+1},y_n) \tag{3.8}$$

(ii) We get an expression similar to (3.8) for the exact solution of (3.5) on the grid (see Definition 2.2):

$$\begin{aligned} u_h(t_{n+1})=u_h(t_n)+\tau Q_h(t_n,u_h(t_n))+\tau r_h(t_n,t_{n+1},u_h(t_n))+ \\ +\tau A_h(t_n,u_h(t_n))u_h(t_{n+1})+(I-\tau A_h(t_n,u_h(t_n)))\beta_{n+1} \end{aligned} \tag{3.9}$$

One easily verifies by Taylor expansion

$$\|(I-\tau A_h(t_n,u_h(t_n)))\beta_{n+1}\|_2 = \mathcal{O}(\tau^2)+\mathcal{O}(\tau h^2) \qquad (\alpha_{h,0}=\mathcal{O}(h^2)).$$

iii) It holds

$$A_h(s,y) = D_h^+ \, E_h(s,y) \, D_h^-$$

with the operators

$$D_h^- := \frac{1}{h}\begin{bmatrix} 1 & 0 & 0 & & 0 & 0 \\ -1 & 1 & 0 & & 0 & 0 \\ 0 & -1 & 1 & & 0 & 0 \\ & & \cdot & \ddots & & \\ 0 & 0 & 0 & & -1 & 1 \\ 0 & 0 & 0 & & 0 & -1 \end{bmatrix} \in \mathbb{R}^{(N+1)xN}, \quad D_h^+ := (-D_h^-)^T$$

approximating $\frac{\partial}{\partial x}$ on the grid and the diagonal matrix

$$E_h(s,y) := \text{diag}\,(d_0(s,y),\dots,d_N(s,y)) \in \mathbb{R}^{(N+1)x(N+1)}.$$

For the exact solution we get on the grid

$$\left.\frac{\partial u}{\partial x}\right|_{x_i} \approx D_h^- \, u_h(t) + \varphi_h(t) \qquad (i=0(1)N,\ x_0:=0)$$

with

$$\varphi_h(t) := \left(-\frac{1}{h}\varphi(0,t),0,0,\ \dots\ ,0,\ \frac{1}{h}\varphi(1,t)\right)^T \in \mathbb{R}^{N+1}.$$

iv) Subtracting (3.8) from (3.9) leads to the following recursion for the full global discretization error

$$\begin{aligned}\varepsilon_{n+1} = \varepsilon_n &+ \tau A_h(t_n,y_n)\varepsilon_{n+1} + \tau(A_h(t_n,u_h(t_n)) - A_h(t_n,y_n))u_h(t_{n+1}) + \\ &+ \tau(Q_h(t_n,u_h(t_n)) - Q_h(t_n,y_n)) + \\ &+ \tau(r_h(t_n,t_{n+1},u_h(t_n)) - r_h(t_n,t_{n+1},y_n)) + (I-\tau A_h(t_n,u_h(t_n)))\beta_{n+1}\end{aligned} \tag{3.10}$$

Now take on both sides of this equation the (discrete $\mathcal{L}^2$-) inner product with ε_{n+1}.

We have (cf. (iii))

$$\begin{aligned}\langle\varepsilon_{n+1},A_h(t_n,y_n)\varepsilon_{n+1}\rangle &= \langle\varepsilon_{n+1},D_h^+E_h(t_n,y_n)D_h^-\varepsilon_{n+1}\rangle \\ &= -\langle D_h^-\varepsilon_{n+1},E_h(t_n,y_n)D_h^-\varepsilon_{n+1}\rangle \le -\alpha\,\|D_h^-\varepsilon_{n+1}\|_2^2 \quad ,\end{aligned}$$

$$\begin{aligned}&\langle\varepsilon_{n+1},(A_h(t_n,u_h(t_n)) - A_h(t_n,y_n))u_h(t_{n+1}) + \\ &\qquad + r_h(t_n,t_{n+1},u_h(t_n)) - r_h(t_n,t_{n+1},y_n)\rangle = \\ &= \langle\varepsilon_{n+1},D_h^+(E_h(t_n,u_h(t_n)) - E_h(t_n,y_n))(D_h^-u_h(t_{n+1})+\varphi_h(t_{n+1}))\rangle \\ &= -\langle D_h^-\varepsilon_{n+1},(E_h(t_n,u_h(t_n)) - E_h(t_n,y_n))(D_h^-u_h(t_{n+1})+\varphi_h(t_{n+1}))\rangle \\ &\le \|D_h^-\varepsilon_{n+1}\|_2\cdot\|D_h^-u_h(t_{n+1})+\varphi_h(t_{n+1})\|_\infty\cdot\left\|\left(d_i(t_n,u_h(t_n)) - d_i(t_n,y_n)\right)_{i=0}^N\right\|_2 \\ &\le \gamma\cdot\|D_h^-\varepsilon_{n+1}\|_2\cdot\|\varepsilon_n\|_2\end{aligned}$$

with γ being the product of the Lipschitz constant L_d of $d(.)$ with $\|D_h^-u_h(t_{n+1})+\varphi_h(t_{n+1})\|_\infty$ (approximating $\|u_x(t)\|_\infty$ on the grid).

From (3.10) we now get

$$\begin{aligned}\|\varepsilon_{n+1}\|_2^2 \le (1+\tau L)\|\varepsilon_{n+1}\|_2\|\varepsilon_n\|_2 - \alpha\tau\|D_h^-\varepsilon_{n+1}\|_2^2 + \gamma\tau\|D_h^-\varepsilon_{n+1}\|_2\|\varepsilon_n\|_2 + \\ + \|\varepsilon_{n+1}\|_2\|(I-\tau A_h(t_n,u_h(t_n)))\beta_{n+1}\|_2\end{aligned}$$

and it follows

$$\begin{aligned}\|\varepsilon_{n+1}\|_2^2 \le (1+\tau L)\|\varepsilon_{n+1}\|_2\cdot\|\varepsilon_n\|_2 + \frac{\gamma^2}{4\alpha}\,\tau\,\|\varepsilon_n\|_2^2 + \\ + \|\varepsilon_{n+1}\|_2\|(I-\tau A_h(t_n,u_h(t_n)))\beta_{n+1}\|_2.\end{aligned}$$

By considering the two possibilities $\|\varepsilon_{n+1}\|\le\|\varepsilon_n\|$ and $\|\varepsilon_{n+1}\|\ge\|\varepsilon_n\|$ separately it is seen that

$$\|\varepsilon_{n+1}\|_2 \le (1+\delta\tau)\|\varepsilon_n\|_2 + \|(I-\tau A_h(t_n,u_h(t_n)))\beta_{n+1}\|_2$$

with $\delta = \frac{\gamma^2}{4\alpha}\cdot L$. Provided that $\varepsilon_0=0$ this yields the convergence result by induction. ■

Remark 3.5.: The proof of Theorem 3.2 was stimulated by LEES [1966] who proved uniform (second order) convergence of a linearly implicit two step method applied to the one dimensional heat flow equation (under the assumption $\tau=\mathcal{O}(h)$, however). ■

4. Numerical Examples

In Chapter 3 we proved uniform convergence of order 1 of the modified linearly implicit Euler method applied to quasi-linear problems. Now we give some numerical examples that illustrate the practical relevance of these results. Numerical examples of linearly implicit one step methods applied to semi-linear problems are given in STREHMEL/WEINER [1987].

All examples are one dimensional, $G = (0,1) \subset \mathbb{R}$. The differential operators have been discretized in space by finite differences on an equidistant grid with distance h. The exact solutions are known, they determine the initial and Dirichlet boundary conditions. We give the full global discretization error for different stepsizes in time (τ) and space (h). To get the numerical uniform order q_{num} of convergence a numerical solution $y_h(\tau_0)$ with a very small stepsize in time ($\tau_0=1/10000$) has been computed for each semi-discrete system thus giving the possibility to eliminate the space discretization error:

$$q_{num} := \log_2 \frac{\|y_h(\tau)-y_h(\tau_0)\|}{\|y_h(2\tau)-y_h(\tau_0)\|}$$

Example 4.1.: BURGER's equation

$$\frac{\partial u}{\partial t} = \nu\,\frac{\partial^2 u}{\partial x^2} - u\cdot\frac{\partial u}{\partial x} \qquad x\in(0,1) \qquad t\in(0,1]$$

This equation has been studied by several authors. For small values of ν steep gradients may exist in the exact solution u. We used the "large" value $\nu=0.1$ (to ensure a "smooth" exact solution) and the solution given by WHITHAM [1974], Chapter 4. This solution has also been used by VERWER [1986].

$$u(x,t) = 1 - 0.9\,\frac{r_1}{r_1+r_2+r_3} - 0.5\,\frac{r_2}{r_1+r_2+r_3}$$

where $r_1=\exp(-\frac{x-0.5}{20\nu} - \frac{99t}{400\nu})$, $r_2=\exp(-\frac{x-0.5}{4\nu} - \frac{3t}{16\nu})$ $r_3=\exp(-\frac{x-0.375}{2\nu})$.

To ensure condition (3.2) (i) to be fulfilled upwind-technique was used for semi-discretization. Thus space discretization is of order 1, only. As in example 3.1 it can be shown that (3.2.) (ii)-(iv) hold in the Chebyshev norm.

Theorem 3.1 yields $\quad \|\varepsilon_{n+1}\|_\infty = \mathcal{O}(\tau) + \mathcal{O}(h)$.

τ	1/10		1/20		1/40		1/80		1/160	
h	$\|\varepsilon_n\|_\infty$	.	$\|\varepsilon_n\|_\infty$	q_{num}	$\|\varepsilon_n\|_\infty$	q_{num}	$\|\varepsilon_n\|_\infty$	q_{num}	$\|\varepsilon_n\|_\infty$	q_{num}
1/10	2.8e-2	.	2.1e-2	0.94	1.7e-2	0.97	1.5e-2	0.99	1.4e-2	1.00
1/20	2.3e-2	.	1.5e-2	0.94	1.1e-2	0.97	9.2e-3	0.99	8.2e-3	1.00
1/40	2.0e-2	.	1.2e-2	0.94	7.9e-3	0.97	5.8e-3	0.99	4.7e-3	1.00
1/80	1.9e-2	.	1.1e-2	0.94	6.3e-3	0.97	4.1e-3	0.99	3.0e-3	1.00
1/160	1.8e-2	.	9.9e-3	0.94	5.5e-3	0.97	3.2e-3	0.99	2.1e-3	1.00

Example 4.2.:

$$\frac{\partial u}{\partial t} = d(u)\,\frac{\partial^2 u}{\partial x^2} + q(t,x,u) \qquad x\in(0,1) \qquad t\in(0,1)$$

exact solution: $\quad u(x,t) = (1 + \sin \pi x)e^{-t}$

source term: $\quad q(t,x,u) = \pi^2 u^2 + (0.1\pi^2 - 1 - \pi^2 e^{-t})u - 0.1\pi^2 e^{-t}$

$\quad d(u) = 0.1 + u$

semi-discretization in space by central differences similar to Example 3.1

Due to Theorem 3.1 it holds $\|\varepsilon_n\|_\infty = \mathcal{O}(\tau) + \mathcal{O}(h^2)$.

τ	1/10		1/20		1/40		1/80		1/160	
h	$\|\|\varepsilon_n\|\|_\infty$	.	$\|\|\varepsilon_n\|\|_\infty$	q_{num}	$\|\|\varepsilon_n\|\|_\infty$	q_{num}	$\|\|\varepsilon_n\|\|_\infty$	q_{num}	$\|\|\varepsilon_n\|\|_\infty$	q_{num}
1/10	3.1e-1	.	2.1e-1	0.68	1.3e-1	0.83	9.1e-2	0.92	6.6e-2	0.97
1/20	2.8e-1	.	1.8e-1	0.70	1.0e-1	0.84	5.9e-2	0.93	3.5e-2	0.97
1/40	2.8e-1	.	1.7e-1	0.70	9.6e-2	0.85	5.2e-2	0.93	2.8e-2	0.97
1/80	2.7e-1	.	1.7e-1	0.70	9.4e-2	0.85	5.0e-2	0.93	2.6e-2	0.98
1/160	2.7e-1	.	1.7e-1	0.70	9.4e-2	0.85	4.9e-2	0.93	2.5e-2	0.98

Example 4.3.:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(d(u)\frac{\partial u}{\partial x}\right) + q(t,x,u) \qquad x\in(0,1) \qquad t\in(0,1)$$

exact solution: $u(x,t) = (1 + \sin \pi x)e^{-t}$

source term: $q(t,x,u) = (0.1\pi^2-1)u-0.1\pi^2e^{-t}-$

$$-\pi^2(\cos 2\pi x - \sin \pi x)\cdot e^{-2t}$$

$d(u) = 0.1 + u$

semi-discretization in space by central differences similar to Chapter 3.2. Due to Theorem 3.2 it holds $||\varepsilon_n||_2 = \mathcal{O}(\tau) + \mathcal{O}(h^2)$.

τ	1/10		1/20		1/40		1/80		1/160	
h	$\|\|\varepsilon_n\|\|_2$	.	$\|\|\varepsilon_n\|\|_2$	q_{num}	$\|\|\varepsilon_n\|\|_2$	q_{num}	$\|\|\varepsilon_n\|\|_2$	q_{num}	$\|\|\varepsilon_n\|\|_2$	q_{num}
1/10	2.4e-2	.	1.4e-2	1.03	8.4e-3	1.02	5.9e-3	1.01	4.7e-3	1.02
1/20	2.1e-2	.	1.1e-2	1.03	5.8e-3	1.02	3.3e-3	1.01	2.1e-3	1.02
1/40	2.1e-2	.	1.0e-2	1.03	5.2e-3	1.02	2.7e-3	1.01	1.4e-3	1.02
1/80	2.0e-2	.	1.0e-2	1.03	5.0e-3	1.02	2.5e-3	1.01	1.3e-3	1.02
1/160	2.0e-2	.	1.0e-2	1.03	5.0e-3	1.02	2.5e-3	1.01	1.2e-3	1.02

The uniform convergence accordingly to Theorem 3.1 and Theorem 3.2 , respectively, is clear to be seen. For "large" values h the space discretization error dominates, especially for space discretizations of lower order (Example 4.1). The main point of our interest is the error caused by discretization in time. For "small" grid distances h (i.e. small space discretization errors) it is approximately equal to the full discretization error. As we would expect uniform order $q_{num} \approx 1$ is observable.

Thus the numerical experiments underline the theoretical results of Chapter 3. The modified linearly implicit Euler method is a suitable method to solve semi-discrete systems of quasi-linear parabolic problems.

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