Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science

H.J.A.M. Heijmans

From binary to grey-level morphology

Department of Analysis, Algebra and Geometry Report AM-R9009 May

The Centre for Mathematics and Computer Science is a research institute of the Stichting Mathematisch Centrum, which was founded on February 11, 1946, as a nonprofit institution aiming at the promotion of mathematics, computer science, and their applications. It is sponsored by the Dutch Government through the Netherlands Organization for the Advancement of Research (N.W.O.).

From Binary to Grey-level Morphology

H.J.A.M. Heijmans
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

This paper is entirely concerned with the construction of morphological operators on spaces of grey-level functions. The approach adopted here is to represent a function by a sequence of threshold sets and to transform this sequence into another one. This can be done by applying the same set operator at any level: the resulting operator is called a flat operator. But one may also apply a different set operator at each level. Or alternatively, one can transform the threshold sets recursively, using the outcome at a certain grey-level for the computation at the next level.

AMS 1980 Mathematics Subject Classification: 68U10, 06A23.

Keywords and phrases: mathematical morphology, image analysis, complete lattice, dilation, erosion, increasing operator, geodesic operator, reconstruction, annular opening, grey-level function, set operator, function operator, threshold set, flat operator, threshold decomposition of grey-level dilations and erosions, order convergence, iteration, idempotence, finite operator.

1. Introduction

Originally, mathematical morphology has been developped for binary images, the main ingredients being set theoretical concepts such as set union, set intersection, and set complement, and derived notions like Minkowski addition and subtraction. There are several ways to extend binary morphology to spaces of grey-level functions. A first approach is to represent a function by its *umbra*, the points on and below the graph of the function. This approach has been chosen by many researchers, see e.g., [2,4,20,21]. Some of the pitfalls of the umbra approach have been pointed out by Ronse in [15]. In this paper we shall follow a different route and represent a function by a (decreasing) family of sets, the so-called threshold sets. Any procedure which transforms a decreasing family of sets into another decreasing family gives rise to a morphological operator for grey-level functions. In this paper we shall describe some of such procedures. An important class

Report AM-R9009
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

of grey-level operators is obtained by applying the same set operator at any threshold level. An operator obtained in such a manner is called a *flat operator.*

To a large extent, mathematical morphology is concerned with morphological operators (or transformations), which transform an image into another one and which have certain properties. A main feature of morphological operators is that they somehow (in a sense which can be made precise) interfere with the underlying order structure of the lattice which represents the images under study. This observation has recently led to a generalization of mathematical morphology to complete lattices. This generalization was initiated by Serra (see [18]) and has been worked out further by Ronse and the author in [7,16]. In Sections 2 and 3 we will briefly recall some of the notions which we need in the sequel. Section 2 is devoted to complete lattices and Section 3 to binary morphology. In Section 4 we give a precise description of the relation between a function and its threshold sets. Grey-level dilations and erosions form the topic of Section 5; among others we present a complete characterization of grey-level dilations and erosions which are invariant under spatial translations. Here we also discuss annular openings for grey-level functions. Section 6 is entirely devoted to flat operators. In a joint paper with Serra [8] we have studied the problem under what conditions iteration of a morphological operator yields an operator which is idempotent. In Section 7 we recall some of the results obtained there and specialize them to the space of grey-level function. Finally, in Section 8 we indicate by means of two or three examples some other ways to build grey-level operators

Throughout this paper we assume that the underlying grey-level set is finite. Thus we avoid some of the technical problems which arise if the grey-level set is continuous: see [6]. On the other hand, it complicates the definition of grey-level dilations and erosions with structuring elements which are not flat. An account of these problems is given in Section 5.

2. Complete lattices

In this section we briefly recall some basic notions concerning complete lattices. For a comprehensive exposition we refer to the excellent monograph of Birkhoff [1].

A partially ordered set $\mathcal{L}$ is called a *complete lattice* if any subset $\mathcal{H}$ of $\mathcal{L}$ has a least upper bound, called the *supremum* of $\mathcal{H}$, and a greatest lower bound, called the *infimum*. The supremum and infimum are denoted by $\bigvee \mathcal{H}$ and $\bigwedge \mathcal{H}$ respectively. The supremum and infimum of the entire lattice $\mathcal{L}$ are called the greatest and least element of $\mathcal{L}$ and denoted by $\mathcal{I}$ and $\mathcal{O}$ respectively. The supremum and infimum of the empty set are defined to be $\mathcal{O}$ and $\mathcal{I}$ respectively.

Example 2.1. *Complete lattices*

(a) The set $\overline{\mathbb{R}}$ consisting of the real numbers and $\pm\infty$ becomes a complete lattice under the usual order. In this case $\mathcal{O} = -\infty$ and $\mathcal{I} = \infty$.

(b) Let E be an arbitrary non-empty set. The power set $\mathcal{P}(E)$ consisting of all subsets of E ordered by set inclusion is a complete lattice, set union being the supremum and set intersection the infimum. Here $\mathcal{I} = E$ and $\mathcal{O} = \emptyset$. This lattice can be used to represent the binary images. E is sometimes called the *support space*.

(c) In this paper, grey-level images will be represented as functions mapping an underlying space E (e.g. $\mathbb{R}^d$ or $\mathbb{Z}^d$) into some set of grey-levels $\mathcal{G}$. For the moment we assume only that $\mathcal{G}$ is a complete lattice. By $\mathrm{Fun}(E;\mathcal{G})$ we denote the space of all functions mapping E into $\mathcal{G}$. Under the pointwise ordering, $F \leq G$ if $F(x) \leq G(x)$ for every $x \in E$, this set becomes a complete lattice, and the supremum (infimum) of a collection of functions is obtained by taking the pointwise supremum (infimum) in the lattice $\mathcal{G}$. In this paper we assume, unless stated otherwise explicitly, that $\mathcal{G}$ is an equal-spaced finite subset of $\mathbb{R}$, say $\mathcal{G} = \{0, 1, \ldots, N\}$.

The lattice $\mathcal{L}$ is called *distributive* if for all $X, Y, Z \in \mathcal{L}$ we have

$$X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z)$$
$$X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z).$$

If, in addition, for any $X \in \mathcal{L}$ there exists an element X^* such that $X \vee X^* = \mathcal{I}$ and $X \wedge X^* = \mathcal{O}$, then $\mathcal{L}$ is called a Boolean lattice. The element X^* is called the *dual* or *complement* of X. The lattice $\mathcal{P}(E)$ of Example 2.1(b) is Boolean; here the complement of an element $X \in \mathcal{P}(E)$ coincides with the usual set complement. The complete lattice $\mathrm{Fun}(E;\mathcal{G})$ is Boolean if and only if the underlying lattice $\mathcal{G}$ is Boolean. It is distributive if and only if $\mathcal{G}$ is distributive, which is e.g. the case if $\mathcal{G} = \{0, 1, \ldots, N\}$.

As we already noticed in the introduction, the design and investigation of morphological operators is one of the major tasks of the "mathematical morphologist". In fact, the present paper will be entirely devoted to this task. Therefore we now consider operators between two complete lattices $\mathcal{L}_1, \mathcal{L}_2$. Here we shall mainly restrict ourselves to increasing operators: an operator $\psi : \mathcal{L}_1 \to \mathcal{L}_2$ is said to be *increasing* if $X \leq Y$ implies that $\psi(X) \leq \psi(Y)$ for all $X, Y \in \mathcal{L}_1$. Throughout this paper we use the adjective "increasing (decreasing)" in the sense of "nondecreasing (nonincreasing)". Obviously, the set of all increasing operators between $\mathcal{L}_1$ and $\mathcal{L}_2$ is a complete lattice under the pointwise ordering: $\phi \leq \psi$ if $\phi(X) \leq \psi(X)$ for every $X \in \mathcal{L}_1$. The operator ψ is called a *dilation* if ψ acts distributively over suprema, that is, $\psi(\bigvee_{i \in I} X_i) = \bigvee_{i \in I} \psi(X_i)$, for every collection $X_i \in \mathcal{L}_1$, $i \in I$. The dual concept is called an *erosion*: the operator ψ is an erosion if $\psi(\bigwedge_{i \in I} X_i) = \bigwedge_{i \in I} \psi(X_i)$ for every collection $X_i \in \mathcal{L}_1$, $i \in I$. In this paper dilations will be denoted by δ or Δ and erosions with ε or $\mathcal{E}$ (and with d and e respectively if the underlying lattice has the interpretation of a grey-level set: see Example 2.2 below). Dilation and erosion always occur in pairs (called *adjunctions*) in the following sense: to every erosion $\varepsilon : \mathcal{L}_1 \to \mathcal{L}_2$ there corresponds a unique dilation $\delta : \mathcal{L}_2 \to \mathcal{L}_1$, called the adjoint of ε. The converse also holds. The pair (ε, δ) is called an adjunction between $\mathcal{L}_1$ and $\mathcal{L}_2$. If ε is identically $\mathcal{I}$ and δ is identically $\mathcal{O}$, then (ε, δ) is called the *trivial adjunction*.

The following properties hold for any adjunction (ε, δ) between $\mathcal{L}_1$ and $\mathcal{L}_2$.

- δ and ε are increasing operators

- $\delta(\mathcal{O}) = \mathcal{O}$ and $\varepsilon(\mathcal{I}) = \mathcal{I}$
- $\delta(X_2) \leq X_1$ if and only if $X_2 \leq \varepsilon(X_1)$, for $X_1 \in \mathcal{L}_1$ and $X_2 \in \mathcal{L}_2$
- $\delta(X_2) = \bigwedge\{X_1 \in \mathcal{L}_1 \mid X_2 \leq \varepsilon(X_1)\}$, $X_2 \in \mathcal{L}_2$
- $\varepsilon(X_1) = \bigvee\{X_2 \in \mathcal{L}_2 \mid \delta(X_2) \leq X_1\}$, $X_1 \in \mathcal{L}_1$
- $\delta\varepsilon\delta = \delta$
- $\varepsilon\delta\varepsilon = \varepsilon$.

For some further results concerning adjunctions we refer to [3,7,18]. We now present an example which we will use in Section 5.

Example 2.2. *Adjunctions on* $\{0, 1, \ldots, N\}$
Consider the complete lattice $\{0, 1, \ldots, N\}$. A mapping $d : \{0, 1, \ldots, N\} \to \{0, 1, \ldots, N\}$ is a dilation if and only if $d(0) = 0$ and d is increasing. The corresponding erosion e is given by $e(n) = \max\{m \mid d(m) \leq n\}$. Two examples are depicted in Figure 1.

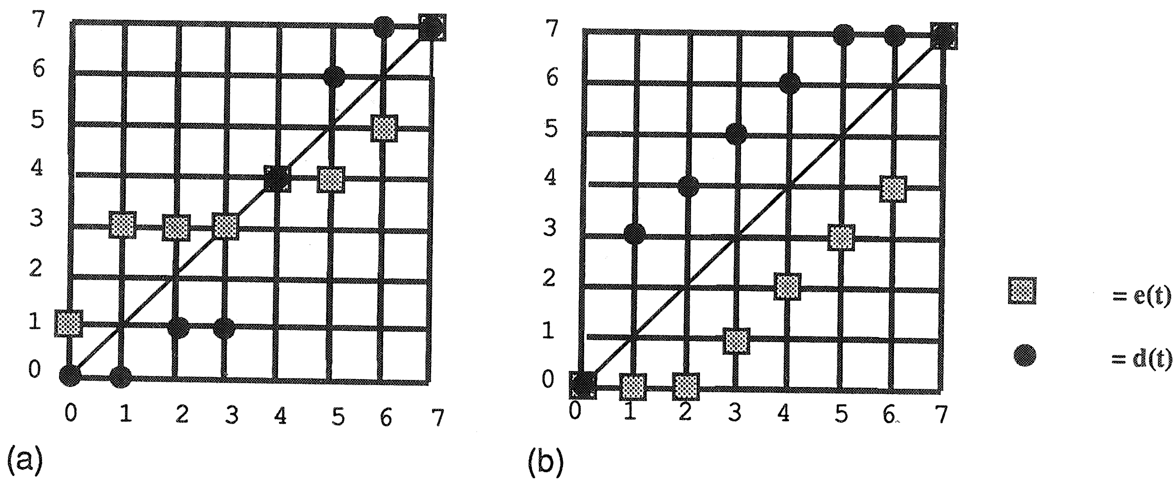


FIGURE 1. *Two adjunctions* (e, d) *on* $\mathcal{G} = \{0, 1, \ldots, 7\}$*: see Example 2.2.*

The e and d in Figure 1(b) are prototype examples of a truncated grey-level translation where 0 and N are "absorbing boundaries" for d and e respectively. For further explanation concerning this example we refer the reader to Section 5 where such grey-level adjunctions play an important role.

Of special interest to us is the case $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$. We denote the identity operator, that is the operator which maps any element of $\mathcal{L}$ onto itself, by **id**. The operator $\psi : \mathcal{L} \to \mathcal{L}$ is said to be (anti–) extensive if $\psi \geq \mathbf{id}$ ($\psi \leq \mathbf{id}$). The operator ψ is called idempotent if $\psi^2 = \psi$. If ψ is increasing and idempotent then ψ is called a *filter*. An (anti–) extensive filter is called a *closing* (*opening*). From the last two properties of adjunctions mentioned above it follows immediately that $\varepsilon\delta$ is a closing on $\mathcal{L}_2$ and that $\delta\varepsilon$ is an opening on $\mathcal{L}_1$ if (ε, δ) is an adjunction between $\mathcal{L}_1$ and $\mathcal{L}_2$.

We conclude this section with a more or less trivial observation, but which nevertheless turns out to be very useful. If $\mathcal{L}$ is a complete lattice with partial order $\leq$ then the set $\mathcal{L}$ with the opposite order $\geq$ is called the *dual* (or *opposite*) lattice of $\mathcal{L}$ and is denoted by $\mathcal{L}'$. So to every definition, property, and statement concerning complete lattices or operators between complete lattices there does exist a dual counterpart obtained by considering the dual lattice(s). In this way one can for instance interrelate dilations and erosions, closings and openings, extensive and anti-extensive operators, etc. In this paper we will repeatedly use this duality principle without mentioning this explicitly.

3. Binary morphology

Originally, mathematical morphology was developed for binary images and extended to grey-level images afterwards. In this paper we will identify a binary image with a subset of the underlying support space E: in the continuous case we have $E = \mathbb{R}^d$ whereas in the discrete case $E = \mathbb{Z}^d$. Now the space of binary images can be identified with the power set $\mathcal{P}(E)$ which is a complete lattice: cfr. Example 2.1(b). In this paper we shall call any operator on $\mathcal{P}(E)$ a *set operator*. If X is a subset of E and h an arbitrary vector in E then we define $X_h := \{x + h \mid x \in X\}$. The translation operator $X \to X_h$ defines an automorphism on $\mathcal{P}(E)$ and can in fact be considered as one of the most fundamental morphological operators. In many applications one is only interested in operators ψ on $\mathcal{P}(E)$ which are translation-invariant, that is,

$$\psi(X_h) = \psi(X)_h.$$

Two well-known translation-invariant increasing operators are the Minkowski addition $\oplus$ and subtraction $\ominus$ with some fixed element $A \subseteq E$, called the *structuring element*, which are respectively given by

$$X \oplus A = \bigcup_{a \in A} X_a$$

$$X \ominus A = \bigcap_{a \in A} X_{-a}.$$

In mathematical morphology, the operator $X \to X \oplus A$ is called *dilation* by A and $X \to X \ominus A$ is called *erosion* by A. This nomenclature is in accordance with the terminology introduced in the previous section. Note in particular that both operators are increasing. For any operator ψ on the Boolean lattice $\mathcal{P}(E)$ we can define its *dual* ψ^* given by

$$\psi^*(X) := (\psi(X^*))^*.$$

If ψ is increasing then so is ψ^*. By taking the dual a dilation becomes an erosion, etc. A straightforward, yet rather important result due to Matheron [14] says that any increasing translation-invariant operator on $\mathcal{P}(E)$ can be decomposed as a union of erosions, or dually, as an intersection of dilations. We refer to [7] for a generalization of this result to complete lattices. In Section 5 we shall formulate a similar representation theorem for function operators.

Composition of dilation and erosion yields the opening and closing, depending on the order in which they are applied. The closing by A is given by

$$X^A = (X \oplus A) \ominus A,$$

and the opening by A is

$$X_A = (X \ominus A) \oplus A.$$

This section is not to be considered as an overview of binary morphology. Rather its principal goal is to recall some of the notions which have been treated elsewhere (mainly in [17,18]) in much greater detail, and which we will use in the forthcoming sections. Two of these notions will be discussed below. The first is the annular opening, a particular kind of opening on binary images introduced by Serra in [18, Section 5.4], and generalized to grey-level functions by Ronse and the author in [16]. Secondly, we will present a short (and incomplete) overview of the so-called *geodesic operators* introduced by Lantuejoul and Beucher in [10].

Let A be a symmetric structuring element (that is, $h \in A$ if and only if $-h \in A$) which does not contain the origin. Then the operator $\alpha(X) = X \cap (X \oplus A)$ defines an opening. The proof of this result, which is rather straightforward, can be found in [18, Section 5.4]. The action of this opening differs from the openings which one usually finds in mathematical morphology. It is not so much the size and shape of a particle which decides if it is removed by application of α but rather the presence or absence of particles in its environment. Usually A has an annular shape whence the name 'annular opening'. In [16] the annular opening has been extende to grey-level functions: there the resulting operator was invariant under grey-level translations. In Section 5 we shall discuss a further generalization dropping the grey-level translation-invariance: in fact, we cannot allow grey-level translations because our grey-level set is assumed to be finite and therefore not closed under translation. For completeness we note that for any symmetric structuring element A, the mapping $X \to X \cup (X \ominus A)$ defines a closing, which we might call the annular closing.

An important class of operators are the so-called *geodesic operators*. These operators are sometimes used to measure certain topological or metrical features of an object. Furthermore they can be applied to find markers required for the segmentation of an image: we refer to [10] for an overview of geodesic methods in mathematical morphology. Here we only discuss some basic geodesic operators. Essentially, one speaks of a geodesic operator if the underlying support space is not the entire space $\mathbb{R}^d$ or $\mathbb{Z}^d$ but only some (usually bounded) subset of it: we call this subset the *mask* and denote it by M.

In what follows we shall restrict to the discrete space $\mathbb{Z}^d$; for the continuous space $\mathbb{R}^d$ the definitons are analogous. Suppose that we have defined some adjacency relation on $\mathbb{Z}^d$ (e.g. 4– or 8–adjacency on $\mathbb{Z}^2$), and that d is a digital distance function on this space. The latter means that for every $x, y, z \in \mathbb{Z}^d$:

$$d(x,x) = 0$$
$$d(x,y) = d(y,x)$$
$$d(x,z) \leq d(x,y) + d(y,z).$$

A sequence $x = x_0, x_1, x_2, \ldots, x_n = y$ in $\mathbb{Z}^d$ is called a path between x and y if x_i and x_{i+1} are adjacent for $i = 0, 1, \ldots, n-1$. The lenght of the path is defined to be $\sum_{i=0}^{n-1} d(x_i, x_{i+1})$. Now let M be a mask in $\mathbb{Z}^d$. The *geodesic distance* $d_M(x,y)$ between two points $x, y \in M$ is defined as the minimum possible length of a path in M connecting x and y. If x and y lie in two distinct connected components of M then their geodesic distance is ∞.

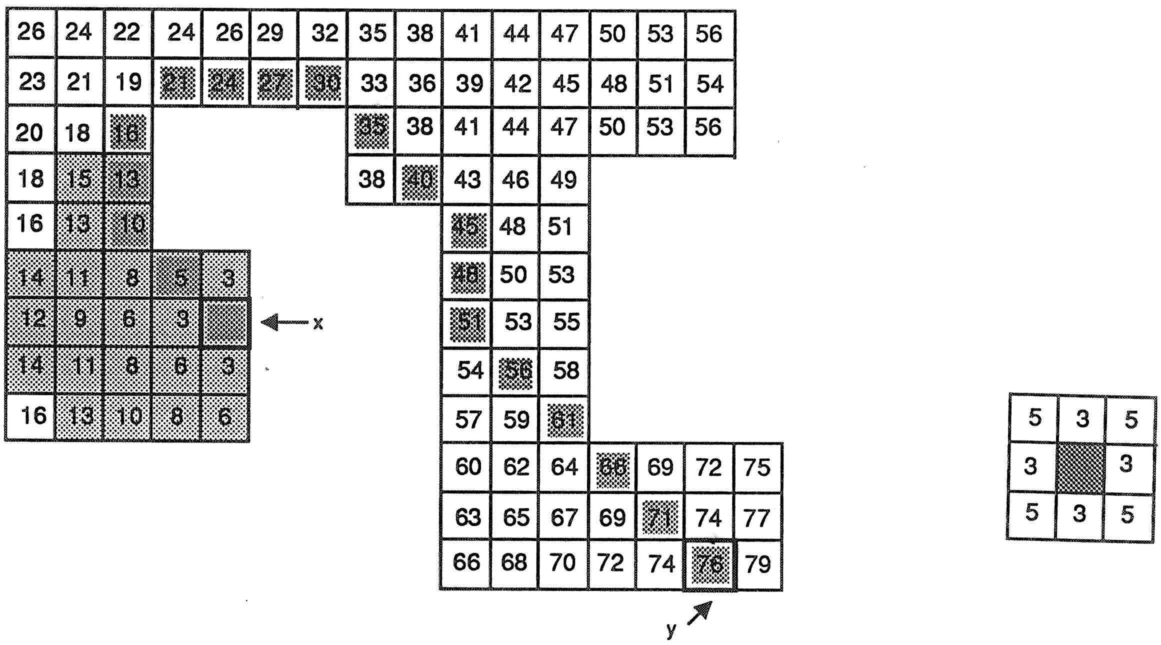


FIGURE 2. *Geodesic distance in $\mathbb{Z}^2$in the case of 8-adjacency. The distance mask used is depicted at the right-hand-side. The geodesic distance between x and y is 76. One of the geodesic shortest paths between x and y is drawn. The region in shaded grey is the geodesic ball with center x and radius 15.*

The *geodesic ball* $B_M(x,r)$ with radius $r \geq 0$ and center $x \in M$ is defined as

$$B_M(x,r) = \{y \in M \mid d_M(x,y) \leq r\},$$

see Figure 2. The *geodesic dilation* with radius r of a subset $X \subseteq M$ is defined as

$$\delta^r(X \mid M) = \bigcup_{x \in X} B_M(x,r) = \{y \in M \mid d_M(x,y) \le r \text{ for some } x \in M\}, \tag{3.1}$$

and similarly, the *geodesic erosion* with radius r is given by

$$\varepsilon^r(X \mid M) = \{x \in M \mid B_M(x,r) \subseteq X\}. \tag{3.2}$$

Note that the geodesic dilation and erosion considered as operators from $\mathcal{P}(M)$ into itself form an adjunction.

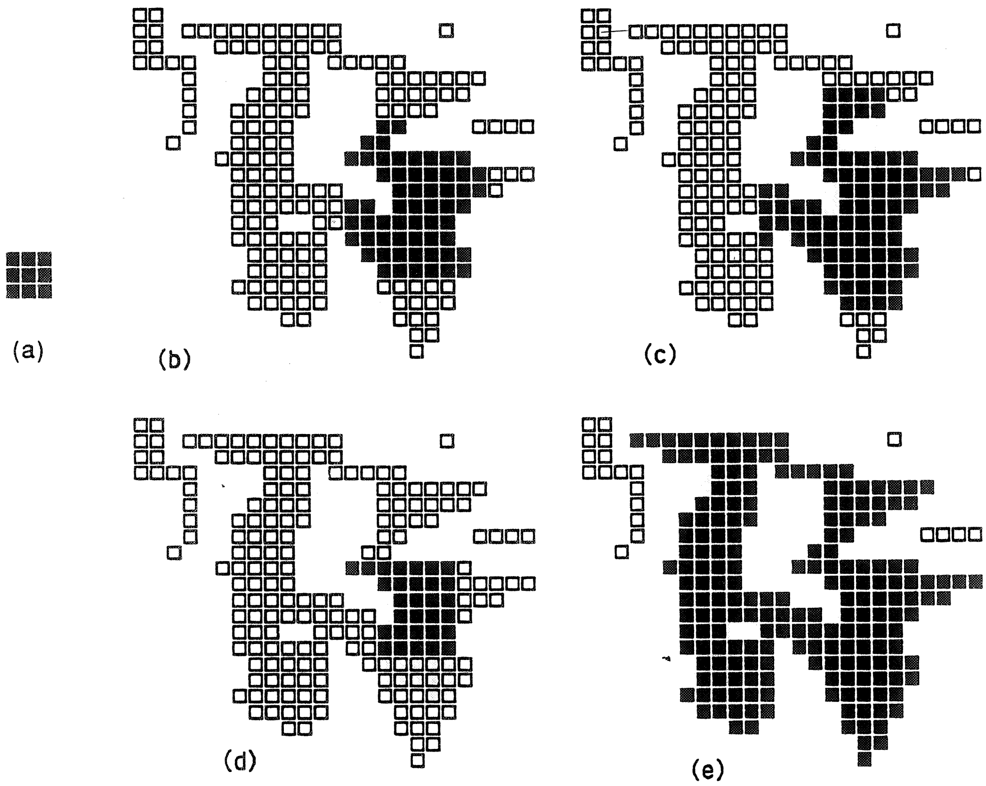


FIGURE 3. *Geodesic operators. (a) The structuring element A. (b) The mask M (white pixels) and the set X (black pixels). (c) The geodesic dilation $\delta^2(X \mid M)$; here $\delta^1(X \mid M) = (X \oplus A) \cap M$. (d) The geodesic erosion $\varepsilon^2(X \mid M)$. (e) The reconstruction $\rho(X \mid M)$.*

Sometimes both adjacency and distance are characterized by the same symmetric structuring element A in the following sense: x and y are adjacent if and only if $x \in A_y$, or equivalently,

$y \in A_x$, and in that case $d(x,y) = 1$. Now, for any pair x, y $(x \neq y)$ we have $d(x,y) = \inf\{n \geq 0 \mid x - y \in A \oplus A \oplus \ldots \oplus A \text{ (n terms)}\}$ and $d(x,x) = 0$. In that case we have

$$\delta^1(X \mid M) = (X \oplus A) \cap M,$$

and δ^r is obtained by r-fold application of δ^1, i.e., $\delta^r = \delta^1 \circ \ldots \circ \delta^1$ (r terms). The corresponding erosion is given by

$$\varepsilon^1(X \mid M) = ((X \cup M^c) \ominus A) \cap M,$$

and $\varepsilon^r = \varepsilon^1 \circ \ldots \circ \varepsilon^1$ (r terms). In Figure 3(c),(d) we have depicted the geodesic dilation and erosion of the set X relative to the mask M both depiceted in (b), and where A is the 3×3-square depicted in (a). Note that $\delta^r(X \mid M)$ is increasing with respect to r. We define the *reconstruction operator* ρ by

$$\rho(X \mid M) = \bigcup_{r \geq 0} \delta^r(X \mid M). \tag{3.3}$$

An example has been depicted in Figure 3(e).

4. From sets to functions

As we already indicated, we shall only consider finite grey-level sets throughout this paper. So let $\mathcal{G} = \{0, 1, \ldots, N\}$ and denote by $\mathrm{Fun}(E)$ the complete lattice of all functions mapping E into $\{0, 1, \ldots, N\}$: see also Example 2.1(c). The range of F, written $\mathrm{Ran}(F)$, is defined to be the set $F(x)$ where x takes values in E. Though $\mathrm{Fun}(R)$ does not define a Boolean lattice, we can still define a mapping $F \to F^*$ which has some of the properties of the complement in a Boolean lattice, namely:

$$F^*(x) = N - F(x), \quad x \in E. \tag{4.1}$$

The operator $F \to F^*$ is called a *dual automorphism*: it is a bijection and maps suprema onto infima and vice versa. A function which only takes the values 0 and t for some fixed $t = 0, 1, \ldots, N$ is called a *flat function*.

For every $t = 0, 1, \ldots, N$ we define the *threshold set* $\mathcal{X}_t(F)$ of F as

$$\mathcal{X}_t(F) = \{x \in E \mid F(x) \geq t\}. \tag{4.2}$$

Obviously, $\mathcal{X}_0(F) = E$. The collection of threshold sets of a function F is decreasing with respect to t: this property is called the *stacking property* by some authors [13,22]. To every decreasing family of sets $\{X_t\}_{t=0}^{N}$ with $X_0 = E$ there corresponds a unique function F with $\mathcal{X}_t(F) = X_t$. This function is given by

$$F(x) = \max\{t = 0, 1, \ldots, N \mid x \in X_t\}.$$

We sometimes identify a set X with its characteristic function, i.e., the function which takes the value 1 on X and 0 outside X. Using this convention we can write

$$F = \sum_{t=1}^{N} X_t.$$

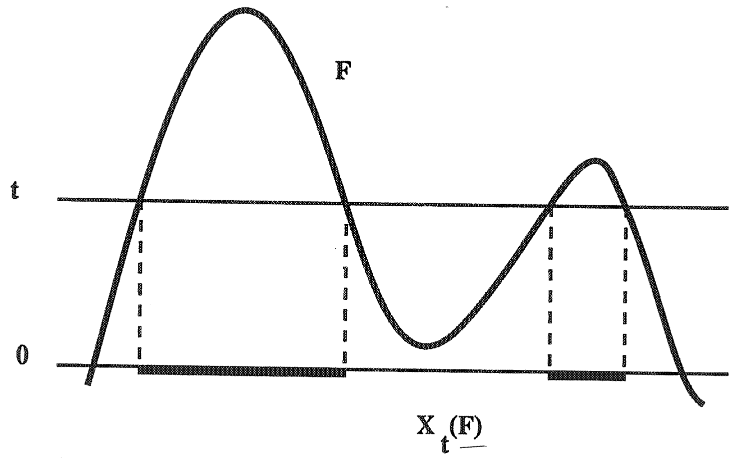


FIGURE 4. *Threshold set*

Let $F_i \in \mathrm{Fun}(E)$ for every i in the index set I. Then the following identities hold.

$$\mathcal{X}_t(\bigvee_{i\in I} F_i) = \bigcup_{i\in I} \mathcal{X}_t(F_i) \tag{4.3}$$

$$\mathcal{X}_t(\bigwedge_{i\in I} F_i) = \bigcap_{i\in I} \mathcal{X}_t(F_i). \tag{4.4}$$

In particular these identities give us that the operator $\mathcal{X}_t : \mathrm{Fun}(E) \to \mathcal{P}(E)$ is both an erosion and a dilation. The operator $\mathcal{F}_t : \mathcal{P}(E) \to \mathrm{Fun}(E)$ given by

$$\mathcal{F}_t(X)(x) = \begin{cases} t, & x \in X \\ 0, & x \notin X, \end{cases} \tag{4.5}$$

is the adjoint of $\mathcal{X}_t$: the pair $(\mathcal{X}_t, \mathcal{F}_t)$ forms an adjunction between $\mathrm{Fun}(E)$ and $\mathcal{P}(E)$. Furthermore it is obvious that $\mathcal{X}_t$ is a dilation. Finally one easily checks that $\mathcal{F}_t$ is distributive over nonempty intersections.

In this paper we are mainly concerned with operators on the complete lattice of grey-level functions. We call such operators *function operators*. A set operator can be considered as a function operator with grey-level set $\{0, 1\}$. Throughout this paper we shall as much as possible denote set operators by lower case Greek symbols such as ψ, ϕ, and function operators by upper case Greek symbols such as Ψ, Φ. If Ψ is a function operator then we define its dual Ψ^* by

$$\Psi^*(F) = (\Psi(F^*))^*.$$

Finally we define the horizontal (or spatial) translation on $\mathrm{Fun}(E)$ by

$$F_h(x) = F(x - h), \quad x \in E.$$

The function operator Ψ is called an *H-operator* if Ψ is invariant under horizontal translations, i.e.,

$$\Psi(F_h) = (\Psi(F))_h,$$

for $F \in \text{Fun}(E)$ and $h \in E$.

5. Grey-level dilations and erosions

In Example 2.2 we have seen that d is a dilation on the complete lattice $\{0, 1, \ldots, N\}$ if d is increasing, and $d(0) = 0$. The dual erosion is then given by the expression $e(n) = \max\{m \mid d(m) \leq n\}$. Such mappings become important if one wants to give a complete characterization of the adjunctions on Fun(E) which are invariant under horizontal translations, the so-called H-adjunctions. The following result was first proved in [7].

Proposition 5.1. *The pair* $(\mathcal{E}, \Delta)$ *is an H-adjunction on* Fun(E) *if and only if for every* $h \in E$ *there exists an adjunction* (e_h, d_h) *on* $\{0, 1, \ldots, N\}$ *such that*

$$\Delta(F)(x) = \bigvee_{h \in E} d_h(F(x-h)) \tag{5.1}$$

$$\mathcal{E}(F)(x) = \bigwedge_{h \in E} e_h(F(x+h)), \tag{5.2}$$

for every $F \in$ Fun(E).

Proof. The reader can easily verify that $(\mathcal{E}, \Delta)$ given by (5.1)-(5.2) forms indeed an adjunction. Here we only show that every dilation is of the form (5.1). Thereto we define for $x \in E$ and $t = 0, 1, \ldots, N$ the function $f_{x,t}$ which takes the value t at the point x and 0 elsewhere. We define $d_h(t) = \Delta(f_{0,t})(h)$. It is clear that $d_h(0) = 0$ (since $\Delta(\mathcal{O}) = \mathcal{O}$) and that d_h is increasing in t. So d_h is a dilation on $\{0, 1, \ldots, N\}$. Now we use that every function F can be written as

$$F = \bigvee_{y \in E} f_{y,F(y)}.$$

Then, by the translation-invariance and the fact that Δ is distributive over suprema we get

$$\begin{aligned}\Delta(F)(x) &= \Delta(\bigvee_{y \in E} f_{y,F(y)})(x) = \bigvee_{y \in E} \Delta(f_{0,F(y)})(x-y) \\ &= \bigvee_{y \in E} d_{x-y}(F(y)) = \bigvee_{h \in E} d_h(F(x-h)),\end{aligned}$$

where we have substituted $h = x - y$. This completes the proof.

∎

If the grey-level set were not finite, but the entire set $\overline{\mathbb{Z}}$, the set of all integers including $\pm\infty$, the same result would still hold, and then we could choose $d_h(t) = t + G(h)$ and $e_h(t) = t - G(h)$, with G some arbitrary $\overline{\mathbb{Z}}$-valued function. In that case (5.1) and (5.2) would reduce to the following well-known expressions:

$$\Delta(F)(x) = \bigvee_{h\in E} [F(x-h) + G(h)]$$

$$\mathcal{E}(F)(x) = \bigwedge_{h\in E} [F(x+h) - G(h)].$$

These operators are also invariant under vertical translations, that is, $\Delta(F+v) = \Delta(F) + v$, for every $F \in \mathrm{Fun}(E)$ and $v \in \mathbb{Z}$, and the same property holds for $\mathcal{E}$. Here $(F+v)(x) = F(x) + v$.

It is tempting to extend the grey-level dilation and erosion given above to the case where the grey-level set is finite, just by truncating those values which lie outside $\{0, 1, \ldots, N\}$. As we already noted in [6] such an approach does lead to wrong results: cfr also Section 4 of [15]. We shall illustrate this by means of an example. We define for $s, t \in \overline{\mathbb{Z}}$,

$$\lfloor s+t \rfloor := \begin{cases} 0, & \text{if } s+t < 0 \\ s+t, & \text{if } s+t \in \{0,1,\ldots,N\} \\ N, & \text{if } s+t > N, \end{cases}$$

and let $\lfloor s-t \rfloor := \lfloor s + (-t) \rfloor$. The operator $\tilde{\Delta}_G$ given by

$$\tilde{\Delta}_G(F)(x) := \bigvee_{h\in\mathrm{dom}(G)} \lfloor F(x-h) + G(h) \rfloor \tag{5.3}$$

defines a dilation in the sense that it distributes over suprema ($\tilde{\Delta}_F(O) \neq O$, however). Similarly we define the operator $\tilde{\mathcal{E}}_G$ by

$$\tilde{\mathcal{E}}_G(F)(x) := \bigwedge_{h\in\mathrm{dom}(G)} \lfloor F(x+h) - G(h) \rfloor. \tag{5.4}$$

Obviously, $\tilde{\mathcal{E}}_G$ is distributive over infima. But as the example depicted in Figure 5 shows, $\tilde{\mathcal{E}}_G$ is not the adjoint of $\tilde{\Delta}_G$.

Note that in (5.3) and (5.4) supremum and infimum is not taken over all $h \in E$ but only over some subset $\mathrm{dom}(G)$, the so-called *domain* of G. In fact both case are equivalent: we can always extend G to the whole set E by putting $G(h) = -\infty$ for h outside $\mathrm{dom}(G)$.

The difficulties noted above can be "solved" by assigning a different status to the minimum and maximum grey-level. They are in fact to be treated as "absorbing barriers": if some function has the value 0 at certain points, then a grey-level translation in positive direction cannot change this value. Similarly a grey-level translation in negative direction cannot change the value of a function at points where it takes the value N. To formalize these ideas we introduce the operations $\dot{+}$ and $\dot{-}$ on $\{0, 1, \ldots, N\}$. For $v \in \mathbb{Z}$ we define the operation $t \to t \dot{+} v$ on $\{0, 1, \ldots, N\}$ as follows:

$$\begin{cases} 0 \dot{+} v = 0, & \text{for every } v \\ t \dot{+} v = 0, & \text{if } t > 0 \text{ and } t + v \leq 0 \\ t \dot{+} v = t + v, & \text{if } t > 0 \text{ and } 0 \leq t + v \leq N \\ t \dot{+} v = N, & \text{if } t > 0 \text{ and } t + v > N. \end{cases}$$

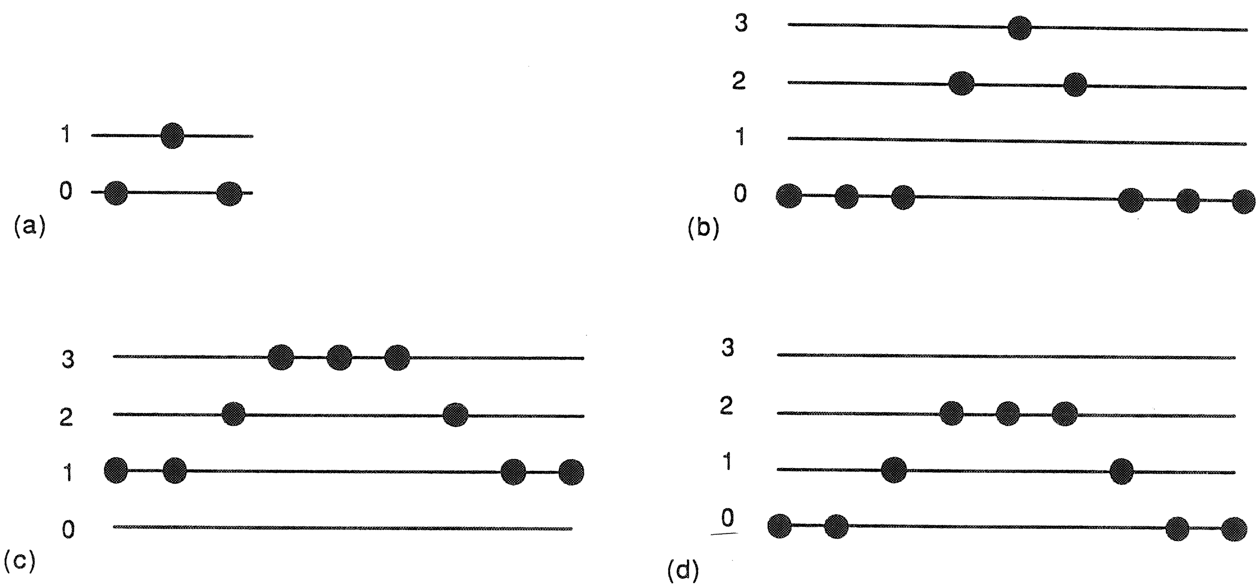


FIGURE 5. *(a) The structuring function G, (b) the function F, and its transforms (c) $\tilde{\Delta}_G(F)$ and (d) $\tilde{\mathcal{E}}_G\tilde{\Delta}_G(F)$. Here $\tilde{\Delta}_G$ and $\tilde{\mathcal{E}}_G$ are given by (5.3) and (5.4) respectively. Since $\tilde{\mathcal{E}}_G\tilde{\Delta}_G(F) \not\geq F$, the pair $(\tilde{\mathcal{E}}_G, \tilde{\Delta}_G)$ does not form an adjunction*

Similarly we define the operation $t \to t \dot{-} v$ on $\{0, 1, \ldots, N\}$ by

$$\begin{cases} t \dot{-} v = 0, & \text{if } t < N \text{ and } t - v \leq 0 \\ t \dot{-} v = t - v, & \text{if } t < N \text{ and } 0 \leq t - v \leq N \\ t \dot{-} v = N, & \text{if } t < N \text{ and } t - v > N \\ N \dot{-} v = N, & \text{for every } v. \end{cases}$$

We emphasize that the operation $\dot{+}$ is not commutative: $s \dot{+} t$ needs not be equal to $t \dot{+} s$. In particular, $0 \dot{+} t = 0$ whereas $t \dot{+} 0 = t$ for any $t = 0, 1, \ldots, N$. An important relation between the operations $\dot{+}$ and $\dot{-}$ is captured by the following lemma.

Lemma 5.2. *Let $d(t) = t \dot{+} v$ and $e(t) = t \dot{-} v$, for some $v \in \mathbb{Z}$. Then (e, d) defines an adjunction on $\{0, 1, \ldots, N\}$.*

The proof is straightforward. An example of such an adjunction with $N = 7$ and $v = 2$ is depicted in Figure 1(b). Combining this result with Proposition 5.1 we obtain a class of grey-level adjunctions $(\mathcal{E}_G, \Delta_G)$ (invariant under horizontal translations) which preserve as much as possible invariance under grey-level translations. Let for any $h \in E$ the adjunction (e_h, d_h) be of the form described by Lemma 5.2, that is $d_h(t) = t \dot{+} G(h)$ and $e_h(t) = t \dot{-} G(h)$. Here G is an arbitrary function with values in $\mathbb{Z}$. Note that (e_h, d_h) is the trivial adjunction if and only if $G(h) \leq -N$. We define the *domain* of G, $\text{dom}(G)$, as the set of $h \in E$ for which $G(h) > -N$. Let Δ_G and $\mathcal{E}_G$ be given by

$$\Delta_G(F)(x) = (F \dot{\oplus} G)(x) := \bigvee_{h \in \text{dom}(G)} (F(x-h) \dot{+} G(h)) \tag{5.5}$$

$$\mathcal{E}_G(F)(x) = (F \dot{\ominus} G)(x) := \bigwedge_{h \in \mathrm{dom}(G)} (F(x+h) \dot{-} G(h)) \tag{5.6}$$

Figure 6 depicts the dilation and erosion given by (5.3)-(5.4) and the dilation and erosion given by (5.5)-(5.6). The dual automorphism $F \to F^*$ yields a duality relation between Δ_G and $\mathcal{E}_G$. To describe this we need some definitions. Recall that for every function $F \in \mathrm{Fun}(E)$ we have defined F^* by $F^*(x) = N - F(x)$. For any grey-level $t = 0, 1, \ldots, N$ we define the complementary grey-level t^* by $t^* = N - t$. One can easily check that for every $t = 0, 1, \ldots, N$ and $v \in Z$,

$$(t^* \dot{+} v)^* = t \dot{-} v, \tag{5.7}$$

For any structuring function G with domain $\mathrm{dom}(G)$ we define the reflected structuring function $\overline{G}$ as the structuring function with domain $\mathrm{dom}(\overline{G}) = \{-x \mid x \in \mathrm{dom}(G)\}$, and $\overline{G}(x) = G(-x)$ for $x \in \mathrm{dom}(\overline{G})$.

Proposition 5.3. *For every $F \in \mathrm{Fun}(E)$ and every structuring function G the following duality relations hold:*

$$(F^* \dot{\oplus} G)^* = F \dot{\ominus} \overline{G}$$
$$(F^* \dot{\ominus} G)^* = F \dot{\oplus} \overline{G}.$$

PROOF. We only prove the first relation.

$$(F^* \dot{\oplus} G)^*(x) = [\bigvee_{h \in \mathrm{dom}(G)} F^*(x-h) \dot{+} G(h)]^* = \bigwedge_{h \in \mathrm{dom}(G)} [F^*(x-h) \dot{+} G(h)]^*$$
$$= \bigwedge_{h \in \mathrm{dom}(G)} [F(x-h) \dot{-} G(h)] = \bigwedge_{h \in \mathrm{dom}(\overline{G})} [F(x+h) \dot{-} \overline{G}(h)] = (F \dot{\ominus} \overline{G})(x).$$

This proves the result. ∎

For any $F \in \mathrm{Fun}(E)$ and $v \in \mathbb{Z}$ we define $F \dot{+} v$ and $F \dot{-} v$ by

$$(F \dot{+} v)(x) = F(x) \dot{+} v,$$
$$(F \dot{-} v)(x) = F(x) \dot{-} v,$$

for $x \in E$. One sees immediately that the dilation Δ_G satisfies

$$\Delta_G(F \dot{+} v) = \Delta_G(F) \dot{+} v \tag{5.8}$$

for $F \in \mathrm{Fun}(E)$ and $v \geq 0$, if and only if G is nonnegative on E. In that case $\mathcal{E}_G$ satisfies

$$\mathcal{E}_G(F \dot{-} v) = \mathcal{E}_G(F) \dot{-} v. \tag{5.9}$$

In [6, Proposition 11.3] we have also proved the converse: let Δ be a H-dilation on $\mathrm{Fun}(E)$ which satisfies (5.8), then there exists a nonnegative function G with $\mathrm{dom}(G) \subseteq E$ such that $\Delta = \Delta_G$. An analogous result holds for erosions. Moreover, we have shown in [6] that any H-operator on $\mathrm{Fun}(E)$ which satisfies this kind of grey-level translation invariance can be obtained as an infimum of dilations of the form $F \dot{\oplus} G$. For completeness we shall give the precise formulation of this result, which can be regarded as a generalization of Matheron's representation theorem for translation invariant set operators: see [14]

Proposition 5.4. *Let $\Psi : \text{Fun}(E) \to \text{Fun}(E)$ be an H-operator which satisfies $\Psi(\mathcal{O}) = \mathcal{O}$ and*

$$\Psi(F \dot{+} v) = \Psi(F) \dot{+} v \tag{5.10}$$

for $F \in \text{Fun}(E)$ and $v \geq 0$. Then Ψ can be written as a infimum of dilations of the form $F \dot{\oplus} G$. Similarly, if $\Psi : \text{Fun}(E) \to \text{Fun}(E)$ is an H-operator with $\Psi(\mathcal{I}) = \mathcal{I}$ and

$$\Psi(F \dot{-} v) = \Psi(F) \dot{-} v \tag{5.11}$$

for $F \in \text{Fun}(E)$ and $v \geq 0$, then Ψ can be written as a supremum of erosions of the form $F \dot{-} G$.

In Section 3 we have briefly discussed annular openings for the binary case. A complete characterization of annular openings on the lattice of grey-level functions can be found in [16, Section 3]. We recall the main results obtained there. Let $\mathcal{G}$ be the infinite grey-level set $\overline{\mathbb{Z}}$ or $\overline{\mathbb{R}}$, and let Δ_G be the dilation given by $\Delta_G(F)(x) = \bigvee_{h \in \text{dom}(G)} [F(x-h) + G(h)]$, where G is a structuring function with domain dom(G): here dom(G) is the set of all x for which $G(x) > -\infty$. Then $\text{id} \wedge \Delta_G$ is an (annular) opening if

(i) dom(G) is symmetric, i.e., $h \in \text{dom}(G)$ if and only if $-h \in \text{dom}(G)$

(ii) $G(h) + G(-h) \geq 0$ for every $h \in \text{dom}(G)$.

Below we shall extend this result for the case where the grey-level set is $\{0, 1, \ldots, N\}$.

Proposition 5.5. *Let d_h be a dilation on $\{0, 1, \ldots, N\}$ for every $h \in E$, and let the dilation Δ be given by (5.1). Then $\text{id} \wedge \Delta$ defines an opening on Fun(E) if for every $h \in E$ one of the following assertions is true*

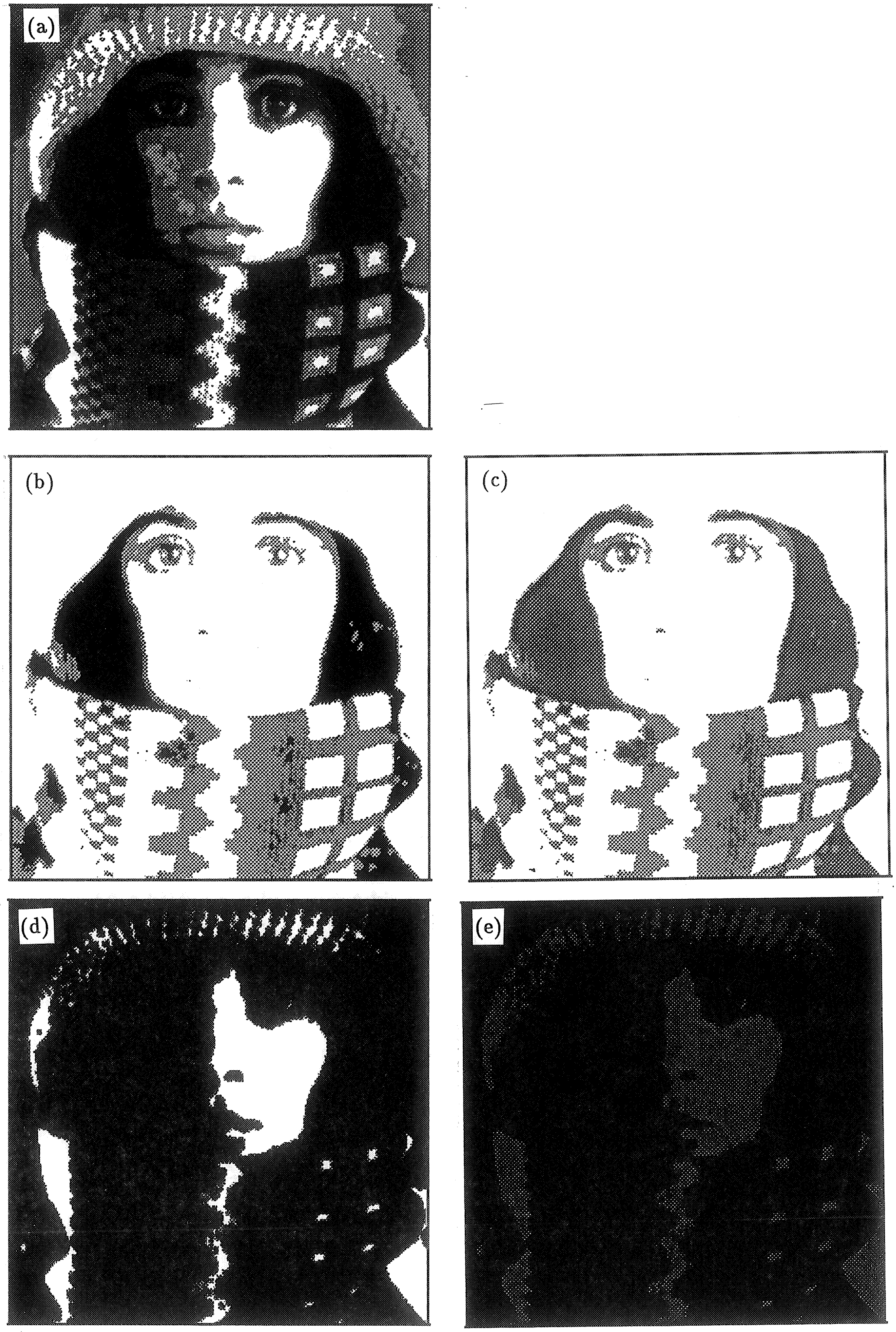
(i) d_h and d_{-h} are identically zero

(ii) $d_{-h} \circ d_h \geq \text{id}$.

Proof. First we observe that any dilation d on $\{0, 1, \ldots, N\}$ has the property that $d(s \wedge t) = d(s) \wedge d(t)$ for $s, t = 0, 1, \ldots, N$. We must show that the operator $\text{id} \wedge \Delta$ is idempotent. Then, since $\text{id} \wedge \Delta$ is anti-extensive, it is an opening. Evidently, $(\text{id} \wedge \Delta)^2 \leq \text{id} \wedge \Delta$. To prove the converse, it sufices to show that $\Delta(\text{id} \wedge \Delta) \geq \text{id} \wedge \Delta$, since then $(\text{id} \wedge \Delta)^2 = \text{id} \wedge \Delta \wedge \Delta(\text{id} \wedge \Delta) \geq \text{id} \wedge \Delta$. We define $H = \{h \in E \mid d_h \text{ is not identically } 0\}$. Let $F \in \text{Fun}(E)$ and $x \in E$. Then

$$\begin{aligned}
\Delta(\text{id} \wedge \Delta)(F)(x) &= \bigvee_{h \in H} d_h\Big(\bigvee_{h' \in H} [d_{h'}(F(x-h-h')) \wedge F(x-h)]\Big) \\
&= \bigvee_{h \in H} \bigvee_{h' \in H} [d_h d_{h'}(F(x-h-h')) \wedge d_h(F(x-h))] \\
(\text{choose } h' = -h) &\geq \bigvee_{h \in H} [d_h d_{-h}(F(x)) \wedge d_h(F(x-h))] \\
&\geq \bigvee_{h \in H} [F(x) \wedge d_h(F(x-h))] = F(x) \wedge \Delta(F)(x).
\end{aligned}$$

This completes the proof. ∎


(a)
(b)
(c)
(d)
(e)

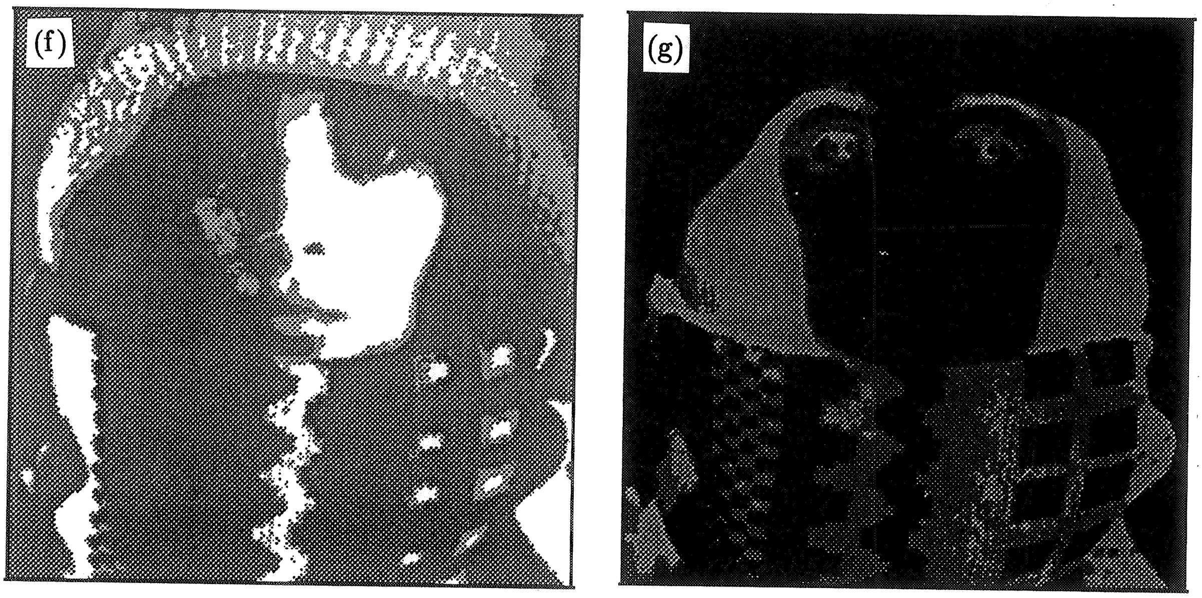


FIGURE 6. *Comparison of the dilations Δ_G and $\tilde{\Delta}_G$ and the erosions $\mathcal{E}_G$ and $\tilde{\mathcal{E}}_G$. The grey-level set is $\{0, 1, \ldots, 7\}$ and the structuring function is the function with domain the 3×3-square $\{(i,j) \mid -1 \leq |i|, |j| \leq 1\}$ and values $G(0,0) = 2$ and $G(1,0) = G(1,1) = \ldots = G(1,-1) = 0$.*
(a) The original image F, (b) the dilation $\Delta_G(F)$, (c) the dilation $\tilde{\Delta}_G(F)$, (d) the erosion $\mathcal{E}_G(F)$, (e) the erosion $\tilde{\mathcal{E}}_G(F)$, (f) the "opening" $\tilde{\Delta}_G\tilde{\mathcal{E}}_G(F)$. (g) The difference $\tilde{\Delta}_G\tilde{\mathcal{E}}_G(F) - F$ is not identically zero, and therefore $\tilde{\Delta}_G\tilde{\mathcal{E}}_G$ is not an opening.

Consider the dilation Δ_G given by (5.5), that is, $d_h(t) = t \dotplus G(h)$. then the assumptions on d_h are satisfied if and only if

(i) $\mathrm{dom}(G)$ is symmetric
(ii) $G(h) \geq 0$ for $h \in \mathrm{dom}(G)$.

Suppose namely that $G(h) < 0$ for some $h \in \mathrm{dom}(G)$. Take $t = \min\{-G(h), N\}$. Then

$$d_{-h}(d_h(t)) = (t \dotplus G(h)) \dotplus G(-h) = 0 \dotplus G(-h) = 0 \not\geq t,$$

hence the condition (ii) in the proposition above is not satisfied. The annular opening given by Proposition 5.5 is non-trivial ($\neq$ id) if and only if $0 \notin \mathrm{dom}(G)$.

In the forthcoming sections we will explain in detail several ways to extend set operators to function operators, in accordance with the title of this paper. Here we shall pay some attention to the reverse step. We will show that the grey-level dilation $F \oplus G$ can be decomposed into binary dilations.

Thereto we need the threshold sets of the functions F and G. The decomposition given here is inspired by the threshold decomposition of grey-level dilations described by Shih and Mitchell in [19]. However, our algorithm slightly differs from the one they give.

We first prove our main results, and we will illustrate them afterwards by means of a (one-dimensional) example.

Proposition 5.6. *Let G be a structuring function which takes values between $-N$ and N on its domain* dom(G). *Then, for any $F \in$ Fun(E) and $x \in E$ we have*

$$(F \dot{\oplus} G)(x) = \bigvee_{-N \le j \le N} \left[\sum_{i=1}^{N} (\mathcal{X}_i(F) \oplus \mathcal{X}_j(G))(x) \dot{+} j\right], \tag{5.12}$$

and also

$$(F \dot{\ominus} G)(x) = \bigwedge_{-N \le j \le N} \left[\sum_{i=1}^{N} (\mathcal{X}_i(F) \ominus \mathcal{X}_j(G))(x) \dot{-} j\right]. \tag{5.13}$$

PROOF. We only prove the second identity. The proof of the first one follows by similar arguments. We "only" have to show that for every $t = 0, 1, \ldots, N$ application of $\mathcal{X}_t$ to the left- and right-hand-side of (5.13) yields the same result. We use that for $t = 0, 1, \ldots, N$, $v \in \mathbb{Z}$ and $F \in$ Fun(E),

$$\mathcal{X}_t(F \dot{-} v) = \mathcal{X}_{t \dot{+} v}(F). \tag{5.14}$$

Namely, $F(x) \dot{-} v \ge t$ if and only if $F(x) \ge t \dot{+} v$ by the fact that $t \to t \dot{-} v$ and $t \to t \dot{+} v$ form an adjunction. We denote the expression which we get by applying $\mathcal{X}_t$ to the right-hand-side of (5.13) by X_t. We must show that $X_t = \mathcal{X}_t(F \dot{\ominus} G)$. Using the fact that $\mathcal{X}_t$ is distributive over infima (see (4.4)), and the identity (5.14) above, we get that

$$\begin{aligned} X_t &= \bigcap_{j=-N}^{N} \mathcal{X}_t\Big(\sum_{i=1}^{N} (\mathcal{X}_i(F) \ominus \mathcal{X}_j(G)) \dot{-} j\Big) \\ &= \bigcap_{j=-N}^{N} \mathcal{X}_{t \dot{+} j}\Big(\sum_{i=1}^{N} (\mathcal{X}_i(F) \ominus \mathcal{X}_j(G))\Big). \end{aligned}$$

Now we use that for any collection of sets (or alternatively, characteristic functions) $H_1 \ge H_2 \ge \ldots \ge H_N$ and any $t \ge 0$

$$\mathcal{X}_t(\sum_{i=1}^{N} H_i) = H_t.$$

Thus we get that

$$
\begin{aligned}
X_t &= \bigcap_{j=-N}^{N} \left(\mathcal{X}_{t\dot{+}j}(F) \ominus \mathcal{X}_j(G)\right) = \bigcap_{j=-N}^{N} \bigcap_{h\in\mathcal{X}_j(G)} [\mathcal{X}_{t\dot{+}j}(F)]_{-h} \\
&= \bigcap_{j=-N}^{N} \bigcap_{h\in\mathrm{dom}(G),\ G(h)\geq j} \mathcal{X}_{t\dot{+}j}(F_{-h}) = \bigcap_{h\in\mathrm{dom}(G)} \mathcal{X}_{t\dot{+}G(h)}(F_{-h}) \\
&= \bigcap_{h\in\mathrm{dom}(G)} \mathcal{X}_t(F_{-h} \dot{-} G(h)) = \mathcal{X}_t\Big(\bigwedge_{h\in\mathrm{dom}(G)} F_{-h} \dot{-} G(h)\Big) \\
&= \mathcal{X}_t(F \dot{\ominus} G).
\end{aligned}
$$

This proves the result. ∎

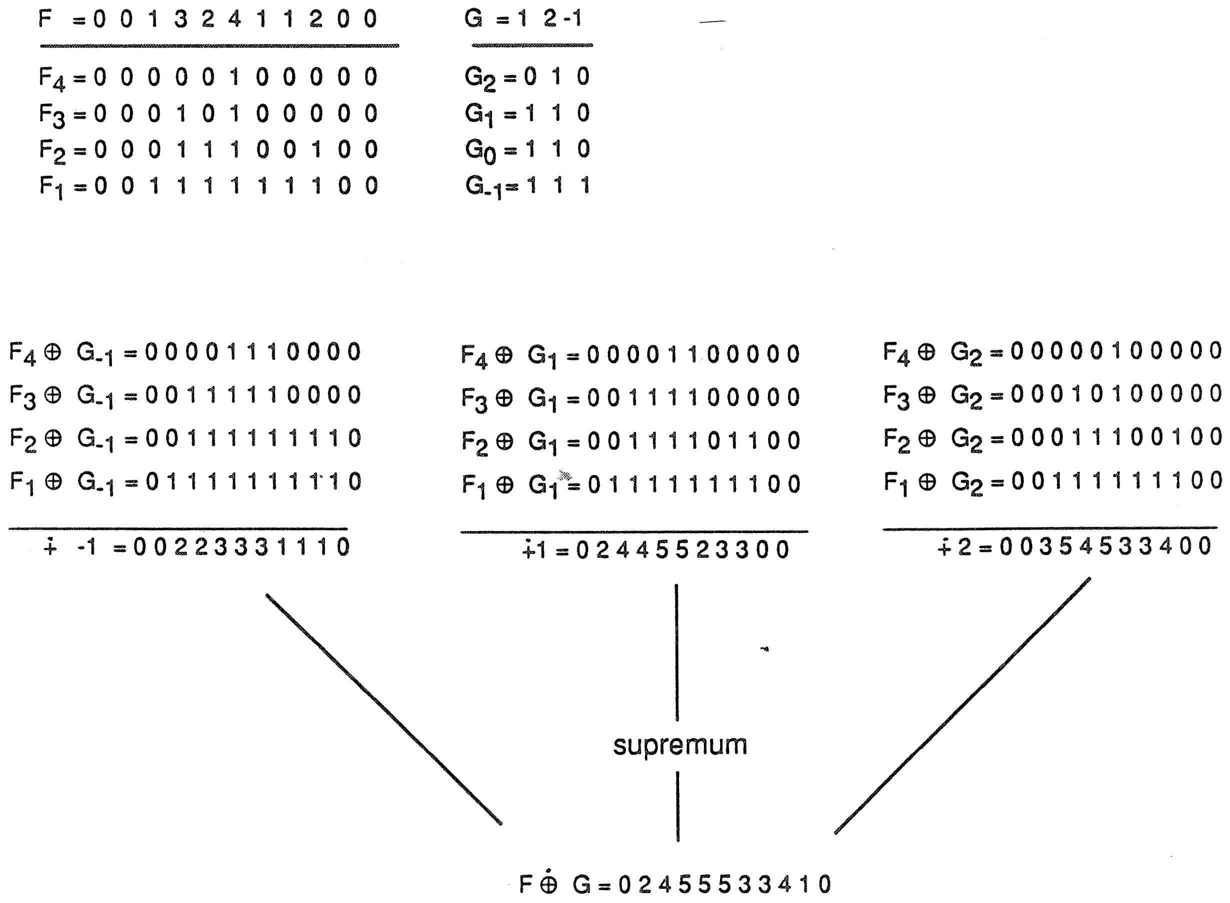


FIGURE 7. *Threshold decomposition of a one-dimensional grey-level dilation*

This result has some important practical implications. Let G be an arbitrary structuring function with domain $\mathrm{dom}(G)$. In practice, $\mathrm{dom}(G)$ will be contained in some small neighbourhood of the origin. Let $G_j := \mathcal{X}_j(G)$, $j = -N, \ldots, N$ be the corresponding binary structuring elements.

Note that $G_j \subseteq \text{dom}(G)$. For any function $F \in \text{Fun}(E)$ we denote by F_i the threshold sets of F, that is $F_i = \mathcal{X}_i(F)$. Now, for any j, we must compute the binary dilations $F_1 \oplus G_j, F_2 \oplus G_j, \ldots, F_N \oplus G_j$. These binary operations can be implemented in parallel. Next we compute the function

$$[(F_1 \oplus G_j) + (F_2 \oplus G_j) + \ldots + (F_N \oplus G_j)] \dot{+} j,$$

for every $j = -N, \ldots, N$, and finally take the supremum over all outcomes. Note that in practical cases the number of grey-levels in G will be rather small, ususally much smaller than $2N+1$, the maximum number possible. In Figure 7 we have worked out a particular one-dimensional case in detail.

6. Flat operators

The contents of this section will form a major justification for the title of this paper. Namely, we shall explain here how any increasing set operator (that is, an operator acting on binary images) can be extended to the space of grey-level functions $\text{Fun}(E)$. In the previous section we have given a complete characterization of grey-level dilations (and erosions) invariant under horizontal translations. An important subclass is formed by the dilations with a *flat structuring function* G, i.e., a function G which takes the value 0 on $\text{dom}(G)$. Defining the structuring set $A_G = \text{dom}(G)$, we get that

$$F \dot{\oplus} G = \bigvee_{h \in A_G} F_h, \quad F \dot{\ominus} G = \bigwedge_{h \in A_G} F_{-h}. \tag{6.1}$$

These expressions are analogous to the Minkowski addition and subtraction for sets. In fact, $F \dot{\oplus} G$ and $F \dot{\ominus} G$ are extensions of respectively the set dilation $X \to X \oplus A_G$ and the set erosion $X \to X \ominus A_G$ to the space of grey-level functions. The main goal of this section is to show that (and how) any increasing set operator can be extended to the space $\text{Fun}(E)$.

Let ψ be an increasing operator on $\mathcal{P}(E)$. We define the function operator $\Psi : \text{Fun}(E) \to \text{Fun}(E)$ by

$$\Psi(F)(x) = \max\{t = 0, 1, \ldots, N \mid x \in \psi(\mathcal{X}_t(F))\}. \tag{6.2}$$

If the set at the right-hand-side is empty, that is, $x \notin \psi(E)$, then we put $\Psi(F)(x) = 0$, which is in accordance with the convention that the maximum of the empty set is 0. We call ψ the *generator* of Ψ. Note that, if Ψ is generated by ψ, then $\Psi(\mathcal{O}) = \mathcal{O}$ if and only if $\psi(\emptyset) = \emptyset$, and $\Psi(\mathcal{I}) = \mathcal{I}$ if and only if $\psi(E) = E$.

We can reformulate (6.2) by considering $\psi(\mathcal{X}_t(F))$ as a Boolean function,

$$\Psi(F) = \sum_{t=1}^{N} \psi(\mathcal{X}_t(F)). \tag{6.3}$$

Here we have used that any increasing operator preserves the stacking property, that is, the property that the threshold sets of a function decrease if t increases. For that reason, function operators which are generated by an increasing set operator are sometimes called *stack filters* in the literature [22]. We shall not use this nomenclature (in fact, we reserve the term "filter" for increasing operators which are idempotent). Instead we shall call any function operator generated by a set operator *flat*. In this terminology, the operator $F \to F \dot{\oplus} G$ on Fun(E) given by (6.1) is called a flat dilation.

A justification for the adjective "flat" is given by our next result which says that a flat operator maps a flat function (a function which can attain at most two different values, namely 0 and t) onto a flat function (with the same t-value). Recall from (4.5) that for every set X, $\mathcal{F}_t(X)$ is the function which is t on X and 0 elsewhere.

Propositon 6.1. *Let Ψ be a flat function increasing operator generated by the set operator ψ, and assume that $\psi(\emptyset) = \emptyset$. Then*

$$\Psi(\mathcal{F}_t(X)) = \mathcal{F}_t(\psi(X)),$$

for $t = 0, 1, \ldots, N$ and $X \in \mathcal{P}(E)$.

PROOF. We use that

$$\mathcal{X}_s(\mathcal{F}_t(X)) = \begin{cases} X, & s \leq t \\ \emptyset, & s > t. \end{cases}$$

For $t = 0$ the assertion is obvious. Now let $t \geq 1$, then

$$\Psi(\mathcal{F}_t(X)) = \sum_{s=1}^{N} \psi(\mathcal{X}_s(\mathcal{F}_t(X))) = \sum_{s=1}^{t} \psi(X) = \mathcal{F}_t(\psi(X)),$$

which proves the result.

■

Another way to express the relation between a flat operator Ψ and its generator ψ is the following:

$$\mathcal{X}_t(\Psi(F)) = \psi(\mathcal{X}_t(F)), \quad t = 1, \ldots, N. \tag{6.4}$$

Note that for $t = 0$ this identity is false if $\psi(E) \neq E$. One can easily show that any flat operator Ψ has a unique generator ψ given by

$$\psi = \mathcal{X}_t \circ \Psi \circ \mathcal{F}_t, \tag{6.5}$$

for $t = 1, \ldots, N$ (the outcome being independent of t).

One of the main features of a flat operator is the fact that it commutes with any grey-level transformation which preserves the ordering.

Proposition 6.2. *Let Ψ be a flat increasing operator and assume that h is an increasing mapping from $\{0, 1, \ldots, N\}$ into itself. Assume furthermore that*

(i) $\Psi(\mathcal{I}) = \mathcal{I}$ if $h(0) > 0$

(ii) $\Psi(\mathcal{O}) = \mathcal{O}$ *if* $h(N) < N$*. Then*

$$\Psi \circ h = h \circ \Psi.$$

The proof of this result can be found in [6]. A similar result has been proved by Janowitz in [9]. Here we mention two important consequences of this result: if Ψ is flat then

$$\Psi(F \dotplus v) = \Psi(F) \dotplus v$$
$$\Psi(F \dot{-} v) = \Psi(F) \dot{-} v,$$

for any function F and any $v \geq 0$. If $\Psi(\mathcal{I}) = \mathcal{I}$ and $\Psi(\mathcal{O}) = \mathcal{O}$ then the result also holds for negative v.

Another consequence of Proposition 6.2 is the property that application of a flat operator Ψ on a grey-level function does not introduce extra grey-levels. To make this precise we define the *range* of a function F by $\mathrm{Ran}(F) = \{F(x) \mid x \in E\}$, that is the set of all grey-levels attained by F.

Proposition 6.3. *Let* Ψ *be a flat increasing operator with* $\Psi(\mathcal{O}) = \mathcal{O}$ *and* $\Psi(\mathcal{I}) = \mathcal{I}$*, then*

$$\mathrm{Ran}(\Psi(F)) \subseteq \mathrm{Ran}(F),$$

for any $F \in \mathrm{Fun}(E)$*.*

For a proof we refer again to [6].

We conclude this section with a number of elementary result concerning flat operators which have been proved in [6].

Proposition 6.4. *Let* Ψ, Ψ_1, Ψ_2 *and* Ψ_i $(i \in I)$ *be flat increasing operators with generators* ψ, ψ_1, ψ_2 *and* ψ_i $(i \in I)$ *respectively.*

(a) $\psi_1 \leq \psi_2$ *if and only if* $\Psi_1 \leq \Psi_2$*.*
(b) $\Psi_2 \circ \Psi_1$ *is a flat operator with generator* $\psi_2 \circ \psi_1$*.*
(c) $\bigvee_{i \in I} \Psi_i$ *and* $\bigwedge_{i \in I} \Psi_i$ *are flat opers with generators* $\bigvee_{i \in I} \psi_i$ *and* $\bigwedge_{i \in I} \psi_i$ *respectively.*
(d) Ψ^* *is a flat operator with generator* ψ^**.*

PROOF. By means of illustration we present a proof of the last statement. We use relation (6.3) and the observation that $\mathcal{X}_t(F^*) = \left(\mathcal{X}_{N-t+1}(F)\right)^*$. Now

$$\begin{aligned}\sum_{t=1}^{N} \psi^*(\mathcal{X}_t(F)) &= \sum_{t=1}^{N} [\psi(\mathcal{X}_t(F)^*]^* = \sum_{t=1}^{N} [1 - \psi(\mathcal{X}_{N-t+1}(F^*))] \\ &= N - \sum_{t=1}^{N} \psi(\mathcal{X}_{N-t+1}(F^*)) = N - \sum_{t=1}^{N} \psi(\mathcal{X}_t(F^*)) \\ &= N - \Psi(F^*) = \Psi^*(F).\end{aligned}$$

This proves the result.

∎

From this result one may conclude that a flat operator Ψ inherits all kind of properties from its generator ψ such as idempotence and (anti-) extensivity. If, e.g., ψ is an opening then Ψ is an opening as well. The following result has also been proved in [6].

Proposition 6.5. *Let $(\mathcal{E}, \Delta)$ be an adjunction on* Fun(E) *and assume that at least one of the two operators $\mathcal{E}, \Delta$ is flat. Then both operators are flat. If ε and δ are the generators of $\mathcal{E}$ and Δ respectively, then (ε, δ) defines an adjunction on $\mathcal{P}(E)$.*

As we already mentioned in Section 3, every translation-invariant increasing set operator can be decomposed as a union of erosions, or alternatively, as an intersection of dilations. Combining this fact with Proposition 6.5 and Proposition 6.4(c) we arrive at the following result.

Proposition 6.6. *Every flat increasing H-operator on* Fun(E) *can be decomposed as a supremum of flat H-erosions, or, alternatively, as an infimum of flat H-dilations.*

7. Continuity, iteration, and idempotence

A major issue in mathematical morphology is the theory of morphological filtering. Recall that an increasing operator ψ on the complete lattice is called a morphological filter if ψ is idempotent.

In [8] we have shown that under certain restrictions iteration of a morphological operator yields one which is idempotent: see also [5,16]. In this section we shall recall some of the results obtained in [8] and see how they apply to flat operators on the space Fun(E).

Let $\mathcal{L}$ be a complete lattice. For a sequence X_n in $\mathcal{L}$ we define

$$\liminf X_n = \bigvee_{N\geq 1} \bigwedge_{n\geq N} X_n$$

$$\limsup X_n = \bigwedge_{N\geq 1} \bigvee_{n\geq N} X_n.$$

It is clear that

$$\liminf X_n \leq \limsup X_n.$$

If equality holds then we say that X_n (order-) converges to X and we denote this as $X_n \to X$. If X_n is decreasing and $X = \bigwedge_{n\geq 1} X_n$ then we write $X_n \downarrow X$. Similarly, if X_n is increasing and $X = \bigvee_{n\geq 1} X_n$ then we write $X_n \uparrow X$. One easily shows that $X_n \downarrow X$ or $X_n \uparrow X$ implies that $X_n \to X$.

It is straightforward to find explicit expressions for the lim sup and lim inf in the lattice $\mathcal{G} = \{0, 1, \ldots, N\}$ of grey-levels. Namely, let t_n be a sequence in $\mathcal{G}$. Then

$$\limsup t_n = \max\{t \mid t_n = t \text{ for infinitely many } t\}$$

$$\liminf t_n = \min\{t \mid t_n = t \text{ for infinitely many } t\}.$$

In particular, $t_n \to t$ if $t_n = t$ eventually. We can use these facts to characterize order convergence on the complete lattice Fun(E). It follows immediately that the lim sup and lim inf in Fun(E) are obtained by taking the pointwise lim sup and lim inf in $\{0, 1, \ldots, N\}$. To be precise,

$$(\limsup F_n)(x) = \limsup F_n(x)$$

$$(\liminf F_n)(x) = \liminf F_n(x),$$

for any sequence F_n in Fun(E) and $x \in E$. Furthermore, $F_n \to F$ if and only if $F_n(x) = F(x)$ eventually, for every $x \in E$. Note that lim sup and lim inf interchange by going from $\mathcal{L}$ to the opposite lattice $\mathcal{L}'$.

Using the notion of order-convergence we can define (semi-) continuity for an operator ψ between two complete lattices.

Definition 7.1. Let $\mathcal{L}_1, \mathcal{L}_2$ be two (possibly identical) complete lattices and let $\psi : \mathcal{L}_1 \to \mathcal{L}_2$ be an arbitrary operator. We say that ψ is $\downarrow$-continuous if $X_n \to X$ implies that $\limsup \psi(X_n) \leq \psi(X)$, and that ψ is $\uparrow$-continuous if $X_n \to X$ implies that $\psi(X) \leq \liminf \psi(X_n)$. If ψ is both $\uparrow$- and $\downarrow$-continuous, that is, $X_n \to X$ implies that $\psi(X_n) \to \psi(X)$, then we say that ψ is (order-) continuous.

Note that if $\psi : \mathcal{L}_1 \to \mathcal{L}_2$ is $\downarrow$-continuous, then ψ is $\uparrow$-continuous as a mapping between the opposite lattices $\mathcal{L}_1$ and $\mathcal{L}_2$.

For increasing operators one can derive an alternative characterization of $\uparrow$- and $\downarrow$-continuity which turns out to be very useful in practice. For a proof we refer to [8]. Since $\downarrow$- and $\uparrow$-continuity form dual notions we shall henceforth restrict ourselves to $\downarrow$-continuity.

Proposition 7.2. *The increasing operator ψ is $\downarrow$-continuous if and only if $X_n \downarrow X$ implies that $\psi(X_n) \downarrow \psi(X)$.*

This result and its counterpart for $\uparrow$-continuous operators implies immediately that every dilation, by the fact that it is distributive over suprema, is $\uparrow$-continuous, and that every erosion is $\downarrow$-continuous. The operator $\mathcal{X}_t$: Fun$(E) \to \mathcal{P}(E)$ is continuous for every $t \geq 0$: this follows immediately from (4.3)-(4.4). From this property the following result is easily verified.

Proposition 7.3. *Let Ψ be a flat function operator generated by the set operator ψ. Then Ψ is $\downarrow$-continuous ($\uparrow$-continuous, continuous) if and only if ψ is $\downarrow$-continuous ($\uparrow$-continuous, continuous).*

The next result shows that (semi-) continuity is a property which is preserved under taking infima and suprema and sometimes under composition.

Proposition 7.4. *Consider the complete lattice* Fun(E).

(a) Every (finite or infinite) infimum of $\downarrow$-continuous operators is $\downarrow$-continuous.

(b) A finite supremum of $\downarrow$-continuous operators is $\downarrow$-continuous.

(c) Composition of increasing $\downarrow$-continuous operators is $\downarrow$-continuous.

PROOF. (a): Let Ψ_i be $\downarrow$-continuous for $i \in I$, and assume that $F_n \to F$. Then

$$\begin{aligned}
\limsup\Big[\bigwedge_{i\in I} \Psi(F_n)\Big] &= \bigwedge_{N\geq 1}\bigvee_{n\geq N}\bigwedge_{i\in I} \Psi_i(F_n) \\
&\leq \bigwedge_{N\geq 1}\bigwedge_{i\in I}\bigvee_{n\geq N} \Psi_i(F_n) \\
&= \bigwedge_{i\in I} \limsup \Psi_i(F_n) \leq \bigwedge_{i\in I} \Psi_i(F),
\end{aligned}$$

whence it follows that $\bigwedge_{i\in I}\Psi_i$ is $\downarrow$-continuous.

(b): Let $\Psi_1,\Psi_2,\ldots,\Psi_p$ be $\downarrow$-continuous, and define $\Psi=\bigvee_{i=1}^p\Psi_i$. We must show that Ψ is $\downarrow$-continuous. Let $F_n\to F$ and let $x\in E$. We are done if we can show that $\Psi(F_n)(x)\le\Psi(F)(x)$ eventually. But this follows immediately from the fact that $\Psi_i(F_n)(x)\le\Psi_i(F)(x)$ eventually, for every $i=1,\ldots,p$.

(c): Follows immediately from Proposition 7.2.

∎

Corollary 7.5. *Let H be a finite subset of E and assume that (e_h,d_h) is an adjunction on $\{0,1,\ldots,N\}$ for every $h\in H$. Then the dilation Δ and the erosion $\mathcal{E}$ given by*

$$\Delta(F)(x)=\bigvee_{h\in H}d_h(F(x-h))$$

$$\mathcal{E}(F)(x)=\bigwedge_{h\in H}e_h(F(x+h))$$

are both continuous.

Proof. We only prove that Δ is continuous. The other result follows by a duality argument. Obviously, like every dilation, Δ is $\uparrow$-continuous. To prove $\downarrow$-continuity we show that the operator $F\to d_h(F_h)$ is $\downarrow$-continuous. Then the assertion follows from Proposition 7.4(b) (note that $\Delta(F)=\bigvee_{h\in H}d_h(F_h)$). Since translation $F\to F_h$ is continuous, because of Proposition 7.4(c) it remains to show that $F\to d_h(F)$ is $\downarrow$-continuous. But this is obvious, and therefore the proof is complete.

∎

The dilation and erosion in this corollary are typical examples of what we shall call *finite operators*. In [8] we have defined and characterized finite operators in the binary case. We shall extend some of the results obtained there to the space of grey-level functions. Before that we give a precise definition we introduce some further notation. Let $F\in\mathrm{Fun}(E)$ and $M\subseteq E$, then we define $F|_M$ as the restriction of F to M, i.e., the function which is $F(x)$ on M and 0 elsewhere.

Definition 7.6. Let for every $h\in E$, $M(h)$ be a subset of E. The function operator Ψ is said to be finite with mask M if for every $h\in E$ and every set N with $M(h)\subseteq N$ we have

$$\Psi(F)(h)=\Psi(F|_N)(h),$$

for every function F.

Proposition 7.7. *Every finite operator on* Fun(E) *is continuous.*

Proof. Suppose that the function operator Ψ is finite. Let $F_n\to F$. We must prove that $\Psi(F_n)\to\Psi(F)$, that is, for every $x\in E$, $\Psi(F_n)(x)=\Psi(F)(x)$ eventually. We fix $x\in E$ and define $M:=M(x)$. Then $\Psi(F_n)(x)=\Psi(F_n|_M)(x)$. From $F_n\to F$ and the fact that M is finite it follows immediately that $F_n|_M=F|_M$ eventually, say for $n\ge N$. So

$$\Psi(F_n)(x)=\Psi(F_n|_M)(x)=\Psi(F|_M)(x)=\Psi(F)(x)$$

for $n \geq N$. This finishes the proof.

∎

In [8] it is explained in detail how one can build idempotent operators, in particular morphological filters, by iteration of an arbitrary one. It turns out that, along with some assumption which guarantees convergence of the sequence of iterates (e.g extensivity or anti-extensivity of the operator under consideration), $\downarrow$- or $\uparrow$-continuity is sufficient to guarantee idempotence of the outcome. Let us quote here one of the results obtained in [8] for openings and closings. Although in our formulation we restrict to the image space $\mathrm{Fun}(E)$ the result holds on any complete lattice $\mathcal{L}$.

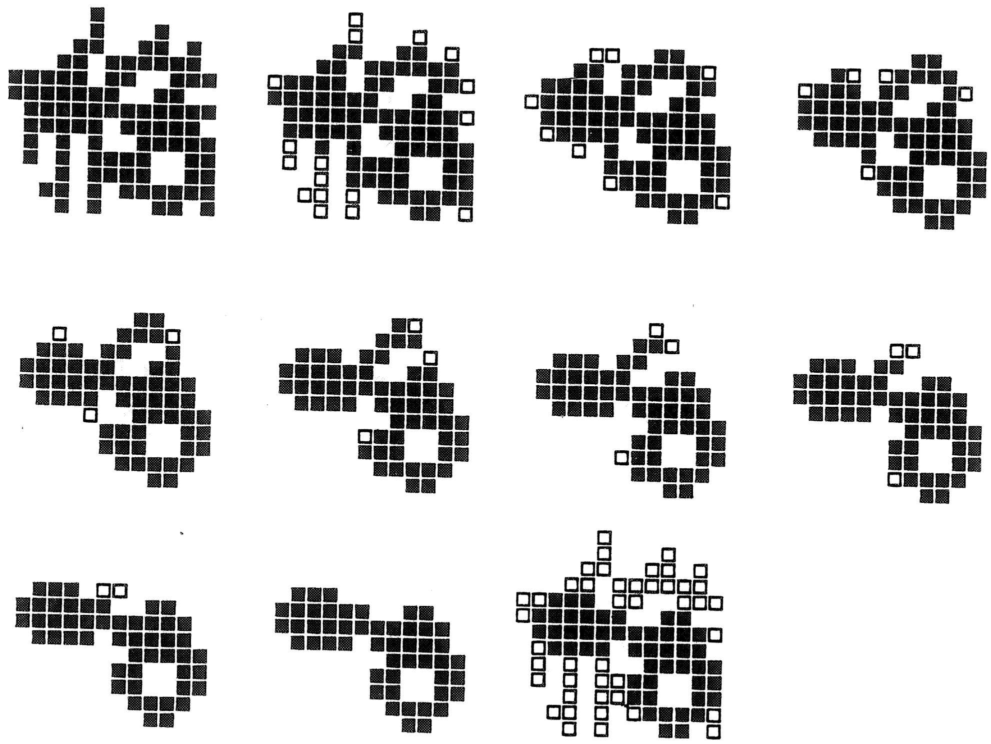


FIGURE 8. *Iteration of the anti-extensive operator $\mu \wedge \mathrm{id}$. The first object is the original image X. The filled squares in the second object represent $\mu(X) \cap X$, the open squares represent the points which have disappeared after the last iteration. The last but one object is $(\mu \wedge \mathrm{id})^\infty(X)$, the median opening of X, and the last object compares the original image X with its median opening.*

Proposition 7.8. *Let Ψ be an increasing operator on* $\mathrm{Fun}(E)$.

(a) If $\Psi \leq \mathrm{id}$ and Ψ is $\downarrow$-continuous, then $\Psi^\infty := \bigwedge_{n\geq 1} \Psi^n$ is an opening.

(b) If $\Psi \geq \mathrm{id}$ and Ψ is $\uparrow$-continuous, then $\Psi^\infty := \bigvee_{n\geq 1} \Psi^n$ is a closing.

PROOF. We only prove (a). Since $\Psi \leq \mathrm{id}$, the sequence Ψ^n is decreasing, whence it follows that $\Psi^n(F) \downarrow \Psi^\infty(F)$ for every $F \in \mathrm{Fun}(E)$. It is clear that $\Psi^\infty \leq \mathrm{id}$, and it remains to show that Ψ^∞ is idempotent. We need only check that $\Psi\Psi^\infty = \Psi^\infty$, since then $\Psi^n\Psi^\infty = \Psi^\infty$ and hence $\Psi^\infty\Psi^\infty = \Psi^\infty$. By the $\downarrow$-continuity of Ψ we get that

$$\Psi\Psi^\infty = \Psi\Big(\bigwedge_{n\geq 1} \Psi^n\Big) = \bigwedge_{n\geq 1} \Psi^{n+1} = \Psi^\infty,$$

which is what we needed to show.

∎

The procedure is best illustrated by an example. We restrict to the binary case because there the changes at each iteration are the most obvious. Recall that the median transform $\mu(X)$ of a discrete binary image $X \subseteq \mathbb{Z}^2$ contains all points $x \in \mathbb{Z}^2$ such that $A_x \cap X$ contains at least five elements: here A is the 3×3-square centered at the origin. Obviously, μ is a finite operator and therefore continuous. Since μ is neither extensive nor anti-extensive we iterate the anti-extensive operator $\mu \wedge \mathrm{id}$. The operator $(\mu \wedge \mathrm{id})^\infty$ is an opening called the *median opening*: see Figure 8.

8. Other construction of function operators

In Section 6 we have seen that every increasing set operator ψ generates an increasing function operator Ψ. This function operator is called a flat operator. The main idea behind this construction is the observation that for every function F, $\psi(\mathcal{X}_t(F))$, $t = 1, \ldots, N$ forms a decreasing sequence of sets and hence corresponds with a function whose threshold sets are precisely $\psi(\mathcal{X}_t(F))$.

This argument remains valid if we do not apply the same set operator ψ at any level t, but allow that ψ depends on t. In order to have that the resulting sequence is decreasing again, we must assume that the set operators ψ_t involved are decreasing with respect to t, that is,

$$\psi_t \leq \psi_s \;\text{ if }\; t \geq s.$$

It follows as before that that to any decreasing sequence of increasing set operators ψ_t there corresponds a unique increasing function operator Ψ given by

$$\Psi(F) = \sum_{t=1}^{N} \psi_t(\mathcal{X}_t(F)),$$

or alternatively,

$$\Psi(F)(x) = \max\{t = 0, 1, \ldots, N \mid x \in \psi_t(\mathcal{X}_t(F))\}.$$

Some theoretical results related to this construction have been discussed in [6]. Here we shall only consider an example.

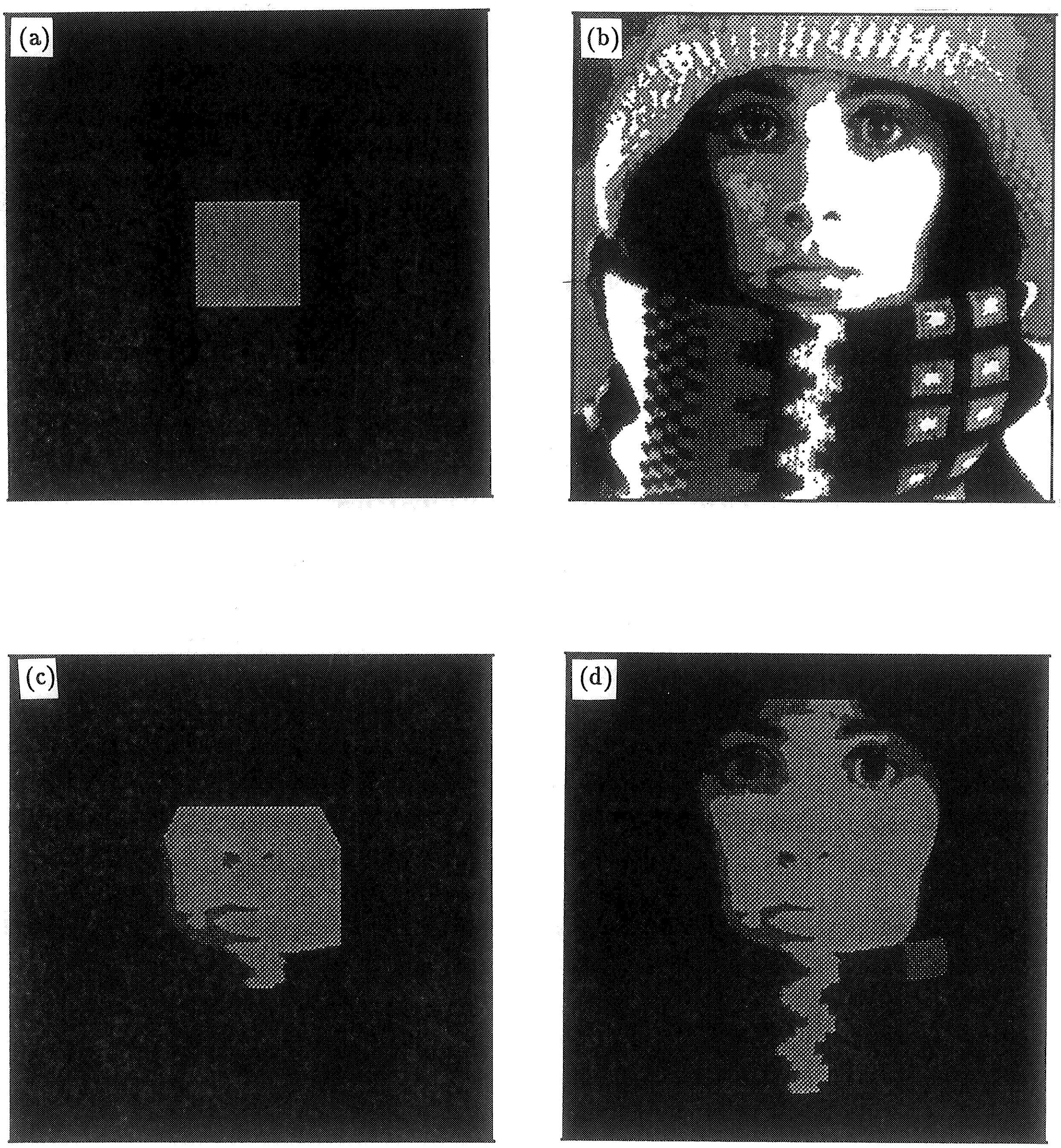

(a)
(b)
(c)
(d)

FIGURE 9. *Geodesic dilation and reconstruction of a grey-level image. (a) Original image (b) mask image (c) geodesic dilation, 20 iterations (d) geodesic dilation, 75 iterations (e) reconstruction.*

Example 8.1. *Geodesic dilation for grey-level images.*
In Section 3 we have defined geodesic operators for binary images. As a special case we have mentioned the geodesic dilation with mask M and structuring element A given by

$$\delta(X \mid M) = (X \oplus A) \cap M.$$

We can easily extend this definition for grey-level images. Let H be the mask function and let A be a flat structuring element. We define the geodesic dilation $\Delta(\cdot \mid H)$ by

$$\Delta(F \mid H) = (F \oplus A) \wedge H,$$

where $F \leq H$. One can easily see that this dilation is generated by a sequence of set operators ψ_t. Namely, if $M_t = \mathcal{X}_t(H)$, then

$$\begin{aligned} \mathcal{X}_t(\Delta(F \mid H)) &= \mathcal{X}_t((F \oplus A) \wedge H) = \mathcal{X}_t(F \oplus A) \cap \mathcal{X}_t(H) \\ &= [\mathcal{X}_t(F) \oplus A] \cap \mathcal{X}_t(H) = [\mathcal{X}_t(F) \oplus A] \cap M_t \\ &= \delta(\mathcal{X}_t(F) \mid M_t). \end{aligned}$$

So if we define $\psi_t(X) = \delta(X \mid M_t)$, then

$$\Delta(F \mid H) = \sum_{t=1}^{N} \psi_t(\mathcal{X}_t(F))$$

for any function $F \leq H$. The reconstruction of F with respect to H is defined as

$$\mathcal{R}(F \mid H) = \bigvee_{r \geq 1} \Delta^r(F \mid H).$$

In Figure 9 one can find an example.

Another way to define function operators will be illustrated by means of two examples. Here the threshold sets of the transformed image are defined recursively, starting at the bottom level $t = 1$ or at the top level $t = N$. We give an example of both cases.

Let $\psi : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathcal{P}(E)$ be a set operator which is increasing in both its variables X, Y. Let $F \in \mathrm{Fun}(E)$ be given and define $X_t = \mathcal{X}_t(F)$. We define the sets Y_t recursively as

$$Y_0 = E$$
$$Y_t = \psi(X_t, Y_{t-1}),\ t = 1, 2, \ldots, N.$$

We show by induction that Y_t is a decreasing sequence of sets. Obviously, $Y_1 \subseteq Y_0$. Now suppose that $Y_t \subseteq Y_{t-1}$. Then, since $X_{t+1} \subseteq X_t$ and since ψ is increasing,

$$Y_{t+1} = \psi(X_{t+1}, Y_t) \subseteq \psi(X_t, Y_{t-1}) = Y_t.$$

We define $\Psi(F)$ as the function with threshold sets Y_t. Then Ψ defines an increasing function operator on $\mathrm{Fun}(E)$. In the figure below we have taken $\psi(X, Y) = X \cap \alpha(Y)$, where α is the opening with the 3×3-square. Then the operator Γ which results from the construction above is also an opening. From the fact that $\mathcal{X}_t(\Gamma(F)) \subseteq \alpha(\mathcal{X}_t(F)) \subseteq \mathcal{X}_t(F)$ it follows that Γ is anti-extensive. So it remains to show that Γ is idempotent. Let $F \in \mathrm{Fun}(E)$ and let X_t, Y_t, Z_t be the threshold sets of $F, \Gamma(F), \Gamma^2(F)$ respectively, then $X_0 = Y_0 = Z_0 = E$. Suppose that $Y_s = Z_s$ for $s \leq t$. Then

$$Z_{t+1} = Y_{t+1} \cap \alpha(Z_t) = X_{t+1} \cap \alpha(Y_t) \cap \alpha(Y_t) = Y_{t+1}.$$

Hence $Y_t = Z_t$ for every t, and therefore $\Gamma(F) = \Gamma^2(F)$.

This operator may be useful if the number of grey-levels N is small. One can show that

$$\mathcal{A}(F) \leq \Gamma(F) \leq \mathcal{A}(F) + 1.$$

Here $\mathcal{A}$ is the extension of α to $\mathrm{Fun}(E)$. Namely, by induction one shows that $\alpha(\mathcal{X}_t(F)) \subseteq Y_t = \mathcal{X}_t(\Gamma(F))$, whence it follows that $\mathcal{A}(F) \leq \Gamma(F)$. On the other hand suppose that $\Gamma(F)(x) = t > 0$. Then $x \in \mathcal{X}_t(\Gamma(F)) = \mathcal{X}_t(F) \cap \alpha(\mathcal{X}_{t-1}(\Gamma(F)))$, hence $x \in \alpha(\mathcal{X}_{t-1}(F) = \mathcal{X}_{t-1}(\mathcal{A}(F))$ and we conclude that $\mathcal{A}(F)(x) \geq t - 1$. This proves the second estimate.

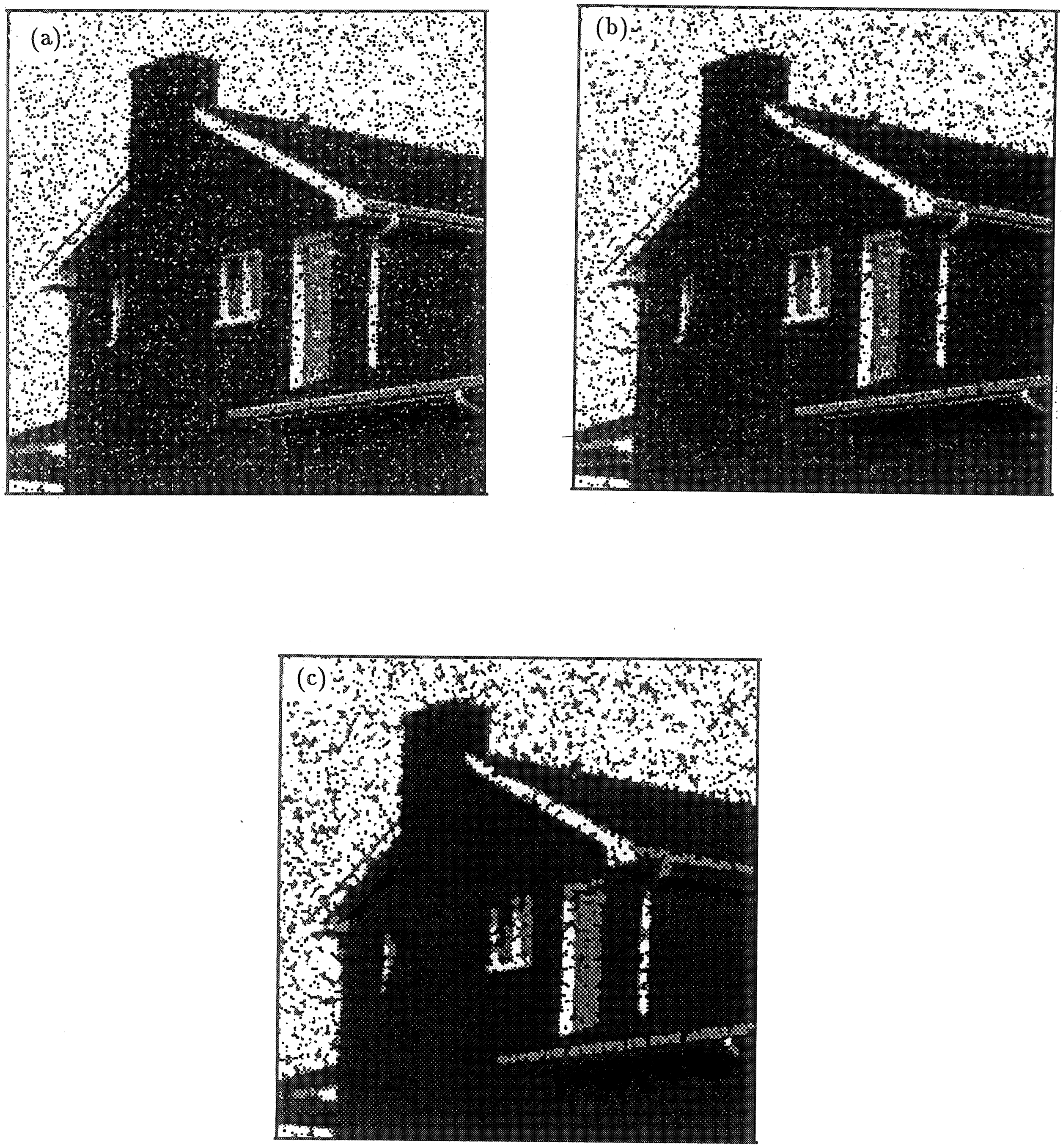


FIGURE 10. *Recursive upward construction of the opening $\mathcal{A}$. (a) Original image F (b) opening $\Gamma(F)$ (c) opening $\mathcal{A}(F)$.*

In Figure 10 we apply the operator $\mathcal{A}$ to an image with only four grey-levels.

We present a second example which illustrates the general idea, but this time we start the processing at the top level N. Recall that $\rho(Y \mid X)$ denotes the reconstruction of Y within X. Let $F \in \mathrm{Fun}(E)$ and $X_t = \mathcal{X}_t(F)$. Assume that m is the maximum grey-level attained by F. We define Y_t as the empty set for $t > m$, and

$$Y_m = X_m$$
$$Y_t = \rho(Y_{t+1} \mid X_t), \ t < m.$$

It is obvious that $Y_{t+1} \subseteq Y_t \subseteq X_t$, and hence the function $\mathcal{B}(\mathcal{F})$ which has threshold sets Y_t satisfies $\mathcal{B}(\mathcal{F}) \leq \mathcal{F}$. Again it is not difficult to prove that $\mathcal{B}$ defines an opening on $\mathrm{Fun}(E)$. To show this, define X_t, Y_t, Z_t as the threshold sets of $F, \mathcal{B}(\mathcal{F}), \mathcal{B}^{\in}(\mathcal{F})$ respectively. We must show that $Y_t = Z_t$ for every t. Obviously this is true for $t \geq m$. Suppose that $Y_s = Z_s$ for $s \geq t+1$. We show that $Y_t = Z_t$.

$$\begin{aligned} Z_t &= \rho(Z_{t+1} \mid Y_t) = \rho(Y_{t+1} \mid Y_t) \\ &= \rho(Y_{t+1} \mid \rho(Y_{t+1} \mid X_t)) = \rho(Y_{t+1} \mid X_t) \\ &= Y_t. \end{aligned}$$

The one but last equality follows easily from the fact that $\rho(Y_{t+1} \mid X_t)$ consists of those points $x \in X_t$ which are connected through some finite path with a point $y \in Y_{t+1}$: hence , if $x \in \rho(Y_{t+1} \mid X_t)$ then also $x \in \rho(Y_{t+1} \mid \rho(Y_{t+1} \mid X_t))$ (the converse is trivial).

Acknowledgment. I would like to thank dr. C. Ronse for thoroughly reading the manuscript and suggesting many improvements.

References.

1 G. Birkhoff (1984). *Lattice Theory*, American Mathematical Society Colloquium Publications, Vol. 25, 3rd edition, Providence, RI.

2 C.R. Giardina and E.R. Dougherty (1988). *Morphological Methods in Image and Signal Processing.* Prentice Hall, Englewood Cliffs, NJ.

3 G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott (1980). *A Compendium of Continuous Lattices*, Springer-Verlag, Berlin.

4 R.M. Haralick, S.R. Sternberg, and X. Zhuang (1987). Image analysis using mathematical morphology, *IEEE Trans. Pattern Analysis & Machine Intelligence* PAMI-9, pp. 532–550.

5 H.J.A.M. Heijmans (1989). Iteration of morphological transformations, *CWI Quarterly* 2, pp. 19–36.

6 H.J.A.M. Heijmans (1990). Grey-level morphology, *CWI Report AM 9003.*

7 H.J.A.M. Heijmans and C. Ronse (1990). The algebraic basis of mathematical morphology. Part I: dilations and erosions, *to appear in Computer Vision, Graphics and Image Processing.*

8 H.J.A.M. Heijmans and J. Serra (1990). Convergence, continuity and iteration in mathematical morphology, *In preparation.*

9 M.F. Janowitz (1986). A model for ordinal filtering of digital images, In: *Statistical Image Processing and Graphics*, E.J. Wegman and D.J. DePriest eds., Marcel Dekker, New York.

10 C. Lantuejoul and S. Beucher (1980). On the use of the geodesic metric in image analysis, *J. Microsc.* 121, pp. 29–49.

11 P. Maragos (1989). A representation theory for morphological image and signal processing, *IEEE Trans. PAMI* 11, pp. 586–599.

12 P. Maragos and R.W. Schafer (1987). Morphological filters – Part I: their set-theoretic analysis and relations to linear shift-invariant filters, *IEEE Trans. Acoustics, Speech and Signal Proc.* 35, pp. 1153–1169.

13 P. Maragos and R.W. Schafer (1987). Morphological filters – Part II: their relations to mean, order-statistic, and stack filters, *IEEE Trans. Acoustics, Speech and Signal Proc.* 35, pp. 1170–1184.

14 G. Matheron (1975). *Random Sets and Integral Geometry*, J. Wiley & Sons, New York.

15 C. Ronse (1988). Why mathematical morphology needs complete lattices, *preprint.*

16 C. Ronse and H.J.A.M. Heijmans (1988). The algebraic basis of mathematical morphology. Part II: openings and closings, *submitted.*

17 J. Serra (1982). *Image Analysis and Mathematical Morphology*, Academic Press, London.

18 J. Serra, ed. (1988). *Image Analysis and Mathematical Morphology, Vol. 2: Theoretical Advances*, Academic Press, London.

19 F.Y. Shih and O.R. Mitchell (1989). Threshold decomposition of gray-scale morphology into binary morphology, *IEEE Trans. Pattern Anal. Mach. Intell.* 11, pp. 31–42.

20 S.R. Sternberg (1982). Cellular computers and biomedical image processing, *Biomedical*

Images & Computers, J. Sklansky & J.C. Bisconte eds., *Lecture Notes in Medical Informatics*, Vol. 17, Springer-Verlag, Berlin, pp. 294–319.

21 S.R. Sternberg (1986). Grayscale morphology, *Computer Vision, Graphics, and Image Processing* 35, pp. 333–355.

22 P.D. Wendt, E.J. Coyle and N.C. Gallagher (1986). Stack Filters, *IEEE Trans. Acoustics, Speech and Signal Proc.* 34, pp. 898–911.