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On Relations between Schmidt Pairs Arising in Robust Control

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Abstract
We present here some generalizations and some simpler derivations of results appeared in Fuhrmann [4] and Fuhrmann and Ober [5]. The main result is that the singular values and Schmidt vectors of the Hankel operator with symbol a normalized coprime factorization of a plant can be given an explicit representation in terms of the plant, of its optimally robust controller and of the Schmidt pairs of another scalar Hankel operator.

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1 Introduction

In spite of the fact that Hankel operators have become, after the seminal paper of Adamjan, Arov and Krein [1], a central tool in systems theory and in interpolation theory, some of its features have been studied in detail only recently in [4]. Among the relevant results are the connection between the singular values and Schmidt vectors of the original operator, those of its Nehari extension, those of the "one step" approximant and those of a Hankel operator obtained from the last Schmidt vector of the original one. In another paper [5], Fuhrmann and Ober derive some connections between the Schmidt vectors of a normalized coprime factorization of a plant and those of a function obtained throught any controller which stabilizes internally the plant. We generalize some of the above results, and present simpler

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proofs. In particular, Theorem 4.2 gives an explicit representation of the Schmidt pairs of the Hankel operator with symbol the normalized coprime factorization of the plant (which is a 2×1 matrix function), in terms of the Schmidt pairs of a scalar Hankel operator, which is simpler to study. Moreover, it also shown how from this Hankel operator we can recover explicitly the plant (Theorem 4.4.

The paper is structured as follows: section two is notation; section three contains some simple proof of results of [4] on Schmidt pairs of a Hankel operator, together with some mild generalizations. Section four is devoted to the Hankel operators arising in the theory of robust control.

2 Preliminaries and notation

We work in the Hilbert space setting of the plane; we define [8] $L^2(\mathbb{I})$ to be the set of the square integrable functions on the imaginary axis, and H^2_+ to be the subspace of L^2 of functions analytic in the right half-plane and s. t.

$$\sup_{x>0} \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x+iy)|^2 \, dy < \infty$$

The inner product in H^2_+ is

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(i\omega) \, \overline{g(i\omega)} \, d\omega$$

where bar denotes conjugate. Analogously, $H^\infty_+$is the subspace of L^2 of functions analytic in the right half-plane and s. t.

$$\sup_{x>0} (\operatorname{ess\,sup}_{y \in \mathbb{R}} |f(x+iy)| \,) < \infty$$

(H^2_- and $H^\infty_-$are defined similarly on the left half-plane). A space $X \in H^2_+$ is *invariant* if $hX \subset X$ for all $h \in H^\infty_+$. If $p(s)$ is a polynomial with real coefficients, we set

$$p^*(s) := p(-s).$$

Let f be function of L^∞: we denote by $\mathbf{P}_+(\mathbf{P}_-)$ the orthogonal projection of L^2 onto $H^2_+(H^2_-)$. The Hankel operator with symbol f is defined as

$$H_f h = \mathbf{P}_- f h \quad h \in H^2_+ \tag{1}$$

A Schmidt pair (ξ,η) of H_f is a pair of vectors $\xi \in H^2_+$, $\eta \in H^2_-$ for which there exists a positive number σ, called singular value, such that

$$H_f\xi = \sigma\eta, \quad H_f^*\eta = \sigma\xi. \tag{2}$$

It can be shown (see [1] or [4]) that if $f \in H^2_-$ and it is rational of degree n, i.e. $f = n^*/d^*$, then there exist n singular values.

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$$

and n linearly independent pairs $(\xi_1,\eta_1)\ldots(\xi_n,\eta_n)$ (Schmidt pairs) satisfying (2).

We assume in the sequel that $\sigma_1 > \sigma_2 > \ldots > \sigma_n$. In this case there exist polynomials $p_1,\ldots,p_n$ of degree at most $n-1$ such that $\xi_i = \frac{p_i}{t}$, $\eta_i = \varepsilon_i\frac{p_i^*}{t^*}$ where ε_i is a real constant of modulus 1. Assume now that n^*/d^* is strictly proper, and define the *signed singular values* of $H_{\frac{n^*}{d^*}}$ as $\lambda_i = \varepsilon_i\sigma_i$. In view of (1), (2) becomes

$$\frac{n^*}{t^*}\frac{p_i}{t} = \lambda_i\frac{p_i^*}{t^*} + \frac{q_i}{t} \tag{3}$$

where q_i are suitable polynomials of degree at most $n-1$. The equation (3) is called *fundamental equation* (see [4]). In what follows we will also assume t monic.

The polynomials p_i are determined up to a (constant) factor by $\frac{n^*}{t^*}$, as well as the values λ_i; moreover, $\frac{p_i}{t}$ is orthogonal to $\frac{p_j}{t}$ for $i \neq j$ (see [1]). It can also be shown that $p_k = u_k s_k$, with u_k stricly unstable of degree $k-1$ and s_k stable of degree $n-k$. In particular, p_1 is stable, i.e. $1/p_1 \in H^\infty_+$ (or equivalently said, p_1/t is outer. We will be interested in the all pass functions $\varphi_k = \lambda_k\frac{s_k^* u_k^* t}{s_k u_k t^*}$. Their relevance in Hankel norm approximation stems from the following lemma (see [1]):

Lemma 2.1 *Two Hankel operators which coincide on a vector $x \in H^2_+$ coincide on the whole invariant subspace generated by x.*

It can be shown that with our assumptions ϕ_k has degree $2n-1$ and that its first $2k-1$ singular values coincide with σ_k. Now, clearly H_{ϕ_1} and H_{ϕ_k} coincide on ξ_k, and therefore they coincide on the invariant subspace X_k generated by ξ_k. Since this function can be written as $\xi_k = \frac{u_k s_k}{t}$, it is easily seen that $X_k = \frac{u_k}{u_k^*}H^2_+$ and that it has codimension $k-1$. The Hankel norm approximant is then defined as $\varphi_1 - \varphi_k$, and since its stable component belongs to the orthogonal complement of X_k it has degree at most $k-1$. It is also easy to see that the Schmidt vectors of H_{ϕ_k} are of the form $\frac{\alpha_k}{t u_k^*}$, where α_k has degree at most $n+k-2$. .

In the next section we will investigate the relation which can be estabished between the Hankel operators generated by these functions.

3 The general structure

We start by examining the relation between the Hankel operators H_{φ_k} and $H_{\varphi_k^*}$. Let first consider the case $\varphi_1 = \lambda_1 \frac{p_1^* t}{p_1 t^*}$. Then the fundamental equation (3) can be written (see [4]) as

$$\lambda_1 \frac{p_1^* t}{p_1 t^*} \cdot \frac{p_i}{t} = \lambda_i \frac{p_i^*}{t^*} + \frac{\pi_i}{p_1}. \tag{4}$$

Obviously, $\pi_1 = 0$. Therefore

$$\begin{aligned} \lambda_1 \frac{p_1 t^*}{p_1^* t} \cdot \frac{\pi_i}{p_1} &= \lambda_1^2 \frac{p_i}{t} - \lambda_i \lambda_1 \frac{p_i^*}{t^*} \frac{p_1 t^*}{p_1^* t} \\ &= \lambda_1^2 \frac{p_i}{t} - \lambda_1^2 \frac{p_i}{t} - \lambda_i \frac{\pi_i^*}{p_1^*} \\ &= -\lambda_i \frac{\pi_i}{p_1^*} + (\lambda_1^2 - \lambda_i^2) \frac{p_i}{t} \end{aligned}$$

In other words, the Hankel operator with symbol $\frac{p_1 t^*}{p_1^* t}$ has signed singular values $-\lambda_2, \ldots, -\lambda_n$ and singular vectors $\frac{\pi_2}{p_1}, \ldots, \frac{\pi_n}{p_1}$. This result has been obtained first by Fuhrmann, and it admits a simple generalization to the all-pass functions generated by any Schmidt pair.

Theorem 3.1 *Let the Schmidt vectors of H_{ϕ_k} be $\frac{\alpha_k}{tu_k^*}$ and its signed singular values $\mu_i, (\mu_1 = \mu_2 \ldots = \mu_{2k-1} > \mu_{2k} > \ldots > \mu_{n+k-1}$; then $H_{\varphi_k^*}$ has signed singular values $\bar{\mu}_i = -\mu_{2k+i-1}$, $1 \leq i \leq n-k$, and Schmidt vectors $\frac{\beta_i}{s_k}$, given by the fundamental equation*

$$\mu_k \varphi_k \frac{\alpha_i}{tu_k^*} = \mu_i \frac{\alpha_i^*}{t^* u_k} + \frac{\beta_i}{s_k} \quad i = 2k, \ldots, n+k-1$$

Proof. From the fundamental equation we get

$$\begin{aligned} \mu_k \varphi_k \frac{\beta_i}{s_k} &= \mu_k^2 \frac{\alpha_i}{tu_k^*} - \mu_k \mu_i \varphi_k \frac{\alpha_i^*}{t^* u_k} = \\ &= \mu_k^2 \frac{\alpha_i}{tu_k^*} - \mu_i^2 \frac{\alpha_i}{tu_k^*} - \mu_i \frac{\beta_i^*}{s_k^*} \\ &= -\mu_i \frac{\beta_i^*}{s_k^*} + (\mu_k^2 - \mu_i^2) \frac{\alpha_i}{tu_k^*} \end{aligned}$$

and hence the result. ■

Another result in [4] admitting a simple generalization is the following: the function $\frac{1}{\lambda_n}\frac{t}{t^*}\frac{p_n}{p_n^*}$ has signed singular values $\frac{1}{\lambda_i}$, and Schmidt vectors $\frac{p_i^*}{t}$.

To see this, from the fundamental equation for $i = n$ we obtain

$$\lambda_1 \frac{p_1^* t}{p_1 t^*} = \lambda_n \frac{p_n^*}{t^*}\frac{t}{p_n} + \frac{\pi_n}{p_1}\frac{t}{p_n} \tag{5}$$

applying this equality to the fundamental equation again we get

$$\begin{aligned}\lambda_1 \frac{p_1^* t}{p_1 t^*}\frac{p_j}{t} &= \lambda_j \frac{p_j^*}{t^*} + \frac{\pi_j^*}{p_1} = \\ &= \lambda_n \frac{p_n^*}{p_n}\frac{t}{t^*}\frac{p_j}{t} + \frac{\pi_n}{p_1}\frac{t}{p_n}\frac{p_j}{t}\end{aligned}$$

Multiplying by $\frac{p_n}{p_n^*}$ we get

$$\lambda_j \frac{t}{t^*}\frac{p_n}{p_n^*}\frac{p_j^*}{t} = \lambda_n \frac{p_j}{t^*} + \frac{t}{p_n^*}\frac{\pi_n}{p_1}\frac{p_j}{t} - \frac{p_n}{p_n^*}\frac{\pi_j}{p_1}$$

division by $\lambda_j\lambda_n$ yields the result. In conclusion, the fundamental equation for $\frac{1}{\lambda_n}\frac{tp_n}{t^*p_n^*}$ reads, for suitable ρ_j:

$$\frac{1}{\lambda_n}\frac{tp_n}{t^*p_n^*}\frac{p_j^*}{t} = \frac{1}{\lambda_j}\frac{p_j}{t^*} + \frac{\rho_j}{p_n^*} \tag{6}$$

The same idea is behind the next theorem:

Theorem 3.2 *Let the Schmidt vectors of H_{ϕ_k} be $\frac{\alpha_k}{tu_k^*}$ and its signed singular values $\mu_i, (\mu_1 = \mu_2 \ldots = \mu_{2k-1} > \mu_{2k} > \ldots > \mu_{n+k-1}$; Then $H_{\frac{\alpha_{k+n-1}^*}{\alpha_{k+n-1}}\frac{tu_k^*}{t^*u_k}}$ has signed singular values $\bar{\mu}_1 = \mu_{k+n-1}, \bar{\mu}_2 = \mu_{k+n-1}, \ldots, \bar{\mu}_{n-k} = \ldots = \bar{\mu}_{k+n-1} = \mu_1^{-1}$ and Schmidt vectors $\frac{\alpha_i^*}{tu_k^*}$*

Proof. Write the fundamental equation for φ_k and $i > k$

$$\varphi_k \frac{\alpha_i}{tu_k^*} = \frac{s_k^* u_k^* t}{s_k u_k t^*}\frac{\alpha_i}{tu_k^*} = \mu_i \frac{\alpha_i^*}{t^* u_k} + \frac{\beta_i}{s_k}$$

yielding

$$\varphi_k = \mu_n \frac{\alpha_{k+n-1}^* u_k^* t}{\alpha_{k+n-1} u_k t^*} + \frac{\beta_{n+k-1}}{s_k}\frac{u_k^* t}{\alpha_{n+k-1}}$$

applying the above to $\frac{\alpha_i}{u_k^* t}$ we obtain

$$\mu_n \frac{\alpha^*_{n+k-1} u^*_k t}{\alpha_{n+k-1} u_k t^*} \frac{\alpha_i}{u^*_k t} + \frac{\beta_{n+k-1}}{s_k} \frac{u^*_k t}{\alpha_{n+k-1}} \frac{\alpha_i}{u^*_k t} =$$

$$= \mu_i \frac{\alpha^*_i}{u_k t^*} + \frac{\beta_i}{s_k}$$

multiplication by $\frac{\alpha_{n+k-1}}{\alpha^*_{n+k-1}}$ gives

$$\mu_i \frac{\alpha_{n+k-1}}{\alpha^*_{n+k-1}} \frac{u^*_k t}{u_k t^*} \frac{\alpha^*_i}{u^*_k t} = \mu_n \frac{\alpha_i}{u_k t^*} + \frac{\beta_{n+k-1}\alpha_i}{s_k \alpha^*_{n+k-1}} - \frac{\alpha_{n+k-1}\beta^*_i}{\alpha^*_{n+k-1} s_k}$$

Division by $\mu_i \mu_n$ yields the result. ■

Another result of [4] which can be easily obtained with this approach is the relation between the Schmidt vectors of the operator $H_{\frac{1}{\lambda_n}\frac{tp^*_1}{t^* p_1}}$ and those of its Hankel-norm approximant of degree $n-1$. The original proof of this result is quite long and involved, and so we present this simpler derivation.

Lemma 3.1 ([4]) *$H_{\frac{1}{\lambda_n}\frac{tp^*_1}{t^* p_1}}$ has signed singular values equal to the first $(n-1)$ signed singular values of H_{φ_1} and its Schmidt vectors are $\lambda_n \frac{\rho^*_j}{p^*_n}$, where the ρ_j are as in (6)*

Proof: From (6) we get

$$\frac{p^*_j}{t^*} = \lambda_n \frac{\rho_j}{p_n} + \frac{\lambda_n}{\lambda_j} \frac{p_j}{t^*} \frac{p^*_n}{p_n} \tag{7}$$

From (5) the $n-1$ approximant has symbol:

$$\frac{p_n t}{\pi_1 p_n} = \lambda_1 \frac{tp^*_1}{t^* p_1} - \lambda_n \frac{tp^*_n}{t^* p_n}$$

Combining the two equalities together:

$$\begin{aligned} &\frac{\pi_n t}{p_1 p_n} \left(\lambda_n \frac{\rho^*_j}{p^*_n} + \frac{\lambda_n}{\lambda_j} \frac{p^*_j}{t} \frac{p_n}{p^*_n} \right) \\ &= \frac{\pi_n t}{p_1 p_n} \frac{p_j}{t} = \left(\lambda_1 \frac{tp^*_1}{t^* p_1} - \lambda_n \frac{tp^*_n}{t^* p_n} \right) \frac{p_j}{t} \\ &= \lambda_j \frac{p^*_j}{t^*} + \frac{\pi_j}{p_1} - \lambda_n \frac{p^*_n p_j}{p_n t^*} \\ &= \lambda_j \lambda_n \frac{\rho_j}{p_n} + \frac{\pi_j}{p_1} \end{aligned}$$

Reordering

$$\frac{\pi_n t}{p_1 p_n} \frac{\rho^*_j}{p^*_n} = \lambda_j \frac{\rho_j}{p_n} + \frac{1}{\lambda_n} \frac{\pi_j}{p_1} - \frac{1}{\lambda_j} \frac{\pi_n p^*_j}{p_1 p^*_n}$$

and this is precisely the claim. Observe that $\lambda_n \rho_1^* = \pi_n$. ■

It should be noticed that (7) actually means that the Schmidt vectors of the approximant are obtained projecting the Schmidt vectors of H_{φ_1} onto the state space of the approximant, $\left(\frac{p_n}{p_n^*} H_+^2\right)^{\perp}$.

Duality. From the above result we obtain a duality structure for a Schmidt vector similar to the one presented in [4].
Let

$$\varphi_k = \frac{p_k^*}{p_k}\frac{t}{t^*} \qquad \tilde{\varphi}_k = \frac{p_k t}{p_k^* t^*}$$

$$\varphi_{k,n} = \frac{\alpha_{n+k-1}}{\alpha_{n+k-1}^*}\frac{u_k^* t}{u t^*} \qquad \tilde{\varphi}_{k,n} = \frac{\gamma_{2n-k-1} s_k^* t^*}{\gamma_{2n-k-1}^* s_k t}$$

where $\frac{\alpha_{n+k-1}}{u_k t^*}$ is the last Schmidt vector of φ_k and $\frac{\gamma_{2n-k-1}}{s_k^* t^*}$ is the last Schmidt vector of $\tilde{\varphi}_k$. Then the above results can be summarized as follows:

$$\sigma_j(\varphi_k) = \sigma_{n+k-j}^{-1}(\varphi_{k,n})$$

$$\sigma_j(\tilde{\varphi}_k) = \sigma_{2n-k-j+1}^{-1}(\tilde{\varphi}_{k,n})$$

It is known that the first (outer) Schmidt vector determines the whole Hankel operator (see [1]) up to a constant factor. Is this true for any Schmidt vector? The answer is affirmative.

Lemma 3.2 *Let $\frac{p_k}{t}$ be a stable and strictly proper rational function of degree n with $k-1$ antistable zeros. Then there is a unique (up to a constant factor) Hankel operator of degree n with $\frac{p_k}{t}$ as its k-th Schmidt vector.*

Proof. We can fix arbitrary λ_k. Then we want to find a real λ_1 and a stable polynomial p_1 such that.

$$\lambda_1 \frac{p_1^*}{p_1}\frac{t}{t^*} = \lambda_k \frac{p_k^* t}{p_k t^*} + \frac{\pi_k t}{p_1 p_k}$$

Multiplying by $\frac{t^*}{t}\lambda_k^{-1}$ we obtain

$$\frac{\lambda_1}{\lambda_k}\frac{p_1^*}{p_1} = \frac{p_k^*}{p_k} + \frac{1}{\lambda_k}\frac{\pi_k t^*}{p_1 p_k}.$$

Computing in the zeros z_i of t^*, we get

$$\frac{\lambda_1}{\lambda_k}\frac{p_1^*}{p_1}(z_i) = \frac{p_k^*}{p_k}(z_i) \quad i = 1, \ldots, n$$

Since t^* is antistable, this is a Pick-Nevanlinna problem in the right half-plane, and it admits a unique solution such that $|\lambda_1|$ is minimal. Thus the conclusion. ∎

4 Schmidt pairs and robust controller

Let now a rational plant e/d be given. We want to study the relations between two Hankel operators which can be canonically associated to the plant and are connected to its optimallty robust stabilizing controller. We say that $(e/t, d/t)$ is a coprime factorization (in H^2_+)of e/d if (e,t) and (d,t) are coprime and t is stable, and $(e/t, d/t)$ is a *normalized* coprime factorizations (in H^2_+) of e/d if also

$$e^*e + d^*d = t^*t \tag{8}$$

holds. Given a coprime factorization $(e/t, d/t)$ of a plant, the coprime factorization $(u/p_1, v/p_1)$ of a controller u/v is said to satisfy the Bezout equation if

$$\frac{d}{t}\frac{v}{p_1} - \frac{e}{t}\frac{u}{p_1} = 1 \tag{9}$$

It is well known (see e.g. [3]) that a controller satisfies the Bezout equation (9) for some coprime factorization if and only if it is internally stabilizing for e/d. We will define now a particular Hankel operator using the following result from [5] :

Lemma 4.1 *Let $(e/t, d/t)$ be a normalized coprime factorizations of e/d, and let $(u/p_1, v/p_1)$ satisfy the Bezout equation (9). Then the strictly proper antistable part of*

$$R^* = \frac{d^*}{t^*}\frac{u}{p_1} + \frac{e^*}{t^*}\frac{v}{p_1}$$

is independent of the choice of u/p_1 and v/p_1.

Therefore also the hankel operator H_{R^*} is independent of the stabilizing controller. In the sequel it will be assumed, as in the general case, that the singular values of H_{R^*} are distict. It is clear that in this case the first Schmidt vector is p_1/t.

Lemma 4.2 *Suppose $(e/t, d/t)$ is a normalized coprime factorization of e/d. Then there exists a unique pair $\frac{u}{p_1}, \frac{v}{p_1} \in H^\infty_+$ and $\lambda \in \mathbb{R}$ such that the Bezout equation (9) is satisfied and*

$$R^* = \frac{d^*}{t^*}\frac{u}{p_1} + \frac{e^*}{t^*}\frac{v}{p_1} = \lambda\frac{tp_1^*}{t^*p_1} \tag{10}$$

i.e. $\lambda^{-1}R^*$ is all-pass of minimal *degree, and $|\lambda|$ is minimal.*

Proof: we could invoke the above lemma, but there is a also the following constructive argument. We seek a solution of

$$dv - eu = tp_1$$

$$d^*u + e^*v = \lambda t p_1^*$$

Multiplying by λp_1^* and p_1 respectively, and regrouping coefficients, we get

$$(\lambda d p_1^* - e^* p_1)v = (d^* p_1 + \lambda e p_1^*)u \tag{11}$$

If we assume v monic, then we obtain $2n$ equations and $2n$ unknowns. We need to check that the above expressions have no common zeros. But rewriting (11) as $\alpha v = \beta u$ we get $\alpha\alpha^* + \beta\beta^* = (1+\lambda^2)p_1 p_1^*$, and therefore α and β are coprime. ■

The controller u/v obtained above is called optimally robust stabilizing controller for g, and it is very relevant, in view of the following well known result by Vidyasagar and Kimura.

Theorem 4.1 ([9]) *Let $g = e/d$ have a normalized coprime factorization like in (8). Let $\mathcal{B}(g,\epsilon) = \{g' : g' = (\frac{d}{t} + \Delta_{\frac{d}{t}})^{-1}(\frac{e}{t} + \Delta_{\frac{e}{t}})\}$ for some pair $(\Delta_{\frac{d}{t}}, \Delta_{\frac{e}{t}})$ such that $\|\Delta_{\frac{d}{t}}, \Delta_{\frac{e}{t}}\|_\infty < \epsilon$ Then k stabilizes all g' in $\mathcal{B}(g,\epsilon)$ if and only if it has a factorization $(\frac{u}{p_1 1}, \frac{v}{p_1 1})$ such that*

a) $\left\| \begin{matrix} \frac{u}{p_1 1} \\ \frac{v}{p_1 1} \end{matrix} \right\|_\infty \leq \frac{1}{\epsilon}$

b) the Bezout equation $\frac{d}{t}\frac{v}{p_1 1} - \frac{e}{t}\frac{u}{p_1 1} = I$ is satisfyed.

The value ϵ depends only on the factorized controller: it can be shown ([9]) that there is a maximal ϵ_{opt} and it is achieved precisely by the controller defined in lemma (4.2). The next result makes precise a connection which was first observed in [5], where it is shown that H_{R^*} and $H_{\left[\begin{smallmatrix} d^*/t^* \\ e^*/t^* \end{smallmatrix}\right]}$ share the same stable Schmidt vectors: the representation of the unstable vectors seems to be new.

Theorem 4.2 *let $(u/p_1, v/p_1)$ be as in lemma 4.2, and let λ_i and p_i/t be the singular values and Schmidt vectors of H_{R^*}. Then the Hankel operator with symbol $\begin{bmatrix} d^*/t^* \\ e^*/t^* \end{bmatrix}$ has signed singular values $\sigma_i = \frac{\lambda_i}{\sqrt{1+\lambda_i^2}}$ and Schmidt vectors*

$$\left\{ \frac{p_i}{t}, \frac{1}{\sqrt{1+\lambda_i^2}} \begin{bmatrix} \frac{u^*}{p_1^*} \\ \frac{v^*}{p_1^*} \end{bmatrix} \frac{p_i^*}{t^*} - \begin{bmatrix} \frac{d^*}{t^*} \\ \frac{e^*}{t^*} \end{bmatrix} \frac{\pi_i^*}{p_1^*} \right\}$$

Proof: in view of (4), (8), (9) and (10), we can write

$$\begin{bmatrix} \frac{d}{t} & \frac{e}{t} \\ -\frac{e^*}{t^*} & \frac{d^*}{t^*} \end{bmatrix} \begin{bmatrix} \frac{u^*}{p_1^*} \\ \frac{v^*}{p_1^*} \end{bmatrix} \frac{p_i^*}{t^*} = \begin{bmatrix} R \\ I \end{bmatrix} \frac{p_i^*}{t^*} = \begin{bmatrix} \lambda_i \frac{p_i}{t} + \frac{\pi_i^*}{p_1^*} \\ \frac{p_i^*}{t^*} \end{bmatrix}$$

and

$$\begin{bmatrix} \frac{d}{t} & \frac{e}{t} \\ -\frac{e^*}{t^*} & \frac{d^*}{t^*} \end{bmatrix} \begin{bmatrix} \frac{d^*}{t^*} - \frac{v}{p_1} \\ \frac{e^*}{t^*} + \frac{u}{p_1} \end{bmatrix} \frac{p_i}{t} = \begin{bmatrix} 0 \\ R^* \end{bmatrix} \frac{p_i}{t} = \begin{bmatrix} 0 \\ \lambda_i \frac{p_i^*}{t^*} + \frac{\pi_i}{p_1} \end{bmatrix}$$

Multiply on both sides by $\begin{bmatrix} \frac{d^*}{t^*} & -\frac{e}{t} \\ \frac{e^*}{t^*} & \frac{d}{t} \end{bmatrix}$ and subtract from the second equation the first multiplied by λ_i:

$$\begin{bmatrix} \frac{d^*}{t^*} - \frac{v}{p_1} \\ \frac{e^*}{t^*} + \frac{u}{p_1} \end{bmatrix} \frac{p_i}{t} - \lambda_i \begin{bmatrix} \frac{u^*}{p_1^*} \\ \frac{v^*}{p_1^*} \end{bmatrix} \frac{p_i^*}{t^*} = \begin{bmatrix} \frac{d^*}{t^*} & -\frac{e}{t} \\ \frac{e^*}{t^*} & \frac{d}{t} \end{bmatrix} \begin{bmatrix} -\lambda_i^2 \frac{p_i}{t} - \lambda_i \frac{\pi_i^*}{p_1^*} \\ \frac{\pi_i}{p_1} \end{bmatrix}$$

or

$$\begin{bmatrix} \frac{d^*}{t^*} \\ \frac{e^*}{t^*} \end{bmatrix} \frac{p_i}{t}(1+\lambda_i^2) = \lambda_i \begin{bmatrix} \frac{u^*}{p_1^*} \\ \frac{v^*}{p_1^*} \end{bmatrix} \frac{p_i^*}{t^*} - \lambda_i \begin{bmatrix} \frac{d^*}{t^*} \\ \frac{e^*}{t^*} \end{bmatrix} \frac{\pi_i^*}{p_1^*} + \begin{bmatrix} \frac{v}{p_1} \\ -\frac{u}{p_1} \end{bmatrix} \frac{p_i}{t} + \begin{bmatrix} -\frac{e}{t} \\ \frac{d}{t} \end{bmatrix} \frac{\pi_i}{p_1} \tag{12}$$

where the first two terms in the second member are in H^2_- and the other two are in H^2_+. Hence we have shown that

$$H_{\begin{bmatrix} \frac{d^*}{t^*} \\ \frac{e^*}{t^*} \end{bmatrix}} \frac{p_i}{t} = \frac{\lambda_i}{1+\lambda_i^2} \left(\begin{bmatrix} \frac{u^*}{p_1^*} \\ \frac{v^*}{p_1^*} \end{bmatrix} \frac{p_i^*}{t^*} - \begin{bmatrix} \frac{d^*}{t^*} \\ \frac{e^*}{t^*} \end{bmatrix} \frac{\pi_i^*}{p_1^*} \right)$$

To conclude the claim, observe that

$$\left[\frac{d}{t}, \frac{e}{t}\right] \left(\begin{bmatrix} \frac{u^*}{p_1^*} \\ \frac{v^*}{p_1^*} \end{bmatrix} \frac{p_i^*}{t^*} - \begin{bmatrix} \frac{d^*}{t^*} \\ \frac{e^*}{t^*} \end{bmatrix} \frac{\pi_i^*}{p_1^*} \right) = R \frac{p_i^*}{t^*} - \frac{\pi_i^*}{p_1^*} = \lambda_i \frac{p_i}{t}$$

and thus

$$H_{\left[\frac{d}{t}, \frac{e}{t}\right]} \frac{1}{\sqrt{1+\lambda_i^2}} \left(\begin{bmatrix} \frac{u^*}{p_1^*} \\ \frac{v^*}{p_1^*} \end{bmatrix} \frac{p_i^*}{t^*} - \begin{bmatrix} \frac{d^*}{t^*} \\ \frac{e^*}{t^*} \end{bmatrix} \frac{\pi_i^*}{p_1^*} \right) = \frac{\lambda_i}{\sqrt{1+\lambda_i^2}} \frac{p_i}{t}$$

which exactly what we wanted. ■

The obvious dual of the previous result, with a similar proof, is the following:

Theorem 4.3 *The Hankel operator* $H_{\left[\frac{d^*}{t^*},\frac{e^*}{t^*}\right]}$ *has Schmidt pairs*

$$\left\{\frac{1}{\sqrt{1+\lambda_i^2}}\left(\begin{bmatrix}\frac{u}{p_1}\\ \frac{v}{p_1}\end{bmatrix}\frac{p_i}{t}-\begin{bmatrix}\frac{d}{t}\\ \frac{e}{t}\end{bmatrix}\frac{\pi_i}{p_1}\right),\frac{p_i^*}{t^*}\right\}$$

with signed singular values $\sigma_i=\dfrac{\lambda_i}{\sqrt{1+\lambda_i^2}}$.

The relevance of the above results lies in the possibility of characterizing completely the operator $H_{\begin{bmatrix}d^*/t^*\\ e^*/t^*\end{bmatrix}}$ in terms of the scalar operator H_{R^*}, whose structure is much simpler to study. An example is the next result:

Corollary 4.1 *The Nehari extension of* $\begin{bmatrix}d^*/t^*\\ e^*/t^*\end{bmatrix}$ *is*

$$\begin{bmatrix}u^*/t^*\\ v^*/t^*\end{bmatrix}\frac{t}{p}\frac{\lambda_1}{1+\lambda_1^2}=\begin{bmatrix}\frac{d^*}{t^*}-\frac{1}{1+\lambda_1^2}\frac{v}{p_1}\\ \frac{e^*}{t^*}+\frac{1}{1+\lambda_1^2}\frac{u}{p_1}\end{bmatrix} \tag{13}$$

Proof: it is sufficient to recall that, if (ξ,η) is the first Schmidt pair of a Hankel operator and ξ is scalar, then the Nehari extension is given by $\Phi=\eta\xi^{-1}$. The second equality is otained then rearranging the terms in (12) ∎

Other relations can be derived from Theorem 4.2:

Corollary 4.2 *The following equalities hold*

$$(1+\lambda^2)d^*p_1=\lambda u^*t+vt^* \tag{14}$$
$$(1+\lambda^2)e^*p_1=\lambda v^*t-ut^* \tag{15}$$

Moreover

$$u^*u+v^*v=(1+\lambda^2)p_1^*p_1 \tag{16}$$

Proof: the only thing to show is that (16) holds. But mupltiplying each side of (14) and (15) by its conjugate and adding each side, we get, in view of (8):

$$\begin{aligned}(1+\lambda^2)^2p_1^*p_1t^*t &= \lambda^2u^*tut^*+\lambda(u^*tv^*t+ut^*vt^*)+vt^*v^*t\\ &+ \lambda^2v^*tvt^*-\lambda(v^*tu^*t+vt^*ut^*)+ut^*u^*t\\ &= (1+\lambda^2)t^*t(u^*u+v^*v)\end{aligned}$$ ∎

The above equalities are used to show that the map from $e/t, d/t$ to R^* is invertible.

Theorem 4.4 *For any strictly proper rational function $p_k/t \in H_{+}^{\infty}$ with $\mathrm{k}-1$ stable zeros and $\lambda_k \in \mathbb{R}$ there exists a unique Hankel operator with a 2×1 inner symbol $\begin{bmatrix} d/t \\ e/t \end{bmatrix}$ such that its $\mathrm{k}-\mathrm{th}$ singular value and Schmidt vector are precisely λ_k and p_k/t.*

Proof: in view of lemma 3.2 and theorem 4.2 we can restrict the proof to p_1. We are looking for functions d, e, u, v such that (14) and (15) are satisfied. We will in fact work with the cojugates:

$$(1+\lambda^2)dp_1^* = \lambda t^* u + tv^*$$

$$(1+\lambda^2)ep_1^* = \lambda t^* v - tu^*$$

. To simplify notation, in this proof we work with $d_1 = (1+\lambda^2)d$ and $e_1 = (1+\lambda^2)e$. We know that there exist unique coprime polynomials (u_1, v_1^*) such that

$$p_1^* = \lambda t^* u_1 + tv_1^* \tag{17}$$

Then, for any d_1 we have $d_1 p_1^* = \lambda t^* d_1 u_1 + t d_1 v_1^*$, and thus

$$d_1 p_1^* = (du_1 - st)t^*\lambda + (dv_1^* + \lambda st^*)t$$

. We can choose s such that $\deg u < \deg t$

$$d_1 u_1 - st = u \qquad\qquad d_1^* v_1 + \lambda s^* t = v \tag{18}$$

and clearly, since $d_1 p_1^* = ut^*\lambda + vt$, it follows that also $\deg v < \deg t$. Multiplying (17) by e_1 we get likewise:

$$e_1 p_1^* = (e_1 v_1^* - \lambda r t^*)t + (e_1 u_1 + rt)\lambda t^*$$

and thus

$$u = -e_1^* v_1 + \lambda r^* t \qquad\qquad v = e_1 u_1 + rt \tag{19}$$

Equating (18) and (19), we get

$$u_1 d_1 - st = -v_1 e_1^* + \lambda r^* t \qquad\qquad v_1 d_1^* + \lambda s^* t = u_1 e_1 + rt$$

and eventually:

$$\begin{bmatrix} u_1 & v_1 \\ v_1^* & -u_1^* \end{bmatrix} \begin{bmatrix} d_1 \\ e_1^* \end{bmatrix} = \begin{bmatrix} st + \lambda r^* t \\ -\lambda st^* + r^* t^* \end{bmatrix}$$

The matrix on the left is nonsingular: in fact its determinant is $-u_1 u_1^* - v_1 v_1^*$, which cannot identically vanish. Inversion and division by $(1+\lambda^2)$ will yield d and e. ■

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