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Counting Interlacing Pairs on the Circle

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Abstract

Let b_{2n} be the number of interlacing chords joining $2n$ points on the circle. Let $a_{2n} = b_{2n} + b_{2n-2} + \cdots + b_2$. Then $a_{2n} = (2n-1)a_{2n-2} + a_{2n-4}$. This formula was conjectured by J. Betramas. The number a_{2n} is also the number of interlacing involutions of $2n$ points on the real line.

Mathematics subject classification 1991: 05A05, 05A15

Key words & phrases: chord diagram, interlacing pairs on the circle, interlacing pairs on the real line, interlacing involutions on the circle, interlacing involutions on the line.

Note. At the time of this work V. V. Kalshnikov was a graduate student at Utrecht University within the framework of Master Class and MRI.

1. Introduction. Consider $2n$ points on the cicle connected pairwise. Such a diagram is sometimes called a chord diagram, e.g. in the theory of singular knots and links. We are interested in chord diagrams such that no neighbours are connected. Let b_{2n} be the number of such diagrams, and let

$$a_{2n} = b_2 + b_4 + \cdots + b_{2n}$$

The first few values of b_{2n} and a_{2n} are as in the table below.

Numbers of interlacing involutions on a cirle							
$2n$	2	4	6	8	10	12	…
b_{2n}	0	1	4	31	293	3362	…
a_{2n}	0	1	5	36	329	3655	…

Jean Betramas of LABRI, Bordeaux, observed numerically that for these values

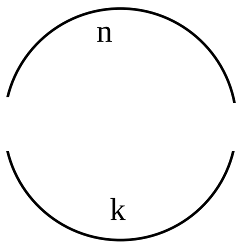
$$a_{2n} = (2n-1)a_{2n-2} + a_{2n-4}, \quad n \ge 3 \tag{1.1}$$

and posed the problem of proving this. In this note we provide a proof.

2. Proof of formula (1.1).

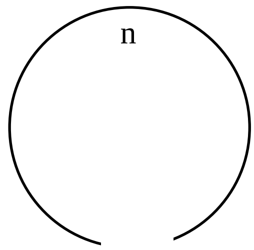
It will convenient to consider the same problem for a circle cut in one or more places. Then of course two points which bracket a cut and are not otherwise neighbours are no longer neighbours.

(n, k) :=

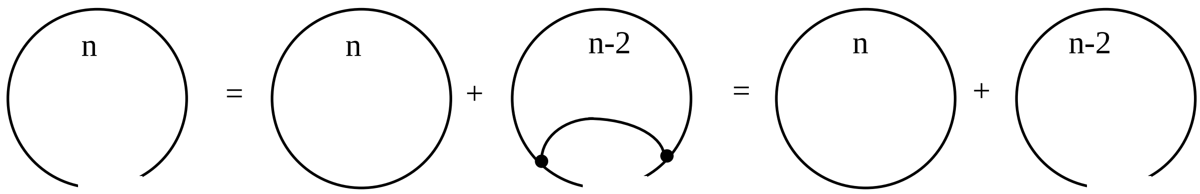


A picture like the one on the left stands for the number of configurations of interlacing pairs of $n+k$ points ($n+k$ even) distributed over the twice cut circle as indicated. Similarly the picture on the right depicts the number of configurations of n (n even) points on the once cut circle (i.e. the real line). Note that the notations are consistent. A diagram with one or more chords drawn already in stands for the number of configurations of interlacing pairs which do contain the particular pairs already indicated.

(n, 0):=



The first thing to notice is that:



where in the middle circle the two points explicitely indicated are the ones that bracket the cut. It follows that

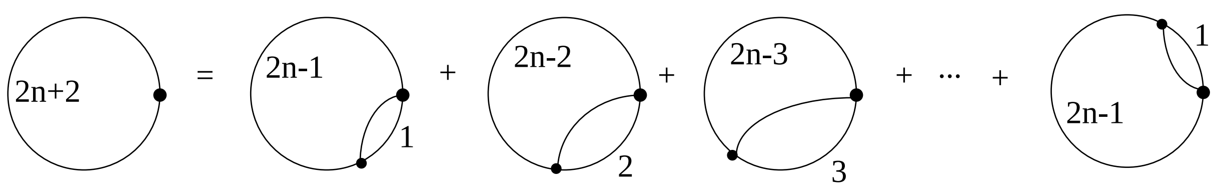
$$(2n,0) = b_{2n} + (2n-2) \tag{2.1}$$

so that

$$a_{2n} = (2n,0) \tag{2.2}$$

providing a direct combinatorial interpretation of the number a_{2n}. And that is really the central idea of this proof.

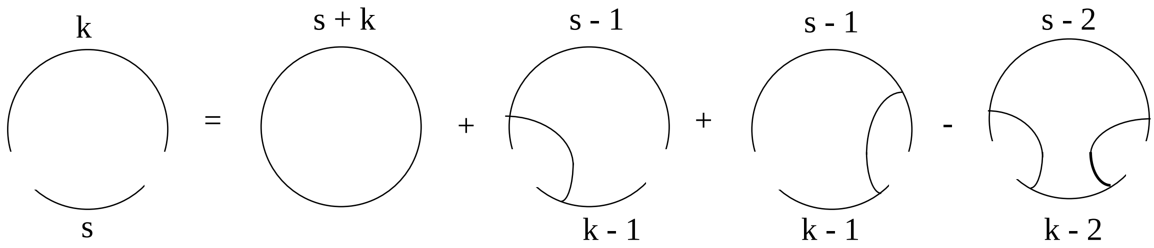
Now take $2n+2$ points on the uncut circle and select any of them. This special point can be paired with any other point except its two neighbours. Pictorially:



In formula this says

$$b_{2n+2} = (2n-1,1) + (2n-2,2) + \cdots + (1,2n-1). \tag{2.3}$$

The next thing to observe is



In formula this says:

$$(s,k) = b_{s+k} + 2(s-1,k-1) - (s-2,k-2) \tag{2.4}$$

Setting $(s,k)=0$ for $s>0$, $k<0$, this equality holds for all s,k such that $s+k\geq 4$. Combining (2.3) and 2.4) we get

$$\begin{aligned}
b_{2n+2} &= \sum_{s=1}^{2n-1}(2n-s,s) = \sum_{s=1}^{2n-1}\{b_{2n} + 2(2n-s-1,s-1) - (2n-s-2,s-2)\}\\
&= (2n-1)b_{2n} + 2\sum_{s=1}^{2n-1}(2n-s-1,s-1) - \sum_{s=1}^{2n-1}(2n-s-2,s-2)\\
&= (2n-1b_{2n} + 2\sum_{s=0}^{2n-2}(2n-2-s,s) - \sum_{s=0}^{2n-4}(2n-4-s,s)\\
&= (2n-1)b_{2n} + 4(2n-2,0) + 2\sum_{s=1}^{2n-3}(2n-2-s,s) - 2(2n-4,0) - \sum_{s=0}^{2n-4}(2n-4-s,s)\\
&= (2n-1)b_{2n} + 4a_{2n-2} + 2b_{2n} - 2a_{2n-4} - b_{2n-2}
\end{aligned}$$

and using $b_{2n} = a_{2n} - a_{2n-2}$ this yields

$$b_{2n+2} = (2n+1)a_{2n} - (2n-2)a_{2n-2} - a_{2n-4}.$$

It is straightforward to check formula (1.1) for $n=6$. With induction assume it holds for $2n\geq 6$. Then

$$\begin{aligned}
a_{2n+2} &= a_{2n} + b_{2n}\\
&= (2n-1)a_{2n-2} + a_{2n-4} + (2n+1)a_{2n} - (2n-2)a_{2n-2} - a_{2n-4}\\
&= (2n+1)a_{2n} + a_{2n-2}
\end{aligned}$$

This finishes the proof.