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Error Analysis and Local Refinement for a One-dimensional, Scalar Conservation Law

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Abstract. For the upwind discretisation of a one-dimensional, scalar conservation law, the local truncation error and its a-posteriori estimation on a locally refined grid are studied. The estimate can be used in a refinement criterion of a solution-adaptive refinement procedure. We consider local refinement in a nonlinear multigrid context and emphasize estimating in the neighbourhood of the interface between a coarse and fine grid. In the neighbourhood of such a grid interface the discretisation scheme used is different from the one used elsewhere. Estimating the local truncation error in such a situation using extrapolation techniques, requires different treatment of the various contributions to the local truncation error. We introduce two discretisations at grid interfaces and a procedure for the estimation of the local truncation error. We show for a model problem that under sufficient smoothness conditions, the estimate is sufficiently accurate.

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1. Introduction. We consider a steady state solution for a one-dimensional, scalar conservation law defined on a bounded, open domain Ω, with appropriate boundary conditions

$$\frac{df(u(x))}{dx} = g(x), \quad x \in \Omega \subset \mathbb{R}, \tag{1.1a}$$

$$u(x_0) = u_0, \text{ on a proper part of } \partial\Omega, \tag{1.1b}$$

where $\partial\Omega \subset \overline{\Omega}$ is the boundary of the domain Ω. We assume smooth data $g(x)$. We investigate the fully one-sided upwind discretisation of (1.1a), augmented with an approximation of (1.1b) and we investigate the solution of the resulting set of equations for an example problem. This one-dimensional conservation law may be considered a model for fluid dynamics problems. We aim at applying the results of our analysis in a method for steady state fluid dynamics computations in more space dimensions. In this method we want to introduce solution-adaptive local refinements in the partitioning of the domain (see [6], [8]) based on an a-posteriori estimate of the local discretisation error. Therefore, we analyse the situation where the partitioning of the domain has a sudden change in mesh size.

We estimate the local discretisation error (a-posteriori) by an extrapolation of the relative truncation error of the difference operator. In general it is only possible to use a standard extrapolation technique if a regular discretisation scheme is used everywhere [3]. Such a regular discretisation scheme admits a simple extrapolation, since asymptotic expansions for the local truncation error can be readily used. For an irregular discretisation scheme, used at boundary interfaces, we propose to split the local truncation error of the discrete operator into two parts. For one part we assume the existence of an asymptotic expansion which can be estimated by extrapolation (the regular or uniform part). The other part (the irregular or non-uniform part) is estimated via estimates of the derivatives which appear in the expression for this part, obtained by Taylor series expansion.

In this paper we consider first-order accurate discretisations only. Extension to second-order discretisations seems straightforward. For a locally non-uniform grid we consider two discretisations: an inaccurate and an accurate one. Analyses are performed for both a *locally*

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uniform grid (application of a regular discretisation scheme) and a *locally non-uniform* grid (application of an irregular discretisation scheme). The notions locally uniform and locally non-uniform are defined in the appropriate sections. They are introduced to facilitate the treatment of the sudden change in coarseness of the grid, due to the use of local refinements.

First we present some definitions and notations and we introduce the discretisation. Then we investigate the local discretisation error, the global discretisation error for an example problem and a method to estimate the local discretisation error of the discretisation of (1.1a) on a locally refined grid. All investigations are supported with results for an example problem.

2. Preliminaries. In this section we give the definitions and notations used in this paper and we define the example problem. Given a partitioning of the domain in *cells*, we distinguish between function values defined in a cell and function values defined at cell end points. A value in a cell is denoted by $(\cdot)_i^l$. A function value at the left end point of a cell is denoted by $(\cdot)_{i,L}^l$ and at the right end point by $(\cdot)_{i,R}^l$.

2.1. Definitions and notations. With a multiple grid solution method in mind, we consider different levels of refinement in the partitioning of the open domain Ω and associate with a level of refinement $l \in \mathbb{Z}$, the partitioning $\Omega^l \subset \Omega$. We call the partitioning Ω^l the grid on level l. The grid on level l is determined by the grid points $x_i^l \in \overline{\Omega^l}$, $i \in \mathbb{Z}$ and it consists of the intervals $\Omega_i^l = (x_i^l, x_{i+1}^l)$, called cells. The grid Ω^l on level l is

$$\Omega^l = \bigcup_i \Omega_i^l. \tag{2.1}$$

The set I is the set of index pairs, defined as

$$I = \{(i,l) \in \mathbb{Z}^2 \mid \exists \Omega_i^l \subset \Omega\}. \tag{2.2}$$

We also define sets of indices I^l by

$$I^l = \{i \in \mathbb{Z} \mid (i,l) \in I\}, \quad \forall l \in \{0, \ldots, L\}, \tag{2.3}$$

where L is the finest level present. Without loss of generality, the sets of indices I^l can be chosen so that on the coarsest level, $l = 0$, the smallest index is zero.

The grid points x_i^l are the boundaries of the cells on level l. A grid point which has only one neighbouring cell and which does not coincide with a physical boundary, is called a *green* boundary. We denote the left and right end points of a cell Ω_i^l also by $x_{i,L}^l$ and $x_{i,R}^l$ respectively. Hence we have the notation

$$x_{i,L}^l = x_i^l, \tag{2.4a}$$

$$x_{i,R}^l = x_{i+1}^l. \tag{2.4b}$$

The size s_i^l of a cell Ω_i^l is defined by

$$s_i^l = x_{i,R}^l - x_{i,L}^l = x_{i+1}^l - x_i^l, \tag{2.5}$$

where we assume $x_{i+1}^l > x_i^l$. We consider the (typical multigrid) situation where cells on a grid are obtained by splitting the cell on the next coarser grid, and the size of a cell on level $l+1$ is approximately half the size of a cell on level l. The grid points do not have to be uniformly distributed over Ω. For all l, we denote by h_l the maximum

$$h_l = \max_{i \in I^l} s_i^l. \tag{2.6}$$

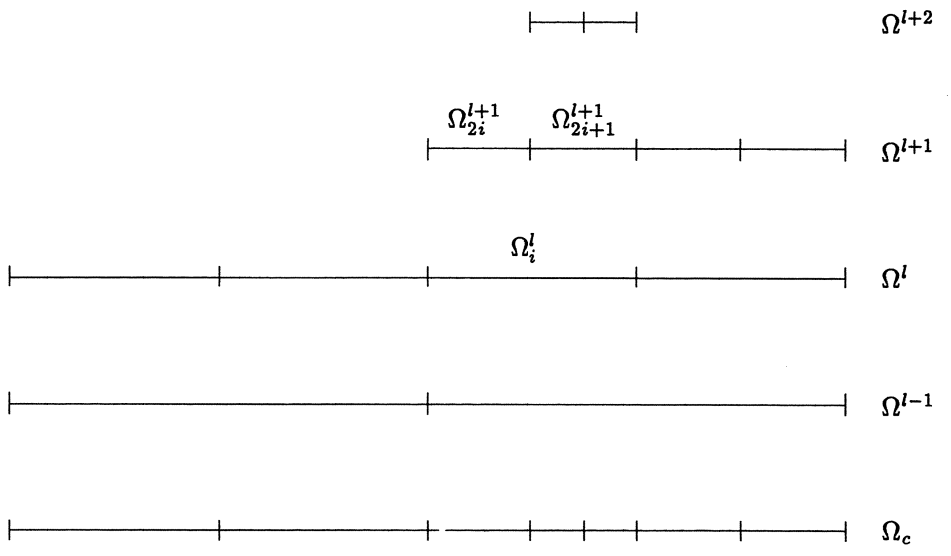

FIG. 2.1. *Example of grids on different levels which result in a locally refined composite grid.*

Furthermore, we consider a grid which is sufficiently smooth, i.e.

$$s_{i+1}^l = s_i^l + \mathcal{O}(h_l^p), \tag{2.7}$$

with $p > 1$ sufficiently large. Consequently we have for all $l \in \{0, \ldots, L-n\}$

$$h_{l+n} = 2^{-n} h_l + \mathcal{O}(h_l^p). \tag{2.8}$$

The refinements of a cell Ω_i^l are denoted by Ω_{2i}^{l+1} and Ω_{2i+1}^{l+1}. The part Ω_f^l of the grid Ω^l consists of all cells on level l which are refined. Thus, Ω^{l+1} consists of all refinements of the cells in Ω_f^l. We denote the non-refined part of Ω^l by Ω_c^l. The collection of all non-refined cells on all levels is called the *composite* grid and is denoted by Ω_c. Thus, we have for all l

$$\Omega^l = \Omega_f^l \cup \Omega_c^l, \tag{2.9}$$

and the composite grid is

$$\Omega_c = \bigcup_{l=0}^{L} \Omega_c^l. \tag{2.10}$$

Similar to (2.3), the sets of indices associated with Ω_f^l and Ω_c^l are denoted by I_f^l and I_c^l respectively. The set I_c of index pairs associated with the composite grid is defined as

$$I_c = \{(i,l) \in I \mid i \in I_c^l, \quad \forall l \in \{0, \ldots, L\}\}. \tag{2.11}$$

In Fig. 2.1 an example is given of different levels of refinement and the associated composite grid.

For all $l \in \{0, \ldots, L\}$ we define two sets of piecewise constant functions on an arbitrary subdomain $\Omega^* \subset \Omega$ by

$$X^l(\Omega^*) = Y^l(\Omega^*) = \{u(x) : \Omega^* \to \mathbb{R} \mid u(x) \text{ is constant in each } \Omega_i^l \cap \Omega^*\}. \tag{2.12}$$

We also define on Ω^* the sets $X(\Omega^*)$ and $Y(\Omega^*)$, which contain $X^l(\Omega^*)$ or $Y^l(\Omega^*)$ as well as all sufficiently smooth functions, for $l \in \{0, \ldots, L\}$. Hence, we have

$$X(\Omega^*) = C^k(\Omega^*) \cup \bigcup_{l=0}^{L} X^l(\Omega^*), \tag{2.13}$$

with k sufficiently large, and similar for $Y(\Omega^*)$. Notice that we have a nested sequence

$$X^l(\Omega^*) \subset X^{l+1}(\Omega^*) \subset X(\Omega^*), \tag{2.14}$$

if $\Omega^* \subset \Omega^{l+1} \subset \Omega^l$. The same applies to Y^l.

By u_i^l we denote the value of the cellwise constant function $u^l \in X^l$, in Ω_i^l. To simplify notation, we introduce restriction operators $\overline{R}^{l,l+n}$ and $R^{l,l+n}$, $n = 0, 1$. With $\Omega^{l+n} \subset \Omega^*$, the restriction operator $\overline{R}^{l,l+n} : X(\Omega^*) \to X^l(\Omega^{l+n})$ maps quite general functions to functions that are piecewise constant on cells in Ω^l, but only defined on Ω^{l+n}. This restriction is defined for $l \in \{0, \ldots, L-n\}$ by

$$\{\overline{R}^{l,l+n}u\}_i^l = \frac{1}{s_i^l} \int_{x_{i,L}^l}^{x_{i,R}^l} u(x)\,dx, \quad \begin{cases} \forall i \in I^l, \text{ if } n = 0, \\ \forall i \in I_f^l, \text{ if } n = 1. \end{cases} \tag{2.15}$$

The other restriction operator is related to (2.15) through the operator $S^{l,l+n} : X(\Omega^{l+n}) \to Y(\Omega^{l+n})$, for $n = 0, 1$, which is defined as

$$S^{l,l+n}u(x) = s_i^l u(x), \tag{2.16}$$

for all $u(x) \in X(\Omega_i^l \cap \Omega^{l+n})$, i.e. $i \in I^l$ if $n = 0$, or $i \in I_f^l$ if $n = 1$. We denote $S^{l,l}$ also by S^l. The restriction operator $R^{l,l+n} : Y(\Omega^{l+n}) \to Y^l(\Omega^{l+n})$, for $n = 0, 1$, is defined by

$$R^{l,l+n} = S^{l,l+n}\overline{R}^{l,l+n}(S^{l+n})^{-1}. \tag{2.17}$$

For example, we define the efflux function of cells on Ω^{l+1} by $F^{l+1} : X(\overline{\Omega}) \to Y^{l+1}(\Omega^{l+1})$, with for Ω_j^{l+1}

$$F_j^{l+1} = f(u(x_{j,R}^{l+1})) - f(u(x_{j,L}^{l+1})), \tag{2.18}$$

where $u(x)$ is continuous in the grid points x_j^{l+1}. Then, we have for all $i \in I_f^l$,

$$\{R^{l,l+1}F^{l+1}\}_i^l = F_{2i}^{l+1} + F_{2i+1}^{l+1} = f(u(x_{2i+1,R}^{l+1})) - f(u(x_{2i,L}^{l+1})), \tag{2.19}$$

which is the cell efflux function for the coarser cell of level l. We denote $\overline{R}^{l,l}$ also by $\overline{R}^l$ and $R^{l,l}$ by R^l.

We denote the restriction (mean value) of the exact solution $\overline{u} \in X(\Omega)$ of (1.1) by the cellwise constant function $\overline{u}^l \in X^l(\Omega^l)$,

$$\overline{u}^l = \overline{R}^l \overline{u}. \tag{2.20}$$

Approximations of $\overline{u}^l$ are denoted by u^l. Such an approximation can be considered as the restriction of some (possibly continuous) approximation u of $\overline{u}$. The $\overline{u}_i^l$ and u_i^l are so-called *cell states* of cell Ω_i^l. The error ϵ^l of an approximation u^l is defined by

$$\epsilon^l = \overline{u}^l - u^l. \tag{2.21}$$

Finally we define the piecewise constant functions on the composite grid, denoted by the subscript c. For example, the piecewise constant u_c is defined by

$$u_c = \{u_i^l \in u^l \mid (i,l) \in I_c\}. \tag{2.22}$$

We consider a finite volume discretisation of the conservation law (1.1a). Integration of (1.1a) over a cell $\Omega_i^l \subset \Omega$, gives

$$\int_{\Omega_i^l} \frac{df(u(x))}{dx}\, dx = f(u(x_{i,R}^l)) - f(u(x_{i,L}^l)) = g_i^l s_i^l. \tag{2.23}$$

In the discretisation, the flux $f(u(x_{i,R}^l)) \equiv f_{i,R}^l$ is approximated through a numerical flux $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$F_{i,R}^l = F\left((u^L)_{i,R}^l, (u^R)_{i,R}^l\right). \tag{2.24}$$

The arguments $(u^L)_{i,R}^l$ and $(u^R)_{i,R}^l$ of the numerical flux function F represent the result of a left and right biased interpolation in u^l, as denoted by the superscripts L and R respectively. Similarly, the approximation of $f_{i,L}^l$ is denoted by $F_{i,L}^l$ and is given by

$$F_{i,L}^l = F\left((u^L)_{i,L}^l, (u^R)_{i,L}^l\right). \tag{2.25}$$

The function F may be considered to be e.g. an approximate Riemann solver. This may describe sophisticated upwind discretisations of the conservation law. Such upwind discretisations for conservation laws are considered in e.g. [5], [10] and [4].

In a local refinement context, some of the values u_i^l required in the computation of the left and right states, $(u^L)_{i,k}^l$ and $(u^R)_{i,k}^l$, $k = L, R$, may not be available. Therefore, we introduce the concept of *virtual states*. A virtual state v_i^l is an interpolant, generally computed from $u^{l-1}, \ldots, u^0$, which is used when Ω_i^l does not exist due to (the lack of) refinement of the grid. This occurs in the neighbourhood of a green boundary. For simplicity, we restrict ourselves to the situation where a virtual state v_i^l depends on u^{l-1} only. We introduce a *virtual cell*, $\omega_i^l \subset \Omega$, $i \notin I^l$, which is defined as the part of Ω which would be exactly Ω_i^l, if the grid would have been sufficiently refined. With the virtual cell ω_i^l we associate the virtual state v_i^l.

The discretisation on level l only defined on the grid Ω^{l+n}, is described by the finite volume operator $N^{l,l+n} : X^l(\Omega^{l+n}) \to Y^l(\Omega^{l+n})$, $l \in \{0, \ldots, L-n\}$ and $n = 0, 1$, which is defined by

$$\{N^{l,l+n}(\overline{R}^{l,l+n} u^l; u^{l-1})\}_i^l = F_{i,R}^l - F_{i,L}^l, \quad \begin{cases} \forall i \in I^l, \text{ if } n = 0, \\ \forall i \in I_f^l, \text{ if } n = 1. \end{cases} \tag{2.26}$$

Here, u^{l-1} acts as a parameter in $N^{l,l+n}$, which determines the possible virtual states. We denote $N^{l,l}$ also by N^l. Following these notations, the discretisation as given by (2.18) for each cell of level $l+1$ is denoted by $R^{l+1}N$. The set of equations on level l is written as

$$N^l(u^l; u^{l-1}) = r^l, \tag{2.27}$$

where the right-hand side represents the source term.

For simplicity we restrict ourselves to the fully one-sided upwind computation of the flux and first-order accuracy. However, the framework described above may be used for more general numerical flux functions and higher-order accuracy. In fact it can be easily extended to a general second-order upwind discretisation of a conservation law in more space dimensions.

For the discretisation of our scalar problem (1.1a) we use the fully one-sided upwind computation of the flux, defined by the numerical flux function

$$F(u^L, u^R) = \begin{cases} f(u^L), & u^L, u^R \geq 0, \\ f(u^R), & u^L, u^R \leq 0, \\ 0, & \text{else.} \end{cases} \tag{2.28}$$

For simplicity and without loss of generality, we consider a problem with $u^L, u^R \geq 0$ only. Since in this case $F(u^L, u^R)$ is a function of u^L only, we may redefine F by $F(u) = f(u)$ to shorten notation. The left state $(u^L)^l_{i,L}$ at the left boundary $x^l_{i,L}$ of Ω^l_i is given by

$$(u^L)^l_{i,L} = \begin{cases} u^l_{i-1}, & \text{if } (i-1) \in I^l, \\ v^l_{i-1}, & \text{if } x^l_{i,L} \text{ is a green boundary.} \end{cases} \tag{2.29}$$

This gives a first-order accurate discretisation if $(i-1) \in I^l$. We consider two interpolations for the virtual state v^{l+1}_{2i-1}: the first-order accurate interpolation $I1$

$$v^{l+1}_{2i-1} = u^l_{i-1}, \tag{2.30}$$

and the second-order accurate interpolation $I2$

$$v^{l+1}_{2i-1} = \frac{3}{4}u^l_{i-1} + \frac{1}{4}u^l_i. \tag{2.31}$$

In section 3 we show that the interpolation $I1$ leads to an $\mathcal{O}(h_l^0)$ and $I2$ to an $\mathcal{O}(h_l)$ local discretisation error.

2.2. The example problem. For our example problem we choose the domain $\Omega = (0,1)$, partitioned by cells of constant size $s^l_i = h_l$. For the flux function we take $f(u) = u^2$, and we choose a source $g(x) = 4x^3$. With a boundary condition $u(x_0) = 0$ at $x_0 = 0$, problem (1.1) does not have a unique solution. However, by imposing the extra condition

$$u(x) \geq 0, \quad \forall x \in \overline{\Omega}, \tag{2.32}$$

we have the unique solution

$$u(x) = x^2, \quad \forall x \in \overline{\Omega}. \tag{2.33}$$

This example problem and its solution are sufficiently simple to allow a detailed analysis. Yet, the problem is sufficiently non-trivial to emulate the properties of a fluid dynamics model such as the Euler equations, with respect to the aspects we focus on.

3. Analysis of accuracy. For the discretisation described above, we investigate the local truncation error and for the example problem, we study the error in the solution, both on a uniform grid (no local refinements) and a locally refined grid. The local discretisation error is $\tau^l(\overline{u})$, where τ^l is the local truncation operator for N^l, defined by

$$\tau^l(u) = (S^l)^{-1}\left(N^l(\overline{R}^l u; \overline{R}^{l-1} u) - R^l N(u)\right). \tag{3.1}$$

Application of the definitions of S^l and N^l in (2.16) and (2.26), gives for all $(i,l) \in I$,

$$\tau^l_i(u) = \frac{1}{s^l_i}(F^l_{i,R} - F^l_{i,L} - f^l_{i,R} + f^l_{i,L}), \tag{3.2}$$

with $F^l_{i,R}$, $F^l_{i,L}$ evaluated using $u^l = \overline{R}^l u$ and $f^l_{i,R}$ and $f^l_{i,L}$ evaluated using u. For our simple upwind discretisation, with

$$f^l_{i,L} = F(u(x^l_{i,L})), \tag{3.3a}$$

$$F^l_{i,L} = F(u^l_{i-1}), \tag{3.3b}$$

we have for a sufficiently smooth $F(u)$

$$\begin{aligned} f^l_{i,L} = F^l_{i,L} &+ \left.\frac{\partial F}{\partial u}\right|_{u(x^l_{i,L})} (u(x^l_{i,L}) - u^l_{i-1}) + \frac{1}{2}\left.\frac{\partial^2 F}{\partial u^2}\right|_{u(x^l_{i,L})} (u(x^l_{i,L}) - u^l_{i-1})^2 \\ &+ \frac{1}{6}\left.\frac{\partial^3 F}{\partial u^3}\right|_{u(x^l_{i,L})} (u(x^l_{i,L}) - u^l_{i-1})^3 + \mathcal{O}\left((u(x^l_{i,L}) - u^l_{i-1})^4\right), \end{aligned} \tag{3.4}$$

and similar expressions for other fluxes. If u^l_{i-1} is not available due to missing refinements, v^l_{i-1} is used instead. Expression (3.4), possibly with virtual states, is used in the following subsections. We also use a Taylor series expansion of a sufficiently smooth $u(x)$ around $x^l_{i,L}$. This expansion is written as

$$u(x) = u_0 + u_1\xi + u_2\xi^2 + u_3\xi^3 + \mathcal{O}(\xi^4), \tag{3.5}$$

where $\xi = x - x^l_{i,L}$.

3.1. Locally uniform grid. In this section we investigate the local truncation error and the global error for the example problem, defined in the previous section, on a locally uniform grid. A locally uniform grid is the collection of grid cells, for which each equation of the discretisation does not involve any virtual state. Such an equation is obtained by application of the regular discretisation scheme (when boundaries are neglected).

3.1.1. Local truncation error. Here we assume that *all* equations are obtained by applying the regular discretisation scheme. On a sufficiently smooth grid, with a maximum size h_l of the cells, using (3.2), (3.5) and (3.4) we derive for $\tau^l(u)$ in Ω^l_i

$$\begin{aligned} \tau^l_i(u) = \frac{1}{s^l_i}\Bigg\{ &\left.\frac{\partial F}{\partial u}\right|_{u_0} (-u_2(s^l_i)^2 - \frac{1}{2}u_3(s^l_i)^3) + \frac{1}{2}\left.\frac{\partial^2 F}{\partial u^2}\right|_{u_0} (-u_1^2(s^l_i)^2 - \frac{4}{3}u_1u_2(s^l_i)^3) \\ &+ \frac{1}{6}\left.\frac{\partial^3 F}{\partial u^3}\right|_{u_0} (-\frac{3}{4}u_1^3(s^l_i)^3)\Bigg\} + \mathcal{O}(h^3_l). \end{aligned} \tag{3.6}$$

Hence, for a differentiable solution of the continuous problem, the discretisation on a locally uniform grid is first-order accurate.

For the example problem, if $i > 0$, (3.2) gives

$$\tau^l_i(\overline{u})s^l_i = (\overline{u}^l_i)^2 - (\overline{u}^l_{i-1})^2 - \overline{u}^2(x^l_{i,R}) + \overline{u}^2(x^l_{i,L}). \tag{3.7}$$

Because of the uniform grid, $s^l_i = h_l$ and hence $x^l_{i,L} = ih_l$. The restriction of the solution of the continuous problem is

$$\overline{u}^l_i = \frac{1}{h_l}\int_{ih_l}^{(i+1)h_l} x^2\,dx = h_l^2(i^2 + i + \frac{1}{3}), \tag{3.8}$$

and hence,

$$(\overline{u}^l_i)^2 = h_l^4(i^4 + 2i^3 + \frac{5}{3}i^2 + \frac{2}{3}i + \frac{1}{9}), \tag{3.9a}$$

$$(\overline{u}^l_{i-1})^2 = h_l^4(i^4 - 2i^3 + \frac{5}{3}i^2 - \frac{2}{3}i + \frac{1}{9}). \tag{3.9b}$$

The square of the solution $\overline{u}(x)$ at the left and right end points of Ω_i^l respectively, is

$$\overline{u}^2(x_{i,L}^l) = h_l^4 i^4, \tag{3.10a}$$

$$\overline{u}^2(x_{i,R}^l) = h_l^4 (i+1)^4. \tag{3.10b}$$

Hence, the local discretisation error is

$$\tau_i^l(\overline{u}) = -h_l^3(6i^2 + \frac{8}{3}i + 1), \quad i > 0. \tag{3.11}$$

A similar result may be obtained from (3.6), which for the example problem reduces to

$$\tau_i^l(u) = -\frac{1}{2}h_l \left\{ \left.\frac{\partial F}{\partial u}\right|_{u_0} \left.\frac{\partial^2 u}{\partial x^2}\right|_{x_{i,L}^l} + \left.\frac{\partial^2 F}{\partial u^2}\right|_{u_0} \left(\left.\frac{\partial u}{\partial x}\right|_{x_{i,L}^l} \right)^2 \right\} + \mathcal{O}(h_l^2). \tag{3.12}$$

For $i = 0$, the boundary condition at $x_0 = 0$ is implemented by defining the flux $F_{0,L}^l$ to be equal to the exact flux, $F_{0,L}^l = f_{0,L}^l = 0$. This gives the local truncation error

$$\tau_0^l(\overline{u}) = -\frac{8}{9}h_l^3. \tag{3.13}$$

Note that $\tau_0^l(u)$ is third-order if $u(0) = 0$, but generally is zeroth-order. We define $0 < \alpha < 1$ through the relation $ih_l = x - \alpha h_l$, for constant x and $i \to \infty$, $h_l \to 0$. Then, the cellwise constant function $\tau^l(\overline{u}(x))$, defined by $\tau^l(\overline{u}(x)) = \tau_i^l(\overline{u})$, for $x \in \Omega_i^l$, is given by

$$\tau^l(\overline{u}(x)) = \begin{cases} -\frac{8}{9}h_l^3, & x \in \Omega_0^l, \\ -6x^2 h_l + (12\alpha x - \frac{8}{3}x)h_l^2 + (-6\alpha^2 + \frac{8}{3}\alpha - 1)h_l^3, & x \in \Omega^l \setminus \Omega_0^l. \end{cases} \tag{3.14}$$

We find that the local discretisation error is first-order in the mesh width.

3.1.2. Error in the solution of example problem. The global discretisation error is defined by $\epsilon^l = \overline{u}^l - u^l$. Here u^l is the solution of (2.27), with $r^l = S^l g^l$, where $g^l = \overline{R}^l g$, with $F_{0,L}^l = 0$ and the extra condition $u_i^l > 0$. The discrete equation for a cell Ω_i^l is given by

$$(u_0^l)^2 = g_0^l s_0^l, \tag{3.15a}$$

$$(u_i^l)^2 - (u_{i-1}^l)^2 = g_i^l s_i^l, \quad i > 0. \tag{3.15b}$$

By induction, it easily follows that

$$(u_i^l)^2 = \sum_{k=0}^{i} g_k^l s_k^l. \tag{3.16}$$

Because we consider a grid which is locally uniform everywhere, N^l is independent of its parameter u^{l-1}. By definition, $\overline{u}^l$ satisfies

$$N^l(\overline{u}^l) = S^l \left(g^l + \tau^l(\overline{u}) \right), \tag{3.17}$$

with the conditions $F_{0,L}^l = 0$ and $\overline{u}_i^l > 0$. Similar to (3.16), the restriction $\overline{u}^l$ of the exact solution satisfies

$$(\overline{u}_i^l)^2 = \sum_{k=0}^{i} \left(g_k^l + \tau_k^l(\overline{u}) \right) s_k^l. \tag{3.18}$$

Subtraction of (3.16) and (3.18) gives

$$(\overline{u}_i^l)^2 - (u_i^l)^2 = \sum_{k=0}^{i} \tau_k^l(\overline{u}) s_k^l. \tag{3.19}$$

With $s_i^l = h_l$, $\overline{u}(x) = x^2$, (3.11) and (3.13) this gives the cumulative truncation error up to and including cell Ω_i^l, given by

$$T_i^l \equiv \sum_{k=0}^{i} \tau_k^l(\overline{u}) s_k^l = h_l^4 \left(\frac{1}{9} - \sum_{k=0}^{i} (6k^2 + \frac{8}{3}k + 1) \right). \tag{3.20}$$

Since

$$\sum_{k=0}^{i} k = \frac{1}{2}i^2 + \frac{1}{2}i, \qquad \sum_{k=0}^{i} k^2 = \frac{1}{3}i^3 + \frac{1}{2}i^2 + \frac{1}{6}i, \tag{3.21}$$

T_i^l, as defined by (3.20), is

$$T_i^l = -h_l^4 (2i^3 + \frac{13}{3}i^2 + \frac{10}{3}i + \frac{8}{9}). \tag{3.22}$$

Expression (3.19) and the definition of ϵ^l (2.21) give

$$(\overline{u}_i^l)^2 - (u_i^l)^2 = (\overline{u}_i^l)^2 - (\overline{u}_i^l - \epsilon_i^l)^2 = T_i^l, \tag{3.23}$$

from which we obtain

$$\epsilon_i^l = \overline{u}_i^l \pm \overline{u}_i^l \sqrt{1 - T_i^l/(\overline{u}_i^l)^2}. \tag{3.24}$$

Since $T_i^l = \mathcal{O}(h_l)$ and both $u_i^l, \overline{u}_i^l > 0$, the minus sign applies. For small $T_i^l/(\overline{u}_i^l)^2$, the global discretisation error may be written as

$$\epsilon_i^l = \frac{1}{2}\frac{T_i^l}{\overline{u}_i^l} + \frac{1}{8}\frac{(T_i^l)^2}{(\overline{u}_i^l)^3} + \frac{1}{16}\frac{(T_i^l)^3}{(\overline{u}_i^l)^5} + \mathcal{O}\left(\frac{(T_i^l)^4}{(\overline{u}_i^l)^7}\right). \tag{3.25}$$

Substitution of the expressions (3.8) and (3.22) for $\overline{u}_i^l$ and T_i^l respectively, gives for the example problem, and small $T_i^l/(\overline{u}_i^l)^2$,

$$\epsilon_i^l = -h_l^2 (i + \frac{2}{3} + \mathcal{O}(i^{-2})) + \mathcal{O}\left(\frac{(T_i^l)^4}{(\overline{u}_i^l)^7}\right). \tag{3.26}$$

The cellwise constant error $\epsilon^l(x)$ is given by

$$\epsilon^l(x) = -x h_l + (-\frac{2}{3} + \alpha) h_l^2 + \mathcal{O}(h_l^4), \tag{3.27}$$

where again α is defined by $ih_l = x - \alpha h_l$, $0 < \alpha < 1$. In Fig. 3.1 a linear interpolation of the error ϵ^l as obtained by numerically computing the solution of the discretisation, is shown for two grids, one with $h_l = 1/32$ and one with $h_l = 1/64$. Equation (3.27) is in agreement with the results in Fig. 3.1. As we expected, the global discretisation error is first-order.

3.2. Locally non-uniform grid. A grid is called locally non-uniform in a cell Ω_i^l, if the discrete equation for Ω_i^l involves a virtual state. This equation is the result of an irregular discretisation scheme. First we look at the local and global error of the irregular discretisation, using the first-order accurate interpolation $I1$ to compute a virtual state. After this we study interpolation $I2$.

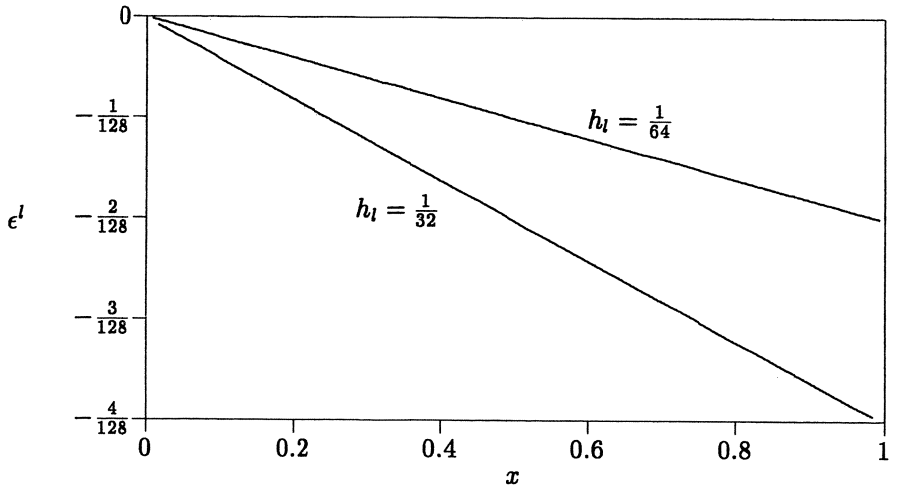

FIG. 3.1. *Error of the solution, for the example problem on uniform grid.*

3.2.1. Local truncation error for interpolation $I1$. We consider a coarse grid, denoted by Ω^l and a fine one, denoted by Ω^{l+1}. The grid Ω^l covers the domain Ω completely and the grid Ω^{l+1} covers only a subdomain. For definiteness, without loss of generality, we take the green boundary of Ω^{l+1} at x_g and $\overline{\Omega^{l+1}} = [x_g, 1]$. Furthermore, we define for the green boundary

$$x_g = x^l_m = x^{l+1}_{2m}. \tag{3.28}$$

The discrete equation for the cell on the locally non-uniform grid involves the virtual state v^{l+1}_{2m-1}. By interpolation $I1$ this virtual state is simply given by

$$v^{l+1}_{2m-1} = u^l_{m-1}. \tag{3.29}$$

With (3.4) and (3.5) we find the expansion of the local truncation error

$$\begin{aligned} \tau^{l+1}_{2m}(u) \;=\; & \frac{1}{s^{l+1}_{2m}} \left\{ \left.\frac{\partial F}{\partial u}\right|_{u_0} \left(\frac{1}{2}u_1 s^{l+1}_{2m} - 2u_2 (s^{l+1}_{2m})^2 + \frac{5}{4}u_3 (s^{l+1}_{2m})^3\right) \right. \\ & + \frac{1}{2}\left.\frac{\partial^2 F}{\partial u^2}\right|_{u_0} \left(-\frac{7}{4}u_1^2 (s^{l+1}_{2m})^2 + u_1 u_2 (s^{l+1}_{2m})^3\right) \\ & \left. + \frac{1}{6}\left.\frac{\partial^3 F}{\partial u^3}\right|_{u_0} \left(\frac{1}{8}u_1^3 (s^{l+1}_{2m})^3\right) \right\} + \mathcal{O}(h_l^3). \end{aligned} \tag{3.30}$$

This shows that interpolation $I1$ to compute a virtual state gives an $\mathcal{O}(h_l^0)$ local truncation error in the locally non-uniform grid cell. For our example problem the local discretisation error $\tau^{l+1}_{2m}(\overline{u})$ in Ω^{l+1}_{2m} is

$$\tau^{l+1}_{2m}(\overline{u}) s^{l+1}_{2m} = (\overline{u}^{l+1}_{2m})^2 - (\overline{u}^l_{m-1})^2 - \overline{u}^2(x^{l+1}_{2m,R}) + \overline{u}^2(x^{l+1}_{2m,L}). \tag{3.31}$$

If we substitute expressions similar to (3.9) and (3.10), into (3.31), we find for the example problem that the local discretisation error of the equation involving the virtual state, is also $\mathcal{O}(h_l^0)$, viz.

$$\tau^{l+1}_{2m}(\overline{u}) = 2x_g^3 - 11x_g^2 h_{l+1} + 2x_g h_{l+1}^2 - \frac{8}{3}h_{l+1}^3. \tag{3.32}$$

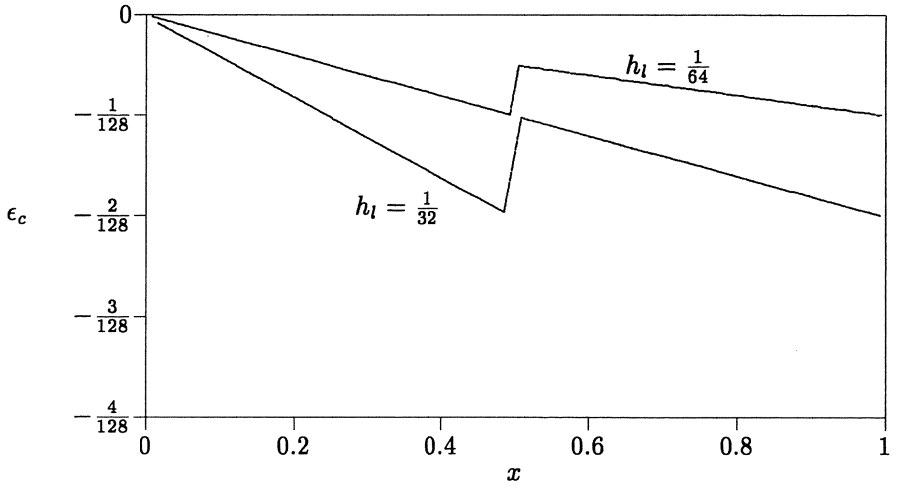

FIG. 3.2. *Error of the solution for the example problem on the composite grid, using interpolation $I1$ for the virtual state and with green boundary $x_g = 1/2$.*

3.2.2. Error in solution of example problem for interpolation $I1$. The global error on the non-refined part Ω_c^l is the same as for the uniform grid situation. For the example problem the global error in the green-boundary cell Ω_{2m}^{l+1} can be found from the equation

$$(\overline{u}_{2m}^{l+1})^2 - (u_{2m}^{l+1})^2 = T_{m-1}^l + \tau_{2m}^{l+1}(\overline{u}) s_{2m}^{l+1} = T_{2m}^{l+1}, \tag{3.33}$$

where T_{2m}^{l+1} is the cumulative truncation error as in (3.20). For the coarse-grid contribution T_{m-1}^l to the cumulative local error, we have from (3.20)

$$T_{m-1}^l = -4x_g^3 h_{l+1} + \frac{20}{3} x_g^2 h_{l+1}^2 - \frac{16}{3} x_g h_{l+1}^3 + \frac{16}{9} h_{l+1}^4. \tag{3.34}$$

Furthermore, (3.32), (3.33) and (3.34) give

$$T_{2m}^{l+1} = -2x_g^3 h_{l+1} - \frac{13}{3} x_g^2 h_{l+1}^2 - \frac{10}{3} x_g h_{l+1}^3 - \frac{8}{9} h_{l+1}^4. \tag{3.35}$$

With expression (3.25) for the error, and proceeding along the same lines as for the uniform grid, we find the global error in Ω_{2m}^{l+1}

$$\epsilon_{2m}^{l+1} = -x_g h_{l+1} - \frac{2}{3} h_{l+1}^2 + \mathcal{O}(h_{l+1}^4). \tag{3.36}$$

It appears that the solution in Ω_{2m}^{l+1} is first-order accurate, although the discrete equation for Ω_{2m}^{l+1} is zeroth-order accurate. For $x_g = 1/2$, a linear interpolation of the global error ϵ_c on the composite grid is shown in Fig. 3.2, for two values of the mesh size: $h_l = 1/32$ and $h_l = 1/64$. Equation (3.36) is in agreement with Fig. 3.2. Notice that for interpolation $I1$ we have

$$\epsilon_{2m}^{l+1} - \epsilon_{m-1}^l = \mathcal{O}(h_l). \tag{3.37}$$

We assume that for $k = l, l+1$ the error can be written as the asymptotic expansion

$$\epsilon^k = h_L \overline{R}^k \epsilon(x) + \mathcal{O}(h_l^2), \tag{3.38}$$

with h_L the maximum mesh width at the highest level and $\epsilon(x) \in X(\Omega)$, independent of h_L. From (3.37) it appears that $\epsilon(x)$ is discontinuous at $x = x_g$, which is also observed in Fig. 3.2.

3.2.3. Local truncation error for interpolation $I2$. In this subsection we consider the more accurate interpolation $I2$. The introduction of this higher-order accurate interpolation is not primarily required to obtain a higher-order error, as was shown in the previous subsection (see also [9]). $I2$ is necessary to obtain a discretisation which has the same order of consistency everywhere. This could make local refinement an effective tool to reduce the local truncation errors. A technique for local refinement based on the local discretisation error, requires that the local truncation error of the equations for the finer cells is smaller than that error for the corresponding coarse-grid cells. This is generally not the case when these equations are inconsistent. In practice, when using adaptive refinement, such inconsistency may result in small, highly refined regions (islands) in Ω (see[7]).

The more accurate interpolation used to obtain the virtual state v_{2m-1}^{l+1}, is given by the second-order accurate interpolation

$$v_{2m-1}^{l+1} = \frac{3}{4}u_{m-1}^{l} + \frac{1}{4}u_{m}^{l}, \tag{3.39}$$

where u_m^l is the restriction of the fine-grid solution

$$u_m^l = \{\overline{R}^{l,l+1}u^{l+1}\}_m^l. \tag{3.40}$$

Following the same lines as in the previous section, we find the expansion of the local truncation error of N^{l+1}

$$\begin{aligned} \tau_{2m}^{l+1}(u) \;=\; & \frac{1}{s_{2m}^{l+1}}\left\{ \left.\frac{\partial F}{\partial u}\right|_{u_0}(-2u_2(s_{2m}^{l+1})^2 + \frac{1}{4}u_3(s_{2m}^{l+1})^3) \right. \\ & + \frac{1}{2}\left.\frac{\partial^2 F}{\partial u^2}\right|_{u_0}(-u_1^2(s_{2m}^{l+1})^2 - \frac{1}{3}u_1u_2(s_{2m}^{l+1})^3) \\ & \left. + \frac{1}{6}\left.\frac{\partial^3 F}{\partial u^3}\right|_{u_0}(-\frac{3}{4}u_1^3(s_{2m}^{l+1})^3)\right\} + \mathcal{O}(h_l^3). \end{aligned} \tag{3.41}$$

Using interpolation $I2$ to obtain a virtual state gives an $\mathcal{O}(h_l)$ local truncation error on the locally non-uniform grid cell. For the example problem with solution $\overline{u}(x) = x^2$, we find the local discretisation error

$$\tau_{2m}^{l+1}(\overline{u}) = -8x_g^2h_{l+1} - \frac{2}{3}x_gh_{l+1}^2 - \frac{8}{3}h_{l+1}^3. \tag{3.42}$$

3.2.4. Error in solution of example problem for interpolation $I2$. Again, on Ω_c^l the global discretisation error is the same as for the uniform grid. The computation of the global error in Ω_{2m}^{l+1} is a little more laborious, due to the less simple interpolation $I2$. The state u_{2m}^{l+1} in the cell adjacent to the green boundary, is part of the solution of the nonlinear system

$$\begin{cases} (u_{2m}^{l+1})^2 - (\frac{3}{4}u_{m-1}^{l} + \frac{1}{4}u_m^l)^2 = g_{2m}^{l+1}s_{2m}^{l+1}, \\ u_m^l = \frac{1}{2}(u_{2m}^{l+1} + u_{2m+1}^{l+1}), \\ (u_{2m+1}^{l+1})^2 - (u_{2m}^{l+1})^2 = g_{2m+1}^{l+1}s_{2m+1}^{l+1}. \end{cases} \tag{3.43}$$

Since u_{m-1}^l and the right-hand sides $g_{2m}^{l+1}s_{2m}^{l+1}$ and $g_{2m+1}^{l+1}s_{2m+1}^{l+1}$ are known, it follows that u_{2m}^{l+1} can be considered as a function of the independent variables u_{m-1}^l, $g_{2m}^{l+1}s_{2m}^{l+1}$ and $g_{2m+1}^{l+1}s_{2m+1}^{l+1}$:

$$u_{2m}^{l+1} = U(u_{m-1}^l, g_{2m}^{l+1}s_{2m}^{l+1}, g_{2m+1}^{l+1}s_{2m+1}^{l+1}), \tag{3.44}$$

where the function $U = U(a, r_1, r_2)$ satisfies

$$\begin{cases} U^2 - (\frac{3}{4}a + \frac{1}{4}b)^2 = r_1, \\ b = \frac{1}{2}(U + c), \\ c^2 - U^2 = r_2. \end{cases} \tag{3.45}$$

The solution of this system is not given in closed form, but we can use (3.45) to estimate the error ϵ_{2m}^{l+1}. Using the definition of the local truncation error, we can write the restriction of the exact solution as

$$\overline{u}_{2m}^{l+1} = U\left(\overline{u}_{m-1}^{l},\, (g_{2m}^{l+1} + \tau_{2m}^{l+1}(\overline{u}))s_{2m}^{l+1},\, (g_{2m+1}^{l+1} + \tau_{2m+1}^{l+1}(\overline{u}))s_{2m+1}^{l+1}\right). \tag{3.46}$$

Assuming that U is sufficiently differentiable and bounded, with bounded derivatives in the limit $h_l \to 0$, it follows from (3.44) and (3.46) that the error in Ω_{2m}^{l+1} can be written as

$$\begin{aligned} \epsilon_{2m}^{l+1} = \overline{u}_{2m}^{l+1} - u_{2m}^{l+1} \;=\; & \frac{\partial U}{\partial a}\epsilon_{m-1}^{l} + \frac{\partial U}{\partial r_1}\tau_{2m}^{l+1}(\overline{u})s_{2m}^{l+1} \\ & + \frac{\partial U}{\partial r_2}\tau_{2m+1}^{l+1}(\overline{u})s_{2m+1}^{l+1} + \frac{1}{2}\frac{\partial^2 U}{\partial a^2}(\epsilon_{m-1}^{l})^2 + \mathcal{O}(h_l^3), \end{aligned} \tag{3.47}$$

where $\frac{\partial U}{\partial a}$, $\frac{\partial U}{\partial r_1}$ and $\frac{\partial U}{\partial r_2}$ are evaluated at

$$\begin{aligned} a &= \overline{u}_{m-1}^{l}, \\ r_1 &= (g_{2m}^{l+1} + \tau_{2m}^{l+1}(\overline{u}))s_{2m}^{l+1}, \\ r_2 &= (g_{2m+1}^{l+1} + \tau_{2m+1}^{l+1}(\overline{u}))s_{2m+1}^{l+1}. \end{aligned} \tag{3.48a}$$

The local discretisation error is $\mathcal{O}(h_l)$ and for twice differentiable U, we find for $h_l \to 0$

$$\epsilon_{2m}^{l+1} = \frac{\partial U}{\partial a}\epsilon_{m-1}^{l} + \mathcal{O}(h_l^2). \tag{3.49}$$

Since $\epsilon_{m-1}^{l} = \mathcal{O}(h_l)$, the approximation u_{2m}^{l+1} of $\overline{u}_{2m}^{l+1}$ is first-order accurate. In Fig. 3.3 the derivatives of $\overline{u}_{2m}^{l+1} = U(a, r_1, r_2)$, with arguments given by (3.48), are given as a function of h_l for the example problem, i.e. with solution $\overline{u}(x) = x^2$ and with a green boundary at $x_g = 1/2$.

Assume an asymptotic expansion of U

$$U = U_0 + U_1 h_l + \mathcal{O}(h_l^2), \tag{3.50}$$

and U_0 and U_1 independent of h_l. Then for $h_l \to 0$ the error ϵ_{2m}^{l+1} in (3.47) can be written as

$$\begin{aligned} \epsilon_{2m}^{l+1} \;=\; & \left(\frac{\partial U_0}{\partial a} + \frac{\partial U_1}{\partial a}h_l\right)\epsilon_{m-1}^{l} + \frac{\partial U_0}{\partial r_1}\tau_{2m}^{l+1}(\overline{u})s_{2m}^{l+1} \\ & + \frac{\partial U_0}{\partial r_2}\tau_{2m+1}^{l+1}(\overline{u})s_{2m+1}^{l+1} + \frac{1}{2}\frac{\partial^2 U_0}{\partial a^2}(\epsilon_{m-1}^{l})^2 + \mathcal{O}(h_l^3), \end{aligned} \tag{3.51}$$

with all derivatives evaluated at (3.48). These derivatives are found by solving the desired derivatives from (3.45), differentiated with respect to a, r_1 or r_2. In the limit $h_l \to 0$, these derivatives are

$$\begin{aligned} &\frac{\partial U_0}{\partial a} = 1, \quad && \frac{\partial^2 U_0}{\partial a^2} = 0, \quad && \frac{\partial U_0}{\partial r_1} = \frac{2}{3}\frac{1}{x_g^2}, \\ &\frac{\partial U_1}{\partial a} = -\frac{3}{2}\frac{1}{x_g}, \quad && \frac{\partial U_0}{\partial r_2} = \frac{1}{12}\frac{1}{x_g^2}. \end{aligned} \tag{3.52}$$

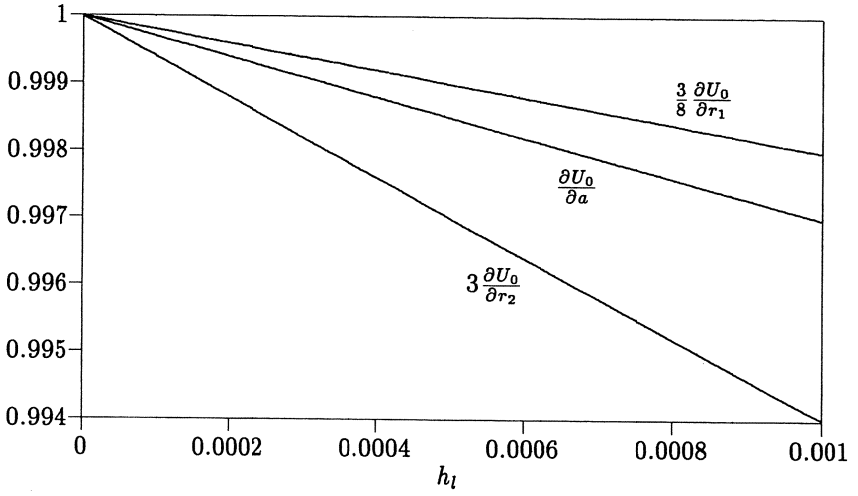

FIG. 3.3. *Derivatives of* $U = \bar{u}_{2m}^{l+1}$ *of the example problem, at green boundary* $x_g = 1/2$.

These results are in agreement with Fig. 3.3, which shows the first derivatives of U_0, for $x_g = 1/2$ and small h_l. Substitution of (3.52) into (3.51) gives for the example problem

$$\epsilon_{2m}^{l+1} = -2x_g h_{l+1} + \frac{3}{2}h_{l+1}^2 + \mathcal{O}(h_l^3). \tag{3.53}$$

In Fig. 3.4 the error is given for two grids, one with $h_l = 1/32$ and one with $h_l = 1/64$, and $x_g = 1/2$. This result is to be compared with Fig. 3.2 for $I1$. Note that in this particular case the discretisation with interpolation $I1$ yields a more accurate solution than $I2$.

For both interpolations the numerical solution is first-order accurate. Opposed to interpolation $I1$, interpolation $I2$ gives

$$\epsilon_{2m}^{l+1} - \epsilon_{m-1}^{l} = \mathcal{O}(h_l^2). \tag{3.54}$$

This means that $\epsilon(x)$ is continuous at $x = x_g$. Interpolation $I2$ to compute the virtual state appears to give a smoother error and hence a smoother approximation of the exact (smooth) solution. A further investigation however, reveals that $\epsilon(x)$ is not differentiable in $x = x_g$, which is also observed in Fig. 3.4. We remind that the use of $I2$ is primarily to obtain a higher-order accurate local truncation error, which can be used in constructing a solution-adaptive, locally refined grid.

4. Estimating the local discretisation error. In the previous sections the exact solution and hence the exact local truncation error is known. In practice we would like to estimate the local discretisation error, while we only have the solution of the discrete problem as an approximation of the exact solution. We consider estimating the local truncation error for a uniform grid and for a locally refined grid. The procedure described in this section is intended to be used in a multigrid context and is based on a classical truncation error extrapolation, [2], [3]. Instead of estimating the local truncation error by extrapolation, it may also be estimated by estimating expression (3.4) for each grid point. For a first-order accurate discretisation this would imply estimating $\partial u/\partial x$, $\partial F/\partial u$ and $\partial^2 F/\partial u^2$ at each grid point. However, we aim at application of our a-posteriori estimation of the local discretisation error to a set of simultaneous conservation laws, defined in two or even three space dimensions. For these kinds of problems, estimating the local truncation error by

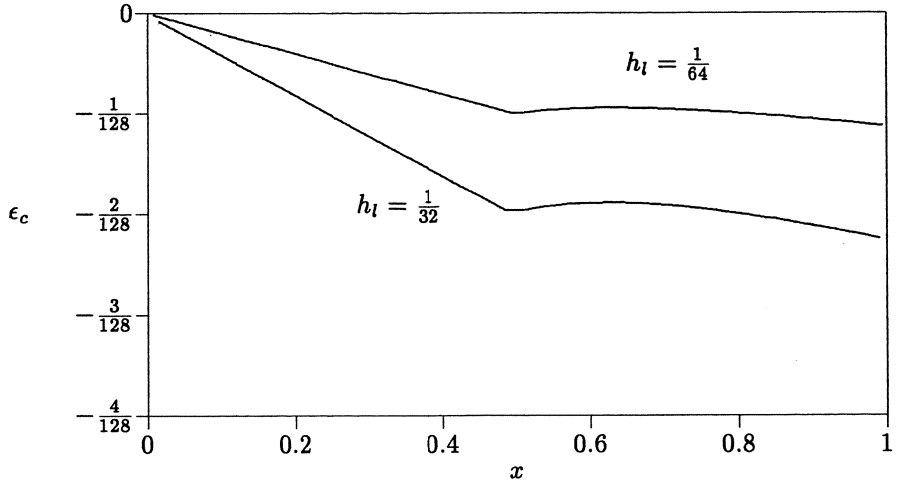

FIG. 3.4. *Error of the solution for the example problem on the composite grid, using interpolation I2 for the virtual state and with green boundary* $x_g = 1/2$.

approximating these derivatives would be very expensive, and even more so for higher-order accurate discretisations. Therefore, we propose to estimate the local discretisation error by application of the extrapolation technique to be considered in sections 4.2 and 4.3.

Since our goal is the use of (an estimate of) the local discretisation error in a refinement criterion, we focus on the situation where the local truncation error is of the same order everywhere. Hence, where an irregular discretisation scheme is applied, we focus on the interpolation $I2$.

4.1. Preliminaries. On the sequence of grids, Ω^l, $l = 0, \ldots, L$, we consider a sequence of locally nested discretisations, denoted by

$$N^l(u^l; u^{l-1}) = r^l, \quad \forall l \in \{0, \ldots, L\}. \tag{4.1}$$

Again, u^{l-1} merely acts as a parameter, since solving the equations for a level l will leave u^{l-1} unchanged. This parameter is superfluous on level $l = 0$, and on any level for which the grid covers Ω completely. As usual for the multigrid FAS procedure (see [1]), the right-hand side of (4.1) is defined as

$$r^l = \begin{cases} S^l g^l, \text{ on } \Omega_c^l, \\ N^{l,l+1}(\overline{R}^{l,l+1} u^{l+1}; u^{l-1}) - R^{l,l+1}\left(N^{l+1}(u^{l+1}; u^l) - r^{l+1}\right), \text{ on } \Omega_f^l. \end{cases} \tag{4.2}$$

The solution of (4.1) is u_c, defined by (2.22) and is associated with the cells on the composite grid Ω_c. The approximation on Ω^l is $\overline{R}^l u$ and on Ω_f^l it is $\overline{R}^{l,l+1} u$. The sequence of (locally) nested discretisations allows us to define the so-called relative discretisation error, similar to the local discretisation error, defined by (3.1). The relative discretisation error is denoted by $\tau_{l+1}^l(\overline{u}^{l+1})$, where $\tau_{l+1}^l : X^{l+1}(\Omega^{l+1}) \to X^l(\Omega^{l+1})$ is defined by

$$\tau_{l+1}^l(u^{l+1}) = (S^{l,l+1})^{-1}\left(N^{l,l+1}(\overline{R}^{l,l+1} u^{l+1}; u^{l-1}) - R^{l,l+1} N^{l+1}(u^{l+1}; u^l)\right). \tag{4.3}$$

With the definition of the local truncation error (3.1) and $R^{l,l+1}$ defined by (2.17), the relative truncation error can be written as

$$\tau_{l+1}^l(\overline{R}^{l+1} u) = (S^{l,l+1})^{-1} N^{l,l+1}(\overline{R}^{l,l+1} u; \overline{R}^{l-1,l} u) \tag{4.4}$$

$$
\begin{aligned}
& - \overline{R}^{l,l+1}(S^{l+1})^{-1}N^{l+1}(\overline{R}^{l+1}u;\overline{R}^{l}u) \\
= \;& (S^{l,l+1})^{-1}\left(N^{l,l+1}(\overline{R}^{l,l+1}u;\overline{R}^{l-1,l}u) - R^{l,l+1}N^{,l+1}(u)\right) \\
& - \overline{R}^{l,l+1}(S^{l+1})^{-1}N^{l+1}(\overline{R}^{l+1}u;\overline{R}^{l}u) + \overline{R}^{l,l+1}(S^{l+1})^{-1}N^{,l+1}(u) \\
= \;& \overline{R}^{l,l+1}\tau^{l}(u) - \overline{R}^{l,l+1}\tau^{l+1}(u).
\end{aligned}
$$

We assume that the global error and the local error can both be written as an asymptotic expansion in the mesh width h_l. By $\epsilon = \epsilon(x) \in X(\Omega)$ we denote a function independent of h_L, so that for a sufficiently smooth grid, the $\mathcal{O}(h_L^q)$ global error on a level l can be written as

$$
\epsilon^l = \overline{R}^l\left(h_L^q\epsilon + \mathcal{O}(h_L^{q+1})\right), \tag{4.5}
$$

where h_L is the mesh width at the finest level present. The solution of the set $N^l(u^l;u^{l-1}) = r^l$ is written as

$$
u^l = \overline{u}^l - \epsilon^l = \overline{R}^l(\overline{u} - h_L^q\epsilon) + \mathcal{O}(h_L^{q+1}). \tag{4.6}
$$

For the asymptotic expansion of the local truncation error, we introduce $\tau : X(\Omega) \to X(\Omega)$ by

$$
\tau(u,u',\ldots) = \tau(u(x),u'(x),\ldots), \tag{4.7}
$$

where the primes denote differentiation with respect to x. We assume that the local truncation error τ^l of N^l can be written as the expansion

$$
\tau^l(u) = h_l^p\overline{R}^l\tau(u,u',\ldots) + \mathcal{O}(h_l^{p+1}). \tag{4.8}
$$

For example, for the model problem (1.1) discretised with the fully one-sided upwind flux, according to (3.12) τ is given by

$$
\tau(u,u',u'') = -\frac{1}{2}\left\{\frac{\partial f}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 f}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2\right\} = -\frac{1}{2}\frac{\partial^2 f}{\partial x^2}. \tag{4.9}
$$

For derivatives we use the notation

$$
\delta_u\tau(u,u',\ldots) = \frac{\partial\tau}{\partial u}(u,u',\ldots). \tag{4.10}
$$

For $u = \overline{u} - h_L^q\epsilon + \mathcal{O}(h_L^{q+1})$ and sufficiently smooth $\overline{u}(x)$ and $\epsilon(x)$, $\tau(u,u',\ldots)$ is related to $\tau(\overline{u},\overline{u}',\ldots)$, through the asymptotic expansion of the error, as

$$
\begin{aligned}
\tau(u,u',\ldots) \;=\; & \tau(\overline{u},\overline{u}',\ldots) - h_L^q\epsilon\delta_u\tau(\overline{u},\overline{u}',\ldots) \\
& - h_L^q\epsilon'\delta_{u'}\tau(\overline{u},\overline{u}',\ldots) - \ldots + \mathcal{O}(h_L^{q+1}) + \mathcal{O}(h_L^{2q}).
\end{aligned} \tag{4.11}
$$

The preliminaries introduced in this section are used to show that our estimates of the local discretisation error are sufficiently accurate, both in a locally uniform and locally non-uniform situation, provided some smoothness conditions are satisfied. Both situations are described in detail in the following two subsections. Results for the example problem are given in section 4.4.

4.2. Estimating on a uniform grid. The restriction of the local truncation error $\tau^{l+1}(u)$ of $N^{l+1}(\overline{R}^{l+1}u)$ on a uniform grid is

$$\begin{aligned}\overline{R}^l\tau^{l+1}(u) &= \overline{R}^l\left(h_{l+1}^p\tau(u)+\mathcal{O}(h_l^{p+1})\right)\\ &= 2^{-p}h_l^p\overline{R}^l\tau(u)+\mathcal{O}(h_l^{p+1})\\ &= 2^{-p}\tau^l(u)+\mathcal{O}(h_l^{p+1}).\end{aligned} \tag{4.12}$$

Hence, we find

$$\tau^l(u) = 2^p\overline{R}^l\tau^{l+1}(u)+\mathcal{O}(h_l^{p+1}). \tag{4.13}$$

Substitution of (4.13) into (4.4) gives

$$\begin{aligned}\tau_{l+1}^l(\overline{R}^{l+1}u) &= \tau^l(u)-\overline{R}^l\tau^{l+1}(u)\\ &= (2^p-1)\overline{R}^l\tau^{l+1}(u)+\mathcal{O}(h_l^{p+1}).\end{aligned} \tag{4.14}$$

Assume we have an interpolation operator $P^{l+1}: X(\Omega^{l+1})\to X^{l+1}(\Omega^{l+1})$, which is used to interpolate a coarse-grid function and which for any sufficiently smooth $u\in X(\Omega^{l+1})$ satisfies

$$P^{l+1}\overline{R}^{l,l+1}\overline{R}^{l+1}u = \overline{R}^{l+1}u+\mathcal{O}(h_l). \tag{4.15}$$

With this interpolation and with (4.11), where $u=\overline{u}-h_L^q\epsilon+\mathcal{O}(h_L^{q+1})$, we find

$$\begin{aligned}\frac{1}{2^p-1}P^{l+1}\tau_{l+1}^l(\overline{R}^{l+1}u) &= P^{l+1}\overline{R}^l\tau^{l+1}(u)+\mathcal{O}(h_l^{p+1})\\ &= \tau^{l+1}(u)+\mathcal{O}(h_l^{p+1})\\ &= \tau^{l+1}(\overline{u})-h_l^ph_L^q\overline{R}^{l+1}\left(\epsilon\delta_u\tau(\overline{u},\overline{u}',\ldots)\right.\\ &\quad\left.+\epsilon'\delta_{u'}\tau(\overline{u},\overline{u}',\ldots)+\ldots\right)\\ &\quad+\mathcal{O}(h_l^ph_L^{q+1})+\mathcal{O}(h_l^ph_L^{2q})+\mathcal{O}(h_l^{p+1}).\end{aligned} \tag{4.16}$$

Hence, with an interpolation operator which satisfies (4.15) and provided ϵ and $\overline{u}$ are sufficiently smooth, the estimate holds

$$\tilde{\tau}^{l+1}(u^{l+1})\equiv\frac{1}{2^p-1}P^{l+1}\tau_{l+1}^l(u^{l+1}) = \tau^l(\overline{u})+\mathcal{O}(h_l^t), \tag{4.17}$$

with $t=\min\{p+q,p+1\}$. According to (4.9) and (4.11), for the the one-sided upwind discretisation of (1.1) we investigate here, it suffices to have differentiable $\overline{u}(x)$ and $\epsilon(x)$.

4.3. Estimating on a locally non-uniform grid. For a locally non-uniform grid, we wish to be able to estimate the local truncation error in a similar way as described in the previous section. As noticed earlier, this is difficult since we have to find asymptotic expansions for the local error of an irregular discretisation scheme. In this subsection we describe how the local error may be split into two parts. This is done so that one part can be estimated by extrapolation. The other part is approximated by approximating the differences that appear in the expression for this local error.

In a locally non-uniform grid situation, (4.12) does not hold, since the discretisation in the neighbourhood of a green boundary differs from the regular discretisation. We split the local discretisation error into two parts. One part, $\tau_u^l(u)$, is the local truncation error in the case where each cell is part of a locally uniform grid (i.e. assume all neighbours exist). The

other part, $\tau_n^l(u)$, is a perturbation due to local refinements and virtual states (i.e. assume fine-grid cells have been deleted and hence some neighbours do not exist any more). This splitting is given by

$$\tau^l(u) = \tau_u^l(u) + \tau_n^l(u). \tag{4.18}$$

The part $\tau_n^l(u)$ is zero in cells which are part of a locally uniform grid (i.e. regular discretisation scheme employed). For the first-order accurate interpolation in (2.29) and fully one-sided upwind computation of the numerical flux, $\tau_n^l(u)$ is unequal zero, only when $x_{i,L}^l$ is a green boundary. We describe estimation of both τ_u^l and τ_n^l in detail, in the next subsections. We consider a locally refined composite grid, so that a coarse grid is locally uniform.

4.3.1. The non-uniform part. The non-uniform part of the local truncation error may be considered as a perturbation of the local truncation error on a locally uniform grid. In the locally non-uniform grid cells, $\tau_n^{l+1}(u)$ results from the difference between a virtual state v_{2m-1}^{l+1} in ω_{2m-1}^{l+1} and the corresponding state u_{2m-1}^{l+1}, if Ω_{2m-1}^{l+1} would exist. We have for $\tau_n^{l+1}(u)$

$$\{\tau_n^{l+1}(u)\}_{2m}^{l+1} = \tau_{2m}^{l+1}(u) - \{\tau_u^{l+1}(u)\}_{2m}^{l+1}. \tag{4.19}$$

By (3.4) and (3.2) we have, for $x_{2m,L}^{l+1}$ a green boundary,

$$\begin{aligned} \{\tau_n^{l+1}\}_{2m}^{l+1} \quad = \quad & \frac{1}{s_{2m}^{l+1}} \left\{ \left.\frac{\partial F}{\partial u}\right|_{u(x_{2m,L}^{l+1})} (u_{2m-1}^{l+1} - v_{2m-1}^{l+1}) \right. \\ & \left. + \mathcal{O}\left(\left(u(x_{2m,L}^{l+1}) - v_{2m-1}^{l+1} \right)^2 - \left(u(x_{2m,L}^{l+1}) - u_{2m-1}^{l+1} \right)^2 \right) \right\}. \end{aligned} \tag{4.20}$$

The second-order accurate interpolation $I2$ to obtain the virtual state v_{2m-1}^{l+1} gives for the non-uniform part $\{\tau_n^{l+1}(u)\}_{2m}^{l+1}$ in Ω_{2m}^{l+1}

$$\{\tau_n^{l+1}(u)\}_{2m}^{l+1} = -\frac{1}{2} s_{2m}^{l+1} \left.\frac{\partial F}{\partial u}\right|_{u(x_{2m,L}^{l+1})} \left.\frac{\partial^2 u}{\partial x^2}\right|_{x_{2m,L}^{l+1}} + \mathcal{O}(h_l^2). \tag{4.21}$$

For a second-order accurate approximation of $\tau_n^{l+1}(u)$, it suffices to approximate $\partial F/\partial u$ at $u(x_{i,L}^l)$ and $\partial^2 u/\partial x^2$ at $x_{i,L}^l$ with first-order accuracy. Let v_{2m-1}^{l+1} be the virtual state obtained by interpolation $I2$ of the restriction of $u(x)$. In addition, let a function $\tilde{u}(x)$ satisfy

$$\frac{\int_{\omega_{2m-1}^{l+1}} \tilde{u}(x)\, dx}{\int_{\omega_{2m-1}^{l+1}} dx} = v_{2m-1}^{l+1}, \tag{4.22a}$$

$$\tilde{u}(x) = u(x), \quad x \geq x_{2m,L}^{l+1}, \tag{4.22b}$$

where ω_{2m-1}^{l+1} is a virtual cell. If the solution of a regular discretisation scheme is the restriction of a differentiable function, then there exists a $\tilde{u}(x)$ which satisfies (4.22) and which is also differentiable. Differentiability is ensured, since for a given v_{2m-1}^{l+1} the scheme may be considered a regular scheme. The regular discretisation is shown to have a solution, which is the restriction of a differentiable function. For this $\tilde{u}(x)$ we have

$$\frac{u_{2m+1}^{l+1} - 2u_{2m}^{l+1} + v_{2m-1}^{l+1}}{(s_{2m}^{l+1})^2} = \left.\frac{\partial^2 \tilde{u}}{\partial x^2}\right|_{x_{2m,L}^{l+1}} + \mathcal{O}(h_l) = \lim_{\xi \downarrow x_{2m,L}^{l+1}} \left.\frac{\partial^2 u}{\partial x^2}\right|_{\xi} + \mathcal{O}(h_l). \tag{4.23}$$

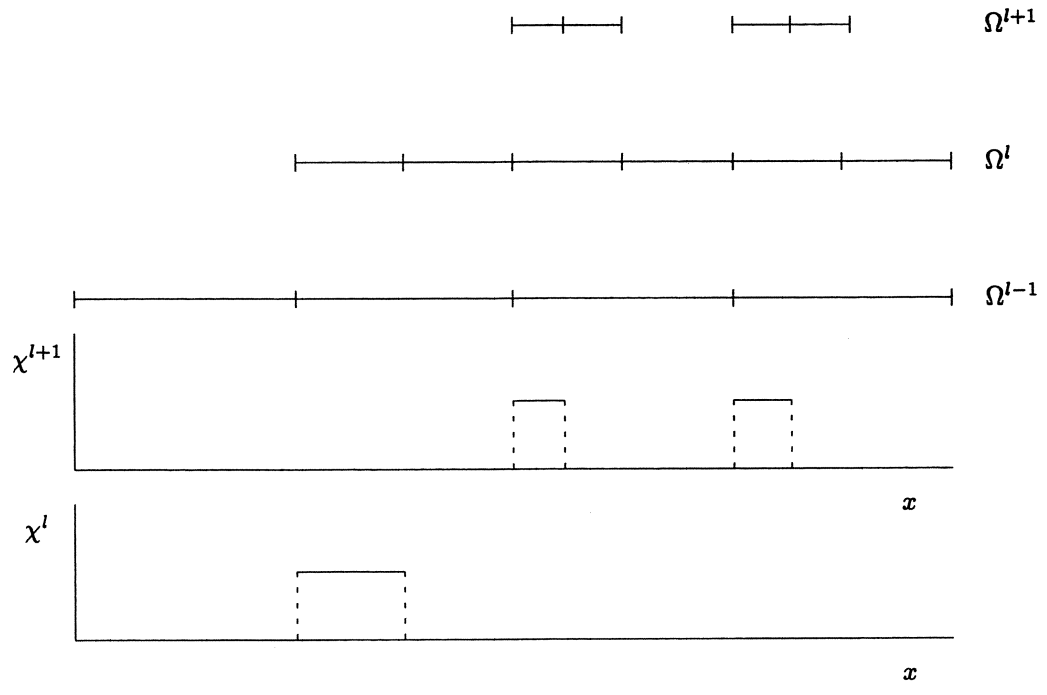

FIG. 4.1. *Example of characteristic function on different levels of refinement.*

Furthermore, it can be easily shown that

$$\left.\frac{\partial F}{\partial u}\right|_{u_{2m}^{l+1}} = \left.\frac{\partial F}{\partial u}\right|_{u(x_{2m,L}^{l+1})} + \mathcal{O}(h_l), \tag{4.24}$$

for $u(x)$ continuous on Ω_{2m}^{l+1}.

For $x_{2m,L}^{l+1}$ a green boundary, we have an a-posteriori estimate for $\tau_n^{l+1}(u)$, denoted by $\tilde{\tau}_n^{l+1}(u^{l+1};u^l)$, given by

$$\begin{aligned}\{\tilde{\tau}_n^{l+1}(u^{l+1};u^l)\}_{2m}^{l+1} &\equiv -\frac{1}{2s_{2m}^{l+1}}\left.\frac{\partial F}{\partial u}\right|_{u_{2m}^{l+1}}(u_{2m+1}^{l+1}-2u_{2m}^{l+1}+v_{2m-1}^{l+1}) \\ &= \{\tau_n^{l+1}(u)\}_{2m}^{l+1}+\mathcal{O}(h_l^2).\end{aligned} \tag{4.25}$$

Otherwise, $\tilde{\tau}_n^{l+1}$ is zero.

4.3.2. The uniform part. Let the characteristic function $\chi^l \in X^l(\Omega^l)$ be defined by

$$\chi^l(x) = \begin{cases} 1, & \text{if } x \in \Omega_i^l \text{ and } x_{i,L}^l \text{ is a green boundary,} \\ 0, & \text{everywhere else and } x \in \Omega^l. \end{cases} \tag{4.26}$$

An example of characteristic functions associated with the locally refined grid, is shown in Fig. 4.1. Similar to the locally uniform case, we introduce functions to be used in asymptotic expansions in terms of h_l, for both τ_u^l and τ_n^l, given by

$$\tau_u(u,u',\ldots) = \tau_u(u(x),u'(x),\ldots) \tag{4.27a}$$

$$\tau_n(u,u',\ldots) = \tau_n(u(x),u'(x),\ldots). \tag{4.27b}$$

An asymptotic expansion of the local truncation error is then written as

$$\tau^l(u) = \overline{R}^l \left\{ h_l^p \tau_u(u, u', \ldots) + \mathcal{O}(h_l^{p+1}) + \chi^l \left(h_l^r \tau_n(u, u', \ldots) + \mathcal{O}(h_l^{r+1}) \right) \right\}, \tag{4.28}$$

where τ_u and τ_n are independent of h_l. The assumption that the coarse grid Ω^l is locally uniform, implies

$$\overline{R}^{l,l+1} \chi^l \tau_n(u, u', \ldots) = 0. \tag{4.29}$$

The local truncation error on the refined part Ω_f^l of the coarse grid is then given by

$$\begin{aligned} \hat{\tau}^l(u) &\equiv \overline{R}^{l,l+1} \tau^l(u) \\ &= \overline{R}^{l,l+1} \left\{ h_l^p \tau_u(u, u', \ldots) + \mathcal{O}(h_l^{p+1}) + \chi^l \left(h_l^r \tau_n(u, u', \ldots) + \mathcal{O}(h_l^{r+1}) \right) \right\} \\ &= h_l^p \overline{R}^{l,l+1} \tau_u(u, u', \ldots) + \mathcal{O}(h_l^{p+1}). \end{aligned} \tag{4.30}$$

The restriction of the local truncation error on the fine grid, can be written as

$$\begin{aligned} \overline{R}^{l,l+1} \tau^{l+1}(u) &= h_{l+1}^p \overline{R}^{l,l+1} \tau_u(u, u', \ldots) + \overline{R}^{l,l+1} \tau_n^{l+1}(u) + \mathcal{O}(h_l^{p+1}) \\ &= 2^{-p} \hat{\tau}^l(u) + \overline{R}^{l,l+1} \tau_n^{l+1}(u) + \mathcal{O}(h_l^{p+1}). \end{aligned} \tag{4.31}$$

From this we see that the local truncation error on the refined part of the coarse grid then may be written as

$$\hat{\tau}^l(u) = 2^p \overline{R}^{l,l+1} \tau^{l+1}(u) - 2^p \overline{R}^{l,l+1} \tau_n^{l+1}(u) + \mathcal{O}(h_l^{p+1}). \tag{4.32}$$

The relative truncation error $\tau_{l+1}^l(u)$ as given by (4.4) can be written as

$$\begin{aligned} \tau_{l+1}^l(\overline{R}^{l+1} u) &= \hat{\tau}^l(u) - \overline{R}^{l,l+1} \tau^{l+1}(u) \\ &= 2^p \overline{R}^{l,l+1} \tau^{l+1}(u) - 2^p \overline{R}^{l,l+1} \tau_n^{l+1}(u) - \overline{R}^{l,l+1} \tau^{l+1}(u) + \mathcal{O}(h_l^{p+1}) \\ &= (2^p - 1) \overline{R}^{l,l+1} \tau^{l+1}(u) - 2^p \overline{R}^{l,l+1} \tau_n^{l+1}(u) + \mathcal{O}(h_l^{p+1}). \end{aligned} \tag{4.33}$$

From (4.28) and the definition of χ^l in (4.26), it is easily seen that $\tau^{l+1}(u)$ cannot be considered to be the restriction of some smooth function. An interpolation operator which satisfies (4.15) for a smooth function, cannot be applied effectively to approximate $\tau^{l+1}(u)$ from (4.33). This would result in errors of $\mathcal{O}(h_l^p)$ in the approximation of the local truncation error, which itself is $\mathcal{O}(h_l^p)$. The part τ_u^{l+1} may be considered as the restriction of a smooth function $h_{l+1}^p \tau_u(u(x)) + \mathcal{O}(h_l^{p+1})$. For the upwind discretisation of the conservation law (1.1a) τ_u is given by (4.9). Introduction of the splitting (4.18) in (4.33), gives

$$\tau_{l+1}^l(\overline{R}^{l+1} u) = (2^p - 1) \overline{R}^{l,l+1} \tau_u^{l+1}(u) - \overline{R}^{l,l+1} \tau_n^{l+1}(u) + \mathcal{O}(h_l^{p+1}). \tag{4.34}$$

Interpolation by using P^{l+1}, which satisfies (4.15), division by $2^p - 1$ and reordering gives

$$\frac{1}{2^p - 1} P^{l+1} \left(\tau_{l+1}^l(\overline{R}^{l+1} u) + \overline{R}^{l,l+1} \tau_n^{l+1}(u) \right) = \tau_u^{l+1}(u) + \mathcal{O}(h_l^{p+1}). \tag{4.35}$$

If $x_{2m,L}^{l+1}$ is a green boundary and the estimate of the non-uniform part is (4.25) with sufficiently smooth $\epsilon(x)$ and $\overline{u}(x)$, then we have, similar to (4.16), an estimate of the non-uniform part of the local truncation error given by

$$\begin{aligned} \tilde{\tau}_u^{l+1}(u^{l+1}; u^l) &\equiv \frac{1}{2^p - 1} P^{l+1} \left(\tau_{l+1}^l(u^{l+1}) + \overline{R}^{l,l+1} \tilde{\tau}_n^{l+1}(u^{l+1}; u^l) \right) \\ &= \tau_u^{l+1}(\overline{u}) + \mathcal{O}(h_l^t), \end{aligned} \tag{4.36}$$

where $t = \min\{p+1, p+q, 2\}$. From (4.9) and (4.11) it follows that $\overline{u}(x)$ and $\epsilon(x)$ have to be differentiable in order to have (4.36) to be true.

4.4. Estimating local discretisation error for example problem. In this section we investigate the a-posteriori estimation of the local error, for the example problem introduced in section 2.2.

4.4.1. Locally uniform grid. In the model problem we use a first-order discretisation and we have a first-order accurate solution, $p = 1$. With the expression 3.11 for the exact local discretisation error of the example problem, we find

$$\tau_{2i}^{l+1}(\overline{u}) = -h_{l+1}^3(24i^2 + \frac{16}{3}i + 1), \tag{4.37a}$$

$$\tau_{2i+1}^{l+1}(\overline{u}) = -h_{l+1}^3(24i^2 + \frac{88}{3}i + \frac{29}{3}). \tag{4.37b}$$

For the problem and discretisation considered here, it suffices to take for P^{l+1} the cellwise constant interpolation

$$\{P^{l+1}(u^l)\}_{2i}^{l+1} = \{P^{l+1}(u^l)\}_{2i+1}^{l+1} = u_i^l, \tag{4.38}$$

for $i \in I_f^l$. This interpolation satisfies (4.15). For the example problem, with solution $\overline{u}(x) = x^2$, the coarse-grid operator N^l, acting on the restriction of the fine-grid solution of the discrete equations, gives

$$\{N^l(\overline{R}^l u^{l+1})\}_i^l = h_{l+1}^4(64i^3 + 48i^2 + 32i + \mathcal{O}(i^0)). \tag{4.39}$$

For the restriction of the fine-grid operator, acting on the fine-grid solution we have

$$\{R^l N^{l+1}(u^{l+1})\}_i^l = h_{l+1}^4(64i^3 + 96i^2 + 64i + \mathcal{O}(i^0)). \tag{4.40}$$

Hence, the relative truncation error is

$$\{\tau_{l+1}^l(u^{l+1})\}_i^l = -h_{l+1}^3(24i^2 + 16i + \mathcal{O}(i^0)). \tag{4.41}$$

The cellwise constant interpolation gives the estimate on the fine grid

$$\tilde{\tau}_{2i}^{l+1}(u^{l+1}) = \tilde{\tau}_{2i+1}^{l+1}(u^{l+1}) = -h_{l+1}^3(24i^2 + 16i + \mathcal{O}(i^0)). \tag{4.42}$$

As expected, this differs $\mathcal{O}(h_l^2)$ from the estimate (4.37).

4.4.2. Locally non-uniform grid. The local discretisation error in the locally non-uniform grid cell Ω_{2m}^{l+1} is

$$\tau_{2m}^{l+1}(\overline{u}) = -8x_g^2 h_{l+1} - \frac{2}{3}x_g h_{l+1}^2 - \frac{8}{3}h_{l+1}^3. \tag{4.43a}$$

For its neighbour we have

$$\tau_{2m+1}^{l+1}(\overline{u}) = -6x_g^2 h_{l+1} - \frac{44}{3}x_g h_{l+1}^2 - \frac{29}{3}h_{l+1}^3. \tag{4.43b}$$

In this locally non-uniform case, the local discretisation error is estimated by estimating τ_u^l and τ_n^l. By application of (4.25), we find for the non-uniform part

$$\{\tilde{\tau}_n^{l+1}(u^{l+1}; u^l)\}_{2m}^{l+1} = -2x_g^2 + \mathcal{O}(h_l^2). \tag{4.44}$$

Furthermore, we find

$$\{N^l(u^l; u^{l-1})\}_m^l = 8x_g^3 h_{l+1} + \frac{44}{3}x_g^2 h_{l+1}^2 + \mathcal{O}(h_l^3), \tag{4.45}$$

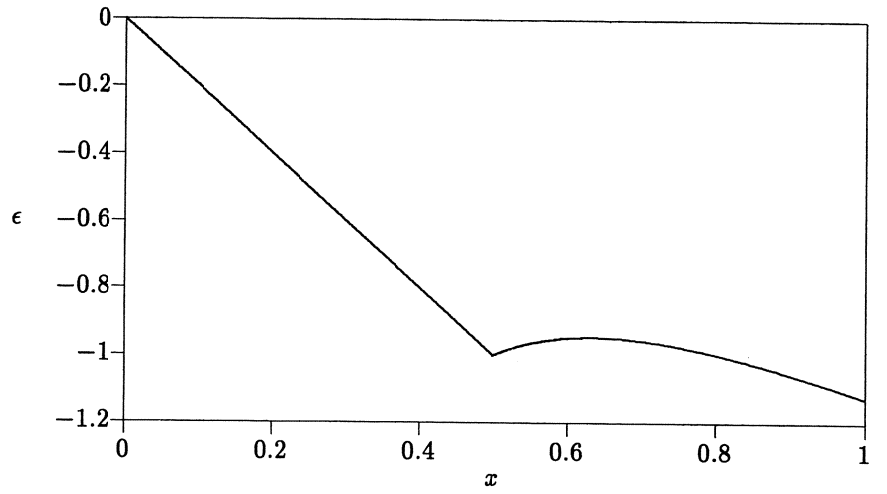

FIG. 4.2. *The function $\epsilon(x)$ for the example problem and green boundary $x_g = 1/2$.*

and

$$\{R^{l,l+1}N^{l+1}(u^{l+1};u^l)\}_m^l = 8x_g^3 h_{l+1} + 24x_g^2 h_{l+1}^2 + \mathcal{O}(h_l^3). \tag{4.46}$$

The estimate of the relative truncation error is

$$\{\tau_{l+1}^l(u^{l+1})\}_m^l = -\frac{14}{3}x_g^2 h_{l+1} + \mathcal{O}(h_l^2). \tag{4.47}$$

Then we have with the cellwise interpolation P^{l+1} and $p = 1$

$$\begin{aligned}\{\tilde{\tau}_u^{l+1}(u^{l+1};u^l)\}_{2m}^{l+1} &= \{\tilde{\tau}_u^{l+1}(u^{l+1};u^l)\}_{2m+1}^{l+1} \\ &= \{\tau_{l+1}^l(u^{l+1})\}_m^l + \{\overline{R}^{l,l+1}\tilde{\tau}_n^{l+1}(u^{l+1};u^l)\}_m^l \\ &= -\frac{17}{3}x_g^2 h_{l+1} + \mathcal{O}(h_l^2),\end{aligned} \tag{4.48}$$

and the estimation of the local truncation error is

$$\{\tilde{\tau}_u^{l+1} + \tilde{\tau}_n^{l+1}\}_{2m}^{l+1} = -\frac{23}{3}x_g^2 h_{l+1} + \mathcal{O}(h_l^2), \tag{4.49a}$$

$$\{\tilde{\tau}_u^{l+1} + \tilde{\tau}_n^{l+1}\}_{2m+1}^{l+1} = -\frac{17}{3}x_g^2 h_{l+1} + \mathcal{O}(h_l^2). \tag{4.49b}$$

Clearly, these are $\mathcal{O}(h_l)$ accurate approximations of the exact local truncation errors, given by (4.43), which itself is $\mathcal{O}(h_l)$. This inaccuracy is caused by the fact that ϵ, which is used in the asymptotic expansion of the error, is not sufficiently smooth. In Fig. 4.2 ϵ is shown for the green boundary at $x_g = 1/2$. As already noted, ϵ is not differentiable in x_g. The inaccuracy of this estimate is in full agreement with the theory developed in section 4.3.2.

An accurate estimate of the local discretisation error can be made, by replacing ϵ with a sufficiently smooth $\tilde{\epsilon}$, and use this in the estimation of the uniform part $\tilde{\tau}_u^{l+1}$. Replacing ϵ is the same as replacing the function u used in the asymptotic expansion of u^l, with a sufficiently smooth and first-order accurate $\tilde{u}$, which satisfies (4.22). Then, the restriction of $\tilde{u}$ may be used in the computation of τ_{l+1}^l.

This restriction of $\tilde{u}$ is easily found by extrapolation of the solution u^{l+1}. We use (4.22) for the mean of $\tilde{u}$ at ω_{2m-1}^{l+1}. The mean at ω_{2m-2}^{l+1} is found from linear extrapolation by

$$\tilde{u}_{2m-2}^{l+1} = \frac{\int_{\omega_{2m-2}^{l+1}} \tilde{u}\,dx}{\int_{\omega_{2m-2}^{l+1}} dx} = 2v_{2m-1}^{l+1} - u_{2m}^{l+1}. \tag{4.50}$$

In Ω_{m-1}^{l} we use, instead of u_{m-1}^{l}, the value of $\frac{1}{2}(\tilde{u}_{2m-2}^{l+1} + \tilde{u}_{2m-1}^{l+1})$. For $\tilde{u}^{l+1}$ we find

$$\tilde{u}_{2m-1}^{l+1} = v_{2m-1}^{l+1} = x_g^2 + x_g h_{l+1} - \frac{1}{6}h_{l+1}^2 + \mathcal{O}(h_l^3), \tag{4.51a}$$

$$\tilde{u}_{2m-2}^{l+1} = x_g^2 - x_g h_{l+1} + \frac{5}{6}h_{l+1}^2 + \mathcal{O}(h_l^3). \tag{4.51b}$$

Then, we also have

$$\tilde{u}_{m-1}^{l} = \frac{1}{2}(\tilde{u}_{2m-2}^{l+1} + \tilde{u}_{2m-1}^{l+1}) = x_g^2 + \frac{1}{3}h_{l+1}^2 + \mathcal{O}(h_l^3). \tag{4.52}$$

The restriction of the fine-grid operator, acting on $\tilde{u}^{l+1}$, is

$$\begin{aligned} \{R^l N^{l+1}(\tilde{u}^{l+1}; \tilde{u}^l)\}_m^l &= (u_{2m+1}^{l+1})^2 - (v_{2m-1}^{l+1})^2 \\ &= 8x_g^3 h_{l+1} + 24x_g^2 h_{l+1}^2 + \mathcal{O}(h_l^3). \end{aligned} \tag{4.53}$$

Furthermore, we have for the coarse-grid operator, acting on the restriction of $\tilde{u}^{l+1}$

$$\begin{aligned} \{N^l(\overline{R}^{l,l+1}\tilde{u}^{l+1}; \tilde{u}^{l+1})\}_m^l &= (\tilde{u}_m^l)^2 - (\tilde{u}_{m-1}^l)^2 \\ &= \frac{1}{4}(u_{2m}^{l+1} + u_{2m+1}^{l+1})^2 - (\tilde{u}_{m-1}^l)^2 \\ &= 8x_g^3 h_{l+1} + 14x_g^2 h_{l+1}^2 + \mathcal{O}(h_l^3). \end{aligned} \tag{4.54}$$

Then, the relative local discretisation error is

$$\{\tau_{l+1}^l(\tilde{u}^{l+1})\}_m^l = -5x_g^2 h_{l+1}\mathcal{O}(h_l^2). \tag{4.55}$$

From this and with P^{l+1} the cellwise constant interpolation and $p = 1$, we find

$$\begin{aligned} \{\tilde{\tau}_u^{l+1}(\tilde{u}^{l+1}; \tilde{u}^l)\}_{2m}^{l+1} &= \{\tilde{\tau}_u^{l+1}(\tilde{u}^{l+1}; \tilde{u}^l)\}_{2m+1}^{l+1} \\ &= \{\tau_{l+1}^l(\tilde{u}^{l+1})\}_m^l + \{\overline{R}^{l,l+1}\tilde{\tau}_n^{l+1}(\tilde{u}^{l+1}; \tilde{u}^l)\}_m^l \\ &= -6x_g^2 h_{l+1} + \mathcal{O}(h_l^2). \end{aligned} \tag{4.56}$$

Then, the estimates of the local discretisation errors on Ω^{l+1} are

$$\{\tilde{\tau}_u^{l+1} + \tilde{\tau}_n^{l+1}\}_{2m}^{l+1} = -6x_g^2 h_{l+1} - 2x_g^2 h_{l+1} + \mathcal{O}(h_l^2) = -8x_g^2 h_{l+1} + \mathcal{O}(h_l^2), \tag{4.57}$$

$$\{\tilde{\tau}_u^{l+1} + \tilde{\tau}_n^{l+1}\}_{2m+1}^{l+1} = -6x_g^2 h_{l+1} + \mathcal{O}(h_l^2). \tag{4.58}$$

These clearly are $\mathcal{O}(h_l^2)$ accurate approximations of the local discretisation errors given by (4.43).

5. Conclusion. For a locally refined grid we have studied a-posteriori estimation of the local discretisation error of a finite volume, upwind discretisation of a one-dimensional conservation law. Two interpolations have been considered for the discretisation at fine-coarse grid interfaces (so-called green boundaries). It appears that, to obtain a first-order

accurate solution, it is sufficient to use a first-order accurate interpolation to compute the so-called virtual states. However, this gives zeroth-order local discretisation errors in the cells neighbouring a green boundary. A second-order accurate interpolation gives a first-order local discretisation error. For both discretisations the solution is first-order accurate. However, the second-order accurate interpolation may yield a smoother solution.

It is shown that extrapolation of the relative local truncation error can be used for the a-posteriori estimation of the local discretisation error, provided that the exact solution and the global discretisation error satisfy certain smoothness conditions. In the neighbourhood of a green boundary, the estimate is based on a splitting of the local discretisation error. In this error we distinguish a part equal to the local discretisation error of the regular scheme (uniform part) and a perturbation, due to the irregularity (non-uniform part). The uniform part is estimated by extrapolation of the relative local truncation error and the non-uniform part by estimating the expression obtained by Taylor series expansion. The estimation technique can be applied (to both discretisations), if the solution of the continuous problem is differentiable. To derive this result, it is assumed that the global error can be written as an asymptotic expansion in the mesh width, with a differentiable function in the lowest-order term. For an example problem it is shown that away from the green boundary, the global error can be considered as the restriction of a differentiable function.

The global error on a locally non-uniform grid may not satisfy the smoothness condition mentioned above. For an example problem it is shown how to accurately estimate the local discretisation error, if this is the case. The estimate is based on extrapolation of the numerical solution across the green boundary. The extrapolant obtained is a sufficiently accurate approximation of the solution, with an error which is the restriction of a sufficiently smooth function.

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