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Counterexamples to Completeness Results for Basic Narrowing

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ABSTRACT

In this paper we analyze completeness results for basic narrowing. We show that basic narrowing is not complete with respect to normalizable solutions for equational theories defined by confluent term rewriting systems, contrary to what has been conjectured. By imposing syntactic restrictions on the rewrite rules we recover completeness. We refute a result of Hölldobler which states the completeness of basic conditional narrowing for complete (i.e. confluent and terminating) conditional term rewriting systems without extra variables in the conditions of the rewrite rules. In the last part of the paper we extend the completeness result of Giovannetti and Moiso for level-confluent and terminating conditional systems with extra variables in the conditions to systems that may also have extra variables in the right-hand sides of the rules.

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1. Introduction

The aim of this paper is to analyze the various completeness results for narrowing in a uniform setting. In order to avoid biting off more than we can chew, we restrict ourselves to ordinary narrowing, basic narrowing, conditional narrowing and basic conditional narrowing. In particular, we do not consider normal narrowing (Fay [15]), the combination of basic and normal narrowing (Réty [40], Nutt *et al.* [38]), narrowing modulo equational theories (Kirchner [29]), nor various narrowing strategies like innermost and lazy narrowing (Fribourg [16] and You [46] respectively, see also Echahed [14]).

Recently there has been much interest in incorporating the logic and functional programming paradigms in a single language. The computational mechanism underlying many of these amalgamated languages is conditional narrowing. Examples include ALF (Hanus [20]), BABEL (Moreno-Navarro and Rodríguez-Artalejo [37]), EQLOG (Goguen and Meseguer [19]), K-LEAF (Giovannetti *et al.* [17]) and SLOG (Fribourg [16]).

Narrowing was first studied in the context of semantic or E-unification. Fay [15] and Hullot [24] showed that narrowing is a complete method for solving equations in the theory defined by a confluent and terminating term rewriting system. Completeness means that for every solution to a given equation, a more general solution can be found by narrowing. It is well-known that the termination requirement can be dropped, provided we restrict ourselves to normalizable solutions. In other words, narrowing is complete for confluent term rewriting systems with respect to normalizable solutions. In order to reduce the search space of narrowing, Hullot [24] introduced the concept of basic narrowing. He showed that basic narrowing is complete for confluent and terminating term rewriting systems. In this paper we show that basic narrowing is not complete for confluent term rewriting systems with respect to normalizable solutions, thereby disproving a conjecture of Yamamoto [45].

Narrowing has been extended to conditional theories by Kaplan [28], Hußmann [26] and Dershowitz and Plaisted [12, 13], among others. Giovannetti and Moiso [18] observed that extra variables in the conditions of the rewrite rules may cause incompleteness (cf. Hußmann [27]). They showed that this incompleteness can be avoided by strengthening confluence to level-confluence. We extend their result to conditional term rewriting systems with extra variables in the right-hand side of the rules. Hölldobler [23] was one of the first to perform a systematic and extensive analysis of various versions of conditional narrowing for conditional term rewriting systems without extra variables. However, we will show that his completeness result for basic conditional narrowing with respect to confluent and terminating conditional term rewriting systems is incorrect. Our counterexample might influence the completeness of ALF (Hanus [20]) since its operational semantics is in essence basic conditional narrowing.

The paper is organized as follows. Section 2 contains a concise introduction to term rewriting and some elementary unification theory. In Section 3 we introduce narrowing and review its completeness. Section 4 is concerned with basic narrowing. We show that completeness is lost if we drop the termination requirement in exchange for the restriction to normalizable solutions, contrary to what is generally believed. In Section 5 we show that orthogonality and right-linearity are sufficient syntactic restrictions for recovering completeness. Conditional narrowing is introduced in Section 6. In Section 7 we show that basic conditional narrowing is not complete for confluent and terminating conditional term rewriting systems. We show that basic conditional narrowing is complete if we strengthen termination to decreasingness, a property of conditional term rewriting systems that implies the decidability of the rewrite relation. In Section 7 we also refute a conjecture of Giovannetti and Moiso [18] about the completeness of basic conditional narrowing for orthogonal conditional term rewriting systems.

Section 8 contains a detailed account of the results of Giovannetti and Moiso [18] concerning the completeness of conditional narrowing for level-confluent systems. In Section 9 we show that conditional narrowing is complete for level-confluent and terminating systems that have extra variables in the right-hand sides of the rewrite rules. Section 10 summarizes the results discussed in detail in previous sections. We mention some open problems and give suggestions for further research.

It is well-known that the correct use of variables and substitutions in completeness proofs requires great care. Several completeness proofs presented in the literature are incorrect with regard to assumptions about variables occurring in narrowing derivations and substitutions. Especially the so-called lifting lemma is notorious in this respect.[3] In the present paper it is our endeavour to give complete and rigorous proofs of the various lifting lemma's and other results. We are aware that easy readability is strained by a fully rigorous treatment of these matters. In order to enhance readability, the proofs of the various lifting lemma's are deferred till Appendix A. Appendix B contains the technical proofs of the propositions that relate certain rewrite sequences to basic narrowing derivations.

2. Preliminaries

In this section we review the basic notions of term rewriting and unification. We refer to Dershowitz and Jouannaud [7] and Klop [32] for extensive surveys.

A *signature* is a set $\mathcal{F}$ of *function symbols.* Associated with every $f \in \mathcal{F}$ is a natural number denoting its arity. Function symbols of arity 0 are called *constants.* The set $\mathcal{T}(\mathcal{F},\mathcal{V})$ of *terms* built from a signature $\mathcal{F}$ and a countably infinite set of *variables* $\mathcal{V}$ is the smallest set such that $\mathcal{V} \subset \mathcal{T}(\mathcal{F},\mathcal{V})$ and if $f \in \mathcal{F}$ has arity n and $t_1,\ldots,t_n \in \mathcal{T}(\mathcal{F},\mathcal{V})$ then $f(t_1,\ldots,t_n) \in \mathcal{T}(\mathcal{F},\mathcal{V})$. We write c instead of $c()$ whenever c is a constant. Identity of terms is denoted by $\equiv$. The set of variables occurring in a term t is denoted by $\mathcal{V}(t)$.

A precise formalism for describing subterm occurrences is obtained through the notion of *position.* The set $O(t)$ of *positions* in a term t is inductively defined as follows:

$$O(t) = \begin{cases} \{\varepsilon\} & \text{if } t \in \mathcal{V}, \\ \{\varepsilon\} \cup \{i.p \mid 1 \leqslant i \leqslant n \text{ and } p \in O(t_i)\} & \text{if } t \equiv f(t_1,\ldots,t_n). \end{cases}$$

So positions are sequences of natural numbers denoting subterm occurrences. If $p \in O(t)$ then $t_{|p}$ denotes the *subterm* of t at position p, i.e.

$$t_{|p} = \begin{cases} t & \text{if } p = \varepsilon, \\ (t_i)_{|q} & \text{if } t \equiv f(t_1,\ldots,t_n) \text{ and } p = i.q. \end{cases}$$

We write $s \subseteq t$ to indicate that s is a subterm of t. If $s \subseteq t$ and $s \not\equiv t$ then s is a *proper* subterm of t. The set $O(t)$ is partitioned into $\overline{O}(t)$ and $O_{\mathcal{V}}(t)$ as follows: $\overline{O}(t) = \{p \in O(t) \mid t_{|p} \notin \mathcal{V}\}$ and $O_{\mathcal{V}}(t) = \{p \in O(t) \mid t_{|p} \in \mathcal{V}\}$. If $p \in O(t)$ then $t[s]_p$ denotes the term that is obtained from t by replacing the subterm at position p by the term s. Formally:

$$t[s]_p = \begin{cases} s & \text{if } p = \varepsilon, \\ f(t_1,\ldots,t_i[s]_q,\ldots,t_n) & \text{if } t \equiv f(t_1,\ldots,t_n) \text{ and } p = i.q. \end{cases}$$

[3] We refer to Appendix A for details.

Positions are partially ordered by the prefix ordering $\leqslant$, i.e. $p \leqslant q$ if there exists an r such that $pr = q$. We write $p < q$ if $p \leqslant q$ and $p \neq q$. Positions p, q are *disjoint*, denoted by $p \perp q$, if neither $p \leqslant q$ nor $q \leqslant p$.

A *substitution* σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F},\mathcal{V})$ such that $\{x \in \mathcal{V} \mid \sigma(x) \not\equiv x\}$ is finite. This set is called the *domain* of σ and denoted by $\mathcal{D}\sigma$. We frequently identify a substitution σ with the set $\{x \mapsto \sigma x \mid x \in \mathcal{D}\sigma\}$. The *empty* substitution will be denoted by ϵ. So $\epsilon = \varnothing$ by abuse of notation. Substitutions are extended to morphisms from $\mathcal{T}(\mathcal{F},\mathcal{V})$ to $\mathcal{T}(\mathcal{F},\mathcal{V})$, i.e. $\sigma(f(t_1,\ldots,t_n)) \equiv f(\sigma(t_1),\ldots,\sigma(t_n))$ for every n-ary function symbol f and terms $t_1,\ldots,t_n$. In the following we write σt instead of $\sigma(t)$. The set of variables *introduced* by σ is denoted by $\mathcal{I}\sigma$, i.e. $\mathcal{I}\sigma = \bigcup_{x\in\mathcal{D}\sigma} \mathcal{V}(\sigma x)$. Composition of substitutions is denoted by '$\circ$', i.e. $(\sigma \circ \tau)x = \sigma(\tau x)$ for all $x \in \mathcal{V}$. We say that a substitution σ is *more general* than a substitution τ, denoted by $\sigma \leqslant \tau$, if there exists a substitution ρ such that $\rho \circ \sigma = \tau$. Let $V \subseteq \mathcal{V}$. The *restriction* $\sigma\restriction_V$ of σ to V is defined as follows:

$$\sigma\restriction_V x = \begin{cases} \sigma x & \text{if } x \in V, \\ x & \text{otherwise.} \end{cases}$$

A *variable renaming* is a bijective substitution. We write $\sigma = \tau\ [V]$ if $\sigma\restriction_V = \tau\restriction_V$ and $\sigma \leqslant \tau\ [V]$ denotes the existence of a substitution ρ such that $\rho \circ \sigma = \tau\ [V]$. Two terms s and t are *unifiable* if there exists a substitution σ, a so-called *unifier* of s and t, such that $\sigma s \equiv \sigma t$. It is well-known that unifiable terms s, t posses a *most general unifier* σ, i.e. $\sigma \leqslant \tau$ for every other unifier τ of s and t. Most general unifiers are unique up to variable renaming.

Let $\sim$ be a binary relation on terms. We say that $\sim$ is *closed under contexts* if $s \sim t$ implies that $u[s]_p \sim u[t]_p$, for all terms u and positions $p \in O(u)$. The relation $\sim$ is *closed under substitutions* if $\sigma s \sim \sigma t$ whenever $s \sim t$, for all substitutions σ.

An *equation* is a pair (s,t) of terms, written as $s = t$. Let E be a set of equations. The smallest symmetric relation that contains E and is closed under contexts and substitutions is denoted by $\leftrightarrow_E$. So $s \leftrightarrow_E t$ if there exist an equation $l = r$ with $l = r \in E$ or $r = l \in E$, a position $p \in O(s)$ and a substitution σ such that $s_{|p} \equiv \sigma l$ and $t \equiv s[\sigma r]_p$. The transitive-reflexive closure of $\leftrightarrow_E$ is denoted by $=_E$. This relation is extended to substitutions as follows: $\sigma =_E \tau$ if $\sigma x =_E \tau x$ for all $x \in \mathcal{V}$. We write $\sigma \leqslant_E \tau$ if there exists a substitution ρ such that $\rho \circ \sigma =_E \tau$. We define $\sigma =_E \tau\ [V]$ and $\sigma \leqslant_E \tau\ [V]$ as above.

Two terms s and t are *E-unifiable* if there exists a substitution σ such that $\sigma s =_E \sigma t$. In the context of a set of equations E, the notion of most general unifier generalizes to *complete sets of E-unifiers.* A set of substitutions Σ is a complete set of E-unifiers of two terms s and t if the following three conditions are satisfied:

- $\mathcal{D}\sigma \subseteq \mathcal{V}(s) \cup \mathcal{V}(t)$ for all $\sigma \in \Sigma$,
- every $\sigma \in \Sigma$ is an E-unifier of s and t,
- if τ is an E-unifier of s and t then there exists a $\sigma \in \Sigma$ such that $\sigma \leqslant_E \tau\ [\mathcal{V}(s) \cup \mathcal{V}(t)]$.

Every set consisting of a most general unifier of terms s and t is a complete set of $\varnothing$-unifiers of s and t.

A *rewrite rule* is a directed equation $l \to r$ satisfying $l \notin \mathcal{V}$ and $\mathcal{V}(r) \subseteq \mathcal{V}(l)$. If $l \to r$ is a rewrite rule and σ a variable renaming then the rewrite rule $\sigma l \to \sigma r$ is called a *variant* of $l \to r$. A *term rewriting system* (TRS for short) is a set of rewrite rules. A rewrite rule $l \to r$ is *left-linear* (*right-linear*) if l (r) does not contain multiple occurrences of the same variable. A *left-linear* (*right-linear*) TRS only contains left-linear (right-linear) rewrite rules.

The *rewrite relation* $\to_\mathcal{R}$ associated with a TRS $\mathcal{R}$ is defined as follows: $s \to_\mathcal{R} t$ if there exist a variant[4] $l \to r$ of a rewrite rule in $\mathcal{R}$, a position $p \in O(s)$ and a substitution σ such that $s_{|p} \equiv \sigma l$ and $t \equiv s[\sigma r]_p$. The term σl is called a *redex* and we say that s rewrites to t by *contracting* redex σl. We call $s \to_{\mathcal{R}} t$ a *rewrite step.* Occasionally we write $s \to_{[p, l \to r, \sigma]} t$ or $s \to_{[p, l \to r]} t$. The transitive-reflexive closure of $\to_{\mathcal{R}}$ is denoted by $\twoheadrightarrow_{\mathcal{R}}$. If $s \twoheadrightarrow_{\mathcal{R}} t$ we say that s *reduces* to t. The transitive closure of $\to_{\mathcal{R}}$ is denoted by $\to^{+}_{\mathcal{R}}$. We write $s \leftarrow_{\mathcal{R}} t$ if $t \to_{\mathcal{R}} s$; likewise for $s \twoheadleftarrow_{\mathcal{R}} t$. The transitive-reflexive-symmetric closure of $\to_{\mathcal{R}}$ is called *conversion* and denoted by $=_{\mathcal{R}}$. If $s =_{\mathcal{R}} t$ then s and t are *convertible.* If E is the set of equations corresponding to $\mathcal{R}$, i.e. $E = \{ l = r \mid l \to r \in \mathcal{R} \}$, then $=_{\mathcal{R}}$ and $=_E$ coincide. Via this correspondence the notion of $\mathcal{R}$-unification is implicitly defined. Two terms t_1, t_2 are *joinable*, denoted by $t_1 \downarrow_{\mathcal{R}} t_2$, if there exists a term t_3 such that $t_1 \twoheadrightarrow_{\mathcal{R}} t_3 \twoheadleftarrow_{\mathcal{R}} t_2$. Such a term t_3 is called a *common reduct* of t_1 and t_2. When no confusion can arise, we omit the subscript $\mathcal{R}$.

A term s is a *normal form* if there is no term t with $s \to t$. A substitution σ is a normal form if σx is a normal form for all $x \in \mathcal{D}\sigma$. A TRS is *weakly normalizing* if every term reduces to a normal form. A TRS is *strongly normalizing* if there are no infinite reduction sequences $t_1 \to t_2 \to t_3 \to \cdots$. In other words, every reduction sequence eventually ends in a normal form. A TRS is *locally confluent* if for all terms s, t_1, t_2 with $t_1 \leftarrow s \to t_2$ we have $t_1 \downarrow t_2$. A TRS is *confluent* or has the *Church-Rosser* property if for all terms s, t_1, t_2 with $t_1 \twoheadleftarrow s \twoheadrightarrow t_2$ we have $t_1 \downarrow t_2$. A well-known equivalent formulation of confluence is that every pair of convertible terms is joinable ($t_1 = t_2 \Rightarrow t_1 \downarrow t_2$). The renowned Newman's Lemma states that every locally confluent and strongly normalizing TRS is confluent. A *complete* TRS is confluent and strongly normalizing. A *semi-complete* TRS is confluent and weakly normalizing. Each term in a (semi-)complete TRS has a unique normal form. The above properties of TRS's specialize to terms in the obvious way. If a term t has a unique normal form then we denote this normal form by $t\downarrow$.

Let $l_1 \to r_1$ and $l_2 \to r_2$ be variants of rewrite rules of a TRS $\mathcal{R}$ without common variables. Suppose $p \in \overline{O}(t)$ such that $(l_1)_{|p}$ and l_2 are unifiable, so $\sigma(l_1)_{|p} \equiv \sigma l_2$ for a most general unifier σ. The term $\sigma l_1 \equiv \sigma l_1[\sigma l_2]_p$ is subject to the reduction steps $\sigma l_1 \to \sigma r_1$ and $\sigma l_1 \to \sigma l_1[\sigma r_2]_p \equiv \sigma(l_1[r_2]_p)$. The pair of reducts $\langle \sigma(l_1[r_2]_p), \sigma r_1 \rangle$ is a *critical pair* of $\mathcal{R}$. If $l_1 \to r_1$ and $l_2 \to r_2$ are variants, we do not consider the case $p = \varepsilon$. A critical pair $\langle s, t \rangle$ is *convergent* if $s \downarrow t$. The well-known Critical Pair Lemma states that a TRS a locally confluent if and only if all its critical pairs are convergent.

A TRS is called *non-ambiguous* or *non-overlapping* if it has no critical pairs. An *orthogonal* TRS is both left-linear and non-ambiguous. For orthogonal TRS's a considerable amount of theory has been developed, see Klop [32] for a comprehensive survey. The most prominent fact is that every orthogonal TRS is confluent. In Section 4 we make use of the work of Huet and Lévy [22] on needed reductions in orthogonal TRS's.

We conclude this section with some information on *multiset orderings.* A *multiset* over a set A is an unordered collection of elements of A in which elements may have multiple occurrences. Every partial order $\succ$ on A can be extended to a partial order $\succ\!\succ$ on the set of finite multisets over A as follows: $M \succ\!\succ N$ if there exists multisets X and Y such that

- $\varnothing \neq X \subseteq M$,
- $N = (M - X) \cup Y$,
- for every $y \in Y$ there exists an $x \in X$ such that $x \succ y$.

The partial order $\succ\!\succ$ is called the *multiset extension* of $\succ$. Dershowitz and Manna [8] showed that the multiset extension of a well-founded order is again well-founded.

[4] The use of variants is not essential for defining the rewrite relation since rewriting is *variant independent*, meaning that if $s \to_{[p, l \to r, \sigma]} t$ and $l' \to r'$ is a variant of $l \to r$ then also $s \to_{[p, l' \to r', \sigma']} t$ for some substitution σ'. However, it states explicitly that we may rename variables when necessary, e.g. when we relate rewriting to narrowing, which is not variant independent in the above sense.

3. Narrowing

In this section we introduce narrowing and review some completeness results. The narrowing relation defined below was introduced by Hullot [24].

DEFINITION 3.1. We say that a term t is *narrowable* into a term t' if there exist a position $p \in \overline{O}(t)$, a variant[5] $l \to r$ of a rewrite rule in $\mathcal{R}$ and a substitution σ such that

- σ is a most general unifier of $t_{|p}$ and l,
- $t' \equiv \sigma(t[r]_p)$.

We write $t \leadsto_{[p,\, l \to r,\, \sigma]} t'$ or simply $t \leadsto_\sigma t'$. The relation $\leadsto$ is called *narrowing.*

NOTATION. We write $t \leadsto_\sigma^* t'$ if there exists a narrowing derivation

$$t \equiv t_1 \leadsto_{\sigma_1} t_2 \leadsto_{\sigma_2} \cdots \leadsto_{\sigma_{n-1}} t_n \equiv t' \qquad ,$$

such that $\sigma = \sigma_{n-1} \circ \cdots \circ \sigma_2 \circ \sigma_1$. If $n = 1$ then $\sigma = \varepsilon$.

In a rewrite step $s \to_{[p,\, l \to r,\, \sigma]} t$ we may always assume that the applied rewrite rule has no variables in common with s and σ is restricted to variables occurring in l. Consequently, σ is a most general unifier of $s_{|p}$ and l, and $t \equiv s[\sigma r]_p \equiv \sigma(s[r]_p)$. Hence rewriting can be viewed as a special case of narrowing.

A nice explanation of the word 'narrowing' can be found in Klop [31]. We now explain how narrowing can be used for equational unification. In order to facilitate the exposition, we extend the set of function symbols with a fresh binary function symbol $=^?$ and a fresh constant *true*. We furthermore assume that $\mathcal{R}$ contains the rewrite rule $x =^? x \to \mathit{true}$.[6] We consider only terms of the following form:

- terms that do not contain any occurrences of $=^?$ and *true*,
- terms $s =^? t$ with s and t satisfying the previous condition,
- the constant *true*.

Terms of the second form are called *goals.* It should be stressed that confluence, completeness and semi-completeness are retained under the addition of the rule $x =^? x \to \mathit{true}$.[7]

EXAMPLE 3.2. Consider the TRS

$$\mathcal{R} = \left\{ \begin{array}{lcl} 0 + x & \to & x \\ S(x) + y & \to & S(x + y). \end{array} \right.$$

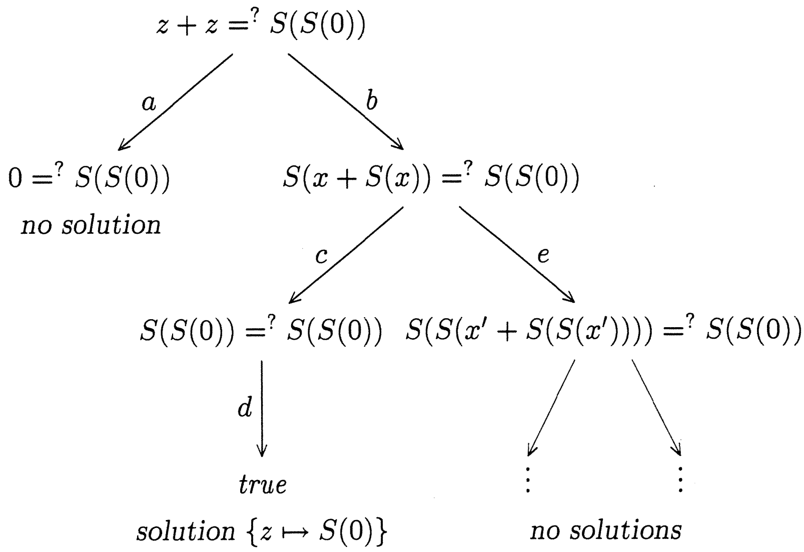
Suppose we want to solve the goal $z + z =^? S(S(0))$. Figure 1 shows that narrowing is able to find the (unique) solution $\{z \mapsto S(0)\}$. This is not a coincidence: below we will see that narrowing is able to find all most general solutions of a given goal, provided $\mathcal{R}$ satisfies certain conditions.

The soundness of narrowing is expressed in the next lemma.

[5] Renaming of rewrite rules is mandatory for ensuring completeness. The idea is to use a single variant of a rewrite rule and a single most general unifier, in order to avoid unnecessary computations. We always require that the rewrite rule has no variables in common with the term to be narrowed, i.e. $\mathcal{V}(l) \cap \mathcal{V}(t) = \varnothing$, but in general this is not sufficient for completeness. From the proof of Lemma 3.4 below, the precise requirements of freshness can be deduced. That proof makes also clear that any idempotent most general unifier is adequate for completeness.

[6] This assumption will not be made when we consider orthogonal TRS's.

[7] This even holds if we would allow unrestricted term formation, due to modularity considerations; see Middeldorp [35].



step	rewrite rule			narrowing substitution
a	$0 + x$	$\rightarrow$	x	$\{x, z \mapsto 0\}$
b	$S(x) + y$	$\rightarrow$	$S(x + y)$	$\{y, z \mapsto S(x)\}$
c	$0 + x'$	$\rightarrow$	x'	$\{x \mapsto 0,\ x' \mapsto S(0)\}$
d	$x =^? x$	$\rightarrow$	*true*	$\{x \mapsto S(S(0))\}$
e	$S(x') + y'$	$\rightarrow$	$S(x' + y')$	$\{x \mapsto S(x'),\ y' \mapsto S(S(x'))\}$

FIGURE 1.

LEMMA 3.3. *Let $\mathcal{R}$ be a TRS. If $s =^? t \rightsquigarrow_\sigma^* true$ then σ is an $\mathcal{R}$-unifier of s and t.*

PROOF. Easy induction on the length of the narrowing derivation $s =^? t \rightsquigarrow_\sigma^* true$, using the observation that $\sigma' s' \rightarrow_{\mathcal{R}} t'$ whenever $s' \rightsquigarrow_{\sigma'} t'$. □

The following lemma of Hullot [24] is the key to completeness. It states that rewrite sequences can be 'lifted' to narrowing derivations. A rigorous proof of this lifting lemma can be found in Appendix A. The proof presented in [24] is incorrect with regard to the normalization of the resulting substitution θ'.

LEMMA 3.4. *Let $\mathcal{R}$ be a TRS. Suppose we have terms s and t, a normalized substitution θ and a set of variables V such that $\mathcal{V}(s) \cup \mathcal{D}\theta \subseteq V$ and $t \equiv \theta s$. If $t \twoheadrightarrow t'$ then there exist a term s' and substitutions θ', σ such that*

- $s \rightsquigarrow_\sigma^* s'$,
- $\theta' s' \equiv t'$,
- $\theta' \circ \sigma = \theta\ [V]$,
- *θ' is normalized.*

Furthermore, we may assume that the narrowing derivation $s \rightsquigarrow_\sigma^* s'$ and the rewrite sequence $t \twoheadrightarrow t'$ employ the same rewrite rules at the same positions. □

THEOREM 3.5 (Hullot [24]). *Let $\mathcal{R}$ be a complete TRS. If $\sigma s =_{\mathcal{R}} \sigma t$ then there exists a narrowing derivation $s =^? t \rightsquigarrow_\tau^* true$ such that $\tau \leq_{\mathcal{R}} \sigma\ [\mathcal{V}(s) \cup \mathcal{V}(t)]$.*

PROOF. Let σ' be the normal form of σ, i.e. $\sigma' = \{x \mapsto (\sigma x){\downarrow} \mid x \in \mathcal{D}\sigma\}$. Notice that $\sigma =_{\mathcal{R}} \sigma'$. Clearly $\sigma' s =_{\mathcal{R}} \sigma' t$. Confluence of $\mathcal{R}$ yields $\sigma' s \downarrow \sigma' t$. Hence there exists a rewrite sequence $\sigma'(s =^? t) \twoheadrightarrow \mathit{true}$. According to Lemma 3.4 there exists a narrowing derivation $s =^? t \rightsquigarrow_\tau^* \mathit{true}$ and a substitution σ'' such that $\sigma'' \circ \tau = \sigma'\ [\mathcal{V}(s) \cup \mathcal{V}(t)]$. Therefore $\tau \leqslant \sigma'\ [\mathcal{V}(s) \cup \mathcal{V}(t)]$. Since $\sigma =_{\mathcal{R}} \sigma'$ we conclude that $\tau \leqslant_{\mathcal{R}} \sigma\ [\mathcal{V}(s) \cup \mathcal{V}(t)]$. $\square$

In the following, statements like Theorem 3.5 will be abbreviated by saying that (a kind of) narrowing is complete for (a class of) TRS's (with respect to certain goals and substitutions). The reason for this terminology becomes apparent in the following equivalent formulation of Theorem 3.5.

COROLLARY 3.6. *Let $\mathcal{R}$ be a complete TRS. The set $\{\sigma\restriction_{\mathcal{V}(s)\cup\mathcal{V}(t)} \mid s =^? t \rightsquigarrow_\sigma^* \mathit{true}\}$ is a complete set of $\mathcal{R}$-unifiers of s and t.* $\square$

From the above proof it is clear that the subscript $\mathcal{R}$ in $\tau \leq_{\mathcal{R}} \sigma$ can be dropped if we only consider normalized substitutions. Strong normalization of $\mathcal{R}$ is only used in the normalization of σ into σ', hence we can strengthen Theorem 3.5 by dropping the strong normalization requirement and restricting ourselves to *normalizable* substitutions.

THEOREM 3.7. *Narrowing is complete for confluent TRS's with respect to normalizable substitutions.* $\square$

Since in a weakly normalizing TRS every substitution is normalizable, we obtain the following result of Yamamoto [45].

COROLLARY 3.8. *Narrowing is complete for semi-complete TRS's.* $\square$

4. Basic Narrowing

The search space of narrowing is quite large. As a matter of fact, the narrowing procedure seldom terminates. Hullot [24] introduced a restricted form of narrowing, the so-called *basic narrowing*, which still is complete for complete TRS's.

DEFINITION 4.1.

(1) A narrowing derivation

$$t_1 \rightsquigarrow_{[p_1, l_1 \to r_1, \sigma_1]} t_2 \rightsquigarrow_{[p_2, l_2 \to r_2, \sigma_2]} \cdots \rightsquigarrow_{[p_{n-1}, l_{n-1} \to r_{n-1}, \sigma_{n-1}]} t_n$$

is *basic* if $p_i \in B_i$ for $1 \leqslant i \leqslant n-1$ where the sets of positions $B_1, \ldots, B_{n-1}$ are inductively defined as follows:

$$\begin{aligned} B_1 &= \overline{O}(t_1), \\ B_{i+1} &= \mathcal{B}(B_i, p_i, r_i) \qquad \text{for } 1 \leqslant i < n-1. \end{aligned}$$

Here $\mathcal{B}(B_i, p_i, r_i)$ abbreviates $(B_i - \{\, q \in B_i \mid p_i \leqslant q \,\}) \cup \{\, p_i q \mid q \in \overline{O}(r_i) \,\}$. If the above derivation is basic then the positions in B_i $(1 \leqslant i \leqslant n-1)$ are referred to as *basic positions.*

(2) A rewrite sequence

$$t_1 \to_{[p_1, l_1 \to r_1, \sigma_1]} t_2 \to_{[p_2, l_2 \to r_2, \sigma_2]} \cdots \to_{[p_{n-1}, l_{n-1} \to r_{n-1}, \sigma_{n-1}]} t_n$$

is *based* on a set of positions $B_1 \subseteq \overline{O}(t_1)$ if $p_i \in B_i$ for $1 \leqslant i \leqslant n-1$ with $B_2, \ldots, B_{n-1}$ defined as above.

So in a basic derivation narrowing is never applied to a subterm introduced by a previous narrowing substitution. It should be noted that the concepts defined above do not depend on the used variants of rewrite rules. It is not difficult to show that the sets B_i defined above are closed under prefix[8] for every rewrite sequence that is based on a set which is closed under prefix. This observation will be used freely in the sequel.

EXAMPLE 4.2. Consider the complete TRS $\mathcal{R} = \{f(f(x)) \rightarrow x\}$. The infinite sequence

$$f(x) =^? x \rightsquigarrow_{\{x \mapsto f(x')\}} x' =^? f(x') \rightsquigarrow_{\{x' \mapsto f(x'')\}} f(x'') =^? x'' \rightsquigarrow_{\{x'' \mapsto f(x''')\}} \cdots$$

is the only narrowing derivation issued from the goal $f(x) =^? x$. It is not basic since the restriction $p_2 \in B_2$ is violated if we take $B_1 = \overline{O}(t_1) = \{\varepsilon, 1\}$ and $B_2 = \{\varepsilon\}$. In later examples we simply state that a given narrowing derivation is basic without giving (the reconstructible) justification, i.e. the sets B_i.

Hullot showed that if all basic narrowing derivations starting at a right-hand side of a rewrite rule terminate, then the search space of basic narrowing is finite for any term. Recently, Chabin and Réty [6] showed that the termination behaviour of basic narrowing can be further improved by adopting a graph representation of the TRS and the goal to be solved.

Herold [21] showed that the sets B_i can be reduced by means of *left-to-right* basic narrowing, without losing completeness. The search space can be further reduced by means of the so-called *selection* narrowing of Bosco *et al.* [5]. In this paper we do not consider these optimizations, but we note that all our results concerning basic (conditional) narrowing—both positive and negative—extend to selection narrowing. Krischer and Bockmayr [33] describe various criteria to detect redundant basic narrowing derivations.

A more elegant formulation of basic narrowing is obtained by partitioning goals into a *skeleton* and *environment part* as in Nutt *et al.* [38] and Hölldobler [23]. In such a formulation narrowing would be defined on pairs $\langle t, \theta \rangle$, consisting of a term t (the skeleton) and a substitution θ (the environment), as follows: $\langle t, \theta \rangle \rightsquigarrow_\sigma \langle t[r]_p, \sigma \circ \theta \rangle$ where $p \in \overline{O}(t)$ and σ is a most general unifier of $(\theta t)_{|p}$ and l for some rewrite rule $l \rightarrow r$. The main reason for adopting the 'standard' definition is that we can still use Lemma 3.4 whereas the above formulation requires a more complicated lifting lemma (in order to ensure completeness of basic narrowing for complete TRS's).

Besides the lifting lemma, the completeness proof of basic narrowing employs Proposition 4.4. A proof of this proposition is given in Appendix B.

DEFINITION 4.3. An *innermost* redex does not contain smaller redexes. In an innermost reduction sequence only innermost redexes are contracted.

PROPOSITION 4.4 (Hullot [25], Yamamoto [45]). *Let $\mathcal{R}$ be a TRS and σ a normalized substitution. Every innermost reduction sequence starting from σt is based on $\overline{O}(t)$.* □

THEOREM 4.5 (Hullot [24, 25]). *Basic narrowing is complete for complete TRS's.*

PROOF. Suppose $\sigma s =_{\mathcal{R}} \sigma t$. Let σ' be the normal form of σ. Just as in the proof of Theorem 3.5 we obtain $\sigma'(s =^? t) \twoheadrightarrow true$. Because $\mathcal{R}$ is complete we may assume that this reduction sequence is innermost. According to the previous proposition the sequence is based on $\overline{O}(s =^? t)$. Since the narrowing derivation constructed by Lemma 3.4 employs the same rewrite rules at the same positions, we know that it is basic. The remainder of the proof follows literally the proof of Theorem 3.5. □

[8] That is, if $p < q$ and $q \in B_i$ then $p \in B_i$.

Several authors (Yamamoto [45], Hölldobler [23]) reported a mistake in the proof of Hullot as given in [24]. Less well-known is the fact that Hullot himself was the first to repair the proof, see Hullot's thesis [25]. Yamamoto observed that strong normalization can be weakened to *weakly innermost normalization.* A TRS is called weakly innermost normalizing if every term has a normal form that can be reached by means of an innermost reduction sequence. More interesting is the following statement.

CONJECTURE 4.6 (Yamamoto [45]). *Basic narrowing is complete for semi-complete TRS's.* □

COUNTEREXAMPLE 4.7. Consider the TRS

$$\mathcal{R} = \left\{ \begin{array}{lcl} f(x) & \to & g(x,x) \\ a & \to & b \\ g(a,b) & \to & c \\ g(b,b) & \to & f(a). \end{array} \right.$$

Induction on the structure of terms and straightforward case analysis reveals that every term has a unique normal form. Hence $\mathcal{R}$ is semi-complete. However, the goal $f(a) =^? c$ cannot be solved by basic narrowing. Figure 2 shows all narrowing derivations starting from this goal. (Since the goal is variable-free, all narrowing steps in the figure are rewrite steps.) The steps

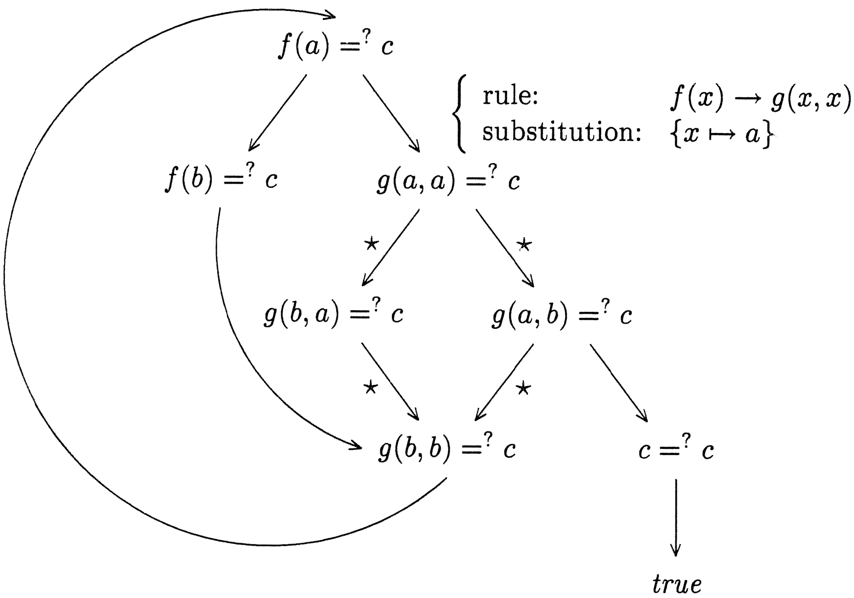


FIGURE 2.

marked with a star are non-basic because each of them rewrites an occurrence of the term a introduced by the substitution $\{x \mapsto a\}$ used in the step from $f(a) =^? c$ to $g(a,a) =^? c$. Since every successful derivation passes through a marked step, basic narrowing is not able to solve the goal $f(a) =^? c$.

In particular basic narrowing is not complete for confluent TRS's with respect to normalizable substitutions, contrary to what is generally believed (as noted by Palamidessi [39]). In the next section we recover the completeness of basic narrowing for semi-complete TRS's by imposing syntactic restrictions on the rewrite rules.

5. Restoring Completeness

Counterexample 4.7 suggests two sufficient conditions for the completeness of basic narrowing for semi-complete TRS's: *orthogonality* and *right-linearity*. We first show the sufficiency of orthogonality. The proof is based on the work of Huet and Lévy [22] on needed reductions. Before stating their main result, we introduce a few preliminary concepts.

DEFINITION 5.1. Let A: $s \rightarrow_{[p, l \rightarrow r]} t$ be a reduction step in a TRS $\mathcal{R}$ and let $q \in O(s)$. The set $q \backslash A$ of *descendants* of q in t is defined as follows:

$$q \backslash A = \begin{cases} \{q\} & \text{if } q < p \text{ or } q \perp p, \\ \{p p_3 p_2 \mid r_{|p_3} \equiv l_{|p_1}\} & \text{if } q = p p_1 p_2 \text{ with } p_1 \in O_{\mathcal{V}}(l), \\ \varnothing & \text{otherwise.} \end{cases}$$

If $Q \subseteq O(s)$ then $Q \backslash A$ denotes the set $\bigcup_{q \in Q} q \backslash A$. The notion of descendant is extended to rewrite sequences in the obvious way. Orthogonal TRS's have the nice property that a descendant of a redex is again a redex (with respect to the same rewrite rule).

DEFINITION 5.2. A redex s in a term t is *needed* if in every reduction sequence from t to normal form a descendant of s is contracted. In a needed reduction sequence only needed redexes are contracted.

THEOREM 5.3 (Huet and Lévy [22]). *Let t be a term in an orthogonal TRS.*

- *If t is not a normal form then t contains a needed redex.*
- *If t has a normal form, repeated contraction of needed redexes leads to that normal form.*

□

PROPOSITION 5.4. *Let $\mathcal{R}$ be an orthogonal TRS and σ a normalized substitution. Every innermost needed reduction sequence starting from σt is based on $\overline{O}(t)$.*

PROOF. See Appendix B. □

The formulation of the completeness theorem for basic narrowing with respect to normalizable solutions in the context of orthogonal TRS's is slightly different than previous completeness results. The reason is that the rewrite rule $x =^? x \rightarrow true$ cannot be used since it disturbs left-linearity. This also explains why we have to require the normalizability of σs and σt.

THEOREM 5.5. *Let $\mathcal{R}$ be an orthogonal TRS. If $\sigma s =_{\mathcal{R}} \sigma t$ and σ, σs and σt are normalizable then there exists a basic narrowing derivation $s =^? t \rightsquigarrow^*_{\tau} s' =^? t'$ and a most general unifier τ' of s' and t' such that $\tau' \circ \tau \leqslant_{\mathcal{R}} \sigma$ $[\mathcal{V}(s) \cup \mathcal{V}(t)]$.*

PROOF. Let σ' be the normal form of σ. By confluence, the terms σs, $\sigma' s$, σt and $\sigma' t$ have the same normal form n. Thus there exists a sequence $\sigma'(s =^? t) \twoheadrightarrow n =^? n$. Due to the absence of the rule $x =^? x \rightarrow true$, the term $n =^? n$ is a normal form. According to Theorem 5.3 we may assume that in the rewrite sequence from $\sigma'(s =^? t)$ to $n =^? n$ only innermost needed redexes are contracted. From Proposition 5.4 we learn that the sequence is based on $\overline{O}(s =^? t)$ and hence the narrowing derivation constructed by Lemma 3.4 is basic. The remainder of the proof follows is similar to the previous completeness proofs. □

The above completeness result has been independently obtained by Giovannetti and Moiso (Moiso [36]).

COROLLARY 5.6. *Basic narrowing is complete for weakly normalizing orthogonal TRS's.* □

The following example shows that the normalizability requirement of σs and σt in Theorem 5.5 is essential.

EXAMPLE 5.7. Consider the orthogonal TRS

$$\mathcal{R} = \begin{cases} f(x) & \rightarrow & h(x,x) \\ g(x) & \rightarrow & h(x,i(x)) \\ a & \rightarrow & i(a). \end{cases}$$

The following narrowing derivation shows that the goal $f(a) =^? g(a)$ can be solved:

$$\begin{aligned} f(a) =^? g(a) & \rightsquigarrow h(a,a) =^? g(a) \\ & \rightsquigarrow h(a,a) =^? h(a,i(a)) \\ & \rightsquigarrow h(a,i(a)) =^? h(a,i(a)). \end{aligned}$$

The third step in the above sequence is non-basic. One easily shows that $f(a)$ and $g(a)$ have no common reduct with respect to basic narrowing. Hence the goal $f(a) =^? g(a)$ cannot be solved by basic narrowing.

Furthermore, orthogonality cannot be weakened to non-ambiguity.

EXAMPLE 5.8. Consider the TRS

$$\mathcal{R} = \begin{cases} f(x) & \rightarrow & g(x,h(x)) \\ g(x,x) & \rightarrow & a \\ b & \rightarrow & h(b). \end{cases}$$

Since there are no critical pairs, $\mathcal{R}$ is non-ambiguous. With some effort we can show that $\mathcal{R}$ is confluent, notwithstanding the presence of the non-left-linear rule $g(x,x) \rightarrow a$. We have $f(b) \twoheadrightarrow_{\mathcal{R}} a$ but basic narrowing is not able to solve the goal $f(b) =^? a$.

Notice that the TRS in Example 5.8 is not weakly normalizing. We conjecture that basic narrowing is complete for semi-complete non-ambiguous TRS's. We now consider the sufficiency of right-linearity. The following useful notion is inspired by a similar notion introduced by Réty [40].

DEFINITION 5.9. Let A: $s \rightarrow_{[p,l \rightarrow r]} t$ be a reduction step in a TRS $\mathcal{R}$. A position $q \in O(s)$ is called an *antecedent* of a position $q' \in O(t)$ if q' is a descendant of q. The set of antecedents of q' in s is denoted by A/q'. This notion is extended to sets of positions in the obvious way.

The next proposition is the key result for proving the sufficiency of right-linearity for the completeness of basic narrowing for confluent TRS's with respect to normalizable solutions. We will make a small concession with regard to our endeavour to rigorous proofs: statements that depend on the easy but tedious interplay between antecedents and basic positions are not proved in full detail. We feel that such detail would veil the structure of the proof. The transformation presented in the proof is illustrated in Example 5.11 below.

Proposition 5.10. *Let $\mathcal{R}$ be a right-linear TRS and σ a normalized substitution. Every reduction sequence starting from σt can be transformed into a reduction sequence that is based on $\overline{O}(t)$.*

Proof. We use induction on the length of the reduction sequence starting from σt. If the length equals zero then we have nothing to prove. Let

$$\sigma t \equiv t_1 \rightarrow_{[p_1, l_1 \rightarrow r_1]} \cdots \rightarrow_{[p_{n-1}, l_{n-1} \rightarrow r_{n-1}]} t_n \rightarrow_{[p_n, l_n \rightarrow r_n]} t_{n+1}$$

be a reduction sequence of n steps. Define $B_1, \ldots, B_n$ as usual. According to the induction hypothesis we may assume that $p_i \in B_i$ for $i = 1, \ldots, n-1$. If $p_n \in B_n$ then the whole sequence is based on $\overline{O}(t)$. So assume that $p_n \notin B_n$. Define sets of positions $V_1, \ldots, V_n, W_1, \ldots, W_n$ as follows:

- $V_n = \{p_n\}$,
- $V_i = A_i/(V_{i+1} - B_{i+1})$ for $i = n-1, \ldots, 1$ (here A_i is the reduction step from t_i to t_{i+1}),
- $W_i = V_i \cap B_i$ for $i = 1, \ldots, n$.

Using the fact that $p_i \in B_i$, it is not difficult to show that $q \not\leqslant p_i$ whenever $q \in V_{i+1} - B_{i+1}$, for $i = 1, \ldots, n-1$. From this we easily obtain that $(t_i)_{|q} \equiv (t_n)_{|p_n}$ for all $q \in V_i$. With some effort we can show that for every $q \in V_i$ either $q \perp p_i$ or q can be written as $q = p_i\, q'\, q''$ for some $q' \in O_{\mathcal{V}}(l_i)$. Moreover, if $q \in W_i$ then only the second case applies. Let m be the smallest index such that $V_m \neq \varnothing$. We now construct the following diagram:

$$\begin{tikzcd}
t_m \arrow[r, "p_m"] \arrow[d, "W_m = V_m"] & t_{m+1} \arrow[r, "p_{m+1}"] \arrow[d, "V_m \backslash A_m"] & t_{m+2} \arrow[r, "p_{m+2}"] \arrow[d, "V_{m+1} \backslash A_{m+1}"] & \cdots \arrow[r, "p_{n-1}"] & t_n \arrow[r, "p_n"] \arrow[d, "V_{n-1} \backslash A_{n-1}"] & t_{n+1} \\
t'_m \arrow[r, "p_m"'] & t'_{m+1} \arrow[d, "W_{m+1}"] & t'_{m+2} \arrow[d, "W_{m+2}"] & \cdots & t'_n \arrow[ur, dashed, no head, "\equiv"'] & \\
 & t''_{m+1} \arrow[ur, "p_{m+1}"'] & t''_{m+2} \arrow[ur, "p_{m+2}"'] & \cdots \arrow[ur, "p_{n-1}"'] & &
\end{tikzcd}$$

A few remarks are in order. First note that $V_m \subseteq B_m$: if $m > 1$ then this follows by definition; the normalization of σ yields $V_1 \subseteq B_1$. Therefore $V_m = W_m$. Right-linearity of $\mathcal{R}$ yields $V_i \backslash A_i \subseteq V_{i+1}$ and hence $V_i \backslash A_i \cup W_{i+1} = V_{i+1}$. Observe that p_i is a redex position in t'_i even if the rewrite rule $l_i \rightarrow r_i$ is non-left-linear. Since $V_{n-1} \backslash A_{n-1} = \{p_n\}$ we have $t'_n \equiv t_{n+1}$. Finally, it is straightforward to show that the reduction sequence

$$t_m \nrightarrow\!\!\!\!\!\!\nrightarrow t'_m \rightarrow t'_{m+1} \nrightarrow\!\!\!\!\!\!\nrightarrow t''_{m+1} \rightarrow t'_{m+2} \nrightarrow\!\!\!\!\!\!\nrightarrow t''_{m+2} \rightarrow \cdots \rightarrow t'_n$$

is based on B_m and thus we succeeded in constructing a reduction sequence from σt to t_{n+1} that is based on $\overline{O}(t)$. $\square$

Example 5.11. Consider the right-linear TRS

$$\mathcal{R} = \left\{ \begin{array}{lcl} f(x,x) & \rightarrow & g(i(b),x) \\ g(x,x) & \rightarrow & h(x) \\ i(x) & \rightarrow & j(x) \\ a & \rightarrow & b \end{array} \right.$$

and the non-basic reduction sequence

$$f(i(a),i(b)) \rightarrow f(i(b),i(b)) \rightarrow g(i(b),i(b)) \rightarrow h(i(b)) \rightarrow h(j(b)).$$

The information extracted from this sequence in the proof of Proposition 5.10 is summarized in the following table:

i	t_i	p_i	B_i	V_i	W_i
1	$f(i(a), i(b))$	1.1	$O(t_1)$	$\varnothing$	$\varnothing$
2	$f(i(b), i(b))$	ε	$O(t_2)$	$\{1, 2\}$	$\{1, 2\}$
3	$g(i(b), i(b))$	ε	$\{\varepsilon, 1, 1.1\}$	$\{1, 2\}$	$\{1\}$
4	$h(i(b))$	1	$\{\varepsilon\}$	$\{1\}$	$\varnothing$
5	$h(j(b))$				

This gives rise to the construction

$$\begin{tikzcd}
f(i(a),i(b)) \arrow[r] & f(i(b),i(b)) \arrow[r] \arrow[d, "W_2"', "\|" marking] & g(i(b),i(b)) \arrow[r] \arrow[d] & h(i(b)) \arrow[d] \\
& f(j(b),j(b)) \arrow[r] & g(i(b),j(b)) \arrow[d, "W_3"'] & h(j(b)) \\
& & g(j(b),j(b)) \arrow[ur] &
\end{tikzcd}$$

from which we obtain the basic reduction sequence

$$f(i(a), i(b)) \to f(i(b), i(b)) \mathrel{\,\|\!\!\!\!\rightarrow} f(j(b), j(b)) \to g(i(b), j(b)) \to g(j(b), j(b)) \to h(j(b)).$$

Notice that there are two further antecedents of $i(b)$, viz. the underlined subterms in $f(\underline{i(a)}, \underline{i(b)})$. These antecedents didn't make their presence into V_1, and with reason: if we start our detour at $f(i(a), i(b))$ instead of $f(i(b), i(b))$ we do not end up with a *basic* sequence.

THEOREM 5.12. *Basic narrowing is complete for confluent right-linear TRS's with respect to normalizable substitutions.*

PROOF. Similar to the proof of Theorem 4.5. The only difference is the replacement of Proposition 4.4 by Proposition 5.10. □

Contrary to the situation in Theorem 5.5 there is no need to restrict ourselves to goals $s =^? t$ and normalizable solutions σ such that σs and σt are normalizable.

COROLLARY 5.13. *Basic narrowing is complete for semi-complete right-linear TRS's.* □

6. Conditional Narrowing

Before introducing conditional narrowing, we give a short review of conditional rewriting.

The rules of a *conditional term rewriting system* (CTRS for short) have the form $l \to r \Leftarrow c$. Here the conditional part c is a (possibly empty) sequence $s_1 = t_1, \ldots, s_n = t_n$ of equations. At present we only require that l is not a single variable. A rewrite rule without conditions will be written as $l \to r$. The rewrite relation associated with a CTRS $\mathcal{R}$ is obtained by interpreting the equality signs in the conditional part of a rewrite rule as joinability. Formally: $s \to_{\mathcal{R}} t$ if there exist a position $p \in O(s)$, a variant $l \to r \Leftarrow c$ of a rewrite rule in $\mathcal{R}$ and a substitution σ such that $s_{|p} \equiv \sigma l$, $t \equiv s[\sigma r]_p$ and $\sigma s' \downarrow_{\mathcal{R}} \sigma t'$ for every equation $s' = t'$ in c. All notions that we defined in Section 2 for TRS's extend to CTRS's.

The various completeness results for conditional narrowing put different restrictions on the distribution of variables among rewrite rules. The next definition makes these restrictions explicit.

DEFINITION 6.1. The set of variables occurring in a conditional rewrite rule R: $l \to r \Leftarrow c$ is denoted by $\mathcal{V}(R)$ and $\mathcal{E}(R)$ denotes the set of *extra* variables occurring in R, i.e. $\mathcal{E}(R) = \mathcal{V}(R) - \mathcal{V}(l)$. Every rewrite rule $l \to r \Leftarrow c$ is classified according to the distribution of variables among l, r and c, as follows:

type	requirement
1	$\mathcal{V}(r) \cup \mathcal{V}(c) \subseteq \mathcal{V}(l)$
2	$\mathcal{V}(r) \subseteq \mathcal{V}(l)$
3	$\mathcal{V}(r) \subseteq \mathcal{V}(l) \cup \mathcal{V}(c)$
4	*no restrictions*

An n-CTRS contains only rules of type n. So a 1-CTRS contains no extra variables, a 2-CTRS may only contain extra variables in the conditions and a 3-CTRS may even have extra variables in the right-hand sides provided these also occur in the corresponding conditional part. A 4-CTRS will simply be called CTRS.

Most of the literature on conditional term rewriting is concerned with 1 and 2-CTRS's. Just as in the unconditional case, we assume that our CTRS's contain the rule $x =^? x \to true$.

NOTATION. If c is the sequence of equations $s_1 = t_1, \ldots, s_n = t_n$ then $\tilde{c}$ denotes the multiset[9] $\{s_1 =^? t_1, \ldots, s_n =^? t_n\}$.

DEFINITION 6.2. Let $\mathcal{R}$ be a CTRS. We inductively define TRS's $\mathcal{R}_n$ for $n \geqslant 0$ as follows:[10]

$$\begin{aligned} \mathcal{R}_0 &= \{\, x =^? x \to \mathit{true} \,\}, \\ \mathcal{R}_{n+1} &= \{\, \sigma l \to \sigma r \mid l \to r \Leftarrow c \in \mathcal{R} \text{ and } \sigma e \twoheadrightarrow_{\mathcal{R}_n} \mathit{true} \text{ for all } e \in \tilde{c} \,\}. \end{aligned}$$

Observe that $\mathcal{R}_n \subseteq \mathcal{R}_{n+1}$ for all $n \geqslant 0$. We have $s \to t$ if and only if $s \to_{\mathcal{R}_n} t$ for some $n \geqslant 0$. The minimum such n is called the *depth* of $s \to t$.

We are now ready to define conditional narrowing. In the literature several different formulations are given (e.g. Kaplan [28], Dershowitz and Plaisted [12], Hußmann [26], Giovannetti and Moiso [18], Bockmayr [2]). In this paper we follow the natural approach of Bockmayr [2]. In this approach narrowing is directly defined on finite multisets[11] of goals, the so-called *goal*

[9] See footnote [12] on page 19.

[10] The usual definition

$$\begin{aligned} \mathcal{R}_0 &= \varnothing, \\ \mathcal{R}_{n+1} &= \{\, \sigma l \to \sigma r \mid l \to r \Leftarrow c \in \mathcal{R} \text{ and } \sigma s \downarrow_{\mathcal{R}_n} \sigma t \text{ for all } s = t \text{ in } c \,\} \end{aligned}$$

does not rely on the presence of the rule $x =^? x \to true$. For terms without occurrences of $=^?$ these relations coincide with the ones of Definition 6.2.

[11] Representing goal clauses by multisets facilitates the definition of basic narrowing in Section 7. As far as conditional narrowing is concerned, we might as well opt for a set representation.

clauses. In examples goal clauses are always presented as sequences of goals, i.e. we omit the curly brackets. The rewrite relation $\to_{\mathcal{R}}$ extends to goal clauses in the obvious way. The extended relation inherits all properties (confluence, strong normalization, ...) of the original $\to_{\mathcal{R}}$. The set of variables occurring in a goal clause T will be denoted by $\mathcal{V}(T)$.

DEFINITION 6.3. Let $\mathcal{R}$ be a CTRS. A goal clause S *conditionally narrows* into a goal clause T if there exist a goal $e \in S$, a position $p \in \overline{O}(e)$, a variant R: $l \to r \Leftarrow c$ of a conditional rewrite rule in $\mathcal{R}$ and a substitution σ such that

- σ is a most general unifier of $e_{|p}$ and l,
- $T = \sigma(S - \{e\} \cup \{e[r]_p\} \cup \tilde{c})$.

We write $S \rightsquigarrow_{[e,p,R,\sigma]} T$ or simply $S \rightsquigarrow_\sigma T$.

EXAMPLE 6.4. Consider the CTRS

$$\mathcal{R} = \begin{cases} even(0) & \to & t & & \\ even(S(x)) & \to & odd(x) & & \\ odd(x) & \to & t & \Leftarrow & even(x) = f \\ odd(x) & \to & f & \Leftarrow & even(x) = t \end{cases}$$

and the goal $even(S(y)) =^? t$. The following derivation shows that the solution $\{y \mapsto S(0)\}$ is found by conditional narrowing:

$$\begin{array}{lll} even(S(y)) =^? t & \rightsquigarrow & odd(y) =^? t \\ & \rightsquigarrow & t =^? t,\ even(y) =^? f \\ & \rightsquigarrow_{\sigma_1} & t =^? t,\ odd(x) =^? f \\ & \rightsquigarrow & t =^? t,\ f =^? f,\ even(x) =^? t \\ & \rightsquigarrow_{\sigma_2} & t =^? t,\ f =^? f \\ & \rightsquigarrow^* & true,\ true. \end{array}$$

Here $\sigma_1 = \{y \mapsto S(x)\}$ and $\sigma_2 = \{x \mapsto 0\}$.

NOTATION. We will use the symbol $\top$ as a generic notation for multisets consisting of a finite number of *true*'s. We write $\mathcal{R} \vdash T$ if $T \twoheadrightarrow_{\mathcal{R}} \top$.

The soundness of conditional narrowing is expressed in the following lemma.

LEMMA 6.5. *Let $\mathcal{R}$ be a CTRS and T a goal clause. If $T \rightsquigarrow_\sigma^* \top$ then $\mathcal{R} \vdash \sigma T$.*

PROOF. Induction on the length of the narrowing derivation from T to $\top$. The case of zero length is trivial. Suppose

$$T \rightsquigarrow_{[e,p,l \to r \Leftarrow c,\sigma_1]} T_1 \rightsquigarrow_{\sigma_2}^* \top.$$

Let $\sigma = \sigma_2 \circ \sigma_2$. By definition $T_1 = \sigma_1(T - \{e\} \cup \{e[r]_p\} \cup \tilde{c})$. The induction hypothesis yields $\mathcal{R} \vdash \sigma_2 T_1$. Hence we have both $\mathcal{R} \vdash \sigma(T - \{e\} \cup \{e[r]_p\})$ and $\mathcal{R} \vdash \sigma\tilde{c}$. From the last observation we infer that $\sigma l \to_{\mathcal{R}} \sigma r$ and therefore

$$\sigma T \to_{\mathcal{R}} \sigma T - \{\sigma e\} \cup \{\sigma e[\sigma r]_p\}.$$

Since $T - \{\sigma e\} \cup \{\sigma e[\sigma r]_p\} = \sigma(T - \{e\} \cup \{e[r]_p\})$ we obtain $\mathcal{R} \vdash \sigma T$. $\square$

In order to compare conditional rewriting and conditional narrowing, Bockmayr [2] introduced a further relation on goal clauses which he called *Reduktion ohne Auswertung der Prämisse* (reduction without evaluating conditions). We will denote a slight variant of this relation by $\rightarrowtail$.

DEFINITION 6.6. Let $\mathcal{R}$ be a CTRS and suppose that S and T are goal clauses. We write $S \rightarrowtail_{\mathcal{R}} T$ if there exist a goal $e \in S$, a position $p \in O(e)$, a variant $l \to r \Leftarrow c$ of a rewrite rule in $\mathcal{R}$ and a substitution σ such that

- $e_{|p} \equiv \sigma l$,
- $T = S - \{e\} \cup \{e[\sigma r]_p\} \cup \sigma\tilde{c}$,
- $\mathcal{R} \vdash \sigma\tilde{c}$.

Occasionally we write $S \rightarrowtail_{[e,p,l\to r\Leftarrow c,\sigma]} T$ or $S \rightarrowtail_{[e,p,l\to r\Leftarrow c]} T$. The transitive-reflexive closure of $\rightarrowtail$ is denoted by $\twoheadrightarrow\!\!\!\!\!\!\rightarrowtail$.

The difference with the definition of Bockmayr is that we require $\mathcal{R} \vdash \sigma\tilde{c}$. For 1-CTRS's the relation $\rightarrowtail$ can be viewed as a special case of the conditional narrowing relation $\rightsquigarrow$, but in general $\rightarrowtail$ is not included in $\rightsquigarrow$ due to extra variables in conditional rewrite rules.

PROPOSITION 6.7 (Bockmayr [2]). *Let $\mathcal{R}$ be a CTRS and T a goal clause. We have $\mathcal{R} \vdash T$ if and only if $T \twoheadrightarrow\!\!\!\!\!\!\rightarrowtail \top$.* □

The proof in [2] still applies since the $\rightarrowtail$-sequences constructed in the "$\Rightarrow$"-direction satisfy the additional constraint. We refrain from giving the proof since it is similar (but easier) than the proof of Proposition 8.10 below, which is spelled out.

In Section 7 we will see that $\rightarrowtail_{\mathcal{R}}$ does not inherit strong normalization of $\mathcal{R}$. Confluence is preserved, provided we are not particular about a few extra *true*'s.

NOTATION. We write $S \simeq T$ if the goal clauses S and T are identical or they differ only in the number of *true*'s, i.e. $S - \top = T - \top$ by abuse of notation.

PROPOSITION 6.8. *Let $\mathcal{R}$ be a CTRS and S a goal clause.*
(1) *If $S \twoheadrightarrow\!\!\!\!\!\!\rightarrowtail_{\mathcal{R}} T$ then T can be partitioned into T_1 and T_2 such that $S \twoheadrightarrow_{\mathcal{R}} T_1$ and $\mathcal{R} \vdash T_2$.*
(2) *If $S \twoheadrightarrow_{\mathcal{R}} T$ then there exists a goal clause T_1 such that $S \twoheadrightarrow\!\!\!\!\!\!\rightarrowtail_{\mathcal{R}} T \cup T_1$ and $\mathcal{R} \vdash T_1$.*
PROOF. Straightforward. □

LEMMA 6.9. *Let $\mathcal{R}$ be a confluent CTRS. If $S \twoheadrightarrow\!\!\!\!\!\!\rightarrowtail_{\mathcal{R}} T_1$ and $S \twoheadrightarrow\!\!\!\!\!\!\rightarrowtail_{\mathcal{R}} T_2$ then there exist goal clauses $T_3 \simeq T_4$ such that $T_1 \twoheadrightarrow\!\!\!\!\!\!\rightarrowtail_{\mathcal{R}} T_3$ and $T_2 \twoheadrightarrow\!\!\!\!\!\!\rightarrowtail_{\mathcal{R}} T_4$.*
PROOF. Proposition 6.8(1) yields goal clauses U_1, V_1, U_2 and V_2 such that $T_1 = U_1 \cup V_1$, $T_2 = U_2 \cup V_2$, $\mathcal{R} \vdash V_1$, $\mathcal{R} \vdash V_2$, $S \twoheadrightarrow_{\mathcal{R}} U_1$ and $S \twoheadrightarrow_{\mathcal{R}} U_2$. Since the relation $\to_{\mathcal{R}}$ is confluent on goal clauses, there exists a goal clause U_3 such that both $U_1 \twoheadrightarrow_{\mathcal{R}} U_3$ and $U_2 \twoheadrightarrow_{\mathcal{R}} U_3$. According to Proposition 6.8(2) there exist goal clauses W_1 and W_2 such that $\mathcal{R} \vdash W_1$, $\mathcal{R} \vdash W_2$, $U_1 \twoheadrightarrow\!\!\!\!\!\!\rightarrowtail_{\mathcal{R}} U_3 \cup W_1$ and $U_2 \twoheadrightarrow\!\!\!\!\!\!\rightarrowtail_{\mathcal{R}} U_3 \cup W_2$. Using Proposition 6.7, we obtain

$$T_1 = U_1 \cup V_1 \twoheadrightarrow\!\!\!\!\!\!\rightarrowtail_{\mathcal{R}} U_3 \cup W_1 \cup V_1 \twoheadrightarrow\!\!\!\!\!\!\rightarrowtail_{\mathcal{R}} U_3 \cup \top$$

and likewise

$$T_2 = U_2 \cup V_2 \twoheadrightarrow\!\!\!\!\!\!\rightarrowtail_{\mathcal{R}} U_3 \cup W_2 \cup V_2 \twoheadrightarrow\!\!\!\!\!\!\rightarrowtail_{\mathcal{R}} U_3 \cup \top.$$

□

In the remainder of this section we show that conditional narrowing is complete for CTRS's without extra variables—the so-called 1-CTRS's. A rigorous proof of the following lifting lemma for 1-CTRS's can be found in Appendix A. The proof presented in [2] is erroneous (see Appendix A for specific details).

LEMMA 6.10. *Let $\mathcal{R}$ be a 1-CTRS. Suppose we have goal clauses S and T, a normalized substitution θ and a set V of variables such that $\mathcal{V}(S) \cup \mathcal{D}\theta \subseteq V$ and $T = \theta S$. If $T \twoheadrightarrow\!\!\!\!\!\!> T'$ then there exist a goal clause S' and substitutions θ', σ such that*
- $S \leadsto_\sigma^* S'$,
- $\theta' S' = T'$,
- $\theta' \circ \sigma = \theta\ [V]$,
- *θ' is normalized.*

Furthermore, we may assume that the narrowing derivation $S \leadsto_\sigma^* S'$ and the rewrite sequence $T \twoheadrightarrow T'$ employ the same rewrite rules at the same positions in the same goals. □

DEFINITION 6.11. A substitution σ is called an *$\mathcal{R}$-solution* of a goal clause S if $\mathcal{R} \vdash \sigma S$.

THEOREM 6.12. *Conditional narrowing is complete for complete 1-CTRS's.*
PROOF. Let σ be a solution of a goal clause T, i.e. $\mathcal{R} \vdash \sigma T$. Let σ' be the normal form of σ. We obtain $\mathcal{R} \vdash \sigma' T$ from the confluence of $\mathcal{R}$. According to Proposition 6.7 there exists a sequence $\sigma' T \twoheadrightarrow \top$. Lemma 6.10 yields a narrowing derivation $T \leadsto_\tau^* \top$ and a substitution σ'' such that $\sigma'' \circ \tau = \sigma'\ [\mathcal{V}(T)]$. Therefore $\tau \leqslant \sigma'\ [\mathcal{V}(T)]$ and hence $\tau \leqslant_{\mathcal{R}} \sigma\ [\mathcal{V}(T)]$. □

In the literature this completeness result is ascribed to different authors. It seems that Kaplan was the first who presented a detailed proof (in a different setting though). As was the case for TRS's, we may drop the requirement of strong normalization in exchange for the restriction to normalizable solutions.

THEOREM 6.13. *Conditional narrowing is complete for confluent 1-CTRS's with respect to normalizable substitutions.* □

COROLLARY 6.14. *Conditional narrowing is complete for semi-complete 1-CTRS's.* □

7. Basic Conditional Narrowing

The formulation of basic narrowing for TRS's (Definition 4.1) does not immediately extend to the conditional case. The reason is that a goal clause consists of several goals, each to be equipped with its own constraint on 'narrowable positions'. In order to keep the administration of these constraints manageable we introduce the following concept.

DEFINITION 7.1. Let T be a goal clause. A *position constraint* for T is a mapping B that assigns to every goal $e \in T$ a subset of $\overline{O}(e)$. The position constraint that assigns to every $e \in T$ the set $\overline{O}(e)$ will be denoted by $\overline{T}$.

DEFINITION 7.2.
(1) A narrowing derivation

$$T_1 \leadsto_{[e_1, p_1, l_1 \to r_1 \Leftarrow c_1, \sigma_1]} \cdots \leadsto_{[e_{n-1}, p_{n-1}, l_{n-1} \to r_{n-1} \Leftarrow c_{n-1}, \sigma_{n-1}]} T_n$$

is *basic* if $p_i \in B_i(e_i)$ for $1 \leqslant i \leqslant n-1$ where the position constraints $B_1, \ldots, B_{n-1}$ are inductively defined by $B_1 = \overline{T}_1$ and

$$B_{i+1}(e) = \begin{cases} B_i(e') & \text{if } e' \in T_i - \{e_i\}, \\ \mathcal{B}(B_i(e_i), p_i, r_i) & \text{if } e' \equiv e_i[r_i]_{p_i}, \\ \overline{O}(e') & \text{if } e' \in \tilde{c}_i \end{cases}$$

for all $1 \leqslant i < n-1$ and $e = \sigma_i e' \in T_{i+1}$.[12]

(2) A sequence

$$T_1 \rightarrowtail_{[e_1, p_1, l_1 \to r_1 \Leftarrow c_1, \sigma_1]} \cdots \rightarrowtail_{[e_{n-1}, p_{n-1}, l_{n-1} \to r_{n-1} \Leftarrow c_{n-1}, \sigma_{n-1}]} T_n$$

is *based* on a position constraint B_1 for T_1 if $p_i \in B_i$ for $1 \leqslant i \leqslant n-1$ with $B_2, \ldots, B_{n-1}$ defined by

$$B_{i+1}(e) = \begin{cases} B_i(e) & \text{if } e \in T_i - \{e_i\}, \\ \mathcal{B}(B_i(e_i), p_i, r_i) & \text{if } e \equiv e_i[\sigma_i r_i]_{p_i}, \\ \overline{O}(e') & \text{if } e = \sigma_i e' \text{ with } e' \in \tilde{c}_i \end{cases}$$

for all $1 \leqslant i < n-1$ and $e \in T_{i+1}$.

Hölldobler [23] showed that basic conditional narrowing is complete for complete 1-CTRS's. This fact is also mentioned in the "summary of completeness results and open problems for conditional narrowing" in Giovannetti and Moiso [18]. However, the following example reveals that this result is incorrect.

COUNTEREXAMPLE 7.3. Consider the 1-CTRS

$$\mathcal{R} = \begin{cases} f(x) & \to & a & \Leftarrow & x = b,\ x = c \\ d & \to & b \\ d & \to & c \\ b & \to & c & \Leftarrow & f(d) = a. \end{cases}$$

Since the recursive path ordering is applicable (with precedence $f \succ a$ and $d \succ b \succ c$) to the unconditional part of $\mathcal{R}$, $\mathcal{R}$ certainly is strongly normalizing. We have $d \to b$ and $d \to c$ and hence $f(d) \to a$ and $b \to c$, which makes the only critical pair $\langle b, c\rangle$ convergent. Local confluence is obtained by some easy case analysis or an appeal to a result of Dershowitz *et al.* [10] which states that the Critical Pair Lemma holds for *overlay* CTRS's. According to Newman's Lemma $\mathcal{R}$ is confluent. However, basic conditional narrowing is not able to solve the goal $f(d) =^? a$ as can be seen from Figure 3 (in this figure trivial goals of the form $t =^? t$ are not shown), while the following non-basic narrowing derivation shows that the goal can be solved:

$$\begin{aligned} f(d) =^? a \quad & \leadsto \quad a =^? a,\ d =^? b,\ d =^? c \\ & \leadsto \quad a =^? a,\ b =^? b,\ d =^? c \\ & \leadsto \quad a =^? a,\ b =^? b,\ c =^? c \\ & \leadsto^* \quad \mathit{true},\ \mathit{true},\ \mathit{true}. \end{aligned}$$

The mistake in Hölldobler [23] is due to the incorrect assumption that the strong normalization of $\rightarrowtail_{\mathcal{R}}$ is implied by the strong normalization of $\mathcal{R}$. We now show that completeness

[12] Recall that $T_{i+1} = \sigma_i(T_i - \{e_i\} \cup \{e_i[r_i]_{p_i}\} \cup \tilde{c}_i)$. If we would have represented goal clauses by sets then the definition of B_{i+1} is ambiguous since $T_i - \{e_i\}$, $\{e_i[r_i]_{p_i}\}$ and $\tilde{c}_i$ do not have to be pairwise disjoint.

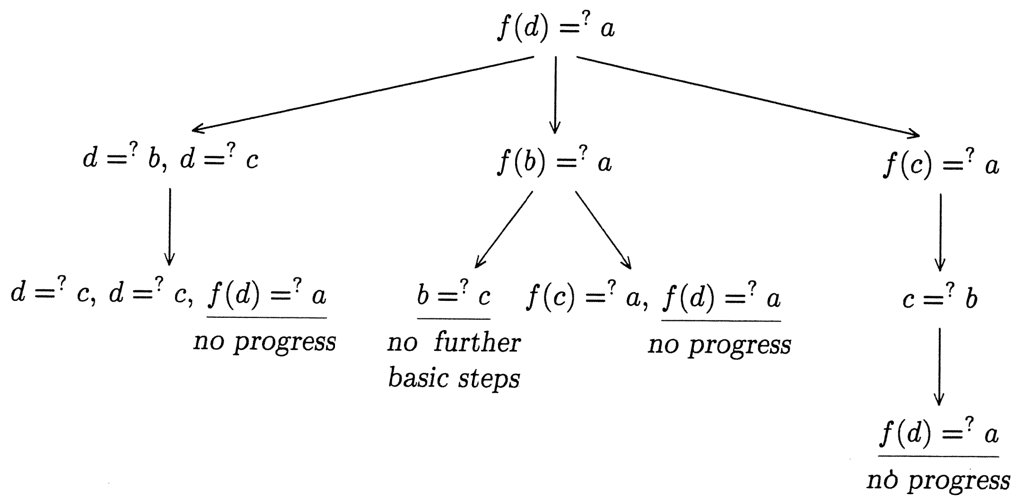


FIGURE 3.

of basic narrowing can be ensured by strengthening strong normalization. In the next section we show that completeness can also be recovered by strengthening confluence. The property defined below originates from Dershowitz *et al.* [11].

DEFINITION 7.4. A 1-CTRS $\mathcal{R}$ is *decreasing* if there exists a well-founded extension $\succ$ of the rewrite relation $\rightarrow_{\mathcal{R}}$ with the following properties:

- $\succ$ has the *subterm property*, i.e. $t \succ t_{|p}$ for all positions $p \in O(t) - \{\varepsilon\}$,
- if $l \rightarrow r \Leftarrow c \in \mathcal{R}$ and σ is a substitution then $\sigma l \succ \sigma s, \sigma t$ for all $s = t$ in c.

Every decreasing 1-CTRS is strongly normalizing and moreover—when there are finitely many rewrite rules—its rewrite relation is decidable.

EXAMPLE 7.5. The CTRS of Counterexample 7.3 is not decreasing: as $f(d) \rightarrow a \Leftarrow d = b, d = c$ is an instance of the first rewrite rule we must have $f(d) \succ b$, but the rule $b \rightarrow c \Leftarrow f(d) = a$ requires $b \succ f(d)$.

LEMMA 7.6. *If $\mathcal{R}$ is a decreasing 1-CTRS then $\rightarrowtail_{\mathcal{R}}$ is strongly normalizing.*

PROOF. With every goal clause S we associate a multiset $m(S)$ by replacing every goal $s =^? t$ in S by the terms s and t. The presence of *true* in S does not contribute to $m(S)$. Using the definition of $\succ$ it is easy to show that $m(S) \succ\!\succ m(T)$ whenever $S \rightarrowtail_{\mathcal{R}} T$. Here $\succ\!\succ$ is the multiset extension of $\succ$. Since the multiset extension of a well-founded ordering is well-founded, the relation $\rightarrowtail_{\mathcal{R}}$ is strongly normalizing. $\square$

The proof of Proposition 7.7 can be found in Appendix B.

PROPOSITION 7.7. *Let $\mathcal{R}$ be a 1-CTRS, T a goal clause and σ a normalized substitution. Every innermost $\rightarrowtail_{\mathcal{R}}$-sequence starting from σT is based on $\overline{T}$.* $\square$

THEOREM 7.8. *Basic conditional narrowing is complete for decreasing and confluent 1-CTRS's.*

PROOF. Suppose σ is a solution of a goal clause T and let σ' be its normal form. We obtain $\sigma' T \twoheadrightarrow \top$ as in the proof of Theorem 6.12. From Lemma's 6.9 and 7.6 we conclude that there exists an innermost $\rightarrowtail$-sequence from $\sigma' T$ to $\top$. According to Proposition 7.7 this sequence is based on $\overline{T}$. It is not difficult to show that the narrowing derivation constructed by Lemma 6.10 is basic. The remainder of the proof follows literally the proof of Theorem 6.12. $\square$

The CTRS of Counterexample 7.3 is not a so-called *normal* CTRS. In a normal CTRS $\mathcal{R}$ every right-hand side of an equation in the conditions of the rewrite rules is a ground normal form with respect to the unconditional TRS obtained from $\mathcal{R}$ by omitting the conditions. One might ask whether this is essential. The following example answers this question negatively.

EXAMPLE 7.9. Consider the normal 1-CTRS

$$\mathcal{R} = \begin{cases} f(x) & \rightarrow & g(x,x) & & \\ a & \rightarrow & b & & \\ g(a,b) & \rightarrow & c & & \\ g(b,b) & \rightarrow & c & \Leftarrow & f(a) = c. \end{cases}$$

Completeness of $\mathcal{R}$ follows as in Counterexample 7.3. (The Critical Pair Lemma however is not applicable.) Notice that $g(b,b) \rightarrow c$ since $f(a) \rightarrow g(a,a) \rightarrow g(a,b) \rightarrow c$. One easily shows that the goal $f(a) =^? c$ cannot be solved by basic conditional narrowing.

We conclude this section with a refutation of the following claim.

CONJECTURE 7.10 (Giovannetti and Moiso [18]). *Basic conditional narrowing is complete for semi-complete orthogonal 1-CTRS's.*[13] □

Since (weakly normalizing) orthogonal CTRS's are in general not confluent (Bergstra and Klop [1]), we cannot replace the phrase "semi-complete orthogonal" by "weakly normalizing orthogonal".

COUNTEREXAMPLE 7.11. Consider the orthogonal 1-CTRS

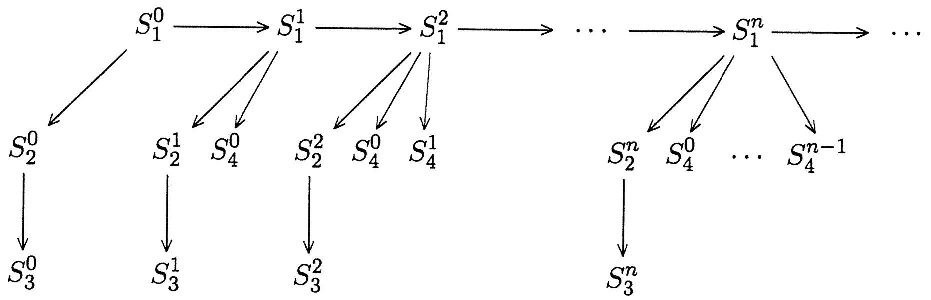
$$\mathcal{R} = \begin{cases} f(x) & \rightarrow & a & \Leftarrow & g(b) = c \\ g(x) & \rightarrow & c & \Leftarrow & x = f(x) \\ b & \rightarrow & f(b). & & \end{cases}$$

With considerable effort we can show that $\mathcal{R}$ is semi-complete. The goal $g(b) =^? c$ can be solved by conditional narrowing as follows:

$$\begin{array}{rll} g(b) =^? c & \leadsto & c =^? c,\ b =^? f(b) \\ & \leadsto & c =^? c,\ f(b) =^? f(b) \\ & \leadsto^* & \mathit{true}. \end{array}$$

In this derivation the second step is not basic. Figure 4 reveals that all basic narrowing derivations issued from $g(b) =^? c$ come across a goal clause that contains the original goal $g(b) =^? c$. Hence basic conditional narrowing is not able to solve the goal $g(b) =^? c$.

[13] Actually, Giovannetti and Moiso conjecture in [18] the completeness of basic conditional narrowing for orthogonal 1-CTRS's with respect to normalized solutions. By refuting the weaker statement, our counterexample becomes stronger.



Abbreviations ($i \geqslant 0$):

S_1^i :	$g(f^i(b)) =^? c$	S_2^i :	$c =^? c,\ f^i(b) =^? f^{i+1}(b)$
S_3^i :	$c =^? c,\ f^i(b) =^? a,\ g(b) =^? c$	S_4^i :	$g(f^i(a)) =^? c,\ g(b) =^? c$
$f^i(t)$:	$\underbrace{f(\ldots f}_{i\ f\text{'s}}(t)\ldots)$		

FIGURE 4.

8. Level-Confluence

Hußmann claimed in [26] that conditional narrowing is also complete for complete CTRS's that have extra variables in the conditions of the rewrite rules, but the following example of Giovannetti and Moiso [18] shows that this is not the case.

EXAMPLE 8.1. Consider the 2-CTRS

$$\mathcal{R} = \begin{cases} a & \rightarrow & b \\ a & \rightarrow & c \\ b & \rightarrow & c & \Leftarrow & x = b,\ x = c. \end{cases}$$

It is easy to show that $\mathcal{R}$ is complete. In particular we have $b \rightarrow_{\mathcal{R}} c$, but all narrowing derivations issued from the goal $b =^? c$ are infinite, e.g.

$$\begin{array}{lll} b =^? c & \rightsquigarrow & c =^? c,\ x =^? b,\ x =^? c \\ & \rightsquigarrow & c =^? c,\ x =^? c,\ x =^? c,\ x' =^? b,\ x' =^? c \\ & \rightsquigarrow & c =^? c,\ x =^? c,\ x =^? c,\ x' =^? c,\ x' =^? c,\ x'' =^? b,\ x'' =^? c \\ & \rightsquigarrow & \cdots \end{array}$$

In order to cope with extra variables in the conditions of the rewrite rules, Giovannetti and Moiso proposed to strengthen confluence.

DEFINITION 8.2. A CTRS $\mathcal{R}$ is called *level-confluent* if each $\mathcal{R}_n$ ($n \geqslant 0$) is confluent. We call $\mathcal{R}$ *level-complete* if each $\mathcal{R}_n$ is complete. This is equivalent to saying that $\mathcal{R}$ is strongly normalizing and level-confluent.

EXAMPLE 8.3. The complete 1-CTRS of Counterexample 7.3 is not level-confluent: we have $d \rightarrow_{\mathcal{R}_1} b$ and $d \rightarrow_{\mathcal{R}_1} c$ but the depth of the joining step $b \rightarrow c$ is 3. Likewise the 2-CTRS of Example 8.1 is not level-confluent.

DEFINITION 8.4. Let $\mathcal{R}$ be a CTRS. We inductively define TRS's $\mathcal{S}_n$ for $n \geqslant 0$ and a TRS $\mathcal{S}$ as follows:

$$\begin{aligned}
\mathcal{S}_0 &= \mathcal{R}_0, \\
\mathcal{S}_{n+1} &= \{\sigma l \to \sigma r \mid R\colon l \to r \Leftarrow c \in \mathcal{R},\ \mathcal{S}_n \vdash \sigma\tilde{c} \text{ and } \sigma\restriction_{\mathcal{E}(R)} \text{ is } \mathcal{S}_n\text{-normalized}\}, \\
\mathcal{S} &= \bigcup_{n\geqslant 0} \mathcal{S}_n.
\end{aligned}$$

LEMMA 8.5 (Giovannetti and Moiso [18]). *If $\mathcal{R}$ is a level-complete 2-CTRS then $\mathcal{R}_n = \mathcal{S}_n$ for all $n \geqslant 0$.*

PROOF. We use induction on n. If $n = 0$ then $\mathcal{R}_n = \mathcal{S}_n$ by definition. Suppose the result holds for n. This immediately gives rise to the inclusion $\mathcal{S}_{n+1} \subseteq \mathcal{R}_{n+1}$. Let $s \to t \in \mathcal{R}_{n+1}$. So there exist a rewrite rule $R\colon l \to r \Leftarrow c$ in $\mathcal{R}$ and a substitution σ such that $s \equiv \sigma l$, $t \equiv \sigma r$ and $\mathcal{R}_n \vdash \sigma\tilde{c}$. Define a substitution τ as follows:

$$\tau x = \begin{cases} (\sigma x)\downarrow_{\mathcal{R}_n} & \text{if } x \in \mathcal{E}(R), \\ \sigma x & \text{otherwise.} \end{cases}$$

The well-definedness of τ follows from the completeness of $\mathcal{R}_n$. We have $\sigma\tilde{c} \twoheadrightarrow_{\mathcal{R}_n} \tau\tilde{c}$. Confluence of $\mathcal{R}_n$ yields $\mathcal{R}_n \vdash \tau\tilde{c}$. From the induction hypothesis we obtain $\mathcal{S}_n \vdash \tau\tilde{c}$. By construction $\tau\restriction_{\mathcal{E}(R)}$ is $\mathcal{R}_n$-normalized and hence $\mathcal{S}_n$-normalized. Since $\sigma l \equiv \tau l$ and $\sigma r \equiv \sigma r$, we conclude that $s \to t \in \mathcal{S}_{n+1}$. $\square$

Since $\mathcal{R}_n$ is closed under substitutions for $n > 0$, i.e. if $l \to r \in \mathcal{R}_n$ then $\sigma l \to \sigma r \in \mathcal{R}_n$ for all substitutions σ, the same holds for $\mathcal{S}_n$. So if $s \to_{\mathcal{S}_n} t$ then there exists a rule[14] $l \to r \in \mathcal{S}_n$ and a position $p \in O(s)$ such that $s_{|p} \equiv l$ and $t \equiv s[r]_p$.

In the following definitions $\mathcal{R}$ denotes an arbitrary CTRS.

DEFINITION 8.6. Let S and T be goal clauses such that $S \rightarrowtail T$ by application of a rewrite rule $R\colon l \to r \Leftarrow c$ with substitution σ. We write $S \rightarrowtail^0 T$ if R is the rule $x =^? x \to \mathit{true}$. We write $S \rightarrowtail^{n+1} T$ if $\mathcal{R}_n \vdash \sigma\tilde{c}$ and $\sigma\restriction_{\mathcal{E}(R)}$ is $\mathcal{R}_n$-normalized.

DEFINITION 8.7. The *level* of a goal clause S is the least n such that $\mathcal{R}_n \vdash S$.

DEFINITION 8.8. A solution σ of a goal clause S is said to be *sufficiently normalized* if $\sigma\restriction_{\mathcal{V}(e)}$ is $\mathcal{R}_n$-normalized where n is the level of σe, for every goal $e \in S$.

DEFINITION 8.9. A sequence $S \twoheadrightarrowtail T$ is called *point-blank* if each step in this sequence can be written as $S' \rightarrowtail^n T'$ for some n that does not exceed the level of the selected goal in S'. Intuitively, a sequence is point-blank if it does not contain unnecessarily complicated steps.

PROPOSITION 8.10. *Let $\mathcal{R}$ be a level-complete 2-CTRS and T a goal clause. We have $\mathcal{R} \vdash T$ if and only if there exists a point-blank sequence $T \twoheadrightarrowtail_{\mathcal{R}} \top$.*

PROOF.

[14] Since $\mathcal{S}_n$ is closed under variable renamings, there is no need to consider variants of rules in $\mathcal{S}_n$.

$\Rightarrow$ By induction on n we will show the existence of a point-blank sequence $T \rightarrowtail\!\!\!\rightarrow \top$ whenever $\mathcal{R}_n \vdash T$. If $n = 0$ then there exists a rewrite sequence from T to $\top$ in which only the rule $x =^? x \to \mathit{true}$ is used. By definition, this sequence is also a $\rightarrowtail^0$-sequence. Suppose $\mathcal{R}_{n+1} \vdash T$. So there exists a $\mathcal{R}_{n+1}$-sequence from T to $\top$. We use induction on the length of this sequence. The case of zero length is trivial. Let $T \to_{\mathcal{R}_{n+1}} T' \twoheadrightarrow_{\mathcal{R}_{n+1}} \top$. We obtain a point-blank sequence $T' \rightarrowtail\!\!\!\rightarrow \top$ from the second induction hypothesis. By definition there exist a goal $e \in T$, a position $p \in O(e)$, a variant R: $l \to r \Leftarrow c$ of a rewrite rule in $\mathcal{R}$ and a substitution σ such that $e_{|p} \equiv \sigma l$, $T' = T - \{e\} \cup \{e[\sigma r]_p\}$, $\mathcal{R}_n \vdash \sigma\tilde{c}$. According to Lemma 8.5 we may assume that $\sigma\lceil_{\mathcal{E}(R)}$ is $\mathcal{R}_n$-normalized. Hence $T \rightarrowtail^{n+1} T' \cup \sigma\tilde{c}$. Applying the first induction hypothesis to $\mathcal{R}_n \vdash \sigma\tilde{c}$ yields a point-blank sequence from $\sigma\tilde{c}$ to $\top$. Let m be the level of e. We distinguish two cases.

$n + 1 \leqslant m$ Combining the step $T \rightarrowtail T' \cup \sigma\tilde{c}$ with the point-blank sequences $T' \rightarrowtail\!\!\!\rightarrow \top$ and $\sigma\tilde{c} \rightarrowtail\!\!\!\rightarrow \top$ yields a point-blank sequence from T to $\top$.

$n + 1 > m$ Applying the first induction hypothesis to $\mathcal{R}_m \vdash \{e\}$ yields a point-blank sequence $\{e\} \rightarrowtail\!\!\!\rightarrow \top$. Since $T - \{e\}$ is a subset of T', we can extract from the point-blank sequence $T' \rightarrowtail\!\!\!\rightarrow \top$ a point-blank sequence from $T - \{e\}$ to $\top$. Combining the resulting sequence with the point-blank sequence $\{e\} \rightarrowtail\!\!\!\rightarrow \top$ yields the desired result.

$\Leftarrow$ We use induction on the length of the sequence $T \rightarrowtail\!\!\!\rightarrow \top$. The case of zero length is trivial. Suppose $T \rightarrowtail T' \rightarrowtail\!\!\!\rightarrow \top$. By assumption this sequence is point-blank and hence there exists an $n \geqslant 0$ such that $T \rightarrowtail^n T'$. This means that there is a goal $e \in T$, a position $p \in O(e)$, a variant R: $l \to r \Leftarrow c$ of a rewrite rule in $\mathcal{R}$ and a substitution σ such that $e_{|p} \equiv \sigma l$, $T' = T'' \cup \sigma\tilde{c}$, $\mathcal{R}_{n-1} \vdash \sigma\tilde{c}$ and $\sigma\lceil_{\mathcal{E}(R)}$ is $\mathcal{R}_{n-1}$-normalized where $T'' = T - \{e\} \cup \{e[\sigma r]_p\}$. Hence $T \to_{\mathcal{R}_n} T''$. Applying the induction hypothesis to the sequence $T' \rightarrowtail\!\!\!\rightarrow \top$ yields $\mathcal{R} \vdash T'$. Since $T'' \subseteq T'$ we also have $\mathcal{R} \vdash T''$ and therefore $\mathcal{R} \vdash T$.

□

For a proof of the following lifting lemma for level-confluent 2-CTRS's we refer to Appendix A.

LEMMA 8.11. *Let $\mathcal{R}$ be a level-confluent 2-CTRS. Suppose we have goal clauses S and T, a sufficiently normalized solution θ of S and a set V of variables such that $\mathcal{V}(S) \cup \mathcal{D}\theta \subseteq V$ and $T = \theta S$. If $T \rightarrowtail\!\!\!\rightarrow T'$ is point-blank then there exist a goal clause S' and substitutions θ', σ such that*

- $S \rightsquigarrow_\sigma^* S'$,
- $\theta' S' = T'$,
- $\theta' \circ \sigma = \theta\ [V]$,
- *θ' is a sufficiently normalized solution of S'.*

Furthermore, we may assume that the narrowing derivation $S \rightsquigarrow_\sigma^* S'$ and the point-blank sequence $T \rightarrowtail\!\!\!\rightarrow T'$ employ the same rewrite rules at the same positions in the same goals. □

THEOREM 8.12 (Giovannetti and Moiso [18]). *Conditional narrowing is complete for level-complete 2-CTRS's.*

PROOF. Let $\mathcal{R}$ be a level-confluent 2-CTRS and suppose that σ is a solution of a goal clause T, i.e. $\mathcal{R} \vdash \sigma T$. Let σ' be the normal form of σ. Confluence of $\mathcal{R}$ yields $\mathcal{R} \vdash \sigma' T$. According to Proposition 8.10 there exists a point-blank sequence $\sigma' T \rightarrowtail\!\!\!\rightarrow \top$. Lemma 8.11 yields a narrowing derivation $T \rightsquigarrow_\tau^* \top$ and a substitution σ'' such that $\sigma'' \circ \tau = \sigma'\ [\mathcal{V}(T)]$. Therefore $\tau \leqslant \sigma'\ [\mathcal{V}(T)]$ and hence $\tau \leqslant_{\mathcal{R}} \sigma\ [\mathcal{V}(T)]$. □

We have seen that in case of complete TRS's and 1-CTRS's strong normalization can be dropped, provided we restrict ourselves to normalizable solutions. This does not hold for level-complete 2-CTRS's as the following example of Giovannetti and Moiso [18] shows.

EXAMPLE 8.13. Consider the level-confluent 2-CTRS

$$\mathcal{R} = \begin{cases} a \;\to\; b & \Leftarrow\; x = f(x) \\ c \;\to\; f(c). & \end{cases}$$

We have $a \to_{\mathcal{R}} b$ because $c \to_{\mathcal{R}} f(c)$, but conditional narrowing is not able to solve the goal $a =^? b$, whose trivial solution ϵ is clearly normalizable.

However, we can strengthen Theorem 8.12 by noting that in the proofs of Lemma 8.5, Proposition 8.10 and Theorem 8.12 it is sufficient to require the weak normalization of every $\mathcal{R}_n$.

DEFINITION 8.14. A CTRS $\mathcal{R}$ is called *level-semi-complete* if each $\mathcal{R}_n$ ($n \geqslant 0$) is semi-complete. Example 8.16 shows that level-semi-completeness is not the same as the combination of level-confluence and weak normalization.

THEOREM 8.15. *Conditional narrowing is complete for level-semi-complete 2-CTRS's.* □

EXAMPLE 8.16. Extend the CTRS of the previous example with the rule

$$f(x) \to d \Leftarrow y = f(y).$$

The new CTRS is level-confluent and weakly normalizing but not level-semi-complete as $\mathcal{R}_1$ is not weakly normalizing. Again the goal $a =^? b$ cannot be solved by conditional narrowing.

In the remainder of this section we prove that basic conditional narrowing is complete for level-complete 2-CTRS. This result is due to Giovannetti and Moiso. The proof sketch in [18] is however not applicable to our setting.

LEMMA 8.17. *If $\mathcal{R}$ is a strongly normalizing 2-CTRS then every point-blank $\rightarrowtail_{\mathcal{R}}$-sequence is finite.*

PROOF. Let T be a goal clause. We will show that there are no infinite point-blank sequences starting from T. Since equations in T without a level do not contribute to a point-blank sequence, we may assume that T has some level n. We use induction on n. If $n = 0$ then only the rule $x =^? x \to$ *true* can be used, and the number of applications of this rule is clearly bounded by the cardinality of T. Suppose the level of T is $n + 1$ and consider an infinite point-blank sequence starting from T. Since point-blank sequences issued from different goals in T do not interfere, we infer from the pigeon-hole principle the existence of a goal $e \in T$ with an infinite point-blank sequence. Consider the first step

$$\{e\} \rightarrowtail_{[e,p,l \to r \Leftarrow c,\sigma]} \{e[\sigma r]_p\} \cup \sigma\tilde{c}$$

in this sequence. Clearly $e \to_{\mathcal{R}} e[\sigma r]_p$. Since the level of $\sigma\tilde{c}$ is less than $n + 1$, an application of the induction hypothesis yields an infinite point-blank sequence starting from $\{e[\sigma r]_p\}$. Hence we can repeat the above process with $e[\sigma r]_p$. We end up with an infinite $\to_{\mathcal{R}}$-sequence, contradicting the strong normalization of $\mathcal{R}$. □

Appendix B contains a proof of the following proposition.

PROPOSITION 8.18. *Let $\mathcal{R}$ be a level-confluent 2-CTRS and σ a sufficiently normalized solution of a goal clause T. Every innermost point-blank $\rightarrowtail_{\mathcal{R}}$-sequence starting from σT is based on $\overline{T}$.* $\square$

THEOREM 8.19. *Basic conditional narrowing is complete for level-complete 2-CTRS's.*

PROOF. Similar to the proof of Theorem 7.8. Suppose σ is a solution of a goal clause T and let σ' be its normal form. We obtain $\sigma'T \rightarrowtail\!\!\!\twoheadrightarrow \top$ as in the proof of Theorem 6.12. With help of Lemma's 6.9 and 8.17 we easily infer the existence of an innermost point-blank sequence from $\sigma'T$ to $\top$. According to Proposition 8.18 this sequence is based on $\overline{T}$. It is not difficult to show that the narrowing derivation constructed by Lemma 8.11 is basic. The proof is completed as usual. $\square$

9. Extra Variables in Right-Hand Sides

In this section we extend the results of the previous section to CTRS's that contain extra variables in the right-hand sides of the rewrite rules. An example of such a CTRS is the following system (inspired by [10]) which specifies the computation of Fibonacci numbers:

$$\left\{\begin{array}{lcll} 0+x & \rightarrow & x & \\ S(x)+y & \rightarrow & S(x+y) & \\ f(0) & \rightarrow & \langle 0, S(0)\rangle & \\ f(S(x)) & \rightarrow & \langle z, y+z\rangle & \Leftarrow\ f(x)=\langle y,z\rangle \\ \mathit{first}(\langle x,y\rangle) & \rightarrow & x & \\ \mathit{fib}(x) & \rightarrow & \mathit{first}(f(x)). & \end{array}\right.$$

We require that extra variables in the right-hand side of a rule occur in its conditional part, i.e. we restrict ourselves to 3-CTRS's. This is not a real restriction as we consider only strongly normalizing CTRS's.

The first problem we encounter when extending the completeness results of the previous section is the correspondence between $\mathcal{R}$ and $\mathcal{S}$: Lemma 8.5 is no longer valid, as shown by the following example.

EXAMPLE 9.1. Consider the CTRS

$$\mathcal{R} = \left\{\begin{array}{lclcl} a & \rightarrow & f(x) & \Leftarrow & x=b \\ b & \rightarrow & c. & & \end{array}\right.$$

The following table, which contains the relevant values[15] of $\mathcal{R}_n$ and $\mathcal{S}_n$ for $n = 1,2$, reveals that $\mathcal{R}_2 \neq \mathcal{S}_2$.

	$b \rightarrow c$	$a \rightarrow f(b)$	$a \rightarrow f(c)$
$\mathcal{R}_1$	✓	✓	✓
$\mathcal{S}_1$	✓	✓	
$\mathcal{R}_2$	✓	✓	✓
$\mathcal{S}_2$	✓		✓

[15] The actual values are obtained by adding all instances of the rule $x =^? x \rightarrow \mathit{true}$.

In the previous section we observed that for level-complete 2-CTRS's the equivalence $s \to_{\mathcal{S}_n} t$ if and only if there exists a rule $l \to r \in \mathcal{S}_n$ and a position $p \in O(s)$ such that $s_{|p} \equiv l$ and $t \equiv s[r]_p$ holds. This important consequence of Lemma 8.5 does not hold for level-complete 3-CTRS's. However, if we *define*[16] $\to_{\mathcal{S}_n}$ as the closure of $\mathcal{S}_n$ under contexts, the equivalence trivializes. Moreover, we will see that this restricted view of $\mathcal{S}_n$ is strong enough to prove all consequences of Lemma 8.5 that are relevant for the completeness of conditional narrowing for level-complete 3-CTRS's. Henceforth we adopt this view. So we no longer consider $\mathcal{S}_n$ ($n \geqslant 0$) and $\mathcal{S}$ as TRS's. When no confusion can arise, we will identify $\mathcal{S}_n$ with $\to_{\mathcal{S}_n}$ in order to avoid excessive use of arrows. For example, we will write[17] "$\mathcal{S}_n$-normalized" instead of "normal form with respect to $\to_{\mathcal{S}_n}$".

LEMMA 9.2. *Let $\mathcal{R}$ be a level-complete 3-CTRS. The following properties hold for all $n \geqslant 0$:*
(1) $\to_{\mathcal{S}_n} \subseteq \to_{\mathcal{R}_n}$,
(2) *$\mathcal{S}_n$ and $\mathcal{R}_n$ define the same normal forms,*
(3) *$\mathcal{S}_n$ and $\mathcal{R}_n$ compute the same normal forms,*
(4) *$\mathcal{S}_n$ is confluent,*
(5) *$\mathcal{R}_n$-convertibility coincides with $\mathcal{S}_n$-convertibility.*

PROOF. We use induction on n. The case $n = 0$ is trivial. Suppose properties (1)–(5) hold for n. We will establish these properties for $n+1$.
(1) If $s \to_{\mathcal{S}_{n+1}} t$ then there exist a position $p \in O(s)$, a rule R: $l \to r \Leftarrow c$ and a substitution σ such that $s_{|p} \equiv \sigma l$, $t \equiv s[\sigma r]_p$, $\mathcal{S}_n \vdash \sigma \tilde{c}$ and $\sigma\lceil_{\mathcal{E}(R)}$ is $\mathcal{S}_n$-normalized. From $\mathcal{S}_n \vdash \sigma\tilde{c}$ we infer that *true* is a normal form of $\sigma\tilde{c}$ with respect to $\mathcal{S}_n$. According to part (3) of the induction hypothesis this implies that *true* is a normal of $\sigma\tilde{c}$ with respect to $\mathcal{R}_n$ and hence $\mathcal{R}_n \vdash \sigma\tilde{c}$. Therefore $s \to_{\mathcal{R}_{n+1}} t$.
(2) Property (1) shows that every normal form of $\mathcal{R}_{n+1}$ is also a normal form of $\mathcal{S}_{n+1}$. Suppose t is $\mathcal{R}_{n+1}$-reducible. We have to show that t is $\mathcal{S}_{n+1}$-reducible. By definition there exist a rule R: $l \to r \Leftarrow c$ and a substitution σ such that σl is a subterm of t and $\mathcal{R}_n \vdash \sigma\tilde{c}$. Define a substitution τ as in the proof of Lemma 8.5. We have $\sigma\tilde{c} \twoheadrightarrow_{\mathcal{R}_n} \tau\tilde{c}$. Confluence of $\mathcal{R}_n$ yields $\mathcal{R}_n \vdash \tau\tilde{c}$. We obtain $\mathcal{S}_n \vdash \tau\tilde{c}$ as in the proof of property (1). By construction $\tau\lceil_{\mathcal{E}(R)}$ is $\mathcal{R}_n$-normalized. According to part (2) of the induction hypothesis $\tau\lceil_{\mathcal{E}(R)}$ is also $\mathcal{S}_n$-normalized. Since $\tau l \equiv \sigma l$ we obtain $\sigma l \to \tau r \in \mathcal{S}_{n+1}$ and hence t is $\mathcal{S}_{n+1}$-reducible.
(3) If t is a normal form of s with respect to $\mathcal{S}_{n+1}$ then, using properties (1) and (2), t is also a normal form of s with respect to $\mathcal{R}_{n+1}$. Suppose t is a normal form of s with respect to $\mathcal{R}_{n+1}$. Let t' be a normal form of s with respect to $\mathcal{S}_{n+1}$. (The existence of t' is guaranteed by the strong normalization of $\mathcal{S}_{n+1}$.) We already showed that t' is also a normal form of s with respect to $\mathcal{R}_{n+1}$. Confluence of $\mathcal{R}_{n+1}$ yields $t \equiv t'$.
(4) Easy consequence of properties (1) and (3).
(5) From property (1) we easily obtain that $=_{\mathcal{S}_n} \subseteq =_{\mathcal{R}_n}$. Suppose $s \to_{\mathcal{R}_n} t$. Let u be the $\mathcal{R}_n$-normal form of s and t. Property (3) shows that $s \twoheadrightarrow_{\mathcal{S}_n} u \twoheadleftarrow_{\mathcal{S}_n} t$. So $\to_{\mathcal{R}_n} \subseteq \downarrow_{\mathcal{S}_n}$. From this we obtain $=_{\mathcal{R}_n} \subseteq =_{\mathcal{S}_n}$ by a straightforward induction argument.

□

[16] Formally, we redefine the relations $\to_{\mathcal{S}_n}$ ($n \geqslant 0$) and $\to_{\mathcal{S}}$ of Definition 8.4 as follows:

$s \to_{\mathcal{S}_0} t$	if $s \to_{\mathcal{R}_0} t$;
$s \to_{\mathcal{S}_{n+1}} t$	if there exist a position $p \in O(s)$, a variant R: $l \to r \Leftarrow c$ of a rewrite rule in $\mathcal{R}$ and a substitution σ such that $s_{\|p} \equiv \sigma l$, $t \equiv s[\sigma r]_p$, $\sigma e \twoheadrightarrow_{\mathcal{S}_n}$ *true* for all $e \in \tilde{c}$ and $\sigma\lceil_{\mathcal{E}(R)}$ is $\mathcal{S}_n$-normalized;
$s \to_{\mathcal{S}} t$	if $s \to_{\mathcal{S}_n} t$ for some $n \geqslant 0$.

[17] We already did so in the previous footnote.

An immediate consequence of Lemma 9.2(3) is the equivalence of the statements $\mathcal{R}_n \vdash T$ and $\mathcal{S}_n \vdash T$.

PROPOSITION 9.3. *Let $\mathcal{R}$ be a level-complete 3-CTRS. If $m \leqslant n$ then*
(1) *every $\mathcal{S}_n$-normal form is a $\mathcal{S}_m$-normal form, and*
(2) *$\mathcal{S}_m$-convertibility is included in $\mathcal{S}_n$-convertibility.*

PROOF. The first part follows from Lemma 9.2(2) and the second part from Lemma 9.2(5). □

LEMMA 9.4. *Let $\mathcal{R}$ be a level-complete 3-CTRS and T a goal clause.*
(1) *If $\mathcal{S} \vdash T$ then $\mathcal{S}_n \vdash T$ for some $n \geqslant 0$.*
(2) *The statements $\mathcal{R} \vdash T$ and $\mathcal{S} \vdash T$ are equivalent.*

PROOF.
(1) We use induction on the length of the sequence $T \twoheadrightarrow_{\mathcal{S}} \top$. If $T = \top$ then $\mathcal{S}_n \vdash T$ for all $n \geqslant 0$. Suppose $T \rightarrow_{\mathcal{S}_m} T' \twoheadrightarrow_{\mathcal{S}} \top$. From the induction hypothesis we obtain a number n such that $\mathcal{S}_n \vdash T'$. Let $k = \max(m, n)$. Proposition 9.3(2) yields $T =_{\mathcal{S}_k} \top$. Using confluence of $\mathcal{S}_k$ we obtain $\mathcal{S}_k \vdash T$.
(2) Easy consequence of Lemma 9.2 and the previous part.
□

A much more complicated problem than the change in behaviour of $\mathcal{S}_n$ is the breakdown of Lemma 8.11.

EXAMPLE 9.5. Consider again the CTRS of Example 9.1 and let $S = \{a =^? f(c)\}$ and $\theta = \varepsilon$. The step

$$\theta S \rightarrowtail \{f(b) =^? f(c),\ b =^? b\} = T'$$

is point-blank. There is only one narrowing step originating from S:

$$S \rightsquigarrow_{\epsilon} \{f(x) =^? f(c),\ x =^? b\} = S'.$$

Every substitution θ' satisfying $\theta' S' = T'$ must have $\theta' x = b$. But this conflicts with the sufficient normalization of θ' since the level of $\theta'(f(x) =^? f(c))$ is 1 and $\theta' x$ is $\mathcal{S}_1$-reducible. Suppose that we extend $\theta S \rightarrowtail T'$ with the step $T' \rightarrowtail \{f(b) =^? f(c), \mathit{true}\} = T''$, i.e. we solve the condition of the rule applied in the previous step. The corresponding narrowing step is

$$S' \rightsquigarrow_{\{x,y \mapsto b\}} \{f(b) =^? f(c),\ \mathit{true}\} = S'',$$

where we used the rule $y =^? y \rightarrow \mathit{true}$. Now the problem has disappeared: every substitution θ'' is sufficiently normalized with respect to S''. By solving the condition $x =^? b$ the problematic term b was transferred from the substitution θ' to the goal S''.

Example 9.5 suggests that Lemma 8.11 may hold if we restrict ourselves to $\rightarrowtail$-sequences that first solve the introduced conditions after every application of a conditional rewrite rule. This indeed turns out to be the case. Observe that $\rightarrowtail$-sequences complying with the obligation to solve conditions immediately after their introduction correspond to ordinary rewrite sequences, the only difference being the introduction of the harmless constant *true* after a condition has been solved in a $\rightarrowtail$-sequence. For that reason we will favour $\rightarrow$ above $\rightarrowtail$ in the following. However, there is no reason to adapt the narrowing relation of Definition 6.3. A single $\rightarrow$-step will simply correspond to a sequence of $\rightsquigarrow$-steps.

DEFINITION 9.6. A sequence $S \twoheadrightarrow_{\mathcal{S}} T$ is *point-blank* if each step in this sequence can be written as $S' \rightarrow_{\mathcal{S}_n} T'$ for some n that does not exceed the level of the selected goal in S'.

The proof of the next proposition is easier than the proof of Proposition 8.10.

PROPOSITION 9.7. *Let T be a goal clause. We have $\mathcal{S} \vdash T$ if and only if there exists a point-blank sequence $T \twoheadrightarrow_{\mathcal{S}} \top$.*

PROOF.

$\Rightarrow$ By induction on n we will show the existence of a point-blank sequence $T \twoheadrightarrow_{\mathcal{S}} \top$ whenever $\mathcal{S}_n \vdash T$. If $n = 0$ then every $\mathcal{S}_n$-sequence starting from T is point-blank. Suppose $\mathcal{S}_{n+1} \vdash T$. So there exists a $\mathcal{S}_{n+1}$-sequence from T to $\top$. We use induction on the length of this sequence. The case of zero length is trivial. Let $T \rightarrow_{\mathcal{S}_{n+1}} T_1 \twoheadrightarrow_{\mathcal{S}_{n+1}} \top$. We obtain a point-blank sequence $T_1 \twoheadrightarrow_{\mathcal{S}} \top$ from the second induction hypothesis. By definition there exist a goal $e \in T$, a position $p \in O(e)$, a variant R: $l \rightarrow r \Leftarrow c$ of a rewrite rule in $\mathcal{R}$ and a substitution σ such that $e_{|p} \equiv \sigma l$ and $T_1 = T - \{e\} \cup \{e[\sigma r]_p\}$. (In this proof we have no need for $\mathcal{S}_n \vdash \sigma\bar{c}$ and the $\mathcal{S}_n$-normalization of $\sigma\restriction_{\mathcal{E}(R)}$.) Let m be the level of e. If $n+1 \leqslant m$ then the step from T to T_1 is point-blank and hence we are done. Suppose $n+1 > m$. We have $\mathcal{R}_m \vdash \{e\}$ and hence $\mathcal{S}_m \vdash \{e\}$ due to Lemma 9.2(3). Applying the first induction hypothesis to $\mathcal{S}_m \vdash \{e\}$ yields a point-blank sequence $\{e\} \twoheadrightarrow_{\mathcal{S}} \top$. From the point-blank sequence $T_1 \twoheadrightarrow_{\mathcal{S}} \top$ we extract a point-blank sequence from $T - \{e\}$ to $\top$. Combining the resulting sequence with the point-blank sequence from $\{e\}$ to $\top$ yields the desired result.

$\Leftarrow$ Trivial.

$\square$

The complicated proof of the following lifting lemma for level-complete 3-CTRS's can be found in Appendix A.

LEMMA 9.8. *Let $\mathcal{R}$ be a level-complete 3-CTRS. Suppose we have a goal clause S, a sufficiently normalized solution θ of S and a set V of variables such that $\mathcal{V}(S) \cup \mathcal{D}\theta \subseteq V$. If $T \simeq \theta S \twoheadrightarrow_{\mathcal{S}} T'$ is point-blank then there exist a goal clause S' and substitutions θ', σ such that*

- $S \rightsquigarrow^*_\sigma S'$,
- $\theta' S' \simeq T'$,
- $\theta' \circ \sigma = \theta\ [V]$,
- *θ' is a sufficiently normalized solution of S'.*

$\square$

THEOREM 9.9. *Conditional narrowing is complete for level-complete 3-CTRS's.*

PROOF. Let σ be a solution of a goal clause T, i.e. $\mathcal{R} \vdash \sigma T$. Let σ' be the normal form of σ. Confluence of $\mathcal{R}$ yields $\mathcal{R} \vdash \sigma' T$. We obtain $\mathcal{S} \vdash \sigma' T$ from Proposition 9.4. According to Proposition 9.7 there exists a point-blank sequence $\sigma' T \twoheadrightarrow_{\mathcal{S}} \top$. Lemma 9.8 yields a narrowing derivation $T \rightsquigarrow^*_\tau \top$ and a substitution σ'' such that $\sigma'' \circ \tau = \sigma'\ [\mathcal{V}(T)]$. Therefore $\tau \leqslant \sigma'\ [\mathcal{V}(T)]$ and hence $\tau \leqslant_{\mathcal{R}} \sigma\ [\mathcal{V}(T)]$. $\square$

In Dershowitz and Okada [9] the impression is created that the above completeness result for level-complete 3-CTRS's is due to Giovannetti and Moiso.[18] This is denied by Moiso [36] in e-mail correspondence.

[18] Actually, Dershowitz and Okada ascribe the completeness of conditional narrowing for level-complete CTRS's (Theorem 5.1 in [9]) to Bosco *et al.* [4]. That paper however is not concerned with *conditional* narrowing.

10. Conclusion

In this paper we have tried to perform a thorough study of the completeness of narrowing and basic narrowing for TRS's and CTRS's. The main results are summarized below. Results preceded with '∘' are new.

Narrowing is complete for

- • complete TRS's,
- • semi-complete TRS's,
- • confluent TRS's with respect to normalizable solutions.

Basic narrowing is complete for

- • complete TRS's,
- ∘ orthogonal TRS's with respect to normalizable solutions and goals,[19]
- ∘ confluent right-linear TRS's with respect to normalizable solutions.
- ∘ weakly normalizing orthogonal TRS's,
- ∘ semi-complete right-linear TRS's.

Basic narrowing is not complete for

- ∘ semi-complete TRS's.

Conditional narrowing is complete for

- • complete 1-CTRS's,
- • semi-complete 1-CTRS's,
- • confluent 1-CTRS's with respect to normalizable solutions,
- • level-complete 2-CTRS's,
- ∘ level-semi-complete 2-CTRS's,
- ∘ level-complete 3-CTRS's.

Conditional narrowing is not complete for

- • complete 2-CTRS's,
- • level-confluent 2-CTRS's with respect to normalizable solutions.

Basic conditional narrowing is complete for

- ∘ decreasing and confluent 1-CTRS's,
- • level-complete 2-CTRS's.

Basic conditional narrowing is not complete for

- ∘ complete 1-CTRS's,
- ∘ semi-complete orthogonal 1-CTRS's.

We expect that the completeness of basic narrowing for level-complete 2-CTRS's carries over to 3-CTRS's. It is less clear whether level-semi-completeness is sufficient for the completeness of conditional narrowing for 3-CTRS's (cf. Theorem 8.15). As a matter of fact, it seems reasonable to conjecture that Theorem 8.15 does not extend to 3-CTRS's since strong normalization is crucial in the proof of the lifting lemma for 3-CTRS's (Lemma 9.8).

Giovannetti and Moiso [18] observed that the confluence proof of Bergstra and Klop [1] for orthogonal and normal 2-CTRS's (III_n systems in the terminology of [1]) actually shows

[19] See Theorem 5.5 for a precise formulation.

level-confluence. Such a result (if at all true) makes less sense for 3-CTRS's since 3-CTRS's typically are not normal, see for example the 3-CTRS at the beginning of Section 9. Thus it is important to develop other criteria that are easy to check and which ensure the level-confluence of 3-CTRS's. Useful techniques which ensure the strong normalization of 2 and 3-CTRS's need also to be developed.

If we extend the set of basic positions as in the combination of basic and normal narrowing (see Réty [40]), our counterexamples (4.7 and 7.3) no longer work. It is worthwhile to investigate whether such a relaxed form of basic narrowing suffices for completeness.

As explained in the introduction, we restricted ourselves in this paper to narrowing and basic narrowing for TRS's and CTRS's. It would be interesting to treat the other variants of narrowing in the same systematic way.

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Appendix A

In this appendix we present full proofs of the lifting lemma for TRS's, 1-CTRS's, level-confluent 2-CTRS's and level-complete 3-CTRS's. In the literature several proofs of the lifting lemma for TRS's and 1-CTRS's are given, but many of them are erroneous. Also some of the lifting lemma's are wrongly stated. Nearly all mistakes are due to incorrect assumptions about variables occurring in narrowing derivations and substitutions. This phenomenon is well-known in logic programming (cf. Shepherdson [44]).

The proof of the lifting lemma for TRS's in Hullot [24] is incorrect with regard to the normalization of the resulting substitution θ'. Moreover, several tedious inductive arguments are left implicit. The proof of the lifting lemma for 1-CTRS's in Kaplan [28] employs false assumptions about narrowing substitutions. Bockmayr [2] presents an incorrect lifting lemma for 1-CTRS's with respect to a single $\rightarrowtail$-step. This 'one step' lifting lemma is not powerful enough to lift rewrite sequences by an inductive proof. These mistakes were confirmed by Bockmayr [3]. The proofs of the lifting lemma for TRS's (modulo some equational theory) presented in Kirchner [29] and Réty [41] are also suspect, as admitted by the respective authors in e-mail correspondence ([30], [42]). Dershowitz and Okada give a rather informal treatment of a lifting lemma for 1-CTRS's and level-confluent 2-CTRS's (Lemma 5.2 in [9]). This lemma is *not* suitable for proving the completeness of conditional narrowing for level-complete 2-CTRS's.[20] Giovannetti and Moiso [18] present a lifting lemma for level-confluent 2-CTRS's without proof. They claim that the proof is strictly analogous to Hullot's proof for TRS's. As a positive exception we mention the proof of the lifting lemma for 1-CTRS's embedded in a logic programming framework by Hölldobler (Lemma 6.5.2 in [23]).

Considering the above, we wil try to give rigorous proofs of the various lifting lemma's. In particular, we take great efforts to motivate all assumptions about variables and substitutions. We start with reviewing some well-known facts about idempotent most general unifiers.

DEFINITION A.1. A substitution σ is *idempotent* if $\sigma \circ \sigma = \sigma$.

PROPOSITION A.2. *A substitution σ is idempotent if and only if $\mathcal{D}\sigma \cap \mathcal{I}\sigma = \varnothing$.*

PROOF.

$\Rightarrow$ Suppose there is a variable $x \in \mathcal{D}\sigma \cap \mathcal{I}\sigma$. Since $x \in \mathcal{I}\sigma$ there exists a variable $y \in \mathcal{D}\sigma$ such that $x \in \mathcal{V}(\sigma y)$. From $x \in \mathcal{D}\sigma$ we obtain $\sigma(\sigma y) \not\equiv \sigma y$. Hence σ is not idempotent.

$\Leftarrow$ If $\mathcal{D}\sigma \cap \mathcal{I}\sigma = \varnothing$ then $\sigma \circ \sigma = \sigma\restriction_{\mathcal{D}\sigma \cap \mathcal{I}\sigma} \circ \sigma \cup \sigma\restriction_{\mathcal{D}\sigma - \mathcal{D}\sigma} = \epsilon \circ \sigma \cup \epsilon = \sigma$.

□

[20] As exemplified by the level-complete 2-CTRS $\{a \to b \Leftarrow x = a\}$.

The unification algorithms of Robinson [43] and Martelli and Montanari [34] produce idempotent most general unifiers that satisfy the property stated in the next theorem.

THEOREM A.3. *If two terms t_1, t_2 are unifiable then they have an idempotent most general unifier σ such that $\mathcal{D}\sigma \cup \mathcal{I}\sigma \subseteq \mathcal{V}(t_1) \cup \mathcal{V}(t_2)$.* □

PROPOSITION A.4. *If σ is an idempotent most general unifier of two terms t_1, t_2 that have no variables in common then $\mathcal{D}\sigma \cup \mathcal{I}\sigma = \mathcal{V}(t_1) \cup \mathcal{V}(t_2)$.*

PROOF.

$\subseteq$ We first show that $\mathcal{D}\sigma \subseteq \mathcal{V}(t_1) \cup \mathcal{V}(t_2)$. Suppose to the contrary that there exists a variable $x \in \mathcal{D}\sigma$ such that $x \notin \mathcal{V}(t_1) \cup \mathcal{V}(t_2)$. Define $\tau = \sigma\restriction_{\mathcal{D}\sigma - \{x\}}$. Clearly τ is a unifier of t_1 and t_2. Let $\theta = \sigma\restriction_{\{x\}}$. We obtain $x \notin \mathcal{I}\sigma$ from the idempotence of σ. Hence $\theta \circ \tau = \sigma$, so τ is more general than σ. But we can show that σ is not more general than τ. For, if there is a substitution θ' with $\theta' \circ \sigma = \tau$ then $\theta'(\sigma x) \equiv \tau x \equiv x$. This is only possible if $\sigma x \equiv y$ for some variable y and $\theta' y \equiv x$. From the idempotence of σ we obtain $y \notin \mathcal{D}\sigma$ and hence $\theta'(\sigma y) \equiv \theta' y \equiv x$ differs from $\tau y \equiv \sigma y \equiv y$. We conclude that $\mathcal{D}\sigma \subseteq \mathcal{V}(t_1) \cup \mathcal{V}(t_2)$. Next we show that $\mathcal{I}\sigma \subseteq \mathcal{V}(t_1) \cup \mathcal{V}(t_2)$. If $\mathcal{I}\sigma \not\subseteq \mathcal{V}(t_1) \cup \mathcal{V}(t_2)$ then there exists a variable $x \in \mathcal{I}\sigma$ which does not occur in t_1 and t_2. According to Theorem A.3 t_1 and t_2 have a most general unifier τ which satisfies $\mathcal{D}\tau \cup \mathcal{I}\tau \subseteq \mathcal{V}(t_1) \cup \mathcal{V}(t_2)$. By definition $\tau = \theta \circ \sigma$ for some substitution θ. If $x \in \mathcal{D}\theta$ then $x \in \mathcal{D}(\theta \circ \sigma)$ and if $x \notin \mathcal{D}\theta$ then $x \in \mathcal{I}(\theta \circ \sigma)$. In both cases we have a contradiction with the fact that $x \notin \mathcal{D}\tau \cup \mathcal{I}\tau$. Therefore $\mathcal{I}\sigma \subseteq \mathcal{V}(t_1) \cup \mathcal{V}(t_2)$.

$\supseteq$ Let $x \in \mathcal{V}(t_1)$. If $x \notin \mathcal{D}\sigma$ then $x \in \mathcal{V}(\sigma t_1)$ and hence $x \in \mathcal{V}(\sigma t_2)$. Since $x \notin \mathcal{V}(t_2)$ this implies that $x \in \mathcal{I}\sigma$.

□

The next four propositions are heavily used in the proofs of the lifting lemma's. Observe that Proposition A.6 is formulated in terms of a relation $\rightarrow_{\mathcal{R}}$ instead of a TRS $\mathcal{R}$. The reason is that in the proof of the lifting lemma for level-complete 3-CTRS's we will use this proposition for the relation $\rightarrow_{\mathcal{S}_n}$ which is not generated by a TRS as it is not closed under substitutions.

PROPOSITION A.5. *If t is a term and σ a substitution then $\mathcal{V}(\sigma t) \subseteq \mathcal{V}(t) - \mathcal{D}\sigma \cup \mathcal{I}\sigma\restriction_{\mathcal{V}(t)}$.*

PROOF. Obvious. □

PROPOSITION A.6. *Let $\rightarrow_{\mathcal{R}}$ be a relation on terms that is closed under contexts and suppose we have sets A, B of variables and substitutions σ, θ, θ' such that the following conditions are satisfied:*

- *$\theta\restriction_A$ is $\mathcal{R}$-normalized,*
- *$\theta' \circ \sigma = \theta\ [A]$,*
- *$B \subseteq A - \mathcal{D}\sigma \cup \mathcal{I}\sigma\restriction_A$.*

Then $\theta'\restriction_B$ is also $\mathcal{R}$-normalized.

PROOF. Let $x \in B$. We have to show that $\theta' x$ is an $\mathcal{R}$-normal form. If $x \in A - \mathcal{D}\sigma$ then $\theta' x \equiv (\theta' \circ \sigma)x \equiv \theta x$ which is an $\mathcal{R}$-normal form by assumption. If $x \in \mathcal{I}\sigma\restriction_A$ then there exists a variable $y \in A$ such that $x \in \mathcal{V}(\sigma y)$. We have $\theta' x \subseteq \theta'(\sigma y) \equiv \theta y$. By assumption θy is an $\mathcal{R}$-normal form and since $\rightarrow_{\mathcal{R}}$ is closed under contexts, $\theta' x$ is also an $\mathcal{R}$-normal form. □

PROPOSITION A.7. *Suppose we have terms t_1, t_2 that have no variables in common and a set A of variables such that $\mathcal{V}(t_1) \subseteq A$. If σ is an idempotent most general unifier of t_1 and t_2 then $\mathcal{I}\sigma \subseteq A - \mathcal{D}\sigma \cup \mathcal{I}\sigma\restriction_A$ and hence $A - \mathcal{D}\sigma \cup \mathcal{I}\sigma = A - \mathcal{D}\sigma \cup \mathcal{I}\sigma\restriction_A$.*

PROOF. According to Proposition A.4 we have $\mathcal{D}\sigma \cup \mathcal{I}\sigma = \mathcal{V}(t_1) \cup \mathcal{V}(t_2)$. In particular $\mathcal{I}\sigma \subseteq$

$\mathcal{V}(t_1) \cup \mathcal{V}(t_2)$. Let $x \in \mathcal{I}\sigma$. Idempotence of σ yields $x \notin \mathcal{D}\sigma$. If $x \in \mathcal{V}(t_1)$ then $x \in A - \mathcal{D}\sigma$. If $x \in \mathcal{V}(t_2)$ then $x \in \mathcal{V}(\sigma t_2) = \mathcal{V}(\sigma t_1)$. Since $x \notin \mathcal{V}(t_1)$ this implies $x \in \mathcal{I}\sigma\restriction_{\mathcal{V}(t_1)} \subseteq \mathcal{I}\sigma\restriction_A$. □

PROPOSITION A.8. *Suppose we have substitutions σ, θ, θ' and sets A, B of variables such that $B - \mathcal{D}\sigma \cup \mathcal{I}\sigma \subseteq A$. If $\theta = \theta'$ $[A]$ then $\theta \circ \sigma = \theta' \circ \sigma$ $[B]$.*

PROOF. We have $(\theta \circ \sigma)\restriction_B = (\theta\restriction_{\mathcal{I}\sigma} \circ \sigma)\restriction_B \cup \theta\restriction_{B-\mathcal{D}\sigma} = (\theta'\restriction_{\mathcal{I}\sigma} \circ \sigma)\restriction_B \cup \theta'\restriction_{B-\mathcal{D}\sigma} = (\theta' \circ \sigma)\restriction_B$. The assumptions are used in the second equality. □

We are now ready for the proof of Lemma 3.4. First we lift a single rewrite step. The extra requirement $\mathcal{V}(s') \cup \mathcal{D}\theta' \subseteq V - \mathcal{D}\sigma \cup \mathcal{I}\sigma$ is needed in order to make the inductive proof of Lemma 3.4 run smoothly.

LEMMA A.9. *Let $\mathcal{R}$ be a TRS. Suppose we have terms s and t, a normalized substitution θ and a set V of variables such that $\mathcal{V}(s) \cup \mathcal{D}\theta \subseteq V$ and $t \equiv \theta s$. If $t \rightarrow_{[p, l \rightarrow r]} t'$ with $\mathcal{V}(l) \cap V = \varnothing$ then there exist a term s' and substitutions θ', σ such that*

- $s \rightsquigarrow_{[p, l \rightarrow r, \sigma]} s'$,
- $\mathcal{V}(s') \cup \mathcal{D}\theta' \subseteq V - \mathcal{D}\sigma \cup \mathcal{I}\sigma$,
- $\theta' s' \equiv t'$,
- $\theta' \circ \sigma = \theta$ $[V]$,
- *θ' is normalized.*

PROOF. We have $(\theta s)_{|p} \equiv \tau l$ for some substitution τ with $\mathcal{D}\tau \subseteq \mathcal{V}(l)$. Since θ is normalized we have $p \in \overline{O}(s)$ and hence $(\theta s)_{|p} = \theta(s_{|p})$. Let $\mu = \tau \cup \theta$. We have $\mu(s_{|p}) \equiv \theta(s_{|p}) \equiv \tau l \equiv \mu l$, so $s_{|p}$ and l are unifiable. Let σ be an idempotent most general unifier of $s_{|p}$ and l. Proposition A.4 yields $\mathcal{D}\sigma \cup \mathcal{I}\sigma = \mathcal{V}(s_{|p}) \cup \mathcal{V}(l)$. Let $s' \equiv \sigma(s[r]_p)$. By definition $s \rightsquigarrow_{[p, l \rightarrow r, \sigma]} s'$. Since $\sigma \leqslant \mu$, there exists a substitution ρ such that $\rho \circ \sigma = \mu$. Let $V' = V - \mathcal{D}\sigma \cup \mathcal{I}\sigma$. Define $\theta' = \rho\restriction_{V'}$. Clearly $\mathcal{D}\theta' \subseteq V'$. We have

$$\mathcal{V}(s') = \mathcal{V}(\sigma(s[r]_p)) \subseteq \mathcal{V}(\sigma(s[l]_p)) = \mathcal{V}(\sigma s) \subseteq V'.$$

The last inclusion follows from Proposition A.5. Using $\theta' = \rho$ $[V']$ we obtain

$$\theta' s' \equiv \rho s' \equiv \rho\sigma(s[r]_p) \equiv \mu(s[r]_p) \equiv \mu s[\mu r]_p.$$

Since $V \cap \mathcal{D}\tau = \varnothing$ we have $\mu = \theta$ $[V]$. Likewise $\mu = \tau$ $[V(r)]$. Hence the term $\mu s[\mu r]_p$ equals $\theta s[\tau r]_p \equiv t'$. We now show that $\theta' \circ \sigma = \theta$ $[V]$. Proposition A.8 yields $\theta' \circ \sigma = \rho \circ \sigma$ $[V]$. We already noticed that $\mu = \theta$ $[V]$. Linking these two equalities via the equation $\rho \circ \sigma = \mu$ yields $\theta' \circ \sigma = \theta$ $[V]$. It remains to show that θ' is normalized. Since $\mathcal{D}\theta' \subseteq V'$ it suffices to show that $\theta'\restriction_{V'}$ is normalized. Let $B = V - \mathcal{D}\sigma \cup \mathcal{I}\sigma\restriction_V$. Proposition A.6 (with $A = V$) yields the normalization of $\theta'\restriction_B$. Since $\mathcal{V}(s_{|p}) \subseteq V$ we obtain $B = V'$ from Proposition A.7. □

LEMMA 3.4. *Let $\mathcal{R}$ be a TRS. Suppose we have terms s and t, a normalized substitution θ and a set V of variables such that $\mathcal{V}(s) \cup \mathcal{D}\theta \subseteq V$ and $t \equiv \theta s$. If $t \twoheadrightarrow t'$ then there exist a term s' and substitutions θ', σ such that*

- $s \rightsquigarrow^*_\sigma s'$,
- $\theta' s' \equiv t'$,
- $\theta' \circ \sigma = \theta$ $[V]$,
- *θ' is normalized.*

Furthermore, we may assume that the narrowing derivation $s \rightsquigarrow^*_\sigma s'$ and the rewrite sequence $t \twoheadrightarrow t'$ employ the same rewrite rules at the same positions.

PROOF. We use induction on the length of the reduction sequence from t to t'. The case of zero length is trivial. Suppose $t \to t_1 \twoheadrightarrow t'$ is a reduction sequence of length $n+1$. Let $t \to_{[p, l \to r]} t_1$. We may assume that $\mathcal{V}(l) \cap V = \varnothing$.[21] According to Lemma A.9 there exist a term s_1 and substitutions θ_1, σ_1 such that

- $s \rightsquigarrow_{[p, l \to r, \sigma_1]} s_1$,
- $\mathcal{V}(s_1) \cup \mathcal{D}\theta_1 \subseteq V - \mathcal{D}\sigma_1 \cup \mathcal{I}\sigma_1$,
- $\theta_1 s_1 \equiv t_1$,
- $\theta_1 \circ \sigma_1 = \theta \ [V]$,
- θ_1 is normalized.

The induction hypothesis yields a term s' and substitutions θ', σ' such that

- $s_1 \rightsquigarrow^*_{\sigma'} s'$,
- $\theta' s' \equiv t'$,
- $\theta' \circ \sigma' = \theta_1 \ [V - \mathcal{D}\sigma_1 \cup \mathcal{I}\sigma_1]$,
- θ' is normalized.

Moreover, we may assume that $s_1 \rightsquigarrow^*_{\sigma'} s'$ and $t_1 \twoheadrightarrow t'$ apply the same rewrite rules at the same positions. Let $\sigma = \sigma' \circ \sigma_1$. Concatenating $s \rightsquigarrow_{\sigma_1} s_1$ and $s_1 \rightsquigarrow^*_{\sigma'} s'$ yields $s \rightsquigarrow^*_{\sigma} s'$. By construction this narrowing derivation employs the same rewrite rules at the same positions as the rewrite sequence $t \twoheadrightarrow t'$. It remains to show that $\theta' \circ \sigma = \theta \ [V]$. Proposition A.8 yields $\theta' \circ \sigma' \circ \sigma_1 = \theta_1 \circ \sigma_1 \ [V]$ and hence $\theta' \circ \sigma = \theta_1 \circ \sigma_1 = \theta \ [V]$. □

The proof of the lifting lemma for 1-CTRS's is almost identical to the one for TRS's.

LEMMA 6.10. *Let $\mathcal{R}$ be a 1-CTRS. Suppose we have goal clauses S and T, a normalized substitution θ and a set V of variables such that $\mathcal{V}(S) \cup \mathcal{D}\theta \subseteq V$ and $T = \theta S$. If $T \rightarrowtail\!\!\!\!\twoheadrightarrow T'$ then there exist a goal clause S' and substitutions θ', σ such that*

- $S \rightsquigarrow^*_{\sigma} S'$,
- $\theta' S' = T'$,
- $\theta' \circ \sigma = \theta \ [V]$,
- *θ' is normalized.*

Furthermore, we may assume that the narrowing derivation $S \rightsquigarrow^*_{\sigma} S'$ and the rewrite sequence $T \rightarrowtail\!\!\!\!\twoheadrightarrow T'$ employ the same rewrite rules at the same positions in the same goals.

PROOF. First we lift a single step as in Lemma A.9 and then we lift sequences as in Lemma 3.4. The only difference with the preceding lemma's is that we are dealing with goal clauses instead of terms. □

Next we consider level-confluent 2-CTRS's. As usual we first lift a single step. The most difficult part is the sufficient normalization of the resulting substitution θ'.

LEMMA A.10. *Let $\mathcal{R}$ be a level-confluent 2-CTRS. Suppose we have a goal clause S, a sufficiently normalized solution θ of S and a set of variables V such that $\mathcal{V}(S) \cup \mathcal{D}\theta \subseteq V$. If $T = \theta S \rightarrowtail_{[\theta e, p, R]} T'$ with $\mathcal{V}(R) \cap V = \varnothing$ is point-blank then there exist a goal clause S' and substitutions θ', σ such that*

- $S \rightsquigarrow_{[e, p, R, \sigma]} S'$,
- $\mathcal{V}(S') \cup \mathcal{D}\theta' \subseteq V - \mathcal{D}\sigma \cup \mathcal{I}\sigma \cup \mathcal{E}(R)$,
- $\theta' S' = T'$,
- $\theta' \circ \sigma = \theta \ [V]$,
- *θ' is a sufficiently normalized solution of S'.*

[21] This is justified by the variant independence of rewriting, cf. footnote 4 on page 5.

PROOF. Let R be the rewrite rule $l \to r \Leftarrow c$. We have $(\theta e)_{|p} \equiv \tau l$ for some substitution τ with $\mathcal{D}\tau \subseteq \mathcal{V}(R)$. We first show that $p \in \overline{O}(e)$. Let n be the level of θe. Since $T \rightarrowtail T'$ is point-blank, there exists a number $m \leqslant n$ such that $T \rightarrowtail^m T'$. This means that $\tau l \to_{\mathcal{R}_m} \tau r$, $\mathcal{R}_{m-1} \vdash \tau\tilde{c}$ and $\tau\restriction_{\mathcal{E}(R)}$ is $\mathcal{R}_{m-1}$-normalized. Because θ is a sufficiently normalized solution of S, $\theta\restriction_{\mathcal{V}(e)}$ is $\mathcal{R}_n$-normalized and hence also $\mathcal{R}_m$-normalized. Thus $p \in \overline{O}(e)$ and so $(\theta e)_{|p} = \theta(e_{|p})$. Let $\mu = \tau \cup \theta$. We have $\mu(e_{|p}) \equiv \theta(e_{|p}) \equiv \tau l \equiv \mu l$. Let σ be an idempotent most general unifier of $e_{|p}$ and l. Proposition A.4 yields $\mathcal{D}\sigma \cup \mathcal{I}\sigma = \mathcal{V}(e_{|p}) \cup \mathcal{V}(l)$. Let $S' = \sigma(S - \{e\} \cup \{e[r]_p\} \cup \tilde{c})$. By definition $S \rightsquigarrow_{[e,p,R,\sigma]} S'$. Let $V' = V - \mathcal{D}\sigma \cup \mathcal{I}\sigma \cup \mathcal{E}(R)$. We now show that $\mathcal{V}(S') \subseteq V'$. Proposition A.5 yields

$$\mathcal{V}(\sigma S) \subseteq V - \mathcal{D}\sigma \cup \mathcal{I}\sigma \subseteq V'. \tag{1}$$

Since $\mathcal{R}$ is a 2-CTRS we have $\mathcal{V}(r) \subseteq \mathcal{V}(l)$ and hence

$$\mathcal{V}(\sigma(e[r]_p)) \subseteq \mathcal{V}(\sigma(e[l]_p)) = \mathcal{V}(\sigma e) \subseteq V - \mathcal{D}\sigma \cup \mathcal{I}\sigma \subseteq V'. \tag{2}$$

The last inclusion follows from (1). It is easy show that $\mathcal{E}(R) \cap \mathcal{D}\sigma = \varnothing$. Hence $\mathcal{V}(\sigma\tilde{c}) \subseteq \mathcal{V}(\sigma l) \cup \mathcal{E}(R)$. Since $\mathcal{V}(l) \subseteq \mathcal{D}\sigma \cup \mathcal{I}\sigma$, Proposition A.5 yields $V(\sigma l) \subseteq (\mathcal{D}\sigma \cup \mathcal{I}\sigma) - \mathcal{D}\sigma \cup \mathcal{I}\sigma = \mathcal{I}\sigma$ and hence

$$V(\sigma\tilde{c}) \subseteq \mathcal{I}\sigma \cup \mathcal{E}(R) \subseteq V'. \tag{3}$$

Combining (1), (2) and (3) yields $\mathcal{V}(S') \subseteq V'$. Since $\sigma \leqslant \mu$, there exists a substitution ρ such that $\rho \circ \sigma = \mu$. Define $\theta' = \rho\restriction_{V'}$. By definition $\mathcal{D}\theta' \subseteq V'$ and $\theta' = \rho\ [V]$. From $\mathcal{E}(R) - \mathcal{D}\sigma = \mathcal{E}(R)$ and $\mathcal{V}(l) - \mathcal{D}\sigma \subseteq \mathcal{I}\sigma$ we infer that $\mathcal{V}(R) - \mathcal{D}\sigma \subseteq V'$ and hence $(V \cup \mathcal{V}(R)) - \mathcal{D}\sigma \cup \mathcal{I}\sigma = V'$. An application of Proposition A.8 yields $\theta' \circ \sigma = \rho \circ \sigma = \mu\ [V \cup \mathcal{V}(R)]$. From $\mu\restriction_V = \theta$ and $\mu\restriction_{\mathcal{V}(R)} = \tau$ we infer that

$$\theta' \circ \sigma = \theta\ [V] \tag{4}$$

and

$$\theta' \circ \sigma = \tau\ [\mathcal{V}(R)]. \tag{5}$$

From these two equalities we obtain

$$\theta' S' = \theta'\sigma(S - \{e\}) \cup \{\theta'\sigma e[\theta'\sigma r]_p\} \cup \theta'\sigma\tilde{c} = \theta(S - \{e\}) \cup \{\theta e[\tau r]_p\} \cup \tau\tilde{c} = T'. \tag{6}$$

We still have to show that θ' is a sufficiently normalized solution of S'. Let $e' \in S'$. First we show that $\theta' e'$ has some level k and then we prove that $\theta'\restriction_{\mathcal{V}(e')}$ is $\mathcal{R}_k$-normalized. By definition there exists an $e'' \in S - \{e\} \cup \{e[r]_p\} \cup \tilde{c}$ such that $e' \equiv \sigma e''$. We distinguish three cases: (a) $e'' \in S - \{e\}$, (b) $e'' \equiv e[r]_p$ and (c) $e'' \in \tilde{c}$.

(a) Since $\mathcal{V}(e'') \subseteq V$ we obtain $\theta' e' \equiv \theta e''$ from (4). Because θ is a solution of S, $\theta e''$ has some level k. Hence $\theta' e'$ has the same level k. By assumption $\theta\restriction_{\mathcal{V}(e'')}$ is $\mathcal{R}_k$-normalized. We have to show that $\theta'\restriction_{\mathcal{V}(e')}$ is also $\mathcal{R}_k$-normalized. Since $\mathcal{V}(e') \subseteq \mathcal{V}(e'') - \mathcal{D}\sigma \cup \mathcal{I}\sigma\restriction_{\mathcal{V}(e')}$ (Proposition A.5) this follows from Proposition A.6.

(b) Let $e'' \equiv e[r]_p$. Formula (6) shows that $\theta' e' \equiv \theta e[\tau r]_p$. Therefore $\theta e \to_{\mathcal{R}_m} \theta' e'$ and because $m \leqslant n$ we also have $\theta e \to_{\mathcal{R}_n} \theta' e'$. Combining this with $\mathcal{R}_n \vdash \theta e$ and the level-confluence of $\mathcal{R}$ yields $\mathcal{R}_n \vdash \theta' e'$ and hence the level of $\theta' e'$ is at most n. So it suffices to show that $\theta'\restriction_{\mathcal{V}(e')}$ is $\mathcal{R}_n$-normalized. Using the fact that $\theta\restriction_{\mathcal{V}(e)}$ is $\mathcal{R}_n$-normalized, we obtain the $\mathcal{R}_n$-normalization of $\theta'\restriction_{\mathcal{V}(e) - \mathcal{D}\sigma \cup \mathcal{I}\sigma}$ from Propositions A.6 and A.7. Formula (2) yields the desired result.

(c) Let $e'' \in \tilde{c}$. Since $\mathcal{V}(e'') \subseteq \mathcal{V}(R)$ we obtain $\theta' e' \equiv \tau e''$ from (5). By definition $\mathcal{R}_{m-1} \vdash \tau\tilde{c}$ and $\tau\restriction_{\mathcal{E}(R)}$ is $\mathcal{R}_{m-1}$-normalized. So the level of $\theta' e'$ does not exceed $m-1$ and hence it suffices to show that $\theta'\restriction_{\mathcal{V}(e')}$ is $\mathcal{R}_{m-1}$-normalized. From (3) we infer that $\mathcal{V}(e') \subseteq \mathcal{I}\sigma \cup \mathcal{E}(R)$. In case (b) we noticed that $\theta'\restriction_{\mathcal{I}\sigma}$ is $\mathcal{R}_n$-normalized and since $\mathcal{R}_{m-1} \subseteq \mathcal{R}_n$, $\theta'\restriction_{\mathcal{I}\sigma}$ is also $\mathcal{R}_{m-1}$-normalized. From $\mathcal{E}(R) \cap \mathcal{D}\sigma = \varnothing$ and (5) we infer that $\theta'\restriction_{\mathcal{E}(R)}$ equals $\tau\restriction_{\mathcal{E}(R)}$, which by definition is $\mathcal{R}_{m-1}$-normalized.

□

LEMMA 8.11. *Let $\mathcal{R}$ be a level-confluent 2-CTRS. Suppose we have goal clauses S and T, a sufficiently normalized solution θ of S and a set V of variables such that $\mathcal{V}(S) \cup \mathcal{D}\theta \subseteq V$ and $T = \theta S$. If $T \twoheadrightarrow T'$ is point-blank then there exist a goal clause S' and substitutions θ', σ such that*

- $S \rightsquigarrow_\sigma^* S'$,
- $\theta' S' = T'$,
- $\theta' \circ \sigma = \theta\ [V]$,
- *θ' is a sufficiently normalized solution of S'.*

Furthermore, we may assume that the narrowing derivation $S \rightsquigarrow_\sigma^* S'$ and the point-blank sequence $T \twoheadrightarrow T'$ employ the same rewrite rules at the same positions in the same goals.

PROOF. Similar to the proof of Lemma 3.4. □

The lifting lemma for level-complete 3-CTRS's is the most complicated. Unlike the preceding lifting lemma's, we cannot first prove a single step version since such a version would depend on the full lifting lemma. Hence we are faced with a double induction argument.

LEMMA 9.8. *Let $\mathcal{R}$ be a level-complete 3-CTRS. Suppose we have a goal clause S, a sufficiently normalized solution θ of S and a set V of variables such that $\mathcal{V}(S) \cup \mathcal{D}\theta \subseteq V$. If $T \simeq \theta S \twoheadrightarrow_{\mathcal{S}} T'$ is point-blank then there exist a goal clause S' and substitutions θ', σ such that*

- $S \rightsquigarrow_\sigma^* S'$,
- $\theta' S' \simeq T'$,
- $\theta' \circ \sigma = \theta\ [V]$,
- *θ' is a sufficiently normalized solution of S'.*

PROOF. We use induction on the level of T. If the level of T equals 0 then only the rule $x =^? x \to \mathit{true}$ is used in the sequence from T to T'. Hence this sequence can be seen as a point-blank $\rightarrowtail$-sequence. Since $\{x =^? x \to \mathit{true}\}$ clearly constitutes a level-confluent 2-CTRS, the result follows from Lemma 8.11. Suppose the level of T equals $n+1$. We will use induction on the length of the sequence from T to T'. The case of zero length is trivial. Suppose $T \to_{\mathcal{S}} T_1 \twoheadrightarrow_{\mathcal{S}} T'$. The structure of the proof is illustrated in Figure 5. Let $\theta e \in T$ be the selected goal in the step $T \to_{\mathcal{S}} T_1$ and let k be its level. Clearly $k \leqslant n+1$. Since the sequence is point-blank we have $T \to_{\mathcal{S}_m} T_1$ for some $m \leqslant k$. So there exist a position $p \in \theta e$, a rewrite rule $R\colon l \to r \Leftarrow c$ and a substitution τ with $\mathcal{D}\tau \subseteq \mathcal{V}(R)$ such that $(\theta e)_{|p} \equiv \tau l$, $T_1 = T - \{\theta e\} \cup \{\theta e[\tau r]_p\}$, $\mathcal{S}_{m-1} \vdash \tau\tilde{c}$ and $\tau\restriction_{\mathcal{E}(R)}$ is $\mathcal{S}_{m-1}$-normalized. We may assume that $\mathcal{V}(R) \cap V = \varnothing$. Because θ is a sufficiently normalized solution of S, $\theta\restriction_{\mathcal{V}(e)}$ is $\mathcal{S}_k$-normalized. From Proposition 9.3(1) we infer that $\theta\restriction_{\mathcal{V}(e)}$ is also $\mathcal{S}_m$-normalized. Thus $p \in \overline{O}(e)$ and so $(\theta e)_{|p} = \theta(e_{|p})$. Let $\mu = \tau \cup \theta$. We have $\mu(e_{|p}) \equiv \theta(e_{|p}) \equiv \tau l \equiv \mu l$. So $e_{|p}$ and l are unifiable. Let σ_1 be an idempotent most general unifier of $e_{|p}$ and l. Proposition A.4 yields $\mathcal{D}\sigma_1 \cup \mathcal{I}\sigma_1 = \mathcal{V}(e_{|p}) \cup \mathcal{V}(l)$. Let $S_1 = \sigma_1(S - \{e\} \cup \{e[r]_p\} \cup \tilde{c})$. Observe that $e \in S$ even if $T \neq \theta S$. By definition $S \rightsquigarrow_{\sigma_1} S_1$. Let $V_1 = V - \mathcal{D}\sigma_1 \cup \mathcal{I}\sigma_1 \cup \mathcal{E}(R)$. The following statements are obtained as in the proof of Lemma A.10:

$$
\begin{tikzcd}
T \arrow[rr] \arrow[dd, dashed, no head, "\theta"] \arrow[dr, dashed, no head] & & T_1 \arrow[r, two heads] \arrow[dd, dashed, no head, "\theta_2"] & T' \arrow[dd, dashed, no head, "\theta'"] \\
& T_1 \cup \tau\tilde{c} \arrow[ur, dashed, no head] \arrow[d, dashed, no head, "\theta_1"] \arrow[r, phantom, "\text{1st induction hypothesis}"] & {} \arrow[r, phantom, "\text{2nd induction hypothesis}"] & {} \\
S \arrow[r, rightsquigarrow, "\sigma_1"'] & S_1 \arrow[r, rightsquigarrow, "{}^{*}_{\sigma_2}"'] & S_2 \arrow[r, rightsquigarrow, "{}^{*}_{\sigma'}"'] & S'
\end{tikzcd}
$$

FIGURE 5.

$$V(\sigma_1 S) \subseteq \mathcal{V}(S) - \mathcal{D}\sigma_1 \cup \mathcal{I}\sigma_1, \tag{1}$$

$$\mathcal{E}(R) \cap \mathcal{D}\sigma_1 = \varnothing, \tag{2}$$

$$V(\sigma_1 l) \subseteq \mathcal{I}\sigma_1, \tag{3}$$

$$V(\sigma_1 \tilde{c}) \subseteq \mathcal{I}\sigma_1 \cup \mathcal{E}(R). \tag{4}$$

From (1), (2) and the inclusion $\mathcal{V}(r) \subseteq \mathcal{V}(l) \cup \mathcal{E}(R)$ we infer that

$$\mathcal{V}(\sigma_1(e[r]_p)) \subseteq \mathcal{V}(\sigma_1 e) \cup \mathcal{E}(R) \subseteq V_1. \tag{5}$$

Hence $\mathcal{V}(S_1) \subseteq V_1$. Since $\sigma_1 \leqslant \mu$, there exists a substitution ρ such that $\rho \circ \sigma_1 = \mu$. Define $\theta_1 = \rho\restriction_{V_1}$. We obtain

$$\theta_1 \circ \sigma_1 = \theta \; [V] \tag{6}$$

and

$$\theta_1 \circ \sigma_1 = \tau \; [\mathcal{V}(R)] \tag{7}$$

as in the proof of Lemma A.10. From these two equations we easily obtain $\theta_1 S_1 \simeq T_1 \cup \tau\tilde{c}$. Next we show that there exist a goal clause S_2 and substitutions θ_2, σ_2 such that $S_1 \rightsquigarrow^*_{\sigma_2} S_2$, $\mathcal{V}(S_2) \cup \mathcal{D}\theta_2 \subseteq V_2$, $\theta_2 S_2 \simeq T_1$, $\theta_2 \circ \sigma_2 = \theta_1 \; [V_1]$ and θ_2 is a sufficiently normalized solution of S_2. Here $V_2 = V_1 - \mathcal{D}\sigma_2 \cup \mathcal{I}\sigma_2$. We distinguish two cases.

$\tilde{c} = \varnothing$ (This means that R is an unconditional rewrite rule.) We define $S_2 = S_1$, $\theta_2 = \theta_1$ and $\sigma_2 = \varepsilon$. Since in this case $\mathcal{V}(r) \subseteq \mathcal{V}(l)$ we can repeat cases (a) and (b) in the proof of Lemma A.10 in order to conclude that θ_2 is a sufficiently normalized solution of S_2. The other requirements are trivially satisfied.

$\tilde{c} \neq \varnothing$ From $\mathcal{S}_{m-1} \vdash \tau\tilde{c}$ we infer that the level of $\tau\tilde{c}$ is less than $n+1$. According to Proposition 9.7 there exists a point-blank sequence $\tau\tilde{c} = \theta_1(\sigma_1\tilde{c}) \twoheadrightarrow_{\mathcal{S}} \top$. Before we can apply the first induction hypothesis, we have to show that θ_1 is a sufficiently normalized solution of $\sigma_1\tilde{c}$. The proof of this fact is the same as case (c) in the proof of Lemma A.10. Applying the first induction hypothesis yields substitutions θ_2 and σ_2 such that

$$\sigma_1\tilde{c} \rightsquigarrow^*_{\sigma_2} \top \tag{8}$$

and

$$\theta_2 \circ \sigma_2 = \theta_1 \; [V_1]. \tag{9}$$

Let $S_2 = \sigma_2\sigma_1(S - \{e\} \cup \{e[r]_p\}) \cup \top$. Clearly $S_1 \rightsquigarrow^*_{\sigma_2} S_2$. Using $\mathcal{V}(S_1) \subseteq V_1$ we easily obtain $\mathcal{V}(S_2) \subseteq V_2$. It is not difficult to show that $\theta_2 S_2 \simeq T_1$. Since $\theta_2{\restriction_{V_2}}$ also satisfies the above requirements (i.e. $\theta_2{\restriction_{V_2}} \circ \sigma_2 = \theta_1\ [V_1]$ and $\theta_2{\restriction_{V_2}} S_2 \simeq T_1$), we may assume that $\mathcal{D}\theta_2 \subseteq V_2$. It remains to show that θ_2 is a sufficiently normalized solution of S_2. Let $e' \in S_2$. There exists an $e'' \in S - \{e\} \cup \{e[r]_p\}$ such that $e' \equiv \sigma_2\sigma_1 e''$. We distinguish two cases: (a) $e'' \in S - \{e\}$ and (b) $e'' \equiv e[r]_p$.

(a) Since $\mathcal{V}(e'') \subseteq V$ and $\mathcal{V}(\sigma_1 e'') \subseteq V_1$ we obtain $\theta_2 e' \equiv \theta e''$ from (6) and (9). Because θ is a sufficiently normalized solution of S, $\theta{\restriction_{\mathcal{V}(e'')}}$ is $\mathcal{S}_l$-normalized where l is the level of $\theta e''$. We have to show that $\theta_2{\restriction_{\mathcal{V}(e')}}$ is also $\mathcal{S}_l$-normalized. This follows from (6), (9) and two applications of Propositions A.5 and A.6.

(b) Let $e'' \equiv e[r]_p$. We have $\theta e \rightarrow_{\mathcal{S}_m} \theta e[\tau r]_p \equiv \theta_2 e'$. The usual reasoning—this time involving Lemma 9.2(4) and Proposition 9.3(2)—shows that the level of $\theta_2 e'$ is at most k and hence it suffices to show that $\theta_2{\restriction_{\mathcal{V}(e')}}$ is $\mathcal{S}_k$-normalized. The crucial observation is that we have the following inclusion:

$$\mathcal{V}(e') = \mathcal{V}(\sigma_2\sigma_1 e[r]_p) \subseteq \mathcal{V}(\sigma_2\sigma_1 e) \cup \mathcal{V}(\sigma_2\sigma_1 l). \tag{10}$$

Suppose to the contrary that there exists a variable $x \in \mathcal{V}(\sigma_2\sigma_1 e[r]_p)$ such that $x \notin \mathcal{V}(\sigma_2\sigma_1 e) \cup \mathcal{V}(\sigma_2\sigma_1 l)$. This implies the existence of a variable $y \in \mathcal{V}(r)$ such that $x \in \mathcal{V}(\sigma_2\sigma_1 y)$. According to Lemma 6.5 we may infer $\mathcal{R} \vdash \sigma_2\sigma_1\tilde{c}$ from (8). Hence $\sigma_2\sigma_1 l \rightarrow_{\mathcal{R}} \sigma_2\sigma_1 r$. Since $x \in \mathcal{V}(\sigma_2\sigma_1 r) - \mathcal{V}(\sigma_2\sigma_1 l)$ and $\rightarrow_{\mathcal{R}}$ is closed under substitutions (in particular under the substitution $\{x \mapsto \sigma_2\sigma_1 l\}$) we obtain a contradiction with the strong normalization of $\mathcal{R}$. Therefore inclusion (10) is valid. Let $A_1 = \mathcal{V}(e)$, $A_2 = \mathcal{V}(\sigma_1 e) \cup \mathcal{V}(\sigma_1 l)$, $B_1 = A_1 - \mathcal{D}\sigma_1 \cup \mathcal{I}\sigma_1{\restriction_{A_1}}$ and $B_2 = A_2 - \mathcal{D}\sigma_2 \cup \mathcal{I}\sigma_2{\restriction_{A_2}}$. By assumption $\theta{\restriction_{A_1}}$ is $\mathcal{S}_k$-normalized. Using (6) and $A_1 \subseteq V$, Proposition A.6 yields the $\mathcal{S}_k$-normalization of $\theta_1{\restriction_{B_1}}$. From Proposition A.7 we obtain $B_1 = A_1 - \mathcal{D}\sigma_1 \cup \mathcal{I}\sigma_1$. Clearly $B_1 \subseteq V_1$. As a consequence of (3) we have $A_2 \subseteq B_1$, so $\theta_1{\restriction_{A_2}}$ is $\mathcal{S}_k$-normalized. Using (9), Proposition A.6 yields the $\mathcal{S}_k$-normalization of $\theta_2{\restriction_{B_2}}$. Observation (10) and Proposition A.5 show that $\mathcal{V}(e') \subseteq B_2$.

Proposition A.8 yields $\theta_2 \circ \sigma_2 \circ \sigma_1 = \theta_1 \circ \sigma_1\ [V]$ and hence $\theta_2 \circ \sigma_2 \circ \sigma_1 = \theta\ [V]$. Now we apply the second induction hypothesis. This yields a goal clause S' and substitutions θ', σ' such that

- $S_2 \rightsquigarrow^*_{\sigma'} S'$,
- $\theta' S' \simeq T'$,
- $\theta' \circ \sigma' = \theta_2\ [V_2]$,
- θ' is a sufficiently normalized solution of S'.

Let $\sigma = \sigma' \circ \sigma_2 \circ \sigma_1$. We clearly have $S \rightsquigarrow^*_{\sigma} S'$. Two applications of Proposition A.8 yield $\theta' \circ \sigma = \theta\ [V]$. □

Appendix B

In this appendix we present proofs of Propositions 4.4, 5.4, 7.7 and 8.18. The proofs are very much alike, but it is difficult to capture the similarities in a separate, general proposition.

PROPOSITION 4.4. *Let $\mathcal{R}$ be a TRS and σ a normalized substitution. Every innermost reduction sequence starting from σt is based on $\overline{O}(t)$.*

PROOF. Suppose

$$\sigma t \equiv t_1 \rightarrow_{[p_1, l_1 \rightarrow r_1, \sigma_1]} \cdots \rightarrow_{[p_{n-1}, l_{n-1} \rightarrow r_{n-1}, \sigma_{n-1}]} t_n$$

is an innermost reduction sequence. Let $B_1 = \overline{O}(t)$ and define $B_2, \ldots, B_{n-1}$ as in Definition 4.1. By induction on i we will show that $(t_i)_{|p}$ is a normal form whenever $p \in O(t_i) - B_i$ for $1 \leqslant i < n$. The case $i = 1$ follows from the normalization of σ. Suppose the statement holds for $i = 1, \ldots, m$ and let $p \in O(t_{m+1}) - B_{m+1}$. We distinguish two cases: $p \perp p_m$ and $p \geqslant p_m$. (The case $p < p_m$ is impossible since this would imply $p \in B_{m+1}$ as we already know that B_m is closed under prefix and $p_m \in B_m$.)

(1) If $p \perp p_m$ then clearly $p \in O(t_m) - B_m$ and $(t_{m+1})_{|p} \equiv (t_m)_{|p}$. The induction hypothesis yields the desired result.

(2) If $p \geqslant p_m$ then there exist positions $q \in O_{\mathcal{V}}(r_m)$ and q' such that $p = p_m q q'$ (otherwise $p \in B_{m+1}$). Hence $(t_{m+1})_{|p} \equiv (\sigma_m r_m)_{|qq'} \equiv (\sigma_m x)_{|q'}$ where x is the variable in r_m at position q. So $(t_{m+1})_{|p}$ is a proper subterm of $\sigma_m l_m$ and because $t_m \rightarrow_{[p_m, l_m \rightarrow r_m, \sigma_m]} t_{m+1}$ is an innermost reduction step, $(t_{m+1})_{|p}$ is a normal form.

□

Before proving Proposition 5.4 we give a few elementary properties of orthogonal TRS's.

Definition B.1. Let $\mathcal{R}$ be a TRS. We write $s \mathrel{\parallel\!\!\!\!\!\rightarrow} t$ if t can be obtained from s by contracting a set of pairwise disjoint redexes in s. The relation $\mathrel{\parallel\!\!\!\!\!\rightarrow}$ is called *parallel reduction.*

The following lemma expresses a famous result in the theory of orthogonal TRS's. Confluence of orthogonal TRS's is an easy consequence of this lemma.

Parallel Moves Lemma. *Let $\mathcal{R}$ be an orthogonal TRS. If $t \mathrel{\parallel\!\!\!\!\!\rightarrow} t_1$ and $t \mathrel{\parallel\!\!\!\!\!\rightarrow} t_2$ then there exists a term t_3 such that $t_1 \mathrel{\parallel\!\!\!\!\!\rightarrow} t_3$ and $t_2 \mathrel{\parallel\!\!\!\!\!\rightarrow} t_3$. Moreover, the redexes contracted in $t_1 \mathrel{\parallel\!\!\!\!\!\rightarrow} t_3$ ($t_2 \mathrel{\parallel\!\!\!\!\!\rightarrow} t_3$) are the descendants in t_1 (t_2) of the redexes contracted in $t \mathrel{\parallel\!\!\!\!\!\rightarrow} t_2$ ($t \mathrel{\parallel\!\!\!\!\!\rightarrow} t_1$).* □

The following consequence of the Parallel Moves Lemma is used in the proof of Proposition 5.4 below.

Proposition B.2. *Let $\mathcal{R}$ be an orthogonal TRS. Suppose s contains a redex r which is not needed. If $s \rightarrow t$ then the descendants of r in t are not needed.*

Proof. Because r is not needed there exists a normalizing reduction sequence

$$s \equiv s_1 \rightarrow s_2 \rightarrow \cdots \rightarrow s_n$$

in which no descendant of r is contracted. Using the Parallel Moves Lemma we construct the following diagram:

$$\begin{tikzcd}
s \equiv s_1 \arrow[r] \arrow[d] & s_2 \arrow[r] \arrow[d, "\parallel" marking] & \cdots \arrow[r] & s_n \arrow[d, "\parallel" marking] \\
t \equiv t_1 \arrow[r, "\parallel" marking] & t_2 \arrow[r, "\parallel" marking] & \cdots \arrow[r, "\parallel" marking] & t_n
\end{tikzcd}$$

The contracted redexes in $t_i \mathrel{\parallel\!\!\!\!\!\rightarrow} t_{i+1}$ are the descendants of the redex contracted in the step $s_i \rightarrow s_{i+1}$. Hence no descendant of r is contracted in the sequence $t_1 \twoheadrightarrow t_n$ and because $s_n \equiv t_n$ no descendant of r in t is needed. □

PROPOSITION 5.4. *Let $\mathcal{R}$ be an orthogonal TRS and σ a normalized substitution. Every innermost needed reduction sequence starting from σt is based on $\overline{O}(t)$.*

PROOF. The proof has the same structure as the proof of Proposition 4.4. Suppose

$$\sigma t \equiv t_1 \rightarrow_{[p_1, l_1 \rightarrow r_1, \sigma_1]} \cdots \rightarrow_{[p_{n-1}, l_{n-1} \rightarrow r_{n-1}, \sigma_{n-1}]} t_n$$

is an innermost needed reduction sequence and define $B_1, \dots, B_{n-1}$ as usual. By induction on i we will show that $(t_i)_{|p}$ contains no needed redexes whenever $p \in O(t_i) - B_i$ for $1 \leqslant i < n$. The case $i = 1$ is trivial. Suppose the statement holds for $i = 1, \dots, m$ and let $p \in O(t_{m+1}) - B_{m+1}$. The case $p \perp p_m$ easily follows from the induction hypothesis. If $p \geqslant p_m$ then $(t_{m+1})_{|p}$ is a proper subterm of $\sigma_m l_m$, just as in the proof of Proposition 4.4. Suppose $(t_{m+1})_{|p}$ contains a redex r. Since $t_m \rightarrow_{[p_m, l_m \rightarrow r_m, \sigma_m]} t_{m+1}$ is an innermost needed reduction step, r is not needed in t_m. Proposition B.2 shows that r is not needed in t_{m+1}. □

PROPOSITION 7.7. *Let $\mathcal{R}$ be a 1-CTRS, T a goal clause and σ a normalized substitution. Every innermost $\rightarrowtail_{\mathcal{R}}$-sequence starting from σT is based on $\overline{T}$.*

PROOF. Suppose

$$\sigma T = T_1 \rightarrowtail_{[e_1, p_1, R_1, \sigma_1]} \cdots \rightarrowtail_{[e_{n-1}, p_{n-1}, R_{n-1}, \sigma_{n-1}]} T_n$$

is an innermost $\rightarrowtail_{\mathcal{R}}$-sequence. Let $B_1 = \overline{T}$ and define the position constraints $B_2, \dots, B_{n-1}$ as in Definition 7.2(2). By induction on i we will show that $e_{|p}$ is a normal form whenever $e \in T_i$ and $p \in O(e) - B_i(e)$ for $1 \leqslant i < n$. For $i = 1$ this is a consequence of the normalization of σ. Suppose the statement holds for $i = 1, \dots, m$. Let R_m be the rule $l_m \rightarrow r_m \Leftarrow c_m$ and take $e \in T_{m+1}$. We distinguish three cases: $e \in T_m - \{e_m\}$, $e \equiv e_m[\sigma_m r_m]_{p_m}$ and $e \in \sigma_m \tilde{c}_m$.

(1) If $e \in T_m - \{e_m\}$ then $B_{m+1}(e) = B_m(e)$ and hence the result follows from the induction hypothesis.
(2) The case $e \equiv e_m[\sigma_m r_m]_{p_m}$ follows as in the proof of Proposition 4.4.
(3) If $e \in \sigma_m \tilde{c}_m$ then $B_{m+1}(e) = \overline{O}(e')$ where $e = \sigma_m e'$. Hence it suffices to show that $\sigma_m \restriction_{\mathcal{V}(e')}$ is normalized. Since $\mathcal{R}$ is a 1-CTRS, we have $\mathcal{V}(e') \subseteq \mathcal{V}(l_m)$ and because $\sigma_m l_m$ is an innermost redex in e_m we know that $\sigma_m \restriction_{\mathcal{V}(l_m)}$ is normalized.

□

PROPOSITION 8.18. *Let $\mathcal{R}$ be a level-confluent 2-CTRS and σ a sufficiently normalized solution of a goal clause T. Every innermost point-blank $\rightarrowtail_{\mathcal{R}}$-sequence starting from σT is based on $\overline{T}$.*

PROOF. Suppose

$$\sigma T = T_1 \rightarrowtail_{[e_1, p_1, R_1, \sigma_1]} \cdots \rightarrowtail_{[e_{n-1}, p_{n-1}, R_{n-1}, \sigma_{n-1}]} T_n$$

is an innermost point-blank $\rightarrowtail_{\mathcal{R}}$-sequence. Let $B_1 = \overline{T}$ and define the position constraints $B_2, \dots, B_{n-1}$ as in Definition 7.2(2). By induction on i we will show that $e_{|p}$ is $\mathcal{R}_k$-normalized whenever $e \in T_i$ and $p \in O(e) - B_i(e)$ for $1 \leqslant i < n$. Here k is the level of e. For $i = 1$ this is a consequence of the sufficient normalization of σ. Suppose the statement holds for $i = 1, \dots, m$. Let R_m be the rule $l_m \rightarrow r_m \Leftarrow c_m$ and take $e \in T_{m+1}$. We distinguish three cases: $e \in T_m - \{e_m\}$, $e \equiv e_m[\sigma_m r_m]_{p_m}$ and $e \in \sigma_m \tilde{c}_m$.

(1) If $e \in T_m - \{e_m\}$ then $B_{m+1}(e) = B_m(e)$ and hence the result follows from the induction hypothesis.
(2) Let $e \equiv e_m[\sigma_m r_m]_{p_m}$ and suppose that the level of e_m equals l. Since the above sequence is point-blank, we have $T_m \rightarrowtail^o T_{m+1}$ for some $o \leqslant l$. Hence $\mathcal{R}_{o-1} \vdash \sigma_m \tilde{c}_m$, $\sigma_m \restriction_{\mathcal{E}(R_m)}$ is $\mathcal{R}_{o-1}$-normalized and $e_m \rightarrow_{\mathcal{R}_o} e$. From $\mathcal{R}_o \subseteq \mathcal{R}_l$, $\mathcal{R}_l \vdash e_m$ and the level-confluence of $\mathcal{R}$

we obtain $\mathcal{R}_l \vdash e$. Therefore the level of e does not exceed l and hence it suffices to show that $e_{|p}$ is $\mathcal{R}_l$-normalized. As in the proof of Proposition 4.4 we distinguish the two cases $p \perp p_m$ and $p \geqslant p_m$.

(a) If $p \perp p_m$ then $e_{|p} \equiv (e_m)_{|p}$ and $p \notin B_m(e_m)$. Hence the result follows from the induction hypothesis.

(b) Let $p \geqslant p_m$. Since $\mathcal{R}$ is a 2-CTRS we have $\mathcal{V}(r_m) \subseteq \mathcal{V}(l_m)$ and hence we infer that $e_{|p}$ is a proper subterm of $\sigma_m l_m$. Because $\sigma_m l_m$ is an innermost $\mathcal{R}_l$-redex in e_m, $\sigma_m\restriction_{\mathcal{V}(l_m)}$ is $\mathcal{R}_l$-normalized.

(3) If $e \in \sigma_m \tilde{c}_m$ then the level e does not exceed $l-1$ and $B_{m+1}(e) = \overline{O}(e')$ where $e = \sigma_m e'$. So it is sufficient to show that $\sigma_m\restriction_{\mathcal{V}(e')}$ is $\mathcal{R}_{l-1}$-normalized. Clearly $\mathcal{V}(e') \subseteq \mathcal{V}(l_m) \cup \mathcal{E}(R_m)$. We already observed that $\sigma_m\restriction_{\mathcal{E}(R_m)}$ is $\mathcal{R}_{l-1}$-normalized. In case (2)(b) we observed that $\sigma_m\restriction_{\mathcal{V}(l_m)}$ is $\mathcal{R}_n$-normalized and hence certainly $\mathcal{R}_{l-1}$-normalized.

□