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N. van den Hijligenberg and R. MartinI

Department of Analysis, Algebra and Geometry

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CWI
P.O. Box 94079
1090 GB Amsterdam
The Netherlands

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N. van den Hijligenberg
CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands
nico@cwi.nl

R. Martini
University of Twente
P.O. Box 217, 7500 AE Enschede, The Netherlands
martini@math.utwente.nl

Abstract

We discuss a method to construct a De Rham complex (differential algebra) of Poincaré-Birkhoff-Witt-type on the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. We determine the cases in which this gives rise to a differential Hopf algebra that naturally extends the Hopf algebra structure of $U(\mathfrak{g})$. The construction of such differential structures is interpreted in terms of colour Lie superalgebras.

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1 Introduction

Recently non-commutative differential geometry has attracted considerable interest, both mathematically and as a framework for certain models in theoretical physics. In particular there is much activity in differential geometry on quantum groups. A non-commutative differential calculus on quantum groups has been developed by Woronowicz [1] following general ideas of Connes [2]. This general theory has been reformulated by Wess and Zumino [3] in a less abstract way. Their approach may be more suitable for specific applications in physics. A large number of papers have been written since and a few other methods to construct a non-commutative differential geometry on a quantum group or to define a differential geometric structure (a De Rham complex) on a given non-commutative algebra have been proposed and discussed by several authors (e.g. [4], [5] and [6]).

In this paper we present a differential calculus on the enveloping algebra of a given Lie algebra. This differential structure turns out to be a differential Hopf algebra, which can be interpreted in a very interesting way in terms of colour Lie superalgebras. The commutative case has been studied previously (see e.g. [7]). Notice that our approach is different from the standard methods to construct non-commutative differential structures on Hopf algebras and quantum groups in the sence that our starting point is not the algebra of functions on the (quantum) group but its dual the (quantized) universal enveloping algebra.

In the classical limit a quantized universal enveloping algebra defines a co-Poisson-Hopf algebra structure on the universal enveloping algebra (see [8]). Further research is in progress concerning the compatibility between the differential calculus and the Poisson co-bracket in order to define a differential Hopf algebra structure on the quantized universal enveloping algebra.

2 A De Rham complex on $U(\mathfrak{g})$

Let A be an associative algebra over the field of complex numbers. A differential algebra on A (or a De Rham complex on A, see [4]) is an $\mathbb{N}$-graded associative algebra Ω equipped with a linear operator d that has the following properties:

1. d is homogeneous of degree 1, i.e. $d(\Omega^p) \subset \Omega^{p+1}$ for all $p \in \mathbb{N}$.
2. d is a differential, i.e. $d^2 = d \circ d = 0$.
3. d is a graded derivation (of degree 1), i.e.

$$d(ab) = d(a)b + (-1)^p a d(b) \quad a \in \Omega^p, b \in \Omega. \tag{2.1}$$

Furthermore, the algebra Ω has to be generated by $\Omega^0 \cup d(\Omega^0)$ where Ω^0 is isomorphic to A. For A we take the universal enveloping algebra of a Lie algebra and discuss the construction of De Rham complexes on such an algebra.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$ with basis $\{x^1, x^2, \dots, x^n\}$ and corresponding structure constants C_k^{ij} which are defined by the property $[x^i, x^j] = C_k^{ij} x^k$. Throughout this article we will make use of the Einstein summation convention. The universal enveloping algebra of $\mathfrak{g}$, which we denote by $U(\mathfrak{g})$, can be viewed as the quotient algebra of the free associative algebra on the alphabet $\{x^1, x^2, \dots, x^n\}$ modulo the ideal generated by the relations

$$x^i x^j - x^j x^i = C_k^{ij} x^k. \tag{2.2}$$

From the Poincaré-Birkhoff-Witt Theorem we know that the monomials $x^{i_1} x^{i_2} \dots x^{i_p}$ with $p \geq 0$ and $i_1 \leq i_2 \leq \dots \leq i_p$ form a basis of $U(\mathfrak{g})$. The main idea behind the construction of Ω on $U(\mathfrak{g})$ is that we demand Ω to be of PBW-type, by this we mean that the monomials

$$dx^{j_1} dx^{j_2} \dots dx^{j_q} x^{i_1} x^{i_2} \dots x^{i_p} \text{ with } j_1 < j_2 < \dots < j_q, i_1 \leq i_2 \leq \dots \leq i_p \quad p, q \geq 0 \tag{2.3}$$

are a basis of the associative algebra Ω. From here on we will write y^j to denote the element $dx^j = d(x^j)$ in Ω^1.

In order to construct Ω we have to impose certain commutation relations between the elements x^i and y^j. On account of the homogeneity and the PBW-property we impose relations of the form

$$x^i y^j = y^k (\Theta_{lk}^{ij} x^l + A_k^{ij}) \tag{2.4}$$

where both Θ_{lk}^{ij} and A_k^{ij} are arbitrary complex numbers. Since we want to obtain an ideal which is invariant under the action of the differential, we adjoin the following commutation relations for the elements y^i and y^j:

$$y^i y^j = -\Theta_{lk}^{ij} y^k y^l. \tag{2.5}$$

This is simply the consequence of applying d to the relations (2.4). We define Ω to be the quotient algebra of the free associative algebra on the alphabet $\{x^1, x^2, .., x^n, y^1, y^2, .., y^n\}$ modulo the ideal J which is generated by the relations (2.2),(2.4) and (2.5). The $\mathbb{N}$-grading of Ω is induced by giving the elements x^i degree 0 and the elements y^j degree 1. The differential d is defined by $d(x^i) = y^i$, $d(y^i) = 0$ and the derivation property (2.1).

The coefficients Θ_{lk}^{ij} and A_k^{ij} should satisfy a number of conditions in order that Ω has the above mentioned properties of a De Rham complex on $U(\mathfrak{g})$. The first condition arises from $d(J) \subset J$, i.e. the d-invariance of the ideal J. By applying d to (2.2) we obtain

$$y^i x^j + x^i y^j - y^j x^i - x^j y^i = C_k^{ij} y^k$$

which, on account of the relations (2.4), can be written as

$$y^i x^j + y^k (\Theta_{lk}^{ij} x^l + A_k^{ij}) - y^j x^i - y^k (\Theta_{lk}^{ji} x^l + A_k^{ji}) = C_k^{ij} y^k.$$

The PBW-property of Ω implies that

$$A_k^{ij} - A_k^{ji} = C_k^{ij} \text{ and } y^i x^j - y^j x^i + (\Theta_{lk}^{ij} - \Theta_{lk}^{ji}) y^k x^l = 0.$$

We can write this in the following compact form

$$(I - P)A = C \text{ and } I - P + \Theta P - P\Theta P = (I - P)(I + \Theta P) = 0. \tag{2.6}$$

In these equations we consider A and C as $n^2 \times n$ and Θ, I and P as $n^2 \times n^2$ matrices over $\mathbb{C}$, the product is just the ordinary matrix multiplication. The permutation matrix P is defined by $P^{ij}_{kl} = \delta^i_l \delta^j_k$ where δ denotes the kronecker delta. Since the relations (2.5) have been found by applying d to the relations (2.4) and since $d^2 = 0$, there are no further conditions arising from the d-invariance.

The PBW-property yields a number of compatibility conditions which are closely related to the well known Diamond Lemma (see e.g. [9]). The ordering is a lexicografic total degree ordering with $y^i < y^j < x^k < x^l$ for all $i < j$ and $k < l$. According to this the monomials of the form (2.3) are precisely the irreducible monomials. The compatibility conditions arise from rewriting terms of the form $x^i x^j x^k, x^i x^j y^k, x^i y^j y^k$ and $y^i y^j y^k$. Evidently rewriting the terms $x^i x^j x^k$ does not lead to any conditions since it simply boils down to the proof of the Poincaré-Birkhoff-Witt Theorem. We can rewrite $x^i x^j y^k$ as

$$x^i x^j y^k = x^i (y^l (\Theta^{jk}_{pl} x^p + A^{jk}_l)) = y^m (\Theta^{il}_{nm} x^n + A^{il}_m)(\Theta^{jk}_{pl} x^p + A^{jk}_l)$$

but also as

$$x^i x^j y^k = (x^j x^i + C^{ij}_l x^l) y^k = x^j y^m (\Theta^{ik}_{nm} x^n + A^{ik}_m) + C^{ij}_l y^p (\Theta^{lk}_{mp} x^m + A^{lk}_p)$$

$$= y^p (\Theta^{jm}_{lp} x^l + A^{jm}_p)(\Theta^{ik}_{nm} x^n + A^{ik}_m) + C^{ij}_l y^p (\Theta^{lk}_{mp} x^m + A^{lk}_p).$$

Due to the PWB-property of Ω the linear part of the expressions should coincide, yielding

$$A^{jk}_l A^{il}_m = A^{ik}_p A^{jp}_m + C^{ij}_l A^{lk}_m$$

We introduce matrices A^i by $(A^i)^j_k = A^{ij}_k$. Thus, the equation above can be written as

$$A^j A^i - A^i A^j = C^{ij}_l A^l. \tag{2.7}$$

The rest of the expressions give rise to the equation

$$y^m x^n (A^{jk}_l \Theta^{il}_{nm} + \Theta^{jk}_{nl} A^{il}_m - A^{ik}_p \Theta^{jp}_{nm} - \Theta^{ik}_{np} A^{jp}_m - C^{ij}_l \Theta^{lk}_{nm})$$

$$+ y^m x^n x^p (\Theta^{jk}_{pl} \Theta^{il}_{nm} - \Theta^{ik}_{pl} \Theta^{jl}_{nm}) = 0.$$

By making use of the following equality

$$x^n x^p = \frac{1}{2}(x^n x^p + x^n x^p) = \frac{1}{2}(x^n x^p + x^p x^n + C^{np}_l x^l)$$

the PBW-basis gives rise to two consistency conditions. The highest order part yields

$$\Theta^{jk}_{pl} \Theta^{il}_{nm} - \Theta^{ik}_{pl} \Theta^{jl}_{nm} + \Theta^{jk}_{nl} \Theta^{il}_{pm} - \Theta^{ik}_{nl} \Theta^{jl}_{pm} = 0$$

which can shortly be written as

$$[\Theta^i_n, \Theta^j_p] = [\Theta^j_n, \Theta^i_p] \tag{2.8}$$

where the matrices Θ^i_j are defined by $(\Theta^i_j)^k_l = \Theta^{ik}_{jl}$. Similarly the second order part coefficients give the condition

$$2([A^j, \Theta^i_n] + [\Theta^j_n, A^i] - C^{ij}_l \Theta^l_n) + (\Theta^j_p \Theta^i_q - \Theta^i_p \Theta^j_q) C^{pq}_n = 0. \tag{2.9}$$

We apply the same method to handle terms like $x^i y^j y^k$. We obtain

$$x^i y^j y^k = y^m (\Theta^{ij}_{lm} x^l + A^{ij}_m) y^k = \Theta^{ij}_{lm} y^m y^p (\Theta^{lk}_{qp} x^q + A^{lk}_p) + A^{ij}_m y^m y^k$$

$$= \Theta^{ij}_{lm} \Theta^{lk}_{qp} y^m y^p x^q + (\Theta^{ij}_{lm} A^{lk}_p - A^{ij}_l \Theta^{lk}_{pm}) y^m y^p$$

and on the other hand we can write

$$x^i y^j y^k = (\Theta^{jk}_{lm} A^{im}_p \Theta^{pl}_{ts} - \Theta^{jk}_{lm} \Theta^{im}_{ps} A^{pl}_t) y^s y^t - \Theta^{jk}_{lm} \Theta^{im}_{qp} \Theta^{ql}_{ts} y^p y^s x^t.$$

We remark that the relations (2.5) can be rewritten in the form

$$(I + S)^{ij}_{kl} y^k y^l = 0$$

where S is a matrix with the property $S^2 = I$. On account of the PBW-property of Ω we obtain the conditions

$$(\Theta^{ij}_{lp}\Theta^{lk}_{ts} + S^{jk}_{ml}\Theta^{im}_{qp}\Theta^{ql}_{ts})y^p y^s x^t = 0 \text{ and } (I+S)^{jk}_{ml}(\Theta^{im}_{sp}A^{sl}_q - A^{im}_s\Theta^{sl}_{qp})y^p y^q = 0.$$

Due to the fact that $\frac{1}{2}(I+S)$ is a projection the preceding equations are equivalent to

$$(2.10) \qquad \begin{aligned} &(I+S)^{jk}_{ml}\Theta^{im}_{qp}\Theta^{ql}_{ts}(I-S)^{ps}_{uv} = 0 \\ &(I+S)^{jk}_{ml}(\Theta^{im}_{sp}A^{sl}_q - A^{im}_s\Theta^{sl}_{qp})(I-S)^{pq}_{uv} = 0. \end{aligned}$$

The first condition can be written in the following elegant form

$$(2.11) \qquad (I+S)_{12}\Theta_{31}\Theta_{32}(I-S)_{12} = 0.$$

The subscripts denote the positions used to embed the $n^2 \times n^2$-matrices into the $n^3 \times n^3$-matrices, e.g. $(\Theta_{12})^{ijk}_{pqr} = \Theta^{ij}_{pq}\delta^k_r$. The consistency condition arising from rewriting the terms $y^i y^j y^k$ is

$$((I+S)_{12}\Theta_{31}\Theta_{32})^{ijk}_{pqr} y^p y^q y^r = 0.$$

This is evidently satisfied if (2.11) holds, so this gives no further conditions for the coefficients.

The conclusion is that the set of conditions (2.6),(2.7),(2.8),(2.9) and (2.10) is sufficient to define a De Rham complex Ω on $U(\mathfrak{g})$ with the PBW-property. In the next section we investigate the possibility to define a differential Hopf algebra on $U(\mathfrak{g})$.

3 A differential Hopf algebra on $U(\mathfrak{g})$

It is well known that the universal enveloping of a Lie algebra $\mathfrak{g}$ has a natural Hopf algebra structure (see e.g. [10]), its comultiplication $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is defined by $\Delta(x) = 1 \otimes x + x \otimes 1$, its counit $\epsilon : U(\mathfrak{g}) \to \mathbb{C}$ by $\epsilon(x) = 0$ for all x in $\mathfrak{g}$. They both are algebra morphisms. The antipode $S : U(\mathfrak{g}) \to U(\mathfrak{g})$ is the unique antialgebra morphism satisfying $S(x) = -x$. In the preceding section we discussed the construction of a differential algebra on $U(\mathfrak{g})$, which essentially is an algebra extension of $U(\mathfrak{g})$ equipped with a differential operator d. In this section we discuss the possibility of extending Δ, ϵ and S from $U(\mathfrak{g})$ to Ω in such a way that Ω becomes a differential Hopf algebra.

Let us first recall the notion of a differential bialgebra, for a complete description we refer to [4]. A differential bialgebra is a differential algebra Ω equipped with a comultiplication Δ and a counit ϵ that are differential algebra morphisms. Note that a differential algebra morphism $\varphi : \Omega \to \Omega'$ simply is an algebra morphism of degree zero that commutes with the differentials, i.e. $\varphi \circ d = d' \circ \varphi$.

In order to write down explicitly the conditions for Δ and ϵ we need to explain the differential algebra structures of $\Omega \otimes \Omega$ and $\mathbb{C}$. The $\mathbb{N}$-grading of $\Omega \otimes \Omega$ is defined by

$$(3.1) \qquad (\Omega \otimes \Omega)^p = \bigoplus_{0 \le q \le p} \Omega^q \otimes \Omega^{p-q}$$

and its multiplication $\mu_\otimes : \Omega \otimes \Omega \otimes \Omega \otimes \Omega \to \Omega \otimes \Omega$ by

$$(3.2) \qquad \mu_\otimes = (\mu \otimes \mu) \circ \sigma_{23}$$

where μ denotes the multiplication of Ω. The linear map σ_{23} denotes the graded flip $\sigma : \Omega \otimes \Omega \to \Omega \otimes \Omega$, which is defined by

$$(3.3) \qquad \sigma(a \otimes b) = (-1)^{pq} b \otimes a \qquad a \in \Omega^p, b \in \Omega^q,$$

applied to the second and third component of the tensor product. The differential $d_\otimes$ of $\Omega \otimes \Omega$ is given by $d_\otimes = d \otimes id + \tau \otimes d$ where $\tau : \Omega \to \Omega$ is the linear map of degree zero satisfying $\tau(a) = (-1)^p a$ for all $a \in \Omega^p$. Written out explicitly this gives

$$(3.4) \qquad d_\otimes(a \otimes b) = d(a) \otimes b + (-1)^p a \otimes d(b) \quad a \in \Omega^p, b \in \Omega.$$

Henceforth, the condition that Δ should be a differential algebra morphism means that Δ is an algebra morphism with the property

$$(3.5) \qquad d_\otimes \circ \Delta = (d \otimes id + \tau \otimes d) \circ \Delta = \Delta \circ d.$$

Note that this is the analogue of the derivation property (2.1) which can be written as $d \circ \mu = \mu \circ d_{\otimes}$. We consider $\mathbb{C}$ to be a differential algebra with $\mathbb{C}^0 = \mathbb{C}$ and $\mathbb{C}^p = 0$ for all $p > 0$. Hence, ϵ is a differential algebra morphism if and only if it is an algebra morphism satisfying $\epsilon \circ d = 0$.

Since Ω is generated by $\Omega^0 \cup d(\Omega^0)$, it suffices to define the actions of Δ and ϵ on $\{x^1, x^2, \dots, x^n, y^1, y^2, \dots, y^n\}$. Naturally the actions on elements x^i coincide with the previously described actions on $U(\mathfrak{g})$. The above mentioned conditions for Δ and ϵ uniquely determine the actions on the elements y^j:

$$\Delta(y^i) = \Delta \circ d(x^i) = d_{\otimes} \circ \Delta(x^i) = d_{\otimes}(x^i \otimes 1 + 1 \otimes x^i) = y^i \otimes 1 + 1 \otimes y^i \tag{3.6}$$

$$\epsilon(y^i) = \epsilon \circ d(x^i) = 0. \tag{3.7}$$

Here we have used that $d(1) = 0$ which is a direct consequence of the derivation property of d.

In order for ϵ and Δ to be well defined, the ideal J needs to be a (two-sided) coideal i.e. $\epsilon(J) = 0$ and $\Delta(J) \subset \Omega \otimes J + J \otimes \Omega$. The first condition is clearly satisfied. To verify the second we apply Δ to the relations (2.2), (2.4) and (2.5). Naturally the relations (2.2) do not give rise to any conditions for the coefficients Θ^{ij}_{lk} and A^{ij}_k. The relations (2.4) yield

$$\Delta(x^i y^j - \Theta^{ij}_{lk} y^k (x^l + A^{ij}_k)) = \Delta(x^i)\Delta(y^j) - \Theta^{ij}_{lk}\Delta(y^k)(\Delta(x^l) + A^{ij}_k 1 \otimes 1) =$$

$$1 \otimes x^i y^j + x^i \otimes y^j + y^j \otimes x^i + x^i y^j \otimes 1 - \Theta^{ij}_{lk}(1 \otimes y^k x^l + x^l \otimes y^k + y^k \otimes x^l + y^k x^l \otimes 1 + A^{ij}_k(1 \otimes y^k + y^k \otimes 1))$$

$$= x^i \otimes y^j + y^j \otimes x^i - \Theta^{ij}_{lk}(y^k \otimes x^l + x^l \otimes y^k) \quad mod \quad (\Omega \otimes J + J \otimes \Omega).$$

Again, on account of the PBW-property of Ω we obtain

$$x^i \otimes y^j = \Theta^{ij}_{lk} x^l \otimes y^k \text{ and } y^j \otimes x^i = \Theta^{ij}_{lk} y^k \otimes x^l \tag{3.8}$$

which evidently implies $\Theta = I$. Similarly we find

$$\Delta(y^i y^j + \Theta^{ij}_{lk} y^k y^l) =$$

$$y^i \otimes y^j - y^j \otimes y^i + \Theta^{ij}_{lk}(y^k \otimes y^l - y^l \otimes y^k) \quad mod \quad (J \otimes \Omega + \Omega \otimes J)$$

which boils down to the condition $(I - \Theta)(I - P) = 0$. Therefore, the conclusion is that Ω can only be a differential bialgebra on $U(\mathfrak{g})$ if $\Theta = I$. In that case the conditions (2.6),(2.7),(2.8) and (2.9) reduce to

$$(I - P)A = C \text{ and } [A^j, A^i] = C^{ij}_l A^l \tag{3.9}$$

due to the fact that $P^2 = I$ and $S^2 = I$. The commutation relations (2.4) and (2.5) take the following form:

$$x^i y^j = y^j x^i + A^{ij}_k y^k \text{ and } y^i y^j = -y^j y^i. \tag{3.10}$$

We remark that we did not check the coassociativity of Δ, i.e. $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$. This is a direct consequence of the coassociativity of Δ on $U(\mathfrak{g}) = \Omega^0$ since

$$(\Delta \otimes id) \circ \Delta \circ d(\Omega^0) = (\Delta \otimes id) \circ d_{\otimes} \circ \Delta(\Omega^0) = (d_{\otimes} \otimes id + \tau \otimes \tau \circ d) \circ (\Delta \otimes id) \circ \Delta(\Omega^0) =$$

$$(d \otimes id \otimes id + \tau \otimes d_{\otimes}) \circ (id \otimes \Delta) \circ \Delta(\Omega^0) = (id \otimes \Delta) \circ d_{\otimes} \circ \Delta(\Omega^0) = (id \otimes \Delta) \circ \Delta \circ d(\Omega^0).$$

By a similar reasoning one can show that the counit property of ϵ on Ω is induced by the same property of ϵ on $U(\mathfrak{g})$.

The next step is to consider a differential Hopf algebra structure on Ω. We recall that Ω is a differential Hopf algebra (see [4]) if it is a differential bialgebra which possesses an antipode S. An antipode S is an element of $End(\Omega)$ satisfying $S \star id = id \star S = \eta \circ \epsilon$, where $\star$ denotes the convolution product on $End(\Omega)$ (see [10]) defined by

$$f \star g = \mu \circ (f \otimes g) \circ \Delta \quad f, g \in End(\Omega). \tag{3.11}$$

By $\eta : \mathbb{C} \to \Omega$ we denote the unit element of Ω, i.e. $\eta(1) = 1$. Since $\eta \circ \epsilon$ is the unit element in $End(\Omega)$ with respect to the convolution product, one can describe the antipode as the unique inverse of the identity. An important property of the antipode is that it is an antialgebra morphism, this means that $S \circ \mu \circ \sigma = (S \otimes S) \circ \mu$ and $S \circ \eta = \eta$ (or equivalenty $S(1) = 1$).

We try to extend the antipode of $U(\mathfrak{g}) = \Omega^0$ to Ω. In order to do that we make the following observation. Suppose that f and g are homogeneous endomorphisms on Ω of degree zero which commute with the differential operator d. Then the properties $d_\otimes \circ \Delta = \Delta \circ d$ and $\mu \circ d_\otimes = d \circ \mu$ imply that

$$d \circ (f \star g) = d \circ \mu \circ (f \otimes g) \circ \Delta = \mu \circ d_\otimes \circ (f \otimes g) \circ \Delta$$

$$= \mu \circ (d \otimes id + \tau \otimes d) \circ (f \otimes g) \circ \Delta = \mu \circ (f \otimes g) \circ (d \otimes id + \tau \otimes d) \circ \Delta$$

$$= \mu \circ (f \otimes g) \circ d_\otimes \circ \Delta = \mu \circ (f \otimes g) \circ \Delta \circ d = (f \star g) \circ d.$$

Hence, the convolution product $f \star g$ also commutes with d. By inductive use of this argument we can conclude that d commutes with $f^n = f \star f \star \ldots \star f$ for all positive values of n if f is a homogeneous endomorphism of degree zero satisfying $d \circ f = f \circ d$. For f we can choose the identity. This makes it plausible that d commutes with $id^{-1} = S$. So, we extend the antipode by demanding $d \circ S = S \circ d$. In particular, this yields

$$S(y^i) = S \circ d(x^i) = d \circ S(x^i) = -d(x^i) = -y^i. \tag{3.12}$$

For S to be a well defined antialgebra morphism on Ω, it must leave the ideal J invariant ($S(J) \subset J$). To verify this condition, it suffices to apply S to the relations (3.10). We will only write out the first, the second can be handled similarly.

$$S(x^i y^j - y^j x^i - A^{ij}_k y^k) = S(y^j)S(x^i) - S(x^i)S(y^j) - A^{ij}_k S(y^k)$$

$$= y^j x^i - x^i y^j + A^{ij}_k y^k = -(x^i y^j - y^j x^i - A^{ij}_k y^k) \in J.$$

In order to check that S is indeed an antipode, we can confine ourselves to verifying the defining property of S for a set of generators of Ω. Since S is the extension of the antipode of $U(\mathfrak{g})$, we only need to compute

$$(S \star id)(y^i) = \mu \circ (S \otimes id)(1 \otimes y^i + y^i \otimes 1) = 1.y^i - y^i.1 = 0 = \eta \circ \epsilon(y^i).$$

So, at the end of this section we come to the following conclusion. In order to obtain a differential Hopf algebra Ω on $U(\mathfrak{g})$ it is necessary that the matrix Θ in (2.4) and (2.5) equals the identity matrix I. In that case necessary and sufficient conditions for the coefficients A^{ij}_k are given by (3.9) and the corresponding Hopf algebraic extension is described by (3.6), (3.7) and (3.12).

4 A Lie algebraic interpretation

The comultiplication of the universal enveloping algebra of a Lie algebra is cocommutative. We can easily verify that the cocommutativity of Δ on Ω^0 gives rise to graded cocommutativity for the extension of Δ to Ω:

$$\sigma \circ \Delta \circ d(\Omega^0) = \sigma \circ d_\otimes \circ \Delta(\Omega^0) = d_\otimes \circ \sigma \circ \Delta(\Omega^0) = d_\otimes \circ \Delta(\Omega^0) = \Delta \circ d(\Omega^0)$$

or more explicitly

$$\sigma \circ \Delta(y^i) = \sigma(1 \otimes y^i + y^i \otimes 1) = y^i \otimes 1 + 1 \otimes y^i = \Delta(y^i).$$

We call the property $\sigma \circ \Delta = \Delta$ graded cocommutativity because σ denotes the graded flip.

It is well known (see e.g. [11]) that a cocommutative Hopf algebra H, which has a compatible filtering, is isomorphic to the Hopf algebra corresponding to the universal enveloping algebra of the Lie algebra of primitive elements of H. Note that the set of primitive elements of H is defined by

$$P(H) = \{x \in H \quad | \quad \Delta(x) = 1 \otimes x + x \otimes 1\} \tag{4.1}$$

which has the structure of a Lie algebra with the commutator $[x, y] = xy - yx$. By a compatible filtering we mean an increasing family of subspaces $(F_p)_{p \in \mathbb{N}}$ of H satisfying

$$\mu(F_p \otimes F_q) \subset F_{p+q} \quad \Delta(F_p) \subset \sum_{0 \le q \le p} F_q \otimes F_{p-q} \quad \bigcup_{p \in \mathbb{N}} F_p = H. \tag{4.2}$$

With respect to the graded cocommutative Hopf algebra Ω we remark that $F_p = \oplus_{0\leq q\leq p}\Omega^q$ defines a compatible filtering. Due to the graded cocommutativity, $P(\Omega)$ does not have the structure of an ordinary Lie algebra. Instead of the usual commutator we define a "colour" commutator on Ω by $[x,y] = xy - (-1)^{pq}yx$ for $x \in \Omega^p$ and $y \in \Omega^q$. The set $P(\Omega)$ is closed under this bracket since

$$\Delta([x,y]) = \Delta(xy - (-1)^{pq}yx) = \Delta(x)\Delta(y) - (-1)^{pq}\Delta(y)\Delta(x) =$$

$$(1\otimes x + x\otimes 1)(1\otimes y + y\otimes 1) - (-1)^{pq}(1\otimes y + y\otimes 1)(1\otimes x + x\otimes 1) =$$
$$1\otimes xy + x\otimes y + (-1)^{pq}y\otimes x + xy\otimes 1 - (-1)^{pq}(1\otimes yx + y\otimes x + (-1)^{pq}x\otimes y + yx\otimes 1)$$
$$= 1\otimes (xy - (-1)^{pq}yx) + (xy - (-1)^{pq}yx)\otimes 1 = 1\otimes [x,y] + [x,y]\otimes 1$$

for all $x \in P(\Omega)^p$ and $y \in P(\Omega)^q$. Note that the $\mathbb{N}$-grading of Ω induces an $\mathbb{N}$-grading on $P(\Omega)$ since Δ is homogeneous of degree zero. By this argument we have derived that $P(\Omega)$ is a colour Lie superalgebra. For the definition of a colour Lie superalgebra we refer to [12]. The corresponding 2-cocycle α is given by

$$\alpha : \mathbb{N}\times\mathbb{N} \to \mathbb{C}^* \qquad \alpha(p,q) = (-1)^{pq}. \tag{4.3}$$

Since $\alpha(q,q) = (-1)^{q^2} = (-1)^q$, the elements of $P(\Omega)^q$ are even (odd) if and only if q is even (odd). As in the case of ordinary Lie algebras, one can define the universal enveloping algebra of a colour Lie superalgebra and a corresponding Hopf algebraic structure on it. Analogous to the above mentioned result of [11], the Hopf algebra Ω is isomorphic to the Hopf algebra $U(P(\Omega))$.

The question is what $P(\Omega)$ explicitly looks like. The set of primitive elements of Ω certainly contains the linear span of $\{x^1, x^2, .., x^n, y^1, y^2, .., y^n\}$ (see (3.6)) which we will denote by L. We consider L as the $\mathbb{N}$-graded vector space $L = \bigoplus_{p\in\mathbb{N}} L^p$ with

$$L^0 = \langle x^i\rangle_{1\leq i\leq n} \quad L^1 = \langle y^i\rangle_{1\leq i\leq n} \quad L^p = 0 \quad (p\geq 2).$$

According to the relations (2.2) and (3.10) we have

$$[x^i, x^j] = C^{ij}_k x^k, \quad [x^i, y^j] = A^{ij}_k y^k \text{ and } \quad [y^i, y^j] = 0 \tag{4.4}$$

so L is a colour Lie supersubalgebra in $P(\Omega)$. Since the elements of L^0 are even and the elements of L^1 are odd, a basis of $U(L)$ can be given by (see e.g. [12])

$$y^{j_1}y^{j_2}\dots y^{j_q}x^{i_1}x^{i_2}\dots x^{i_p} \text{ with } j_1 < j_2 < \dots < j_q, i_1 \leq i_2 \leq \dots \leq i_p \text{ and } p, q \geq 0.$$

A comparison of this basis with the described basis of Ω (see (2.3)), proves that the Hopf algebra Ω is isomorphic to the Hopf algebra $U(L)$. Hence, $P(\Omega) = L$.

The preceding reasoning enables us to interprete the construction of a differential Hopf algebra on the universal enveloping of a Lie algebra $\mathfrak{g}$ in terms of a colour Lie superalgebraic extension of $\mathfrak{g}$. The problem to be solved can be reformulated as follows. Given is a Lie algebra $\mathfrak{g}$ with basis $\{x^1, x^2, \dots, x^n\}$. Let L be the $\mathbb{N}$-graded vector space given by $L^0 = \mathfrak{g}$, $L^1 = \langle y^i\rangle_{1\leq i\leq n}$ and $L^p = 0$ for all $p \geq 2$. Define a bilinear operation [,] on L of degree zero extending the commutator of $\mathfrak{g}$ in such a way that $[x^i, y^j] = A^{ij}_k y^k$. This bracket expresses the commutation relations (3.10). The conditions (3.9) for the coefficients A^{ij}_k are equivalent to demanding that L equipped with this commutator becomes a colour Lie superalgebra with 2-cocyle α given by (4.3), satisfying the additional property that the linear map d defined by $d(x^i) = y^i$ and $d(y^i) = 0$ is a graded derivation of degree 1 on L. The corresponding differential Hopf algebra on $U(\mathfrak{g})$ is then given by the Hopf algebra $U(L)$ equipped with the unique extension of the derivation d from L to $U(L)$.

5 Concluding Remarks

We have presented a framework to construct a De Rham complex on the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. The fundamental property of the differential algebra is that it possesses a so-called PBW-basis. We have proven that the differential algebra can be given a Hopf algebra structure extending the natural Hopf algebra $U(\mathfrak{g})$. In our presentation we assumed $\mathfrak{g}$ to be finite dimensional. This is definitely not a necessary condition, one can easily see that this framework can also be applied in the infinite dimensional case. For more details on this and some explicit examples we refer to [13]. Naturally, a De Rham complex on $U(\mathfrak{g})$ brings to surface the notion of cohomology. It would be very interesting to investigate this De Rham cohomology.

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