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Abstract

This paper discusses some axioms from the literature which have been used to define properties of timed transition systems. The *additivity* axiom proposed by (amongst others) Wang, and Nicollin and Sifakis is compared with the *trajectory* axiom of Lynch and Vaandrager. Some conditions for an additive transition system to be trajectoried are discussed. These are proved sufficient by using some simple terminology from category theory to show how this problem about timed transition systems can be turned into an equivalent problem about monotone functions on partially ordered sets. We also discuss *trajectory (bi)simulation*, which is a variant of Ho-Stuart's *path bisimulation*, and use similar techniques to discuss when (bi)simulation is equivalent to trajectory (bi)simulation.

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1. Introduction

Timed transition systems [MoT90, Wan91] are commonly used in the specification of time-critical systems. They describe a system in terms of the states it can be in, and circumstances under which it can move from one state to the next, either by allowing time to pass or by performing some transition. There are a wide variety of timed transition systems, which are broadly similar but which are presented with different axioms or properties.

In this paper we compare two axioms concerned with the passage of time. One states that if a process allows an interval of time to pass to evolve from one state to another, then there must be an intermediate state at any point along that interval; the other says that there must be a trajectory of consistent states through the interval. The first of these is more commonly given; the second was first introduced in [VaL92]. Although the second is strictly stronger than the first, we explore conditions where they are equivalent. In fact, both properties hold for every timed transition system the authors have encountered in practice.

The notion of a trajectory naturally induces associated forms of simulation and bisimulation, in which it is required that each trajectory from a given state can be simulated by a trajectory from any related state such that intermediate states on the trajectories are related. We explore conditions under which (bi)simulation and trajectory (bi)simulation are equivalent.

The structure of the paper is as follows: the two axioms of timed transition systems are defined; their presence in a transition system is shown to correspond to particular properties of monotone functions between partially ordered sets, so we investigate conditions under which these properties both hold, and translate these back to conditions on transition systems. Two forms of (bi)simulation analogous to the two axioms are then considered, and conditions for their equivalence are obtained by again considering an equivalent problem in a partial order setting.

2. Additivity axioms

In this section, we will define two additivity axioms for timed transition systems, and show why we would like to know when they are equivalent. We can translate the problem of equivalence of these axioms into a problem about monotone functions on partially ordered sets (posets). Finding a solution to the poset problem allows us to find sufficient conditions for the two axioms for transition systems to be equivalent.

2.1 A problem about posets

A *poset* is a set X with a partial ordering relation $\leq$. Given a poset X, define:

- $X' \subseteq X$ is a *chain* (or is *total*) iff $\forall x, y \in X' \,.\, x \leq y$ or $y \leq x$.
- A chain $X' \subseteq X$ is *from* x iff $x \in X'$ and $\forall y \in X' \,.\, x \leq y$.
- A chain $X' \subseteq X$ is *to* x iff $x \in X'$ and $\forall y \in X' \,.\, y \leq x$.
- Given chains $X', X'' \subseteq X$, $X' < X''$ iff $\forall x' \in X', x'' \in X'' \,.\, x' < x''$.
- Given $x, y \in X$, the *interval* from x to y is $[x, y] = \{z \in X \mid x \leq z \leq y\}$.

Given posets X and Y, and a monotone map $f : X \to Y$, define:

- If $X' \subseteq X$ then $f[X'] = \{fx \mid x \in X'\}$.
- f is *finite-to-one* iff for any $y \in Y$ the set $\{x \mid fx = y\}$ is finite.

- f is *onto* iff for any $y \in Y$ there is an $x \in X$ such that $fx = y$.
- f is *chained* iff for any chain Y' in Y there is a chain X' in X such that $f[X'] = Y'$.
- f is *interval-ϕ* iff for all $x \leq y \in X$, $f : [x, y] \to [fx, fy]$ is ϕ.

Any chained function is onto, and so any interval-chained function is interval-onto. The question we would like to consider is this: under what conditions are interval-onto functions interval-chained?

2.2 Time domains

A *monoid* is a set T with an associative operator $+$ with unit 0. Given a monoid $(T, +, 0)$, define:

- T is *left-cancellative* iff $t + u = t + v \Rightarrow u = v$.
- T is *anti-symmetric* iff $t + u = 0 \Rightarrow t = u = 0$.

In this paper, a *time domain* is a left-cancellative anti-symmetric monoid. Examples of time domains include:

- The singleton set $(\mathbf{1}, +, 0)$.
- The natural numbers $(\mathbf{N}, +, 0)$.
- The non-negative rationals $(\mathbf{Q}^+, +, 0)$.
- The non-negative reals $(\mathbf{R}^+, +, 0)$.
- The countable ordinals $(\omega_1, +, 0)$.
- Strings with concatenation $(\Sigma^*, \cdot, \epsilon)$.

Given a time domain $(T, +, 0)$ and a set Σ we define the set of timed strings $TS(T, \Sigma)$ as $(T\Sigma)^* T$, with the concatenation operation $\bullet$ on timed strings given by:

$$(\sigma \cdot t) \bullet (u \cdot \rho) \quad = \quad \sigma \cdot (t + u) \cdot \rho$$

we obtain another example of a time domain:

- Timed strings with concatenation $(TS(T, \Sigma), \bullet, 0)$.

If $\tau \in \Sigma$ then the strings are termed *strong* strings, otherwise they are *weak*.

Each time domain $(T, +, 0)$ induces a binary order of *precedence*:

$$t \leq u \quad \Leftrightarrow \quad \exists v \,.\, t + v = u$$

Note that the ordering on timed strings induced in this way is not just the prefix order on strings, for example $1 \leq 2 \bullet 1$, but $1 \not\leq 2 \cdot 1$.

Proposition 1 *Let $(T, +, 0)$ be a time domain with precedence relation $\leq$. Then:*

1. *$\leq$ is a partial ordering with unique minimal element* 0.
2. *Relation $\leq$ has no maximal elements, unless T is the trivial one-point time domain.*

Proof. The proof of (1) is routine. For (2), suppose $(T,+,0)$ has a maximal element t. Then $t+t=t$, otherwise t would not be maximal. But this implies that $t+t=t+0$, and hence, since T is left-cancellative, $t=0$. Thus T is the one-point time domain. □

As a consequence all nontrivial time domains are infinite. When we talk about chains, intervals, etc. of a time domain, this will always be with respect to its precedence relation. We define a subtraction operator on times related by $\leq$:

- $t-u=v$ iff $u+v=t$

Then $t-u$ is well-defined when $u \leq t$ by left-cancellativity; and is undefined otherwise.

We want to stress that the notion of time domain that we propose here has been chosen to suit the purposes of this paper, and that there are many other definitions occurring in the literature (see [Ben89] for a series of examples). In all definitions we know of the set of time points is equipped with a partial order of precedence, but there are plenty of examples where the precedence relation does not have a minimal element, and there are also nontrivial examples of time domains with a maximal element. Hehner [Heh93], for instance, reduces time to a single bit that distinguishes between finite and infinite execution time of a program.

In this paper, we view time as an attribute of transitions rather than of states, so we think of the elements of our time domains as durations. As a consequence it becomes natural to equip the set of time points with a monoidal structure.

2.3 A problem about timed transition systems

A *timed transition system* $(N, T, \longrightarrow)$ consists of a set N of *states*, a time domain T, and a set $\longrightarrow \subseteq N \times T \times N$ of *transitions* such that:

- $p \xrightarrow{0} p$.
- If $p \xrightarrow{0} q \xrightarrow{0} p$ then $p=q$.
- If $p \xrightarrow{t}\xrightarrow{u} q$ then $p \xrightarrow{t+u} q$.

If we interpret $p \xrightarrow{t} q$ as the statement that it is possible to go from state p to state q in time t, then the first and the last conditions are obvious: if the system is in state p and 0 time elapses then it is still possible to be in state p; and if it is possible to go from p to r in time t, and from r to q in time u, then it is possible to go from p to q in time $t+u$. The middle axiom says that we are viewing states up to an equivalence class, where any states in a tight loop $p \xrightarrow{0} q \xrightarrow{0} p$ are identified.

Examples of timed transition systems include:

- The transition system $(CCS, \Sigma_\tau^*, \longrightarrow)$ of CCS [Mil89].

- The weak transition system $([CCS]_\equiv, \Sigma^*, \Longrightarrow)$ of CCS, where we consider CCS expressions up to the equivalence given by $p \equiv q$ iff $p \stackrel{\epsilon}{\Longrightarrow} q \stackrel{\epsilon}{\Longrightarrow} p$.
- The transition system $(tCCS, TS(\Sigma_\tau, T), \longrightarrow)$ of timed CCS.
- The weak transition system $([tCCS]_\equiv, TS(\Sigma, T), \Longrightarrow)$ of timed CCS.

In fact, the reflexive transitive closure of any conventional (timed or untimed) transition system may be considered as a timed transition system, by considering states up to the equivalence $\equiv$.

In the process algebra community people often tend to think of time as something "new" that has to be added to the classical "untimed" theories. We rather like the view that also the classical process algebras are timed, only with a more abstract notion of time. Timed transition systems offer a uniform semantical basis for describing both classical "untimed" process algebras, and the recent "timed" process algebras. The results of this paper however will primarily be of interest for algebras that aim at describing real-time.

Given a timed transition system $(N, T, \longrightarrow)$:

- A *trajectory* through $T' \subseteq T$ is a vector $\vec{p} = \langle p_t \mid t \in T' \rangle$ such that $\forall t, t+u \in T' \,.\, p_t \xrightarrow{u} p_{t+u}$.
- A trajectory $\vec{p}$ through a chain T' from 0 to t is *from* p_0 *to* p_t.
- $(N, T, \longrightarrow)$ is *additive* iff, whenever $p \xrightarrow{t+u} q$ then $p \xrightarrow{t}\xrightarrow{u} q$.
- $(N, T, \longrightarrow)$ is *trajectoried* iff, whenever $p \xrightarrow{t} q$, $t \neq 0$, and T' is a chain from 0 to t, then there is a trajectory through T' from p to q.

Any trajectoried timed transition system is additive. The question we would like to consider is this: under what conditions are additive timed transition systems trajectoried?

2.4 Why the timed transition system problem is interesting

Additivity has been considered by many authors [Ho+92, Jef91, NiS90, Sch92, Wan91] as essential to modeling the behaviour of timed systems. The more powerful trajectory axiom—that a timed transition system must be trajectoried—was developed by Vaandrager and Lynch [VaL92] to reason about system behaviour when it is necessary to reason about the computation that resulted in a particular behaviour.

For example, given a transition system $(N, TS(\Sigma, \mathbf{R}^+), \longrightarrow)$, we can define a trajectory $\vec{p}$ through $[s, s']$ to be *a-maximal* iff $s \leq s'' < (s'' \bullet t) < s'$ for some $t \in \mathbf{R}^+$ implies $p_{s''} \stackrel{a}{\not\longrightarrow}$. This states that time ($\mathbf{R}^+$) can progress only when event a is not possible. Then we can define the operational semantics of the timed CSP [Sch92] hiding operator as $p \setminus a \xrightarrow{s \setminus a} p' \setminus a$ iff $p \xrightarrow{s} p'$ and there is an a-maximal trajectory through $[0, s]$ from p to p'. This definition only makes sense when the transition system is trajectoried.

There are other examples of the usefulness of the trajectory axiom in [LyV93].

Although being trajectoried is a useful property to have of a timed transition system, it is very difficult to prove, since it relies on proving properties of infinite computations. The additivity property is much simpler to prove, but is not as powerful. Therefore we would like to know some sufficient conditions for additivity to imply trajectoried, to keep the simplicity of proving additivity and the power of having proved trajectoried.

These conditions are not always equivalent, for example in the transition system $(\mathbf{Q}, \mathbf{R}^+, \longrightarrow)$ given by:

$$\frac{}{p \xrightarrow{0} p} \qquad \frac{p < q \quad x > 0}{p \xrightarrow{x} q}$$

This transition system is additive but not trajectoried, as this would imply there was an embedding of the real interval $[0, 1]$ into the rational interval $[0, 1]$, which is impossible for cardinality reasons. However, we will show that all of the timed transition systems known to the authors are trajectoried.

2.5 A categorical view of timed transition systems

A (*small*) *category* $\mathbf{C}$ is:

- A set of *objects*.
- A set of *arrows*, where each arrow f has an object *domain* A and *codomain* B, written '$A \xrightarrow{f} B$ in $\mathbf{C}$'.
- Identity arrows $A \xrightarrow{1_A} A$ in $\mathbf{C}$ for each object A.
- Composite arrows $A \xrightarrow{f;g} C$ in $\mathbf{C}$ for each $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbf{C}$, where ; is associative, with unit 1.

In analogy with the monoid case, define

- $\mathbf{C}$ is *left-cancellative* iff $f; g = f; h \Rightarrow g = h$.
- $\mathbf{C}$ is *anti-symmetric* iff $f; g = 1 \Rightarrow f = g = 1$.

Many common mathematical structures are examples of categories:

- A *monoid* is a category with only one object.
- A *pre-order* is a category where any objects A and B have at most one arrow $A \xrightarrow{f} B$.
- A *poset* is an anti-symmetric pre-order.
- A *timed transition system* is a category with states as objects and transitions $p \xrightarrow{t} q$ as arrows.

A *functor* is a structure-preserving function between categories, so $\mathbf{C} \xrightarrow{F} \mathbf{D}$ iff:

- For any object A in $\mathbf{C}$, there is an object FA in $\mathbf{D}$.
- For any arrow $A \xrightarrow{f} B$ in $\mathbf{C}$, there is an arrow $FA \xrightarrow{Ff} FB$ in $\mathbf{D}$.
- $F1_A = 1_{FA}$ and $F(f; g) = Ff; Fg$.

An *isomorphism* from $\mathbf{C}$ to $\mathbf{D}$ is a functor $\mathbf{C} \xrightarrow{F} \mathbf{D}$ which is a bijection, both on objects and on arrows. A functor F is *faithful* iff whenever $A \xrightarrow{f} B$, $A \xrightarrow{g} B$ and $Ff = Fg$ then $f = g$. Many common functions are examples of functors:

- If $\mathbf{C}$ and $\mathbf{D}$ are monoids then $\mathbf{C} \xrightarrow{F} \mathbf{D}$ is a functor iff F is a monoid homomorphism.
- If $\mathbf{C}$ and $\mathbf{D}$ are posets then $\mathbf{C} \xrightarrow{F} \mathbf{D}$ is a functor iff F is a monotone function.

Proposition 2 *A category* $\mathbf{C}$ *is isomorphic to a timed transition system iff* $\mathbf{C}$ *is left-cancellative, anti-symmetric, and there is a time domain* $\mathbf{D}$ *and faithful functor* $\mathbf{C} \xrightarrow{F} \mathbf{D}$.

Proof.

⇒ If there is an isomorphism Θ from $\mathbf{C}$ to a timed transition system $(N, T, \longrightarrow)$, then it follows that $\mathbf{C}$ is left-cancellative and anti-symmetric. Let $\mathbf{D} = T$, and let F be the function that maps each object in $\mathbf{C}$ to the unique object of $\mathbf{D}$ (recall that $\mathbf{D}$ is a monoid), and each arrow f of $\mathbf{C}$ to the label of the transition Θf. It is routine to check that F is a faithful functor from $\mathbf{C}$ to $\mathbf{D}$.

⇐ If $\mathbf{C}$ is left-cancellative and anti-symmetric, there is a faithful functor $\mathbf{C} \xrightarrow{F} \mathbf{D}$, and $\mathbf{D}$ is a time domain, then let N be the set of objects of $\mathbf{C}$, $T = \mathbf{D}$, and $\longrightarrow \subseteq N \times T \times N$ be the relation given by $p \xrightarrow{t} q$ iff $Ff = t$, for some arrow $p \xrightarrow{f} q$ of $\mathbf{C}$. It is routine to show that $(N, T, \longrightarrow)$ is a timed transition system. Let Θ be the function that maps each object of $\mathbf{C}$ to itself, and each arrow $p \xrightarrow{f} q$ of $\mathbf{C}$ to the triple $p \xrightarrow{Ff} q$. Then Θ is an isomorphism from $\mathbf{C}$ to $(N, T, \longrightarrow)$. □

Proposition 2 says that essentially a timed transition system is a faithful functor F from a left-cancellative anti-symmetric category $\mathbf{C}$ to a time domain $\mathbf{D}$.

2.6 Why the timed transition system problem is an example of the poset problem

Given a category $\mathbf{C}$, define the category $\triangle\mathbf{C}$ as:

- Objects are arrows from $\mathbf{C}$.
- Arrows are triples (f, g, h) of arrows in $\mathbf{C}$ with $f; g = h$. The domain of (f, g, h) is f and the codomain h.
- For $A \xrightarrow{f} B$ an object of $\triangle\mathbf{C}$, the identity arrow 1_f is $(f, 1_B, f)$.
- Composition is given by $(f, g, h); (h, k, l) = (f, g; k, l)$

Given a functor $\mathbf{C} \xrightarrow{F} \mathbf{D}$, we can define the functor $\triangle\mathbf{C} \xrightarrow{\triangle F} \triangle\mathbf{D}$ by $\triangle F f = Ff$ and $\triangle F(f, g, h) = (Ff, Fg, Fh)$. (Category theorists will note that $\triangle$ is a functor $\mathbf{Cat} \xrightarrow{\triangle} \mathbf{Cat}$.)

Category $\triangle\mathbf{C}$ is of interest, as it is a poset of the arrows in a left-cancellative anti-symmetric category. For example, for each of the following time domains $\triangle\mathbf{C}$ generates the natural ordering:

- The singleton set $(\mathbf{1}, +, 0)$ has the trivial ordering.
- The natural numbers $(\mathbf{N}, +, 0)$ has the number ordering.
- The non-negative rationals $(\mathbf{Q}^+, +, 0)$ has the number ordering.
- The non-negative reals $(\mathbf{R}^+, +, 0)$ has the number ordering.
- The countable ordinals $(\omega_1, +, 0)$ has the ordinal ordering.
- Strings $(\Sigma^*, \cdot, \epsilon)$ has the prefix ordering.
- Timed strings $(TS(\Sigma, T), \bullet, 0)$ has the timed prefix ordering.

However:

- The two-point domain $(\mathbf{2}, \vee, 0)$ is not left-cancellative, since $1 \vee 0 = 1 \vee 1$ but $0 \neq 1$. Thus $\triangle(\mathbf{2}, \vee, 0)$ is not a poset, since there are two arrows $1 \xrightarrow{(1,0,1)} 1$ and $1 \xrightarrow{(1,1,1)} 1$. In general, the only left-cancellative monoid with a zero is the trivial one-point monoid.

- The integers $(\mathbf{Z}, +, 0)$ are not anti-symmetric, since $1 + -1 = 0$ but $1 \neq 0$. Thus $\triangle(\mathbf{Z}, +, 0)$ is not a poset, since $0 \xrightarrow{(0,1,1)} 1 \xrightarrow{(1,-1,0)} 0$. In general, the only anti-symmetric group is the trivial one-point group.

The following proposition generalizes Proposition 1(1).

Proposition 3 *$\triangle\mathbf{C}$ is a poset iff $\mathbf{C}$ is left-cancellative and anti-symmetric.*

Proof. Routine. □

This means that if $\mathbf{C} \xrightarrow{F} \mathbf{D}$ is a timed transition system, $\triangle\mathbf{C} \xrightarrow{\triangle F} \triangle\mathbf{D}$ is a monotone function between posets, and so we can show that our timed transition system problem is a specific instance of our poset problem.

Proposition 4 *Let $\mathbf{C} \xrightarrow{F} \mathbf{D}$ be a timed transition system. Then*

- *$\mathbf{C} \xrightarrow{F} \mathbf{D}$ is additive iff $\triangle\mathbf{C} \xrightarrow{\triangle F} \triangle\mathbf{D}$ is interval-onto.*
- *$\mathbf{C} \xrightarrow{F} \mathbf{D}$ is trajectoried iff $\triangle\mathbf{C} \xrightarrow{\triangle F} \triangle\mathbf{D}$ is interval-chained.*

Proof.

- Suppose $\mathbf{C} \xrightarrow{F} \mathbf{D}$ is additive. In order to prove that $\triangle\mathbf{C} \xrightarrow{\triangle F} \triangle\mathbf{D}$ is interval-onto, suppose that $\triangle\mathbf{C}$ has an arrow $f \xrightarrow{k} g$, and $\triangle\mathbf{D}$ has arrows $t \xrightarrow{l} v, v \xrightarrow{l'} u$ such that $\triangle Fk = l; l'$. We must prove the existence of arrows $f \xrightarrow{m} j, j \xrightarrow{m'} g$ in $\triangle\mathbf{C}$ such that $k = m; m'$, $\triangle Fm = l$ and $\triangle Fm' = l'$. Since $f \xrightarrow{k} g$ is an arrow of $\triangle\mathbf{C}$, $\mathbf{C}$ has an arrow f' such that $k = (f, f', g)$ and hence $f; f' = g$. Let $Ff' = t'$. Then $u = F(g) = F(f; f') = F(f); F(f') = t; t'$. Since $t \xrightarrow{l} v$ is an arrow of $\triangle\mathbf{D}$, $\mathbf{D}$ has an arrow v' such that $l = (t, v', v)$ and hence $v = t; v'$. Since $v \xrightarrow{l'} u$ is an arrow of $\triangle\mathbf{D}$, $\mathbf{D}$ has an arrow u' such that $l' = (v, u', u)$ and hence $u = v; u'$. Now observe that $t;t' = u = v;u' = (t;v');u' = t;(v';u')$. Since $\mathbf{D}$ is left-cancellative, this implies $t' = v';u'$. Hence $Ff' = t' = v';u'$. Since $\mathbf{C} \xrightarrow{F} \mathbf{D}$ is additive, $\mathbf{C}$ has arrows h, h' such that $f' = h;h'$, $Fh = v'$ and $Fh' = u'$. Define $i = f;h$, $m = (f,h,i)$ and $m' = (i,h',g)$. Then i is an object of $\triangle\mathbf{C}$, and m, m' are arrows of $\triangle\mathbf{C}$ (use $i;h' = (f;h);h' = f;(h;h') = f;f' = g$). We derive $m;m' = (f,h,i);(i,h',g) = (f,h;h',g) = (f,f',g) = k$. Since $Fi = F(f;h) = Ff;Fh = t;v' = v$, $\triangle Fm = (Ff,Fh,Fi) = (t,v',v) = l$ and $\triangle Fm' = (Fi,Fh',Fg) = (v,u',u) = l'$.

- Suppose $\triangle\mathbf{C} \xrightarrow{\triangle F} \triangle\mathbf{D}$ is interval-onto. To show that $\mathbf{C} \xrightarrow{F} \mathbf{D}$ is additive, suppose that $\mathbf{C}$ has an arrow $A \xrightarrow{f} B$ and $\mathbf{D}$ has arrows t, t', such that $Ff = t;t'$. We must prove the existence of arrows g, g' in $\mathbf{C}$ such that $f = g;g'$, $Fg = t$ and $Fg' = t'$. $\triangle\mathbf{C}$ has an arrow $k = (1_A, f, f)$, and $\triangle\mathbf{D}$ has arrows $l = (0,t,t)$ and $l' = (t,t',t;t')$. Since:

$$\triangle Fk = (F1_A, Ff, Ff) = (0, t;t', t;t') = (0,t,t);(t,t',t;t') = l;l'$$

  and $\triangle\mathbf{C} \xrightarrow{\triangle F} \triangle\mathbf{D}$ is interval-onto, $\triangle\mathbf{C}$ has arrows $1_A \xrightarrow{m} g$ and $g \xrightarrow{m'} f$ such that $k = m;m'$, $\triangle Fm = l$ and $\triangle Fm' = l'$. Let $m' = (g,g',f)$. It follows that $f = g;g'$, $Fg = t$ and $Fg' = t'$, as required.

- Suppose $\mathbf{C} \xrightarrow{F} \mathbf{D}$ is trajectoried. To show that $\triangle\mathbf{C} \xrightarrow{\triangle F} \triangle\mathbf{D}$ is interval-chained, suppose that $\triangle\mathbf{C}$ has an arrow $f \xrightarrow{k} g$, $\triangle\mathbf{D}$ has an arrow $t \xrightarrow{l} u$ with $\triangle Fk = l$, and Y is a chain in $\triangle\mathbf{D}$ from t to u. We must prove the existence of a chain X in $\triangle\mathbf{C}$ from f to g such that $F[X] = Y$. Let $k = (f,h,g)$, for some $p \xrightarrow{h} q$, and $l = (t,v,u)$. Then $g = f;h$ and $u = t;v$. Since $t;v = u = F(g) = F(f;h) = Ff;Fh = t;Fh$ and $\mathbf{D}$ is left-cancellative, $Fh = v$. If $v = 0$ then $t = u$ and $Y = \{t\}$. In this case $X = \{f,g\}$ gives the required chain in $\triangle\mathbf{C}$. So assume that $v \neq 0$. Let Z be the collection of arrows u of $\mathbf{D}$ for which $t;u$ is in Y. Then Z is a chain of $\mathbf{D}$ from 0 to v. Now we use that $\mathbf{C} \xrightarrow{F} \mathbf{D}$ is trajectoried: there is a function τ that associates to each element of Z an object of $\mathbf{C}$ such that $\tau(0) = p$, $\tau(t) = q$, and, for all $w, w;w'$ in Z, there is an arrow $\tau(w) \xrightarrow{m} \tau(w;w')$ with $Fm = w'$. In particular, for all z in Z, there is an arrow $p \xrightarrow{m_z} \tau(z)$ with $Fm_z = z$. Let $X = \{f;m_z | z \in Z\}$. Since F is faithful, $m_0 = 1_p$ and $m_v = h$. For all $w, w' \in Z$, $m_w \leq m_{w'} \Leftrightarrow w \leq w'$. Proof:

  - $\Rightarrow$ Suppose $m_w \leq m_{w'}$. Then there is an m such that $m_{w'} = m_w;m$. Hence $w' = Fm_{w'} = F(m_w;m) = F(m_w);Fm = w;Fm$, and therefore $w \leq w'$.
  - $\Leftarrow$ Suppose $w \leq w'$. Then there is a w'' with $w;w'' = w'$ and an arrow $\tau(w) \xrightarrow{m} \tau(w')$ with $Fm = w''$. Since F is faithful, $m_w;m = m_{w'}$. Hence $m_w \leq m_{w'}$.

  Using the fact that Z is totally ordered, this implies that X is a chain in $\triangle\mathbf{D}$ from t to u.

- Suppose $\triangle\mathbf{C} \xrightarrow{\triangle F} \triangle\mathbf{D}$ is interval-chained. To show that $\mathbf{C} \xrightarrow{F} \mathbf{D}$ is trajectoried, suppose that $p \xrightarrow{f} q$ is an arrow of $\mathbf{C}$ with $Ff = t \neq 0$, and Y is a chain of $\mathbf{D}$ from 0 to t. We must prove the existence of a function τ that associates to each element of Y an object of $\mathbf{C}$ such that $\tau(0) = p$, $\tau(t) = q$, and, for all $v, v;v'$ in Y, there is an arrow $\tau(v) \xrightarrow{g} \tau(v;v')$ with $Fg = v'$. $\triangle\mathbf{C}$ has an arrow $k = (1_p, f, f)$, and $\triangle\mathbf{D}$ has an arrow $l = (0,t,t)$ with $Fk = l$. Further Y is a chain in $\triangle\mathbf{D}$ from 0 to t. Because

$\triangle\mathbf{C} \xrightarrow{\triangle F} \triangle\mathbf{D}$ is interval-chained, there exists a chain X in $\triangle\mathbf{C}$ from 1_p to f with $F[X] = Y$. Hence there exists a function τ that associates to each $u \in Y$ an object of $\mathbf{C}$ that is the codomain of an arrow h in X with $Fh = u$, and with in particular $\tau(0) = p$ and $\tau(t) = q$ (here we need $t \neq 0$). Now suppose v and $v; v'$ are elements of Y. Let $r = \tau(v)$ and $r' = \tau(v; v')$. Then X has arrows $p \xrightarrow{m} r$ and $p \xrightarrow{m'} r'$ with $Fm = v$ and $Fm' = v; v'$. Since X is a chain, m and m' are ordered. There are two possibilities:

- There is a g such that $m' = m; g$. Then $v; v' = Fm' = F(m; g) = Fm; Fg = v; Fg$. Because $\mathbf{D}$ is left-cancellative, $v' = Fg$. Further $\tau(v) \xrightarrow{g} \tau(v; v')$.
- There is a g such that $m = m'; g$. Then $v; 0 = v = Fm = F(m'); F(g) = v; (v'; F(g))$. Because $\mathbf{D}$ is left-cancellative, $0 = v'; F(g)$. Hence, because $\mathbf{D}$ is also anti-symmetric, $v' = F(g) = 0$. This implies $r = \tau(v) = \tau(v; v') = r'$. Therefore $\tau(v) \xrightarrow{g} \tau(v; v')$. $\square$

2.7 Some sufficient conditions

In this section, we shall present some sufficient conditions for an interval-onto function to be interval-chained. We shall use some concepts from set theory: see a textbook such as [Joh87] for details. Given posets X, Y and a monotone map $f : X \to Y$, define:

- $X' \subseteq X$ is a κ-chain iff X' is a chain of cardinality less than κ.
- $f : X \to Y$ is *limited* iff for all $y \in Y$ and $|Y|$-chains $X' < X''$ in X if $f[X'] < \{y\} < f[X'']$ then $\exists x \in f^{-1}[y] \,.\, X' < \{x\} < X''$.

It is simple to show that any limited function is onto (since $X' = X'' = \emptyset$ has $f[X'] < \{y\} < f[X'']$ so $\exists x \in f^{-1}[y]$) and interval-onto, and we can show that any limited function is chained.

Proposition 5 *Every limited function is chained*

Proof (using the axiom of choice). Let $f : X \to Y$ be limited, and let Y' be a chain in Y. By the axiom of choice let $\{y_\alpha \mid \alpha < \kappa\}$ be a well-ordering of Y' in a cardinal κ. Then for each $\alpha < \kappa$, define by transfinite induction the chains X_α and X'_α as:

$$\begin{aligned} X_\alpha &= \{x_\beta \mid \beta < \alpha, y_\beta < y_\alpha\} \\ X'_\alpha &= \{x_\beta \mid \beta < \alpha, y_\beta > y_\alpha\} \end{aligned}$$

where by the axiom of choice, for each $\alpha < \kappa$, x_α is defined (since f is limited) to be such that $X_\alpha < x_\alpha < X'_\alpha$ and $fx_\alpha = y_\alpha$. Then $X' = \{x_\alpha \mid \alpha < \kappa\}$ is a chain, and $f[X'] = Y'$. Thus f is chained. $\square$

In fact, two weaker forms of the axiom of choice are sufficient for this proof: that any well-ordered set of sets admits a choice function; and that any total order can be well-ordered. Furthermore, this result itself is as least as strong as the axiom of choice for totally ordered sets of sets—that any total order of non-empty sets has a choice function.

We can now present some conditions for an interval-onto function to be interval-chained. Define:

- The *coverage* of $X' \subseteq X$ is $\{x \in X \mid \forall x' \in X' . x \leq x' \text{ or } x \geq x'\}$.
- X is *κ-covered* iff there is a κ-chain X' in X with totally ordered coverage.
- X is *interval-ϕ* iff for all $x \leq y \in X$, $[x, y]$ is ϕ.

Proposition 6 *Each of the following is a sufficient condition for an interval-onto function $f : X \to Y$ to be interval-chained:*

1. *X is interval-ω-covered and Y is interval-total.*
2. *Y is interval-countable.*
3. *f is interval-finite-to-one.*

Proof.

1. For any $x \leq x' \in X$, $[x, x']$ is ω-covered, so we can find $x = x_0 \leq x_1 \leq \cdots \leq x_n = x'$ such that the coverage of $\{x_i \mid 0 \leq i \leq n\}$ is total. This implies that, for all i, $[x_i, x_{i+1}]$ is a chain of X. Suppose Y' be a chain in $[fx, fx']$. Let $X' = ([x_0, x_1] \cup \cdots \cup [x_n, x_{n+1}]) \cap f^{-1}[Y']$. Then X' is a chain of X. In order to see that $f[X'] = Y'$, suppose $y \in Y'$. Because Y is interval-total, we can find, $0 \leq i < n$ such that $fx_i \leq y \leq fx_{i+1}$. Since f is interval-onto, there exists an $x'' \in [x_i, x_{i+1}]$ such that $fx'' = y$. Clearly $x'' \in X'$. Thus f is interval-chained.

2. Suppose Y is interval-countable. By Proposition 5 it is enough to prove that f is interval-limited. For this, suppose that $x' \leq x'' \in X$, $y \in [fx', fx'']$, and $X' < X''$ $|[fx', fx'']|$-chains in $[x', x'']$ such that $f[X'] < \{y\} < f[X'']$. We must find an x in $f^{-1}[y] \cap [x', x'']$ such that $X' < \{x\} < X''$. If $y = fx'$ then we can take $x = x'$, and if $y = fx''$ then we can take $x = x''$. So assume that $fx' < y < fx''$. Because Y is interval-countable, X', X'' are finite chains. Thus we can take z' to be the maximal element of $X' \cup \{x'\}$, and z'' to be the minimal element of $X'' \cup \{x''\}$. Then $z' < z''$ and $fz' < y < fz''$. Since f is interval-onto, there exists an $x \in [x', x'']$ such that $x \in [z', z'']$ and $fx = y$. Clearly $x \in f^{-1}[y] \cap [x', x'']$ and $X' < \{x\} < X''$.

3. Similar to the proof of (2). □

Finally, we can translate these conditions back to conditions on timed transition systems. Define:

- A timed transition system is *image-finite* if for every p and t, there are at most finitely many q such that $p \xrightarrow{t} q$.
- A transition $p \xrightarrow{t} q$ of a timed transition system is *deterministic* iff for any $u + v = t$, there is at most one r such that $p \xrightarrow{u} r \xrightarrow{v} q$.
- An additive timed transition system is *finitely variable* iff for any $p \xrightarrow{t} q$ we can find p_i and t_i such that $p = p_0 \xrightarrow{t_0} p_1 \xrightarrow{t_1} \cdots \xrightarrow{t_n} p_{n+1} = p'$, $t = t_0 + \cdots + t_n$, and each transition $p_i \xrightarrow{t_i} p_{i+1}$ is either deterministic or has $t_i = 0$.

Proposition 7 *Let* $\mathbf{C} \xrightarrow{F} \mathbf{D}$ *be a timed transition system. Then*

1. *If* $\mathbf{C} \xrightarrow{F} \mathbf{D}$ *is finitely variable and* $\mathbf{D}$ *is interval-total, then* $\triangle\mathbf{C}$ *is interval-*ω*-covered.*
2. *If* $\mathbf{C} \xrightarrow{F} \mathbf{D}$ *is image-finite then* $\triangle F$ *is interval-finite-to-one.*

Proof. Routine. □

Proposition 8 *Each of the following three conditions on timed transition systems is sufficient to guarantee the equivalence of additivity and trajectoried:*

1. *Finite variability and interval-totality of the time domain.*
2. *Interval-countability of the time domain.*
3. *Image-finite.*

Proof. Immediate from Propositions 4, 6, and 7. □

For example, for both strong and weak strings:

- Real-timed strings $(TS(\Sigma, \mathbf{R}^+), \bullet, 0)$ are interval-total, so any finitely variable real-time transition system is trajectoried.
- Strings $(\Sigma^*, \cdot, \epsilon)$ are interval-countable, so any untimed transition system is trajectoried.
- Discrete-timed strings $(TS(\Sigma, \mathbf{N}), \bullet, 0)$ are interval-countable, so any discrete-time transition system is trajectoried.
- Rational-timed strings $(TS(\Sigma, \mathbf{Q}^+), \bullet, 0)$ are interval-countable, so any rational-time transition system is trajectoried.
- Any image-finite real-time transition system is trajectoried.
- Any finitely variable real-time transition system is trajectoried.

3. (Bi)simulations

In this section, we will apply the same techniques to bear on a related problem. We can define two notions of bisimulation: Milner's [Mil89] definition which says that two terms are *bisimilar* if they can match transitions, and a form of Ho Stuart's [Ho+92] *path bisimulation* which says that two terms are *trajectory bisimilar* if they can match trajectories. Then the question is under what conditions these two notions agree. Again, we will reduce this (and also the corresponding question for *simulations*) to a problem about monotone functions on posets, and use the poset problem to find sufficient conditions.

3.1 Another problem about posets

Given posets X and Y, and a monotone map $f : X \to Y$, define:

- $\uparrow x = \{y \in X \mid x \leq y\}$.

- A monotone function $f : X \to Y$ is *upper-ϕ* iff for all $x \in X$, the function $f : \uparrow x \to \uparrow(fx)$ is ϕ.

Then any upper-chained function is upper-onto. The question we would now like to consider is this: under what conditions are upper-onto functions upper-chained?

3.2 Another problem about timed transition systems

One preorder between transition systems is *simulation.* A *simulation* between timed transition systems $(M, T, \longrightarrow)$ and $(N, T, \longrightarrow)$ is a relation R between M and N such that whenever $p \; R \; q$:

- If $p \xrightarrow{t} p'$ then we can find q' with $q \xrightarrow{t} q'$ and $p' \; R \; q'$.

A *bisimulation* is a relation R such that R and R^{-1} are simulations. Let $\sqsubseteq$ be the largest simulation, and let $\sim$ be the largest bisimulation. Note that this definition includes both strong bisimulation on $(CCS, \Sigma_\tau^*, \longrightarrow^*)$ and weak bisimulation on $([CCS]_\equiv, \Sigma^*, \Longrightarrow)$.

We may define another notion of simulation, based on the *path bisimulation* from [Ho+92]: a *trajectory simulation* between $(M, T, \longrightarrow)$ and $(N, T, \longrightarrow)$ is a relation R between M and N such that whenever $p_0 \; R \; q_0$:

- If T' is a chain through T from 0 and $\vec{p}$ is a trajectory through T' and $p \xrightarrow{0} p_0$ then there is a trajectory $\vec{q}$ through T' such that $q \xrightarrow{0} q_0$ and $\forall u \in T'.p_u \; R \; q_u$

A *trajectory bisimulation* is a relation R such that R and R^{-1} are trajectory simulations. Let $\vec{\sqsubseteq}$ be the largest trajectory simulation, and let $\vec{\sim}$ be the largest trajectory bisimulation.

In a timed transition system, any trajectory simulation is a simulation. The question we would now like to consider is this: under what conditions is a simulation a trajectory simulation?

3.3 Why the trajectory simulation problem is interesting

For timed transition systems, the notion of trajectory simulation is strictly stronger than simulation. For example, let $\mathbf{R}^\infty$ be the reals with a new top element ∞, and let the timed transition system $(\{p_x \mid x \in \mathbf{R}^\infty\}, \mathbf{Q}^+, \longrightarrow)$ be given by:

$$\frac{}{p_x \xrightarrow{0} p_x} \qquad \frac{t > 0 \quad x < y}{p_x \xrightarrow{t} p_y} \qquad \frac{x \in \mathbf{Q}}{p_x \xrightarrow{0} p_\infty}$$

Thus p_x can perform any rational number and increase x, and if x is rational, then p_x can also reach a *time-stop* state p_∞ which only has the transition $p_\infty \xrightarrow{0} p_\infty$. Then let $(\{q_x \mid x \in \mathbf{R}^\infty\}, \mathbf{Q}^+, \longrightarrow)$ be given by:

$$\frac{}{q_x \xrightarrow{0} q_x} \qquad \frac{t > 0 \quad x > y}{q_x \xrightarrow{t} q_y} \qquad \frac{x \in \mathbf{Z}}{q_x \xrightarrow{0} q_\infty}$$

This transition system is the same, except that q_x can only time-stop whenever x is an integer. Then we have the bisimulation:

$$\{(p_x, q_y) \mid x \in \mathbf{Q}, y \in \mathbf{Z}\} \cup \{(p_x, q_y) \mid x \in \mathbf{R} \setminus \mathbf{Q}, y \in \mathbf{R} \setminus \mathbf{Z}\} \cup \{(p_\infty, q_\infty)\}$$

and so $p_0 \sim q_0$. However, there is no trajectory bisimulation R such that $p_0 \; R \; q_0$, since there is a trajectory $\vec{p}$ through $[0,1]$ from p_0 to p_1 such that infinitely many p_x can time-stop, and any trajectory $\vec{q}$ through $[0,1]$ from q_0 to some q_y will only have finitely many q_x which can time-stop.

Another example can be given in the timed CSP notation [Sch92]:

$$\textstyle\bigsqcap_{t>0}(\text{WAIT}\, t; a \to \text{STOP}) \qquad\qquad \bigsqcap_{t>0}(\text{WAIT}\, t; a \to \text{WAIT}\, t; \text{STOP})$$

These processes are weakly bisimilar, but not weak trajectory bisimilar: the first has a trajectory with $a \to \text{STOP}$ at every $t > 0$, which cannot be matched by the second. Similar examples may be defined in the PARTY language [Ho+92].

This last example indicates that for languages like timed CSP trajectory bisimulation is *not* an interesting equivalence by itself, since it is *not* a congruence:

$$\text{STOP} \; \vec{\sim} \; \text{WAIT}\, t; \text{STOP}$$

but:

$$\textstyle\bigsqcap_{t>0}(\text{WAIT}\, t; a \to \text{STOP}) \not\vec{\sim} \bigsqcap_{t>0}(\text{WAIT}\, t; a \to \text{WAIT}\, t; \text{STOP})$$

However, trajectory bisimulation *is* useful when we can show that it coincides with bisimulation, that is $p \sim q$ iff $p \; \vec{\sim} \; q$. For example, to show that timed CSP hiding preserves bisimulation, we have to show that whenever $p \sim q$ then $p \setminus a \sim q \setminus a$. Without trajectory bisimulation, this requires complex *ad hoc* reasoning, but it is simple to show that if $p \; \vec{\sim} \; q$ then $p \setminus a \; \vec{\sim} \; q \setminus a$.

3.4 Why the trajectory simulation problem is an example of the second poset problem
Given posets X_1, X_2 and Y, and monotone functions $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$:

- A *relator* between f_1 and f_2 is a poset X with monotone maps $\pi_1 : X \to X_1$, $\pi_2 : X \to X_2$ such that $\pi_1; f_1 = \pi_2; f_2 : X \to Y$.
- A *simulator* is a relator where π_1 is upper-onto.
- A *trajectory simulator* is a relator where π_1 is upper-chained.

We define a relation $\triangle R$ between arrows of categories in terms of a relation R between objects of categories:

- If $\mathbf{C}_1 \xrightarrow{F_1} \mathbf{D}$, $\mathbf{C}_2 \xrightarrow{F_2} \mathbf{D}$, and R is a relation between the objects of $\mathbf{C}_1$ and the objects of $\mathbf{C}_2$ then $\triangle R$ is a relation between the arrows of $\mathbf{C}_1$ and $\mathbf{C}_2$ defined $f_1 \triangle R \; f_2$ iff $A_1 \xrightarrow{f_1} B_1$, $A_2 \xrightarrow{f_2} B_2$, $F_1 f_1 = F_2 f_2$ and $B_1 \; R \; B_2$.

Proposition 9 *In timed transition systems $\mathbf{C}_1 \xrightarrow{F_1} \mathbf{D}$ and $\mathbf{C}_2 \xrightarrow{F_2} \mathbf{D}$:*

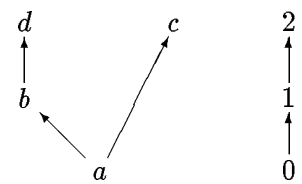
1. *R is a simulation between $\mathbf{C}_1$ and $\mathbf{C}_2$ iff $\triangle R$ is a simulator between $\triangle F_1$ and $\triangle F_2$.*
2. *R is a trajectory simulation between $\mathbf{C}_1$ and $\mathbf{C}_2$ iff $\triangle R$ is a trajectory simulator between $\triangle F_1$ and $\triangle F_2$.*

Proof. Routine. $\square$

Thus we have reduced the problem of showing that a simulation R is a trajectory simulation to showing that the upper-onto function π_1 is upper-chained.

3.5 A sufficient condition

We already know (assuming the axiom of choice) that any upper-limited function is upper-chained, so we now have to find a sufficient condition for an upper-onto function to be upper-limited. As we would expect from our two counter-examples, neither an interval-countable labeling nor a finitely variable transition system are sufficient. The reason why we cannot apply the techniques of Section 2.7 is that not every upper-onto function is limited, for example the function $f : \{a, b, c, d\} \to \{0, 1, 2\}$ given by:



$$\begin{array}{rcl} fa & = & 0 \\ fb & = & 1 \\ fc & = & 2 \\ fd & = & 2 \end{array}$$

is upper-onto but not interval-onto (since $f[a, c] = \{0, 2\}$) and so is not limited. However, given an upper-onto $f : X \to Y$, we can define a partial order $\leq_f \subseteq \leq$, and show a sufficient condition for $f : (X, \leq_f) \to Y$ to be upper-limited. We can then apply the techniques from Section 2.7 to show that $f : (X, \leq_f) \to Y$ is upper-chained, and so $f : X \to Y$ is upper-chained.

Informally, the partial order $\leq_f$ is defined to be the largest partial order smaller than $\leq$ such that $f : (X, \leq_f) \to Y$ is interval-onto. That is, if $x \leq_f x'$ and $fx \leq y \leq fx'$ then $\exists x'' \in f^{-1}[y] \,.\, x \leq_f x'' \leq_f x'$. However, this definition is cyclic, so we shall formally define it in a similar fashion to Milner's [Mil89] definition of bisimulation. For each ordinal α, define $x \leq_f^\alpha x'$ iff $x \leq x'$ and

$$\forall y \in [fx, fx'] \,.\, \exists x'' \in f^{-1}[y] \,.\, \forall \beta < \alpha \,.\, x \leq_f^\beta x'' \leq_f^\beta x'$$

Then $x \leq_f x'$ iff $\forall \alpha \,.\, x \leq_f^\alpha x'$.

Proposition 10 *If $f : X \to Y$ and Y is interval-total then*

1. *$\leq_f$ is a partial order, and*
2. *$f : (X, \leq_f) \to Y$ is interval-onto.*

Proof. The proof of (1) is routine. For (2), suppose $x \leq_f x'$ and $y \in [fx, fx']$. For each ordinal α, define the set X_α as:

$$X_\alpha = \{x'' \in f^{-1}[y] \mid \forall \beta < \alpha \,.\, x \leq_f^\beta x'' \leq_f^\beta x'\}$$

Then by the definition of $\leq_f$, $X_\alpha \neq \emptyset$, and, for all $\gamma < \alpha$, $X_\gamma \supseteq X_\alpha$. Since each X_α is a set, it follows that there is some x'' in all of the X_α. Then $x \leq_f x'' \leq_f x'$, and $fx'' = y$. □

Proposition 11 *If $f : X \to Y$ is upper-onto, upper-finite-to-one, and Y is interval-total then $f : (X, \leq_f) \to Y$ is upper-limited.*

Proof. This proof uses some concepts from lattice theory. See a textbook such as [DP90] for details.

First, we shall show by induction on α that $f : (X, \leq_f^\alpha) \to Y$ is upper-onto, that is for any $x \in X$ and $y' \geq y = fx$ there is an $x' \geq_f^\alpha x$ such that $fx' = y'$. If $\alpha = 0$ then the result follows immediately, since $\leq_f^0 = \leq$. Otherwise, for each chain $Y = y_0 \leq \cdots \leq y_n$ in $[y, y']$ and $\beta < \alpha$ define X_Y^β as:

$$\begin{aligned} X_Y^\beta = \ & \{x' \in f^{-1}[y] \mid \exists x_0 \in f^{-1}[y_0], \ldots, x_n \in f^{-1}[y_n] \,. \\ & x \leq_f^\beta x_0 \leq_f^\beta \cdots \leq_f^\beta x_n \leq_f^\beta x'\} \end{aligned}$$

By induction X_Y^β is non-empty. Since f is upper-finite-to-one, each X_Y^β is finite. Since $X_{Y \cup Y'}^{\beta \vee \gamma} \subseteq X_Y^\beta \cap X_{Y'}^\gamma$, the X_Y^β form a non-empty $\supseteq$-directed set of finite non-empty sets. Any such set has a top, which is finite and non-empty, and we let x' be any member of that top. It is easy to show that $fx' = y'$ and that $x \leq_f^\alpha x'$. Thus $f : (X, \leq_f^\alpha) \to Y$ is upper-onto.

Second, we shall show that $f : (X, \leq_f) \to Y$ is upper-onto, that is for any $x \in X$ and $y' \geq y = fx$ there is an $x' \geq_f x$ such that $fx' = y'$. For each α, define:

$$X_\alpha = \{x' \in f^{-1}[y'] \mid x \leq_f^\alpha x'\}$$

By the first part, each X_α is non-empty. Since f is upper-finite-to-one, each X_α is finite. Since $X_{\alpha \vee \beta} = X_\alpha \cap X_\beta$, the X_α form a non-empty $\supseteq$-directed set of finite non-empty sets. Any such set has a top, which is finite and non-empty, and we let x' be any member of that top. It is easy to show that $fx' = y'$ and that $x \leq_f x'$. Thus $f : (X, \leq_f) \to Y$ is upper-onto.

Finally, we shall show that $f : (X, \leq_f) \to Y$ is upper-limited, that is for any $x \in X$, chains $X' <_f X''$ in $\uparrow_f x$, and $f[X'] < \{y'''\} < f[X'']$ in $\uparrow(fx)$ there is an $x''' \in f^{-1}[y''']$ such that $X' <_f \{x'''\} <_f X''$. Wlog, we can assume that $x \in X'$ and so X' is non-empty. We then have two cases, depending on whether X'' is empty:

- If X'' is empty, then for each $x' \in X'$, define $X_{x'}$ as:

$$X_{x'} = \{x''' \in f^{-1}[y'''] \mid x' \leq_f x'''\}$$

Since $f : (X, \leq_f) \to Y$ is upper-onto, each $X_{x'}$ is non-empty. Since f is upper-finite-to-one, each $X_{x'}$ is finite. Since $X_{x' \vee x''} = X_{x'} \cap X_{x''}$, the $X_{x'}$ form a non-empty $\supseteq$-directed set of finite non-empty sets. Any such set has a top, which is finite and non-empty, and we let x''' be any member of that top.

- If X'' is non-empty, then for each $x' \in X'$ and $x'' \in X''$, define $X^{x'}_{x''}$ as:

$$X^{x'}_{x''} = \{x''' \in f^{-1}[y'''] \mid x' \leq_f x''' \leq_f x''\}$$

Since $f : (X, \leq_f) \to Y$ is interval-onto, the $X^{x'}_{x''}$ form a non-empty $\supseteq$-directed set of non-empty finite sets, and we let x''' be any member of its top.

It is easy to show that $fx''' = y'''$ and $X' <_f \{x'''\} <_f X''$. Thus $f : (X, \leq_f) \to Y$ is upper-limited. □

We are now in a position to prove the main result of this section.

Proposition 12 *If $f : X \to Y$ is upper-onto, and upper-finite-to-one then $f : X \to Y$ is upper-chained.*

Proof. Consider $x \in X$, and a chain $Y' \subseteq \uparrow fx$; then $Y'' = Y' \cup \{fx\}$ is also a chain. Let $X'' = f^{-1}[Y'']$, and let $f' : X'' \to Y''$ be the function f domain restricted to X''. Then f' is upper-onto and upper-finite-to-one, since f is. Finally, Y'' is total, since it is a chain in Y. It follows from Proposition 11 that $f' : (X'', \leq_{f'}) \to Y''$ is upper-limited, and thus, by Proposition 5, upper-chained. Since $x \in X''$ and Y' is a chain in $\uparrow f'x$, it follows that there is a $\leq_{f'}$-chain $X' \subseteq \uparrow_{\leq_{f'}} x$ such that $f'[X'] = Y'$. As X' is totally ordered under $\leq_{f'}$, it is also totally ordered under $\leq$. Clearly $X' \subseteq \uparrow x$. Hence $f : X \to Y$ is upper-chained. □

The condition that $f_2 : X_2 \to Y$ is upper-finite-to-one is precisely the condition that the corresponding timed transition system is image-finite.

Thus we have shown that any simulation between timed transition systems $(M, T, \longrightarrow)$ and $(N, T, \longrightarrow)$ is a trajectory simulation if $(N, T, \longrightarrow)$ is image-finite.

4. Discussion

Many timed transition systems which the authors are familiar with use the natural numbers as their underlying time domain. The domain of timed strings corresponding to the transitions that may be performed will be interval-finite, and so these transition systems will be trajectoried. These include one of the versions of timed LOTOS [BoL92], the Temporal Process Language of [HeR91], the Algebra of Timed Processes ATP [NiS90], the process algebra described in [Ort92], and the algebra for time and probabilities [Han91]. Also, the transition systems discussed in [ClZ92] are required to be image-finite, which is enough to ensure that they are trajectoried.

The majority of timed transition systems which the authors are familiar with are image-finite: only finitely many results are possible from any state through any particular (delay or action) transition. As well as the transition systems mentioned above, these include a number of algebras using the reals as the underlying time domain: the three versions of Timed CCS [MoT90, Wan91, Che91], Timed ACP [BaB91], APA [Jef91], and a different version of timed LOTOS [QAF89]. These are all additive, so it follows that they are trajectoried.

Timed CSP [Sch92] and PARTY [Ho+92] are not image-finite; they both allow infinite choice. However, they are both forward and backward deterministic under delay transitions, and so they are finitely variable on both weak and strong timed strings, which is enough to ensure that they are trajectoried on both strong and weak timed strings.

In the framework investigated in [VaL92], no assumptions are made about the transition system, or the time domain, and the authors make explicit their requirement that the transition systems are trajectoried; additivity without any other assumptions is not enough.

In addition, we have shown that for image-finite processes from any of these algebras, simulation is equivalent to trajectory simulation.

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REFERENCES

[BaB91] J. C. M. Baeten and J. A. Bergstra, Real time process algebra, *Formal Aspects of Computing* 3(2), pp 142–188, 1991.

[Ben89] J. van Benthem. Time, logic and computation. pp 1–49, LNCS 354, 1989.

[BoL92] T. Bolognesi and F. Lucidi, Timed process algebras with urgent interactions and a unique powerful binary operator, pp 124–147, LNCS 600, 1992.

[ClZ92] R. Cleaveland and A. E. Zwarico, *A theory of testing for real-time*, North Carolina State University report, 1992.

[Che91] Liang Chen, *An interleaving model for real-time systems*, report ECS-LFCS-91-184, University of Edinburgh, 1991.

[DP90] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, 1990.

[Han91] H. Hansson, *A calculus for communicating systems with time and probabilities*, Ph.D. thesis, University of Uppsala, 1991.

[Heh93] E. C. R. Hehner, *Abstractions of time*, Manuscript, June 1993.

[HeR91] M. Hennessy and T. Regan, *A process algebra for timed systems*, report 5/91, University of Sussex, 1991.

[Ho+92] C. Ho-Stuart, H. S. M. Zedan, M. Fang, M, and C. M. Holt, *PARTY: A process algebra with real-time from York*, report YCS 177, University of York, 1992.

[Jef91] A. S. A. Jeffrey, *Observation spaces and timed processes*, D.Phil thesis, Oxford University, 1991.

[Joh87] P. T. Johnstone, *Notes on logic and set theory*, Cambridge University Press, 1987.

[LyV93] N. A. Lynch and F. W. Vaandrager. *Forward and backward simulations Part II: Timing-based systems*, Report CS-R9314, CWI, Amsterdam, 1993.

[Mil89] R. Milner, *Communication and Concurrency*, Prentice-Hall, 1989.

[MoT90] F. Moller and C. Tofts, A temporal calculus of communicating systems, pp 401–415, LNCS 458, 1990.

[NiS90] X. Nicollin and J. Sifakis, *The algebra of timed processes ATP: theory and application*, RT-C26, Project SPECTRE, Laboratoire de Génie Informatique de Grenoble, 1991 .

[Ort92] Y. Ortega-Mallén, Operational semantics for timed observations, pp 507–528, LNCS 571, 1992.

[QAF89] J. Quemada, A. Azcorra, and A. Fernandez, TIC: A timed calculus for LOTOS, in *Formal Description Techniques II, FORTE '89* (Son T. Vuong, ed), North Holland 1990.

[Sch92] S. A. Schneider, An operational semantics for timed CSP, *Information and Computation*, to appear.

[VaL92] F. W. Vaandrager and N. A. Lynch, Action transducers and timed automata, pp 436–455, LNCS 630, 1992.

[Wan91] Wang Yi, *A calculus of real time systems,* Ph.D. Thesis, Department of Computer Sciences, Chalmers University of Technology, 1991.