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A New Strategy for Proving ω-Completeness applied to Process Algebra

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Abstract

A new technique for proving ω-completeness based on proof transformations is presented. In the first place we apply this technique to axiom systems for finite, concrete and sequential processes. It turns out that the number of actions is important for these sets to be ω-complete. For the axiom systems for bisimulation and completed trace semantics one action suffices and for the trace axioms 2 actions are enough. The ready, failure, ready trace and failure trace axioms are only ω-complete if an infinite number of actions is available. In the second place we consider process algebra with parallelism and show several axiom sets (containing the axioms of standard concurrency) ω-complete.

Key Words & Phrases: ω-completeness, process algebra, bisimulation semantics, ready semantics, failure semantics, ready trace semantics, failure trace semantics, completed trace semantics, trace semantics, parallel operator.

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1 Introduction

An equational theory E over a signature Σ is called *ω-complete* iff for all open terms t_1, t_2:

$$E \vdash t_1 = t_2 \quad \Leftrightarrow \quad E \vdash \sigma(t_1) = \sigma(t_2) \text{ for all closed substitutions } \sigma.$$

Not all equational theories are ω-complete: a well known example is the commutativity of the + in Peano arithmetic: all closed instances are derivable from any standard aximatization, but the law itself is not. Another example is the three-element groupoid of MURSKIĬ [13], who showed that for an ω-complete specification of the groupoid an infinite number of equations is necessary[1].

Several process algebra theories are ω-incomplete, too, and up till now this was more or less ignored (exceptions are MILNER [11] and MOLLER [12]). But there are several reasons why ω-completeness should not be neglected. In the first place equations between open terms play an important role in process algebra. For instance, processes are often described with sets of (open) equations. A complete set of axioms (not necessarily ω-complete) gives no guarantee that such sets of equations can be dealt with in a satisfactory manner. An example of this situation are the so-called 'axioms of standard concurrency' [2] in ACP, which had to be introduced in addition to the 'complete' set of axioms in order to prove the expansion theorem [3]. The status of these axioms became clear only after

[1] Recently, BERGSTRA and HEERING have shown that with hidden sorts and functions every recursively enumerable equational theory has an ω-complete axiomatisation [1].

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MOLLER [12] showed that in CCS with interleaving, but without communication, some of the axioms of standard concurrency are required for ω-completeness.

Furthermore, ω-completeness is also useful for theorem provers [8, 9, 15]. In [14] the so-called method 'proof by consistency' is introduced which can be applied to show inductive theorems equationally provable if ω-completeness of the axioms has been shown. In HEERING [6] it is argued that ω-completeness is desirable for the partial evaluation of programs. If $P(x,y)$ is a program with parameters x and y, and x has fixed value c, then the program $P_c(y)$ $(=P(c,y))$ should be evaluated as far as possible. In general this can only be achieved if the evaluation rules are ω-complete.

A more or less standard technique for proving ω-completeness is the following: given a set of axioms E over a signature Σ, find 'normal forms' and show that every open term is provably equal to a normal form. Then prove that for all pairs of different normal forms, closed instantiations can be found that differ in a model $\mathcal{M}$ for E. E does not necessarily have to be complete with respect to $\mathcal{M}$. This last step shows that the equivalence of these instantiations cannot be derived from E. From this ω-completeness of E follows directly. We prove the ω-completeness of the trace and completed trace axioms in this way. This technique has some disadvantages. The proofs are in general quite long and it is often difficult to find a suitable normal form.

In this paper we present an alternative technique that employs transformations of proofs. It is explained in section 3. This technique cannot always be applied, as shown by an example, but if applicable, proofs of ω-completeness turn out to be shorter and for the major part straightforward. Moreover, no reference to a model is necessary. We apply our method to five sets of axioms, which are taken from [4], for finite, concrete, sequential processes. Only for the set of bisimulation axioms ω-completeness had been shown before by MOLLER [12] using a longer proof. It turns out that the number of actions is important for the axiom sets to be ω-complete. We need an infinite number of actions for the ready trace, failure trace, ready and failure axioms. For the bisimulation and the completed trace axioms at least one action is required whereas for the trace axioms two actions are necessary. Then we study axiom sets for finite, concrete process algebra with interleaving without communication (also done in [12]) and interleaving with communication. We give straightforward proofs of the ω-completeness of these sets.

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2 Preliminaries

Throughout this text we assume the existence of a countably infinite set V of variables with typical elements x, y, z. A (one sorted) *signature* $\Sigma = (F, rank)$ consists of a set of *function names* F, disjunct with V, and a rank function $rank : F \longrightarrow \mathbb{N}$, denoting the arity of each function name in F. $T(\Sigma)$ is the set of *closed* or *ground terms* over signature Σ and $\mathbb{T}(\Sigma)$ is the set of *open terms* terms over Σ and V. We use the symbol $\equiv$ for syntactic equality between terms. Furthermore, we have *substitutions* $\sigma, \rho : V \longrightarrow \mathbb{T}(\Sigma)$ mapping variables to terms. Substitutions are in the standard way extended to functions from terms to terms. An expression of the form $t = u$ $(t, u \in \mathbb{T}(\Sigma))$ is called an *equation* over Σ. The letter e is used to range over equations. An expression of the form

$$\frac{e_1, \ldots, e_n}{e}$$

is called an *inference rule*. We call $e_1, \ldots, e_n$ the *premises* and e the *conclusion* of the inference rule. Substitutions are extended to equations and inference rules as expected.

An *equational theory* over a signature Σ is a set E of equations over Σ. These equations are called *axioms*. An equation e can be *proved* from a theory E, notation $E \vdash e$, if e is an instantiation of an axiom in E or if e is the conclusion of an instantiation of an inference rule r in table 1 of which all (instantiated) premises can be proved. If it is clear from the context what E is, we sometimes write

$x = x$ (reflexivity)	$\dfrac{x = y}{y = x}$ (symmetry)	$\dfrac{x = y \quad y = z}{x = z}$ (transitivity)
$\dfrac{x_i = y_i \quad 1 \leq i \leq rank(f)}{f(x_1, \ldots, x_{rank(f)}) = f(y_1, \ldots, y_{rank(f)})}$ for all $f \in F$ (congruence)		

Table 1: The inference rules of equational logic.

only e instead of $E \vdash e$. We write $E_1 \vdash E_2$ if $E_1 \vdash e$ for all $e \in E_2$. Note that if $E \vdash t = u$ for $t, u \in T(\Sigma)$, then $t = u$ can be proved using ground axioms and inference rules only.

An equational theory E is *ω-complete* if for all equations e: $E \vdash e$ iff $E \vdash \sigma(e)$ for all substitutions $\sigma : V \to T(\Sigma)$. Note that the implication from left to right is trivial. So, in general we only prove the implication from the right-hand side to the left-hand side.

3 The general proof strategy

Let $\Sigma = (F, rank)$ be a signature and let E be an equational theory over Σ. We present a technique to show that E is ω-complete. Assume $t = t'$ is an equation between open terms that can be proved for all its closed instantiations by the axioms of E. We transform $t = t'$ to a closed equation by a substitution $\rho : V \to T(\Sigma)$ that maps each variable in t and t' to a unique closed (sub)term representing this variable. By assumption $E \vdash \rho(t) = \rho(t')$. We transform the proof of this fact to a proof for $E \vdash t = t'$ by a translation R which replaces each subterm representing a variable by the variable itself. This transformation yields the desired proof if requirements (1), (2) and (3) below are satisfied. (1) says that the translation of $\rho(t) = \rho(t')$ must yield $t = t'$ (or something provably equivalent). In general this only works properly if each subterm representing a variable is unique for that variable and cannot be confused with other subterms. Requirements (2) and (3) guarantee that the transformed proof is indeed a proof. This is most clearly stated in (5), which is a consequence of (2) and (3).

- For $u \equiv t$ or $u \equiv t'$:

$$E \vdash R(\rho(u)) = u. \tag{1}$$

- For each $f \in F$ with $rank(f) > 0$ and $u_1, \ldots, u_{rank(f)}, u'_1, \ldots, u'_{rank(f)} \in T(\Sigma)$:

$$E \cup \{u_i = u'_i,\ R(u_i) = R(u'_i) | 1 \leq i \leq rank(f)\} \vdash \tag{2}$$

$$R(f(u_1, \ldots, u_{rank(f)})) = R(f(u'_1, \ldots, u'_{rank(f)})).$$

- For each axiom $e \in E$ and closed substitution $\sigma : V \to T(\Sigma)$:

$$E \vdash R(\sigma(e)). \tag{3}$$

Theorem 3.1. *Let E be an equational theory over signature Σ. If for each pair of terms $t, t' \in \mathbb{T}(\Sigma)$ that are provably equal for all closed instantiations, there exist a substitution $\rho : V \to T(\Sigma)$ and a mapping $R : T(\Sigma) \to \mathbb{T}(\Sigma)$ satisfying (1),(2) and (3), then E is ω-complete.*

Proof. Let $t, t' \in \mathbb{T}(\Sigma)$ such that for each substitution $\sigma : V \to T(\Sigma)$:

$$E \vdash \sigma(t) = \sigma(t'). \tag{4}$$

We must prove that $E \vdash t = t'$. This is an immediate corollary of the following statement:

$$E \vdash u = u' \text{ for } u, u' \in T(\Sigma) \quad \Rightarrow \quad E \vdash R(u) = R(u'). \tag{5}$$

It follows from (4) that $E \vdash \rho(t) = \rho(t')$. Using (5) this implies $E \vdash R(\rho(t)) = R(\rho(t'))$. By (1) it follows that $E \vdash t = t'$.

Statement (5) is shown by induction on the proof of $E \vdash u = u'$. As u and u' are closed terms, we may assume that the whole proof of $E \vdash u = u'$ consists of closed terms. First we consider the inference rules without premises. There are two possibilities. In the first case $u = u'$ has been shown by the inference rule $x = x$, i.e. $u \equiv \sigma(x) \equiv u'$ for some substitution $\sigma : V \rightarrow T(\Sigma)$. Clearly, $E \vdash R(u) = R(u')$ using the same inference rule and a substitution $\sigma' : V \rightarrow \mathbb{T}(\Sigma)$ defined by $\sigma'(x) = R(\sigma(x))$. Otherwise, $u = u'$ is an instantiation $\sigma(e)$ of an axiom $e \in E$. Using (3) it follows immediately that $E \vdash R(\sigma(e))$.

We check here the inference rules with premises. First we deal with the rule for transitivity. So assume $E \vdash u = u'$ has been proved using $E \vdash u = u''$ and $E \vdash u'' = u'$. By induction we know that there are proofs for $E \vdash R(u) = R(u'')$ and $E \vdash R(u'') = R(u')$. Applying the inference rule for transitivity again we have that $E \vdash R(u) = R(u')$. The rule for symmetry can be dealt with in the same way. Now suppose that $E \vdash f(u_1, ..., u_{rank(f)}) = f(u'_1, ..., u'_{rank(f)})$ has been proved using $E \vdash u_i = u'_i$ $(1 \leq i \leq rank(f))$. By induction we know that $E \vdash R(u_i) = R(u'_i)$. Using (2), it follows immediately that $E \vdash R(f(u_1, ..., u_{rank(f)})) = R(f(u'_1, ..., u'_{rank(f)}))$. □

This proof strategy cannot always be applied. This is illustrated by the following example.

Example 3.2. Suppose we have an axiomatisation for the natural numbers with a function max giving the maximum of any pair of numbers. In the signature we have a 0, a successor function S and max. The following set E_{max} of axioms is easily seen to be complete with respect to the standard interpretation.

$$\begin{aligned} &max(x, 0) = x, \\ &max(0, x) = x, \\ &max(S(x), S(y)) = S(max(x, y)). \end{aligned}$$

Clearly, E_{max} is not ω-complete as for instance associativity and commutativity of max are not derivable in general although each closed instance of them is.

It is impossible to use our technique to prove any extension of E_{max} ω-complete. This can be seen by considering the following two terms:

$$\begin{aligned} &t_1 = max(S(0), x) \text{ and} \\ &t_2 = x. \end{aligned}$$

We can see that these terms are not provably equal because with $x = 0$, the first term is equal to $S(0)$ and the second is equal to 0. Note that this is the only way to see the difference. If any term that is not equal to 0 is substituted for x then both terms are equivalent.

Suppose we would like to apply our technique in this case. If we take ρ such that $\rho(x) = 0$ then we must define the translation R such that $R(\rho(x)) = R(0) = x$. But then $R(\rho(0)) = x$ which is not (provably) equivalent to 0, violating (1). Suppose ρ is chosen such that $\rho(x) \neq 0$ and R could be defined such that (1) holds, i.e. $E_{max} \vdash R(\rho(t_i)) = t_i$ $(i = 1, 2)$. This implies that (5), which follows from (2) and (3), cannot hold because it implies that $E_{max} \vdash t_1 = t_2$.

So, this example shows that the new technique is not generally applicable, but as will be shown in the next sections, there are enough cases where application of this technique leads to attractive proofs.

4 Applications in finite, concrete, sequential process algebra

In the remainder of this paper we apply our technique to prove completeness of several axiom systems. In this section sets given for BCCSP in [4] are studied. BCCSP is a basic CCS and CSP-like language for finite, concrete, sequential processes. It is parameterized by a set *Act* of actions representing the elementary activities that can be performed by processes. We write $|Act|$ for the number of elements in *Act* ($|Act| = \infty$ if *Act* has an infinite number of elements). The language BCCSP contains a constant δ, which is comparable to **0** or NIL in CCS and to STOP in CSP. We call δ *inaction* or sometimes *deadlock*. There is an *alternative composition* operator $+$ with its usual meaning and, furthermore, there is an *action prefix* operator $a:$ for each action a in *Act*.

In the sequel we will often use sums of arbitrary finite size. It is convenient to have a notation for these. Therefore we introduce the abbreviation:

$$\sum_{i \in I} t_i = t_{i_1} + \ldots + t_{i_n}$$

where $I = \{i_1, \ldots, i_n\}$ is a finite index set and $t_i \in \mathbb{T}(\text{BCCSP})$ $(i \in I)$. We take $\sum_{i \in \emptyset} t_i = \delta$. Note that this notation is only justified if $+$ is commutative and associative. We only use this notation when this is the case.

The depth $|t|$ of a term $t \in \mathbb{T}(\text{BCCSP})$ is inductively defined as follows:

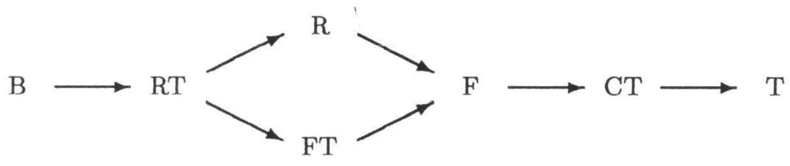
$$|\delta| = 0, \qquad |x| = 0 \text{ for all } x \in V,$$
$$|a : t| = 1 + |t| \text{ for all } a \in Act, \qquad |t_1 + t_2| = \max(|t_1|, |t_2|).$$

In table 2 we present several axiom systems, taken from [4], corresponding to several semantics in process algebra. We will investigate the ω-completeness of these sets. On the top line of this table we find their abbreviations: B stands for *Bisimulation*, RT for *Ready Trace*, FT for *Failure Trace*, R for *Ready*, F for *Failure*, CT for *Completed Trace* and finally T for *Trace* semantics. The axioms that are necessary for ready trace semantics (besides the axioms for bisimulation) are given by the following scheme:

$$a : (\sum_{i \in I} a_i : x_i + y) + a : (\sum_{i \in J} a_i : x_i + y) = a : (\sum_{i \in I \cup J} a_i : x_i + y) \tag{6}$$

where $\{a_i | i \in I\} = \{a_i | i \in J\}$, and $x_i, y \in V$ $(i \in I \cup J)$. This scheme differs from the axiomatisation given in [4], where an additional function name I and a conditional axiom were used to axiomatize ready trace semantics. We do not want to introduce these concepts here, although we show in lemma 4.2.2 that in both cases the same equations between terms over the signature BCCSP can be proved.

Let X stand for any of the semantics B,RT,... The symbol 'v' in a column of semantics X indicates that an axiom is derivable from the other axioms valid for X. The symbol '+' means that the axiom is required for a complete axiomatisation of the models given in [4] and 'ω' means that the axiom is only necessary for an ω-complete axiomatisation. It follows immediately that:



where the semantics to the left are finer than the semantics to the right. The semantics FT and R are incomparable. The abbreviation for a semantics will also be used to denote the set of axioms necessary for its ω-complete axiomatisation.

Lemma 4.1. *Let $t, u \in \mathbb{T}(BCCSP)$. If $T \vdash t = u$, then $|t| = |u|$.*

	B	RT	FT	R	F	CT	T
$x + y = y + x$	+	+	+	+	+	+	+
$(x + y) + z = x + (y + z)$	+	+	+	+	+	+	+
$x + x = x$	+	+	+	+	+	+	+
$x + \delta = x$	+	+	+	+	+	+	+
(see (6) in text)		+	+	v	v	v	v
$a : x + a : y = a : x + a : y + a : (x + y)$			+		v	v	v
$a(b : x + u) + a : (b : y + v) =$ $a : (b : x + b : y + u) + a : (b : x + b : y + v)$				+	+	v	v
$a : x + a : (y + z) = a : x + a : (x + y) + a : (y + z)$					+	ω	v
$a : (b : x + u) + a : (c : y + v) = a : (b : x + c : y + u + v)$						+	v
$a : x + a : y = a : (x + y)$							+

Table 2: Axioms for several process algebra semantics.

Proof. Direct with induction on the proof of $t = u$. □

As $\mathrm{T} \vdash \mathrm{B}$, $\mathrm{T} \vdash \mathrm{RT}$ etc. it immediately follows from the last lemma that '$\mathrm{X} \vdash t = u \Rightarrow |t| = |u|$', where X is any of the sets B, RT, etc.

4.1 The semantics B

We start considering the axioms for bisimulation semantics. If *Act* contains at least one element, then B is ω-complete. This fact has already been shown in [12] where a traditional technique was used. Note that it makes no sense to investigate the situation where $Act = \emptyset$, because in that case all closed terms will have the form δ, $\delta + \delta$, $\delta + \delta + \ldots$ and therefore they are equal and we only require the axiom $x = y$ for an ω-complete axiomatisation.

Theorem 4.1.1. *If $|Act| \geq 1$, then the axiom system B is ω-complete.*
Proof. As $|Act| \geq 1$, *Act* contains at least one action a. This action will play an important role in this proof. We follow the lines set out in theorem 3.1. So, assume we have two terms $t, t' \in \mathbb{T}(\mathrm{BCCSP})$. Select a natural number $m > \max(|t|, |t'|)$ and define $\rho : V \rightarrow T(\mathrm{BCCSP})$ by:

$$\rho(x) = a^{n(x) \cdot m} : \delta$$

where $a^k : \delta$ is an abbreviation of k applications of $a :$ to δ and $n : V \rightarrow \mathbb{N} \setminus \{0\}$ is a function assigning a unique natural number to each variable in x. Define $R : T(\mathrm{BCCSP}) \rightarrow \mathbb{T}(\mathrm{BCCSP})$ as follows:

$R(\delta) = \delta$,
$R(t + u) = R(t) + R(u)$,
$R(b : t) = b : R(t)$ if $b \neq a$ or $|b : t| \neq m \cdot n(x)$ for all $x \in V$,
$R(a : t) = x$ if $|a : t| = m \cdot n(x)$ for some $x \in V$.

We will now check conditions (1), (2) and (3) of theorem 3.1. We prove (1) with induction on a term $u \in \mathbb{T}(\mathrm{BCCSP})$ provided $|u| < m$. Note that this is sufficient as $|t| < m$ and $|t'| < m$.

$R(\rho(\delta)) = \delta$,
$R(\rho(x)) = R(a^{n(x) \cdot m} : \delta) = x$,
$R(\rho(u_1 + u_2)) = R(\rho(u_1)) + R(\rho(u_2)) = u_1 + u_2$,
$R(\rho(b : u)) = b : R(\rho(u)) = b : u$ if $b \neq a$,
$R(\rho(a : u)) = R(a : \rho(u)) =^* a : R(\rho(u)) = a : u$.

$=^*$ follows directly from the observation that $|a : \rho(u)| \neq m \cdot n(x)$ for all $x \in V$. In order to see this, first note that $1 \leq |a : u| < m$. If u does not contain variables, it is clear that $1 \leq |a : \rho(u)| < m$ and hence, $|a : \rho(u)| \neq m \cdot n(x)$. So, suppose u contains variables. By applying ρ to u each variable x is replaced by $a^{n(x)\cdot m} : \delta$. So $|a : \rho(u)| = p + n(x) \cdot m$ where x is a variable in u such that there is no other variable y in u with $n(y) > n(x)$ and p $(1 \leq p < m)$ is the 'depth' of the deepest occurrence of x in u. As $1 \leq p < m$, $|a : \rho(u)| \neq n(x) \cdot m$ for each $x \in V$.

Now we check (2). Assume $\mathrm{B} \vdash u_i = u_i'$ and $\mathrm{B} \vdash R(u_i) = R(u_i')$ for $u_i, u_i' \in T(\mathrm{BCCSP})$ and $i = 1, 2$. We find that:

$$\mathrm{B} \vdash R(u_1 + u_2) = R(u_1) + R(u_2) = R(u_1') + R(u_2') = R(u_1' + u_2').$$
$$\mathrm{B} \vdash R(b : u_1) = b : R(u_1) = b : R(u_1') = R(b : u_1') \text{ if } b \neq a.$$
$$\mathrm{B} \vdash R(a : u_1) =^* a : R(u_1) = a : R(u_1') =^+ R(a : u_1')$$
$$\text{if } |a : u_1| \neq m \cdot n(x) \text{ for all } x \in V.$$

$=^*$ follows directly from the condition. As $\mathrm{B} \vdash u_1 = u_1'$ it follows that $|a : u_1| = |a : u_1'|$ (cf. lemma 4.1) and hence, $|a : u_1'| \neq m \cdot n(x)$ for all $x \in V$. This justifies $=^+$.

$$\mathrm{B} \vdash R(a : u_1) = x =^* R(a : u_1') \text{ if } |a : u_1| = m \cdot n(x) \text{ for some } x \in V.$$

It follows that $|a : u_1'| = m \cdot n(x)$ explaining $=^*$.

Finally, we must check (3). This is trivial as the axioms do not contain actions. We only check the axiom $x + y = y + x$. The other axioms can be dealt with in the same way. Let $\sigma : V \to T(\mathrm{BCCSP})$ be a substitution, then:

$$\mathrm{B} \vdash R(\sigma(x + y)) = R(\sigma(x)) + R(\sigma(y)) = R(\sigma(y)) + R(\sigma(x)) = R(\sigma(y + x)).$$

□

4.2 The semantics RT,FT,R and F

We will show that the sets of axioms RT,FT,R and F are all ω-complete in case *Act* is infinite. If *Act* is finite, we have the following identity:

$$a : \sum_{i \in J} a_i : \delta + a : (x + \sum_{i \in J} a_i : \delta) = a : (x + \sum_{i \in J} a_i : \delta) \tag{7}$$

where $\{a_i | i \in J\} = Act$. Each closed instance of this identity is derivable from the axioms of RT,FT,R or F. However, (7) is not derivable in its general form: if (7) were derivable, then it also holds if *Act* is extended by a 'fresh' action $b \notin \{a_i | i \in J\}$. Define a substitution σ satisfying $\sigma(x) = b : \delta$. Applying σ to (7) yields:

$$a : \sum_{i \in J} a_i : \delta + a : (b : \delta + \sum_{i \in J} a_i : \delta) = a : (b : \delta + \sum_{i \in J} a_i : \delta).$$

but this equation does not hold in the failure model [4]. Hence, it is not derivable from F and therefore it can certainly not be derived from RT,FT or R.

So, in order to prove RT,FT,R and F ω-complete, *Act* must at least be countably infinite. The following theorem shows that this condition is also sufficient.

Theorem 4.2.1. *If $|Act|$ is infinite, then the axiom sets RT,FT,R and F are ω-complete.*
Proof. Take two terms t, t'. Define a substitution $\rho : V \to T(\mathrm{BCCSP})$ by:

$$\rho(x) = a_x : \delta$$

where a_x is a unique action for each $x \in V$ and a_x must not occur in either t or t'. Note that these actions can always be found as $|Act| = \infty$. Define $R : T(\mathrm{BCCSP}) \to \mathbb{T}(\mathrm{BCCSP})$ as follows:

$R(\delta) = \delta,$
$R(a : u) = a : R(u)$ if $a \neq a_x$ for each $x \in V,$
$R(a_x : u) = x,$
$R(u_1 + u_2) = R(u_1) + R(u_2).$

Condition (1) of theorem 3.1 can be checked by induction on the structure of open terms not containing action prefix operators a_x :.

$R(\rho(\delta)) = \delta,$
$R(\rho(x)) = R(a_x : \delta) = x,$
$R(\rho(a : u)) = R(a : \rho(u)) = a : R(\rho(u)) = a : u$ as $a \neq a_x$ for each $x \in V,$
$R(\rho(u_1 + u_2)) = R(\rho(u_1)) + R(\rho(u_2)) = u_1 + u_2.$

Condition (2) can be checked in the same straightforward manner. Suppose X $\vdash R(u_i) = R(u_i')$ for $u_i, u_i' \in T(\text{BCCSP})$ and $i = 1, 2$ where X may be replaced by either RT,FT,R or F. Then:

X $\vdash R(a : u_1) = a : R(u_1) = a : R(u_1') = R(a : u_1')$ if $a \neq a_x$ for each $x \in V$.
X $\vdash R(a_x : u_1) = x = R(a_x : u_1')$.
X $\vdash R(u_1 + u_2) = R(u_1) + R(u_2) = R(u_1') + R(u_2') = R(u_1' + u_2')$.

Finally, we check (3). We restrict ourselves to the ready trace axiom scheme. All other axioms can be dealt with in the same way. First we assume that $a = a_x$. Let $\sigma : V \to T(\text{BCCSP})$ be a substitution. Then RT $\vdash$:

$$R(a_x : (\sum_{i \in I} a_i : \sigma(x_i) + \sigma(y)) + a_x : (\sum_{i \in J} a_i : \sigma(x_i) + \sigma(y))) =$$
$$x + x = x =$$
$$R(a_x : (\sum_{i \in I \cup J} a_i : \sigma(x_i) + \sigma(y))).$$

In case $a \neq a_x$ for each $x \in V$, we have that RT proves:

$$R(a : (\sum_{i \in I} a_i : \sigma(x_i) + \sigma(y)) + a : (\sum_{i \in J} a_i : \sigma(x_i) + \sigma(y))) =$$
$$a : (\sum_{i \in I} R(a_i : \sigma(x_i)) + R(\sigma(y))) + a : (\sum_{i \in J} R(a_i : \sigma(x_i)) + R(\sigma(y))) =$$
$$a : (\sum_{i \in I \setminus \{i \in I | a_i = a_x\}} a_i : R(\sigma(x_i)) + \sum_{x \in \{x | a_x = a_i \wedge i \in I\}} x + R(\sigma(y))) +$$
$$a : (\sum_{i \in J \setminus \{i \in J | a_i = a_x\}} a_i : R(\sigma(x_i)) + \sum_{x \in \{x | a_x = a_i \wedge i \in J\}} x + R(\sigma(y))) =^*$$
$$a : (\sum_{i \in (I \cup J) \setminus \{i \in I \cup J | a_i = a_x\}} a_i : R(\sigma(x_i)) + \sum_{x \in \{x | a_x = a_i \wedge i \in J\}} x + R(\sigma(y))) =$$
$$R(a : (\sum_{i \in I \cup J} a_i : \sigma(x_i) + \sigma(y))).$$

$=^*$ follows from the observations that $\{a_i | i \in I,\ a_i \neq a_x \text{ for some } x \in V\} = \{a_i | i \in J,\ a_i \neq a_x \text{ for some } x \in V\}$ and $\{x | a_x = a_i \wedge i \in I\} = \{x | a_x = a_i \wedge i \in J\}$ which follow directly from the fact that $\{a_i | i \in I\} = \{a_i | i \in J\}$. □

In [4] the semantics for RT is characterized by the following conditional axiom:

$$I(x) = I(y) \;\Rightarrow\; a : x + a : y = a : (x + y). \tag{8}$$

It may be used in equational proofs in the same way as the inference rules in table 1, where $I(x) = I(y)$ is the premise and $a : x + a : x = a : (x + y)$ is the conclusion. The auxiliary function name I gives the initial actions of a term. It is defined by the following axioms:

$I(\delta) = \delta,$
$I(a : x) = a : \delta,$
$I(x + y) = I(x) + I(y).$

We call this alternative set of axioms RT$'$. For terms not including I, RT$'$ and RT are equally powerful, which we show in the next lemma. As a result, we find that RT$'$ is ω-complete for BCCSP-expressions.

Lemma 4.2.2. *If $t, t' \in \mathbb{T}(BCCSP)$, then:*

$$RT \vdash t = t' \quad \Leftrightarrow \quad RT' \vdash t = t'.$$

Proof. (Sketch) The implication from the left-hand side to the right-hand side is straightforward. The other implication follows by constructing a proof for RT $\vdash t = t'$ from the proof of RT$' \vdash t = t'$. Only rule (8) is non-trivial to handle. Here, we use the observation that if RT$' \vdash u = u'$, then the initial actions and the initial variables in u and u' are the same. In the definition of these concepts I is transparent. Hence, if RT$' \vdash I(u) = I(u')$, then u and u' have the same initial actions and variables. Therefore rule (6) can be applied. □

4.3 The completed trace axioms

We now show the ω-completeness for the axiom set CT. However, it is not possible to use the technique presented in the beginning. This will be shown in example 4.3.4. Therefore, we will use a more traditional technique. Hence, it is necessary to explicitly define the completed trace semantics for BCCSP. In CT the meaning of a process is its set of traces that end in inaction.

Definition 4.3.1. The interpretation $[\![.]\!]_{\mathrm{CT}} : T(\mathrm{BCCSP}) \rightarrow 2^{Act^*}$ (the set of subsets of strings over *Act*) is defined as follows:

$[\![\delta]\!]_{\mathrm{CT}} = \emptyset,$
$[\![a : t]\!]_{\mathrm{CT}} = \{a \star s | s \in [\![t]\!]_{\mathrm{CT}}\} \cup \{a | [\![t]\!]_{\mathrm{CT}} = \emptyset\},$
$[\![t_1 + t_2]\!]_{\mathrm{CT}} = [\![t_1]\!]_{\mathrm{CT}} \cup [\![t_2]\!]_{\mathrm{CT}}.$

We say that $t_1, t_2 \in T(\mathrm{BCCSP})$ are *completed trace equivalent*, notation $t_1 =_{\mathrm{CT}} t_2$, iff $[\![t_1]\!]_{\mathrm{CT}} = [\![t_2]\!]_{\mathrm{CT}}$.

Lemma 4.3.2. *(Soundness) Let $t_1, t_2 \in T(BCCSP)$:*

$$\mathrm{CT} \vdash t_1 = t_2 \quad \Rightarrow \quad t_1 =_{\mathrm{CT}} t_2.$$

Proof. Straightforward using the definitions. □

The following lemma gives some equations that are derivable from CT.

Lemma 4.3.3. CT $\vdash$

(a) $a : x + a : (x + y) = a : x + a : y + a : (x + y),$

(b) $a : (x + y) + a : x + a : z = a : (x + y + z) + a : x + a : z,$

(c) $a : (b : x + y) + a : z = a : (b : x + y + z) + a : z.$

Moreover, B+(b)+(c) $\vdash$ CT. Hence, B+(b)+(c) is an alternative ω-complete axiomatisation for completed trace semantics.

For completed trace semantics the following theorem states the completeness of the axioms with respect to the given model. Moreover, as t_1 and t_2 may be open terms, ω-completeness is implied also. The technique as set out in theorem 3.1 does not work. This is shown in the following example.

Example 4.3.4. Consider the following two BCCSP-terms.

$$t_1 = a : x + a : (a : \delta + x),$$
$$t_2 = a : (a : \delta + x).$$

These two terms are clearly different in CT as for a substitution σ with $\sigma(x) = \delta$, $\sigma(t_1)$ has a completed trace a which is not available in $\sigma(t_2)$. For every substitution σ' with $\sigma'(x) \neq \delta$, $\sigma'(t_1) =_{CT} \sigma'(t_2)$. Hence, using the same arguments as in example 3.2, we cannot apply our new technique.

The next theorem states that CT is ω-complete.

Theorem 4.3.5. *If $|Act| \geq 1$, then for all $t_1, t_2 \in \mathbb{T}(BCCSP)$, we have that:*

$$\forall \sigma : V \to T(BCCSP)\ \sigma(t_1) =_{\mathrm{CT}} \sigma(t_2) \quad \Rightarrow \quad \mathrm{CT} \vdash t_1 = t_2.$$

Proof. We write $\vec{x}$ for $\sum_{x \in W} x$ where $W \subseteq V$ is a finite set of variables. $\vec{x}$ is called a *sequence of variables*. We call a term t a *CT-normal form* iff

$$t \equiv \sum_{a \in A}(a : (t_a + \vec{y}_a) + \sum_{j \in J_a} a : \vec{x}_j)$$

satisfying for each $a \in A$:

(a) $A \subseteq Act$ is a finite set of actions,

(b) t_a is a CT-normal form,

(c) for each $j \in J_a$, it holds that all variables in $\vec{x}_j$ also appear in $\vec{y}_a$,

(d) if t_a has no initial actions (see below), then for each $j \in J_a$, there is a variable x_j in $\vec{y}_a$ that is not present in $\vec{x}_j$,

(e) for $j_1, j_2 \in J_a$ and $j_1 \neq j_2$, $\vec{x}_{j_1}$ contains a variable not present in $\vec{x}_{j_2}$.

If for a CT-normal form t: $|A| > 0$, then we say that t has *initial actions*. Note that if t has no initial actions, then t is equal to δ.

Fact 1. *For each term $t \in \mathbb{T}(BCCSP)$, there is a term u in CT-normal form such that* $\mathrm{CT} \vdash t = u + \vec{x}$ *for some sequence of variables $\vec{x}$.*
Proof of fact. This proof is based on induction on the structure of t. Let $t \equiv \delta$. If we take $A = \emptyset$ and $\vec{x} \equiv \delta$, we obtain the CT-normal form $\delta + \delta$, which is clearly provably equal to δ. Let $t \equiv x$. Then we can take the CT-normal form $\delta + x$.

If $t \equiv a : t'$, then, by induction, there is a CT-normal form u', such that $u' + \vec{x}'$, for some sequence of variables $\vec{x}'$, is provably equal to t'. Hence:

$$\mathrm{CT} \vdash t = a : t' = a : (u' + \vec{x}') + \delta$$

and $a:(u'+\vec{x}')+\delta$ is a CT-normal form (conditions (a),(b),(c),(d) and (e) can easily be checked).

Suppose $t \equiv t_1 + t_2$. Then, by induction, there are CT-normal forms u_1 and u_2 such that for sequences $\vec{z}_1$ and $\vec{z}_2$:

$$\mathrm{CT} \vdash t_1 = u_1 + \vec{z}_1 \text{ and } \mathrm{CT} \vdash t_2 = u_2 + \vec{z}_2.$$

The term u_l $(l = 1,2)$ can be written as:

$$u_l \equiv \sum_{a \in A^l} (a:(t_a^l + \vec{y}_a^l) + \sum_{j \in J_a^l} a:\vec{x}_j).$$

We assume that $J_a^1 \cap J_a^2 = \emptyset$ for $a \in A^1 \cap A^2$. The term $t_1 + t_2$ is provably equal to $u_1 + u_2 + \vec{z}_1 + \vec{z}_2$, which is, using CT, equal to:

$$\sum_{a \in A^1 \cap A^2} (a:(t_a + \vec{y}_a^1) + a:(t_a^2 + \vec{y}_a^2) + \sum_{j \in J_a^1 \cup J_a^2} a:\vec{x}_j) + \tag{9}$$
$$\sum_{a \in A^1 \setminus A^2} (a:(t_a^1 + \vec{y}_a^1) + \sum_{j \in J_a^1} a:\vec{x}_j) +$$
$$\sum_{a \in A^2 \setminus A^1} (a:(t_a^2 + \vec{y}_a^2) + \sum_{j \in J_a^2} a:\vec{x}_j) + \vec{z}_1 + \vec{z}_2.$$

Now note that the summands in (9) are in CT-normal form for $a \in A^1 \setminus A^2$ and $a \in A^2 \setminus A^1$. In order to prove fact 1, it is sufficient to only transform the first summand into CT-normal form. Therefore we consider for each $a \in A_1 \cap A_2$:

$$a:(t_a^1 + \vec{y}_a^1) + a:(t_a^2 + \vec{y}_a^2) + \sum_{j \in J_a^1 \cup J_a^2} a:\vec{x}_j. \tag{10}$$

First we deal with the possibility that both t_a^1 and t_a^2 in (10) have initial actions. In this case we can apply the axiom $a:(b:x+u)+a:(c:y+v) = a:(b:x+c:y+u+v)$ to rewrite (10) to:

$$a:(t_a^1 + t_a^2 + \vec{y}_a^1 + \vec{y}_a^2) + \sum_{j \in J_a^1 \cup J_a^2} a:\vec{x}_j. \tag{11}$$

This term is a CT normal form, satisfying conditions (a) and (b). Later on we show how conditions (c), (d) and (e) are satisfied.

Now, suppose that t_a^1 has no initial actions (the case where t_a^2 has no initial actions is symmetric and therefore skipped). Then t_a^1 is equal to δ. Hence (10) has the form:

$$a:(t_a^2 + \vec{y}_a^2) + (a:\vec{y}_a^1 + \sum_{j \in J_a^1 \cup J_a^2} a:\vec{x}_j) \tag{12}$$

and this term satisfies conditions (a) and (b).

From (11) and (12) we may assume that (10) is provably equal to a term of the form:

$$a:(t+\vec{y}) + \sum_{j \in J} a:\vec{x}_j$$

which is a CT normal form satisfying conditions (a) and (b). Now, assume condition (c) does not hold. This means that there is a $k \in J$ and a variable x in $\vec{x}_k$ such that x is not in $\vec{y}$. We can prove from CT using the typical RT axiom that:

$$a:(t+\vec{x}) + a:\vec{x}_k + \sum_{j \in J \setminus \{k\}} a:\vec{x}_j =$$
$$a:(t+\vec{x}+\vec{x}_k) + a:(t+\vec{x}) + a:\vec{x}_k + \sum_{j \in J \setminus \{k\}} a:\vec{x}_j.$$

If t has initial actions then this is equal to:

$$a:(t+\vec{x}+\vec{x}_k)+(a:\vec{x}_k+\sum_{j\in J\setminus\{k\}} a:\vec{x}_j),$$

otherwise, it is equal to:

$$a:(\vec{x}+\vec{x}_k)+a:\vec{x}+a:\vec{x}_k+\sum_{j\in J\setminus\{k\}} a:\vec{x}_j.$$

As J is finite and each $\vec{x}_j$ ($j \in J$) contains a finite number of variables, we can repeat this step a finite number of times and satisfy condition (c).

Hence, we may assume that we have a term $a:(t+\vec{y})+\sum_{j\in J} a:\vec{x}_j$ satisfying conditions (a), (b) and (c). Suppose t has no initial actions and for some $j \in J$, $\vec{x}_j$ and $\vec{y}$ contain the same variables. Apply axiom $x = x + x$ to fulfill condition (d). Note that the conditions (a), (b) and (c) are not invalidated by this operation. Now we consider a term:

$$a:(t+\vec{y})+\sum_{j\in J} a:\vec{x}_j$$

which satisfies condition (a), (b), (c) and (d), but for which (e) does not hold. This means that there are sequences of variables $\vec{x}_{j_1}$ and $\vec{x}_{j_2}$ ($j_1, j_2 \in J$ and $j_1 \neq j_2$) such that all variables in $\vec{x}_{j_1}$ are also present in $\vec{x}_{j_2}$. Hence, there is a sequence of variables $\vec{x}$ such that $\vec{x}_{j_1} + \vec{x} = \vec{x}_{j_2}$ such that $\vec{x}$ and $\vec{x}_{j_1}$ do not have variables in common. Now we apply lemma 4.3.3(a) to show:

$$a:(t+\vec{y})+a:(\vec{x}_{j_1}+\vec{x})+a:\vec{x}_{j_1}+\sum_{j\in J\setminus\{j_1,j_2\}} a:\vec{x}_j =$$
$$a:(t+\vec{y})+a:(\vec{x}_{j_1}+\vec{x})+a:\vec{x}_{j_1}+a:\vec{x}+\sum_{j\in J\setminus\{j_1,j_2\}} a:\vec{x}_j =^*$$
$$a:(t+\vec{y})+\sum_{j\in J\setminus\{j_2\}} a:\vec{x}_j+a:\vec{x}.$$

For $=^*$ we use condition (c) and the typical FT axiom. It can be seen that this step can also only be applied a finite number of times. So after some time we achieve a term satisfying all conditions for a CT-normal form. □

Let in the sequel for $l = 1, 2$:

$$t_l \equiv \sum_{a\in A^l}(a:(t_a^l+\vec{y}_a^l)+\sum_{j\in J_a^l} a:\vec{x}_j).$$

We say that t_1 and t_2 are *different* if one of the following holds:

(1) $A^1 \neq A^2$,

(2) $A^1 = A^2$ and for some $a \in A^1$, t_a^1 and t_a^2 are different,

(3) $A^1 = A^2$ and for some $a \in A^1$, $\vec{y}_a^1$ and $\vec{y}_a^2$ do not contain the same variables.

(4) $A^1 = A^2$ and for some $a \in A^1$, there is a $j_1 \in J_a^1$ such that for each $j_2 \in J_a^2$, $\vec{x}_{j_1}$ and $\vec{x}_{j_2}$ do not contain the same variables or, symmetrically, there is a $j_2 \in J_a^2$ such that for each $j_1 \in J_a^1$, $\vec{x}_{j_2}$ and $\vec{x}_{j_1}$ do not contain the same variables.

Fact 2. *If two CT-normal forms t_1 and t_2 are not different, then* $\mathrm{B} \vdash t_1 = t_2$ *(and thus* $\mathrm{CT} \vdash t_1 = t_2$*).*
Proof of fact. Straightforward. □

Fact 3. *Let t be a CT-normal form. Let $m \in \mathbb{N}$ be selected such that $m > |t|$. Let $\sigma : V \to T(BCCSP)$ be a substitution such that $\sigma(x) = \delta$ or $\sigma(x) = b^m : \delta$ where $b \in Act$. For each $s \in [\![\sigma(t)]\!]_{\mathrm{CT}}$: $1 \leq |s| \leq |t|$ or $|s| > m$.*
Proof of fact. By definition $|s| \geq 1$. The remainder of this fact follows directly by induction on the structure of t. □

An important corollary of this fact is that $|s| \neq m$.

Fact 4. *Let t_1 and t_2 be two different CT-normal forms. Let $m \in \mathbb{N}$ be selected such that $m > \max(|t_1|, |t_2|)$ and let $b \in Act$. There is a substitution $\sigma : V \to T(BCCSP)$ with for each $x \in V$: $\sigma(x) = \delta$ or $\sigma(x) = b^m : \delta$ such that $\sigma(t_1) \neq_{\mathrm{CT}} \sigma(t_2)$.*
Proof of fact. This proof is given by induction on $|t_1| + |t_2|$. As t_1 and t_2 are different, one of the following must be the case:

(1) $A^1 \neq A^2$. Then there is an $a \in Act$ such that $a \in A^1 \setminus A^2$ or $a \in A^2 \setminus A^1$. We only consider the first case. Define $\sigma(x) = \delta$ for all $x \in V$. If $a \in A^1 \setminus A^2$, then there is a completed trace $a \star s \in [\![\sigma(t_1)]\!]_{\mathrm{CT}}$ for some $s \in Act^\star$ and for any $s' \in Act^\star$, $a \star s' \notin [\![\sigma(t_2)]\!]_{\mathrm{CT}}$. Hence, $\sigma(t_1) \neq_{\mathrm{CT}} \sigma(t_2)$.

(2) If $A^1 = A^2$ then it can be that for some $a \in A^1$, t_a^1 and t_a^2 are different. By induction there is a substitution σ such that for all $x \in V$, $\sigma(x) = \delta$ or $\sigma(x) = b^m : \delta$ and $\sigma(t_a^1) \neq_{\mathrm{CT}} \sigma(t_a^2)$. Hence, there is some $s \in [\![\sigma(t_a^1)]\!]_{\mathrm{CT}} \setminus [\![\sigma(t_a^2)]\!]_{\mathrm{CT}}$ or $s \in [\![\sigma(t_a^2)]\!]_{\mathrm{CT}} \setminus [\![\sigma(t_a^1)]\!]_{\mathrm{CT}}$. Again we only consider the first case. It is obvious that $a \star s \in [\![\sigma(t_1)]\!]_{\mathrm{CT}}$. We will now show that $a \star s \notin [\![\sigma(t_2)]\!]_{\mathrm{CT}}$. Note that this immediately implies $\sigma(t_1) \neq_{\mathrm{CT}} \sigma(t_2)$. By fact 3 we know that $|s| \neq m$. As for all x in $\vec{y}_a^2$ or in $\vec{x}_j$ $(j \in J_a^2)$, $\sigma(x) = \delta$ or $\sigma(x) = a^m : \delta$, $s \notin [\![\sigma(\vec{y}_a^2)]\!]_{\mathrm{CT}}$ and $s \notin [\![\sigma(\vec{x}_j)]\!]_{\mathrm{CT}}$. As already stated, $s \notin [\![\sigma(t_a^2)]\!]_{\mathrm{CT}}$. Hence, $a \star s \notin [\![\sigma(t_2)]\!]_{\mathrm{CT}}$.

(3) If $A^1 = A^2$, then it can be that for some $a \in A^1$, $\vec{y}_a^1$ and $\vec{y}_a^2$ do not contain the same variables. This means that there is a variable $x \in \vec{y}_a^1$ such that $x \notin \vec{y}_a^2$ or vice versa. It is sufficient to consider only the first case. Define a substitution σ by:

$$\sigma(y) = \begin{cases} b^m : \delta & \text{if } x = y, \\ \delta & \text{otherwise.} \end{cases}$$

Clearly, $a \star b^m \in [\![\sigma(t_1)]\!]_{\mathrm{CT}}$. We show that $a \star b^m \notin [\![\sigma(t_2)]\!]_{\mathrm{CT}}$. By definition of σ, $b^m \notin [\![\sigma(\vec{y}_a^2)]\!]_{\mathrm{CT}}$. Also, $b^m \notin [\![\sigma(t_a^2)]\!]_{\mathrm{CT}}$ because, by fact 3, for no $s \in [\![\sigma(t_a^2)]\!]_{\mathrm{CT}}$: $|s| = m$. By condition (c) of a CT normal form $a \star b^m \notin [\![\sigma(\sum_{j \in J_a^2} a : \vec{x}_j)]\!]_{\mathrm{CT}}$. So we may conclude $a \star b^m \notin [\![\sigma(t_2)]\!]$.

(4) Assume none of the three cases above apply. Then it must be the case that for some $a \in A^1 (= A^2)$ there is a $j_1 \in J_a^1$ such that for each $k_2 \in J_a^2$, $\vec{x}_{j_1}$ and $\vec{x}_{k_2}$ do not contain the same variables, or there is a $j_2 \in J_a^2$ such that for each $k_1 \in J_a^1$, $\vec{x}_{j_2}$ and $\vec{x}_{k_1}$ do not contain the same variables. Consider all $\vec{x}_{j_1}$ and $\vec{x}_{j_2}$ with the above mentioned property and select one, say $\vec{x}_j$ such that $|\vec{x}_j|$ is minimal. For symmetry we may assume that $j \in J_a^1$. Define σ by:

$$\sigma(x) = \begin{cases} \delta & \text{if } x \text{ in } \vec{x}_j, \\ b^m : \delta & \text{otherwise.} \end{cases}$$

Note that $a \in [\![\sigma(t_1)]\!]_{\mathrm{CT}}$. We will show that $a \notin [\![\sigma(t_2)]\!]_{\mathrm{CT}}$. From the definition of completed traces it follows that if $a \in [\![\sigma(t_2)]\!]_{\mathrm{CT}}$ then

for each $k \in J_a^2$, $[\![\sigma(\vec{x}_k)]\!]_{\mathrm{CT}} = \emptyset$ or
t_a^2 has no initial actions and $[\![\sigma(\vec{y}_a^2)]\!]_{\mathrm{CT}} = \emptyset$.

But neither of these holds. Consider some $\vec{x}_k$ ($k \in J_a^2$). As $\vec{x}_j$ is minimal, it follows by condition (e) that there is some x in $\vec{x}_k$, but x is not in $\vec{x}_j$. Hence $\sigma(x) = b^m : \delta$ and thus $[\![\sigma(\vec{x}_k)]\!]_{\mathrm{CT}} \neq \emptyset$.

By condition (d), either t_a^1 has initial variables or y_a^1 contains a variable which is not present in $\vec{x}_j$. Because none of the cases (1), (2), (3) above applied, t_a^2 and t_a^1 have the same initial actions and y_a^1 and y_a^2 contain the same variables. Hence, $[\![\sigma(t_a^2 + \vec{y}_a^2)]\!]_{\mathrm{CT}} \neq \emptyset$. So we may conclude that $a \notin [\![\sigma(t_2)]\!]_{\mathrm{CT}}$.

□

With these facts, it is straightforward to finish the proof. Suppose $t_1, t_2 \in \mathbb{T}(\mathrm{BCCSP})$ and $\sigma(t_1) =_{\mathrm{CT}} \sigma(t_2)$ for each closed substitution σ. By fact 1 there are CT-normal forms u_1, u_2 and sequences of variables $\vec{y}_1$, $\vec{y}_2$ such that ($l = 1, 2$):

$$\mathrm{CT} \vdash t_l = u_l + \vec{y}_l.$$

There are three possible cases that must be considered.

1. u_1 and u_2 are different. By fact 4 there is a trace $s \in Act^*$ with length $|s| \neq m$ ($m > \max(|t_1|, |t_2|)$) such that $s \in [\![\sigma(u_1)]\!]_{\mathrm{CT}}$ and $s \notin [\![\sigma(u_2)]\!]_{\mathrm{CT}}$ or vice versa for a closed substitution σ. For symmetry we only consider the first case. Hence $s \in [\![\sigma(u_1 + \vec{y}_1)]\!]_{\mathrm{CT}}$. By fact 4 we also know that for each $x \in V$, $\sigma(x) = \delta$ or $\sigma(x) = b^m : \delta$. Hence, $s \notin [\![\sigma(u_2 + \vec{y}_2)]\!]_{\mathrm{CT}}$ and therefore $\sigma(t_1) \neq_{\mathrm{CT}} \sigma(t_2)$ which contradicts the assumption.

2. There is a variable x in $\vec{y}_1$ that is not available in $\vec{y}_2$ (or vice versa). Define a substitution σ by:

$$\sigma(y) = \begin{cases} b^m : \delta & \text{if } y = x, \\ \delta & \text{otherwise.} \end{cases}$$

   Hence, $b^m \in [\![\sigma(u_1 + \vec{y}_1)]\!]_{\mathrm{CT}}$. As for each $s \in [\![\sigma(u_2)]\!]_{\mathrm{CT}}$: $|s| \neq m$ and x does not appear in $\vec{y}_2$, $s \notin [\![\sigma(u_2 + \vec{y}_2)]\!]_{\mathrm{CT}}$. Hence, $\sigma(t_1) \neq_{\mathrm{CT}} \sigma(t_2)$. Contradiction.

3. Suppose the cases above do not apply. Then by fact 2:

$$\mathrm{B} \vdash u_1 + \vec{y}_1 = u_2 + \vec{y}_2.$$

   Hence, $\mathrm{CT} \vdash t_1 = t_2$.

□

4.4 The trace axioms

Again we do not use the new technique as in this case the 'standard' technique is more convenient to use. Therefore, we give the trace semantics explicitly. In trace semantics each process is characterized by its set of prefix closed traces:

Definition 4.4.1. The interpretation $[\![.]\!]_{\mathrm{T}} : T(\mathrm{BCCSP}) \to 2^{Act^*}$ is defined as follows:

$[\![\delta]\!]_{\mathrm{T}} = \emptyset$,
$[\![a : t]\!]_{\mathrm{T}} = \{a \star \sigma | \sigma \in [\![t]\!]_{\mathrm{T}}\} \cup \{a\}$,
$[\![t_1 + t_2]\!]_{\mathrm{T}} = [\![t_1]\!]_{\mathrm{T}} \cup [\![t_2]\!]_{\mathrm{T}}$.

We say that $t_1, t_2 \in T(\mathrm{BCCSP})$ are *trace equivalent*, notation $t_1 =_{\mathrm{T}} t_2$, iff $[\![t_1]\!]_{\mathrm{T}} = [\![t_2]\!]_{\mathrm{T}}$.

Lemma 4.4.2. *(Soundness) Let $t_1, t_2 \in T(BCCSP)$:*

$$\mathrm{T} \vdash t_1 = t_2 \;\Rightarrow\; t_1 =_{\mathrm{T}} t_2.$$

Proof. Straightforward using the definitions. □

For trace semantics we need two actions in order to prove T ω-complete. If $|Act| = 1$ then the following axiom is valid:

$$x + a : x = a : x.$$

This can easily be seen by proving $\mathrm{T} \vdash t + a : t = a : t$ for all $t \in T(\text{BCCSP})$ with induction on t if $|Act| = 1$. The axiom $x + a : x = a : x$ is in general not derivable from T, because instantiating x with $b : \delta$ yields $b : \delta + a : b : \delta \neq_{\mathrm{T}} a : b : \delta$ where $a, b \in Act$ are two different actions. In the next theorem we show that if $|Act| \geq 2$ then the axiom set T is ω-complete. First we define the notion of a syntactic summand. This notion is only used in this section.

Definition 4.4.3. Let $t, u \in \mathbb{T}(\text{BCCSP})$. t is a *syntactic summand* of u, notation $t \sqsubseteq u$ if:

- $t \equiv a : t'$ and $u \equiv a : t'$ for some $t' \in \mathbb{T}(\text{BCCSP})$ or,
- $u \equiv u_1 + u_2$ and $t \sqsubseteq u_1$ or $t \sqsubseteq u_2$.

Lemma 4.4.4. *Let $t_1, t_2 \in \mathbb{T}(BCCSP)$. If for each syntactic summand $u \in \mathbb{T}(BCCSP)$,*

$$u \sqsubseteq t_1 \quad \Leftrightarrow \quad u \sqsubseteq t_2$$

then $B \vdash t_1 = t_2$.
Proof. Straightforward. □

The next theorem says that T is ω-complete if two actions are available.

Theorem 4.4.5. *If $|Act| \geq 2$, then for each $t_1, t_2 \in \mathbb{T}(BCCSP)$, we have that:*

$$\forall \sigma : V \to T(BCCSP) : \ \sigma(t_1) =_{\mathrm{T}} \sigma(t_2) \quad \Rightarrow \quad \mathrm{T} \vdash t_1 = t_2.$$

Proof. We use the abbreviation $a_1 \star ... \star a_n : t$ with $a_1 \star ... \star a_n \in Act^\star$ for $a_1 : ... : a_n : t$. For $s \in Act^\star$, we define $|s|$ to be $|s : \delta|$, i.e. the length of trace s. For traces $s_1, s_2 \in Act^\star$ we write $s_1 \leq s_2$ if for some $r \in Act^\star$, $s_1 \star r = s_2$ or $s_1 = s_2$. In this case s_1 is a *prefix* of s_2.

First we define a T-*normal form*, which plays a crucial role in this proof. A term $t \in \mathbb{T}(\text{BCCSP})$ is a T-normal form if

$$t \equiv \sum_{i \in I} s_i : \delta + \sum_{i \in J} s_i : x_i$$

with $s_i \in Act^\star$ $(i \in I \cup J)$, satisfying:

(1) for each s_j $(j \in I \cup J)$ with $|s_j| > 1$, there is a $i \in I$ such that $s_i \star a = s_j$ for some $a \in Act$.

(2) for each s_j $(j \in J)$ with $|s_j| > 0$, there is a $i \in I$ such that $s_j = s_i$.

Fact 1. *Let $t \in \mathbb{T}(BCCSP)$. Then there is a T-normal form t' such that:*

$$\mathrm{T} \vdash t = t'.$$

Proof of fact. Straightforward with induction on t. □

Fact 2. *Let t and t' be two T-normal forms such that for some u, $u \sqsubseteq t$, $u \not\sqsubseteq t'$ or vice versa. Then there is a substitution $\sigma : V \longrightarrow T(BCCSP)$ such that:*

$$\sigma(t) \neq_{\mathrm{T}} \sigma(t').$$

Proof of fact. For symmetry it is sufficient to consider only the case where $u \sqsubseteq t$ and $u \not\sqsubseteq t'$. We can distinguish between:

(1) $u \equiv s : \delta$ with $s \in Act^{\star}$. Define $\sigma(x) = \delta$ for all $x \in V$. Note that $s \in [\![\sigma(t)]\!]_{\mathrm{T}}$. Moreover, it holds that $s \in [\![\sigma(t')]\!]_{\mathrm{T}}$ iff $s : \delta \sqsubseteq t'$; conditions (1) and (2) are required to prove this. Hence, as $s : \delta \not\sqsubseteq t'$, $s \notin [\![\sigma(t')]\!]_{\mathrm{T}}$.

(2) $u \equiv s : x$ for some $x \in V$ and $s \in Act^{\star}$. Let m be a natural number such that $m > \max(|t|, |t'|)$. Define $\sigma(x) = a^m : b : \delta$ where $a, b \in Act$ are two different actions and $\sigma(y) = \delta$ if $y \not\equiv x$. Clearly, $s \star a^m \star b \in [\![\sigma(t)]\!]_{\mathrm{T}}$. We will show that $s \star a^m \star b \notin [\![\sigma(t')]\!]_{\mathrm{T}}$. Therefore we write $t' \equiv \sum_{i \in I} s_i : \delta + \sum_{i \in J} s_i : y_i$ in the following way:

$$\sum_{i \in I} s_i : \delta + \sum_{i \in K_1} s_i : y_i + \sum_{i \in K_2} s_i : x + \sum_{i \in K_3} s_i : x + \sum_{i \in K_4} s_i : x$$

where

$$K_1 = \{i | i \in J \text{ and } y_i \not\equiv x\},$$
$$K_2 = \{i | i \in J,\ y_i \equiv x \text{ and } |s_i| < |s|\},$$
$$K_3 = \{i | i \in J,\ y_i \equiv x \text{ and } |s_i| = |s|\},$$
$$K_4 = \{i | i \in J,\ y_i \equiv x \text{ and } |s_i| > |s|\}.$$

Note that $J = K_1 \cup K_2 \cup K_3 \cup K_4$. We will show that $s \star a^m \star b$ cannot originate from any of these components. We deal with all five cases separately:

(a) For any $r \in [\![\sum_{i \in I} s_i : \delta]\!]_{\mathrm{T}}$, $|r| < m$ and therefore $r \neq s \star a^m \star b$.

(b) For any $r \in [\![\sum_{i \in K_1} s_i : \sigma(y_i)]\!]_{\mathrm{T}}$, $|r| < m$ because $\sigma(y_i) = \delta$. Hence, $r \neq s \star a^m \star b$.

(c) For any $r \in [\![\sum_{i \in K_2} s_i : \sigma(x)]\!]_{\mathrm{T}}$, $|r| \leq |s_i| + m + 1 < |s| + m + 1 = |s \star a^m \star b|$. Hence, $r \neq s \star a^m \star b$.

(d) For any $r \in [\![\sum_{i \in K_3} s_i : \sigma(x)]\!]_{\mathrm{T}}$, $r \leq s_i \star a^m \star b$ for some $i \in K_3$. If $|r| < |s| + m + 1$, clearly, $r \neq s \star a^m \star b$. If $|r| = |s| + m + 1$, then $r = s_i \star a^m \star b$. As $s : x \not\sqsubseteq t'$, $s_i \neq s$. Therefore $r \neq s \star a^m \star b$.

(e) Let for some $r \in Act^{\star}$, $r[i]$ be the i^{th} symbol in r. For any $r \in [\![\sum_{i \in K_4} s_i : \sigma(x)]\!]_{\mathrm{T}}$, $r \leq s_i \star a^m \star b$ for some $i \in K_4$. If $|r| \leq |s| + m$, then clearly $r \neq s \star a^m \star b$. If $|r| > |s| + m$, consider $r[|s| + m + 1]$. As $|s_i \star a^m \star b| > |s \star a^m \star b| > |s_i|$, $r[|s| + m + 1] = a$. But, $s \star a^m \star b[|s| + m + 1] = b$. Hence, if $|r| > |s| + m$, it also holds that $r \neq s \star a^m \star b$.

This finishes the proof of the second fact. □

Using both facts it follows almost immediately that T is ω-complete with respect to $=_{\mathrm{T}}$. Suppose $t, t' \in \mathbb{T}(\text{BCCSP})$ such that for each substitution $\sigma : V \longrightarrow T(\text{BCCSP})$, it holds that $\sigma(t) =_{\mathrm{T}} \sigma(t')$. Both t and t' are provably equal to T-normal forms u and u' (fact 1). If u and u' have different syntactic summands, then by the second fact $\rho(u) \neq_{\mathrm{T}} \rho(u')$ for some substitution $\rho : V \longrightarrow T(\text{BCCSP})$. This is a contradiction. Hence, by lemma 4.4.4, $B \vdash u = u'$ and therefore:

$$\mathrm{T} \vdash t = u = u' = t'.$$

□

5 Extensions with the parallel operator

We extend the signature BCCSP with operators for parallelism.

5.1 Interleaving without communication

First, we study BCCSP with the merge and the leftmerge, but without communication. The resulting signature is called $\text{BCCSP}_{\parallel\!\lfloor}$. We will study $\text{BCCSP}_{\parallel\!\lfloor}$ in the setting of bisimulation where $|Act| = \infty$. The upper half of table 3 contains a complete set of axioms. The completeness follows immediately from the completeness of the axiom set B for BCCSP because any closed term over the signature $\text{BCCSP}_{\parallel\!\lfloor}$ can be rewritten to a term over the signature BCCSP.

In order to have an ω-complete set of axioms, we add two new axioms (see the lower squares of table 3). These axioms are derivable for all closed instances. Therefore they are valid in bisimulation semantics. The complete set of axioms in table 3 is called $\text{B}_{\parallel\!\lfloor}$. The following theorem states the ω-completeness of $B_{\parallel\!\lfloor}$.

$x + y = y + x$	$x \parallel y = x \parallel\!\lfloor y + y \parallel\!\lfloor x$
$(x + y) + z = x + (y + z)$	$\delta \parallel\!\lfloor x = \delta$
$x + x = x$	$a : x \parallel\!\lfloor y = a : (x \parallel y)$
$x + \delta = x$	$(x + y) \parallel\!\lfloor z = x \parallel\!\lfloor z + y \parallel\!\lfloor z$
$x \parallel\!\lfloor \delta = x$	$x \parallel\!\lfloor (y \parallel z) = (x \parallel\!\lfloor y) \parallel\!\lfloor z$

Table 3: The axioms for BCCSP with the leftmerge.

Theorem 5.1.1. *The set of axioms in table 3 is ω-complete if Act contains an infinite number of actions.*

Proof. We use the technique presented in section 3. Suppose two terms $t, t' \in \mathbb{T}(\text{BCCSP}_{\parallel\!\lfloor})$ are given. Define $\rho : V \to T(\text{BCCSP}_{\parallel\!\lfloor})$ by $\rho(x) = a_x : \delta$ where a_x is a unique action for each $x \in V$ and a_x does neither occur in t nor in t'. Define $R : T(\text{BCCSP}_{\parallel\!\lfloor}) \to \mathbb{T}(\text{BCCSP}_{\parallel\!\lfloor})$ as follows:

$R(\delta) = \delta$,
$R(a : t) = a : R(t)$ where $a \neq a_x$ for all $x \in V$,
$R(a_x : t) = x \parallel\!\lfloor R(t)$,
$R(t + u) = R(t) + R(u)$,
$R(t \parallel u) = R(t) \parallel R(u)$,
$R(t \parallel\!\lfloor u) = R(t) \parallel\!\lfloor R(u)$.

In order to show the axioms in table 3 ω-complete we must check properties (1), (2) and (3) of theorem 3.1.

(1) We show that $\text{B}_{\parallel\!\lfloor} \vdash R(\rho(u)) = u$ with induction on $u \in \mathbb{T}(\text{BCCSP}_{\parallel\!\lfloor})$, provided u does not contain actions of the form a_x.
$R(\rho(x)) = x \parallel\!\lfloor \delta = x$,
$R(\rho(\delta)) = \delta$,
$R(\rho(t + u)) = R(\rho(t)) + R(\rho(u)) = t + u$,
$R(\rho(a : t)) = R(a : \rho(t)) =^* a : R(\rho(t)) = a : t$.
$=^*$ follows from the fact that $a \neq a_x$ for all $x \in V$.

(2) For the $+$-operator the proof is straightforward: $\text{B}_{\parallel\!\lfloor} \cup \{R(t_i) = R(u_i) | i = 1, 2\} \vdash R(t_1 + t_2) = R(t_1) + R(t_2) = R(u_1) + R(u_2) = R(u_1 + u_2)$. The function names $\parallel\!\lfloor$ and $\parallel$ can be dealt with in the same way. The action prefix case is slightly more complicated. $R(t_1) = R(u_1) \vdash R(a : t_1) = a : R(t_1) = a : R(u_1) = R(a : u_1)$ if $a \neq a_x$ for all $x \in V$. If $a = a_x$ for some $x \in V$, then $R(t_1) = R(u_1) \vdash R(a_x : t_1) = x \parallel\!\lfloor R(t_1) = x \parallel\!\lfloor R(u_2) = R(a_x : u_1)$.

(3) It is straightforward to check the axioms that do not explicitly refer to actions. So we only check the axiom $a : x \mathbin{\|\!\!\_} y = a : (x \parallel y)$. Let $\sigma : V \rightarrow T(\Sigma)$ be defined such that $\sigma(x) = t$ and $\sigma(y) = u$. $\mathrm{B}_{\|\!\!\_} \vdash R(a : t \mathbin{\|\!\!\_} u) = a : R(t) \mathbin{\|\!\!\_} R(u) = a : (R(t) \parallel R(u)) = R(a : (t \parallel u))$ if $a \neq a_x$ for all $x \in V$. In the other case $\mathrm{B}_{\|\!\!\_} \vdash R(a_x : t \mathbin{\|\!\!\_} u) = (x \mathbin{\|\!\!\_} R(t)) \mathbin{\|\!\!\_} R(u) = x \mathbin{\|\!\!\_} (R(t) \parallel R(u)) = R(a_x : (t \parallel u))$.

□

In many cases it is easy to show the ω-completeness of the axioms of new features introduced in $\mathrm{BCCSP}_{\|\!\!\_}$. As examples we introduce the silent step τ into $\mathrm{BCCSP}_{\|\!\!\_}$ and we will consider $\mathrm{BCCSP}_{\|\!\!\_}$ in trace semantics.

Example 5.1.2. We add a constant τ (*the silent step* or *internal move*) to $\mathrm{BCCSP}_{\|\!\!\_}$. The new signature is called $\mathrm{BCCSP}^{\tau}_{\|\!\!\_}$. The internal step has been axiomatized in different ways. In [10] τ is characterized by three τ-laws. This characterization is often called *weak bisimulation.*

$$\begin{aligned} a : \tau : x &= a : x, \\ \tau : x + x &= \tau : x, \\ a : (\tau : x + y) &= a : (\tau : x + y) + a : x. \end{aligned}$$

If one adds these laws to $\mathrm{B}_{\|\!\!\_}$, obtaining $\mathrm{B}^{\tau}_{\|\!\!\_}$, we have to add the following two axioms in order to make $\mathrm{B}^{\tau}_{\|\!\!\_}$ ω-complete. Axioms of this form already appeared in [7].

$$\begin{aligned} x \mathbin{\|\!\!\_} \tau : y &= x \mathbin{\|\!\!\_} y, \\ x \mathbin{\|\!\!\_} (\tau : y + z) &= x \mathbin{\|\!\!\_} (\tau : y + z) + x \mathbin{\|\!\!\_} y. \end{aligned}$$

Both new axioms are derivable for all closed instances, and therefore valid in any model for $\mathrm{B}^{\tau}_{\|\!\!\_}$.

In [5] τ is axiomatized by the single equation:

$$a : (\tau : (x + y) + x) = a : (x + y).$$

This variant is called *branching bisimulation.* The set $\mathrm{B}_{\|\!\!\_}$, together with this axiom is called $\mathrm{B}^{b}_{\|\!\!\_}$. The single axiom:

$$x \mathbin{\|\!\!\_} (\tau : (y + z) + y) = x \mathbin{\|\!\!\_} (y + z)$$

suffices to make $\mathrm{B}^{b}_{\|\!\!\_}$ ω-complete. This axiom is derivable for all closed instances, and therefore it holds in any model for $\mathrm{B}^{b}_{\|\!\!\_}$.

We do not give the ω-completeness proofs as they can easily be given along the lines of the proof of theorem 5.1.1. In fact it suffices to only check condition (3) for the new axioms, because conditions (1) and (2) are provable in exactly the same way.

Example 5.1.3. Here we study the ω-completeness of $\mathrm{BCCSP}_{\|\!\!\_}$ in trace semantics. As any term over the signature $\mathrm{BCCSP}_{\|\!\!\_}$ can be rewritten to a term over the signature BCCSP by the axioms in $\mathrm{B}_{\|\!\!\_}$, and T is complete for the signature BCCSP in trace semantics, $\mathrm{B}_{\|\!\!\_} \cup \mathrm{T}$ is complete for $\mathrm{BCCSP}_{\|\!\!\_}$ in trace semantics. For ω-completeness we must add the equation:

$$x \mathbin{\|\!\!\_} y + x \mathbin{\|\!\!\_} z = x \mathbin{\|\!\!\_} (y + z),$$

which is derivable from $\mathrm{B}_{\|\!\!\_} \cup \mathrm{T}$ for all its closed instances. The proof of this fact follows the lines of the proof of theorem 5.1.1.

$x+y=y+x$ $(x+y)+z=x+(y+z)$ $x+x=x$ $x+\delta=x$	
$x \parallel y = x \mathbin{\lfloor\!\lfloor} y + y \mathbin{\lfloor\!\lfloor} x + x \mid y$ $a:x \mathbin{\lfloor\!\lfloor} y = a:(x \parallel y)$ $\delta \mathbin{\lfloor\!\lfloor} x = \delta$ $(x+y) \mathbin{\lfloor\!\lfloor} z = x \mathbin{\lfloor\!\lfloor} z + y \mathbin{\lfloor\!\lfloor} z$	$x \mid y = y \mid x$ $a:x \mid b:y = (a \mid b):(x \parallel y)$ $\delta \mid x = \delta$ $(x+y) \mid z = x \mid z + y \mid z$
$(x \mathbin{\lfloor\!\lfloor} y) \mathbin{\lfloor\!\lfloor} z = x \mathbin{\lfloor\!\lfloor} (y \parallel z)$ $x \mathbin{\lfloor\!\lfloor} \delta = x$	$(x \mid y) \mid z = x \mid (y \mid z)$ $x \mid (y \mathbin{\lfloor\!\lfloor} z) = (x \mid y) \mathbin{\lfloor\!\lfloor} z$

Table 4: The axioms for $\mathrm{BCCSP}_{\mid}$.

5.2 Interleaving with communication

In this section the signature BCCSP is extended with the merge, the leftmerge and the *communication merge* ($\mid$). The signature obtained in this way is called $\mathrm{BCCSP}_{\mid}$. Its properties are described by the axioms in table 4 which are taken from [2]. In order to represent communication, we have an operator $\mid$ on actions. The only identities on $(Act, \mid)$ are commutativity and associativity of $\mid$. The axioms in the upper two squares of table 4 combined with the condition that $\mid$ on actions is commutative and associative, are already complete for $\mathrm{BCCSP}_{\mid}$-terms in the bisimulation model. This can again easily be seen by the fact that any term over the signature $\mathrm{BCCSP}_{\mid}$ can be rewritten to a term over BCCSP. For BCCSP the four axioms in the left upper corner of table 4 are complete in the bisimulation model. The axioms in the lower squares are necessary for an ω-complete axiomatisation. We call the axiom system in table 4 $\mathrm{B}_{\mid}$.

Example 5.2.1. The following facts are derivable from $\mathrm{B}_{\mid}$. We leave the proofs to the reader.

$x \parallel y = y \parallel x,$
$(x \parallel y) \parallel z = x \parallel (y \parallel z),$
$(a_1 \mid \ldots \mid (a_i \mid a_{i+1}) \mid \ldots \mid a_n) : x = (a_1 \mid \ldots \mid (a_{i+1} \mid a_i) \mid \ldots \mid a_n) : x,$
$(a_1 \mid \ldots \mid (a_i(a_{i+1} \mid a_{i+2})) \mid \ldots \mid a_n) : x = (a_1 \mid \ldots \mid ((a_i \mid a_{i+1}) \mid a_{i+2}) \mid \ldots \mid a_n) : x.$

The last two identities show that it is not necessary to include axioms for the commutativity and the associativity of $\mid$ on actions in $\mathrm{B}_{\mid}$.

Theorem 5.2.2. $\mathrm{B}_{\mid}$ *is ω-complete if Act contains an infinite number of actions.*
Proof. This proof has the same structure as the proof of theorem 5.1.1. We will only give the non-trivial steps. Suppose two terms $t, t' \in \mathbb{T}(\mathrm{BCCSP}_{\mid})$ are given. Define $\rho : V \to T(\mathrm{BCCSP}_{\mid})$ as follows:

$$\rho(x) = a_x : \delta$$

where a_x is unique for each $x \in V$ and does not occur in t or t'. We define $R : T(\mathrm{BCCSP}_{\mid}) \to \mathbb{T}(\mathrm{BCCSP}_{\mid})$ by:

$R(\delta) = \delta,$
$R((a_1 \mid \ldots \mid a_n) : t) = (a_1 \mid \ldots \mid a_n) : R(t)$ if $a_i \neq a_x$ for $1 \leq i \leq n$ and $x \in V$,
$R(a_x : t) = x \mathbin{\lfloor\!\lfloor} R(t),$
$R((a_x \mid a_1 \mid \ldots \mid a_n) : t) = x \mid R((a_1 \mid \ldots \mid a_n) : t)$ for $n \geq 1$,
$R((a_1 \mid a_2 \mid \ldots \mid a_n) : t) = R(a_2 \mid \ldots \mid a_n \mid a_1) : t)$ for $n \geq 2$ provided $a_1 \neq a_x$ for all $x \in V$,
$R(t+u) = R(t) + R(u),$
$R(t \parallel u) = R(t) \parallel R(u),$

$R(t \mathbin{⌊\!⌊} u) = R(t) \mathbin{⌊\!⌊} R(u)$,
$R(t \mid u) = R(t) \mid R(u)$.

For ρ and R we now check properties (1), (2) and (3) of theorem 3.1.

(1) Straightforward. In this step the axiom $x \mathbin{⌊\!⌊} \delta = x$ plays an essential role.

(2) Straightforward for almost all cases, the only exception being the action prefix operator $(a_1 \mid \ldots \mid a_n) : x$ where for some a_i $(1 \leq i \leq n)$, $a_i = a_x$ with $x \in V$. Assuming that $B_| \vdash R(t) = R(u)$ for $t, u \in T(\mathrm{BCCSP}_|)$, we show that $B_| \vdash R((a_1 \mid \ldots \mid a_n) : t) = R((a_1 \mid \ldots \mid a_n) : u)$.

$R((a_1 \mid \ldots \mid a_n) : t) =$

(a) $x_j \mid (\ldots \mid (x_{j'} \mid ((a_k \mid \ldots \mid a_{k'}) : R(t)))\ldots) =$
$x_j \mid (\ldots \mid (x_{j'} \mid ((a_k \mid \ldots \mid a_{k'}) : R(u)))\ldots) =$
$R((a_1 \mid \ldots \mid a_n) : u)$ if there is a $1 \leq i \leq n$ such that $a_i \neq a_x$ for all $x \in V$.

(b) $x_1 \mid (\ldots \mid (x_{n-1} \mid (x_n \mathbin{⌊\!⌊} R(t)))\ldots) =$
$x_1 \mid (\ldots \mid (x_{n-1} \mid (x_n \mathbin{⌊\!⌊} R(u)))\ldots) = R((a_1 \mid \ldots \mid a_n) : u)$, otherwise.

(3) Only the axioms containing occurrences of the action prefix operator are non trivial to check. So we consider the axioms $a : (x \mathbin{⌊\!⌊} y) = a : (x \parallel y)$ and $a : x \mid b : y = (a \mid b) : (x \parallel y)$. We start off with the first one. Let $a = (a_1 \mid \ldots \mid a_n)$ and let σ be a closed substitution such that $\sigma(x) = t$ and $\sigma(y) = u$. Three cases must be considered.

(a) $a_i \neq a_x$ for all $1 \leq i \leq n$ and $x \in V$.
$B_| \vdash R(a : t \mathbin{⌊\!⌊} u) = a : R(t) \mathbin{⌊\!⌊} R(u) = a : (R(t) \parallel R(u)) = R(a : (t \parallel u))$.

(b) $a_i = a_{x_i}$ for each $1 \leq i \leq n$ and $x_i \in V$.
$R((a_1 \mid \ldots \mid a_n) : t \mathbin{⌊\!⌊} u) =$
$(x_1 \mid (\ldots \mid (x_{n-1} \mid (x_n \mathbin{⌊\!⌊} R(t)))\ldots)) \mathbin{⌊\!⌊} R(u) =$
$(((x_1 \mid \ldots \mid x_{n-1}) \mid x_n) \mathbin{⌊\!⌊} R(t)) \mathbin{⌊\!⌊} R(u) =$
$((x_1 \mid \ldots \mid x_{n-1}) \mid x_n) \mathbin{⌊\!⌊} (R(t) \parallel R(u)) =$
$(x_1 \mid (\ldots \mid (x_{n-1}) \mid |(x_n \mathbin{⌊\!⌊} (R(t) \parallel R(u))))\ldots)) =$
$R((a_1 \mid \ldots \mid a_n) : (t \parallel u))$.

(c) For some $1 \leq i \leq n$, $a_i \neq a_x$ for all $x \in V$ and for some $1 \leq i \leq n$, $a_i = a_x$.
$R((a_1 \mid \ldots \mid a_n) : t \mathbin{⌊\!⌊} u) =$
$(x_j \mid (\ldots \mid (x_{j'} \mid ((a_{k_1} \mid \ldots \mid a_{k'}) : R(t)))\ldots)) \mathbin{⌊\!⌊} R(u) =$
$(x_j \mid \ldots \mid x_{j'}) \mid ((a_{k_1} \mid \ldots \mid a_{k'}) : R(t) \mathbin{⌊\!⌊} R(u)) =$
$(x_j \mid \ldots \mid x_{j'}) \mid ((a_k \mid \ldots \mid a_{k'}) : (R(t) \parallel R(u))) =$
$x_j \mid (\ldots \mid (x_{j'} \mid ((a_k \mid \ldots \mid a_{k'}) : (R(t) \parallel R(u))))\ldots) =$
$R((a_1 \mid \ldots \mid a_n) : (t \parallel u))$.

We now check the axiom $a : x \mid b : y = (a \mid b) : (x \parallel y)$. We can distinguish 9 cases (cf. checking the axiom $a : x \mathbin{⌊\!⌊} y = a : (x \parallel y)$). We will not discuss all of these, but restrict ourselves to the case where some of the actions, but not all, in a and b have the form a_x.

$R(a : t \mid b : u) =$
$(x_{j_1} \mid (\ldots \mid (x_{j_1'} \mid (a_{k_1} \mid \ldots \mid a_{k_1'}) : R(t))\ldots)) \mid (y_{j_2} \mid (\ldots \mid (y_{j_2'} \mid (b_{k_2} \mid \ldots \mid b_{k_2'}) : R(u))\ldots)) =$
$(x_{j_1} \mid \ldots \mid x_{j_1'} \mid y_{j_2} \mid \ldots \mid y_{j_2'}) \mid ((a_{k_1} \mid \ldots \mid a_{k_1'}) : R(t) \mid (b_{k_2} \mid \ldots \mid b_{k_2'}) : R(u)) =$
$(x_{j_1} \mid (\ldots \mid (x_{j_1'} \mid (y_{j_2} \mid (\ldots \mid (y_{j_2'} \mid ((a_{k_1} \mid \ldots \mid a_{k_1'}) \mid (b_{k_2} \mid \ldots \mid b_{k_2'})) : (R(t) \parallel R(u)))\ldots)))\ldots)) =$
$R((a_1 \mid \ldots \mid a_n \mid b_1 \mid \ldots \mid b_n) : (t \parallel u))$.

In the last step we used example 5.2.1 to rearrange the actions.

□

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