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ABSTRACT

We show for the M/G/1 processor sharing queue that the service time distribution is regularly varying of index $-\nu$, ν non-integer, iff the sojourn time distribution is regularly varying of index $-\nu$. This result is derived from a new expression for the Laplace-Stieltjes transform of the sojourn time distribution. That expression also leads to other new properties for the sojourn time distribution. We show how the moments of the sojourn time can be calculated recursively and prove that the k-th moment of the sojourn time is finite iff the k-th moment of the service time is finite. In addition, we give a short proof of a heavy traffic theorem for the sojourn time distribution, prove a heavy traffic theorem for the moments of the sojourn time, and study the properties of the heavy traffic limiting sojourn time distribution when the service time distribution is regularly varying. Explicit formulas and multiterm expansions are provided for the case that the service time has a Pareto distribution.

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1. Introduction

In this paper, we investigate asymptotic properties of the sojourn time distribution in the stable M/G/1 processor sharing (PS) queue. In the (egalitarian) processor sharing service discipline every customer is being served with rate $1/X$ with X the number of customers in the system. An extensive overview on processor sharing queues can be found in the surveys [41, 42].

In particular, we are interested in the tail behaviour of the sojourn time distribution when the service time distribution $B(x)$ has a heavy tail, i.e.

$$1 - B(x) \sim h_\nu x^{-\nu}, \tag{1.1}$$

if $x \to \infty$ (with $f(x) \sim g(x)$ we mean $f(x)/g(x) \to 1$), and $1 < \nu < 2$. Queueing systems in which the tail of the service time behaves like (1.1) have recently become important in the performance modelling and analysis of communication traffic. The main reason for this is that extensive traffic measurements for traffic in Ethernet Local Area Networks [39], Wide Area Networks [33], and VBR video [7], exhibit phenomena like self-similarity and long-range

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dependence – phenomena that can be explained by the occurrence of service requirements like in (1.1).

A convenient class of distribution functions that satisfies (1.1) and enables a tractable asymptotic analysis is the class of distribution functions with *regularly varying tails*, see the appendix.

The main result in this study is presented in Theorem 4.1, which states that the sojourn time distribution of the M/G/1 processor sharing queue is regularly varying of index $-\nu$ if and only if the service time distribution is regularly varying of index $-\nu$, ν not an integer. More precisely, Theorem 4.1 provides explicit equivalents of the former in terms of the latter. To the best of the authors' knowledge, this is the first asymptotic expansion for the sojourn time distribution in the M/G/1 processor sharing queue – even the asymptotic behaviour of the sojourn time distribution in the M/M/1 processor sharing queue seems to be unknown.

Theorem 4.1 reveals a crucial property of the processor sharing service discipline. For the $GI/G/1$ queue in which the service discipline is first come first served (FCFS), Cohen [17] has shown that the waiting time distribution is regularly varying of index $1-\nu$ iff the service time distribution is regularly varying of index $-\nu$. This implies that if the latter is the case, also the sojourn time is regularly varying of index $1-\nu$, which means that the tail of the sojourn time distribution is even fatter than the tail of the service time distribution. This is due to the FCFS discipline, in which long jobs are favoured, and short jobs can be held up by long jobs. Theorem 4.1 implies that the processor sharing discipline is more effective in handling heavy-tailed service times: In a processor sharing system, short jobs can overtake long jobs, so the influence of long jobs on the sojourn time of short jobs is limited. This consideration has formed the main motivation for the present study.

To establish Theorem 4.1, we will use the Tauberian theorem due to Bingham and Doney [9] and the expression for the Laplace-Stieltjes transform (LST) of the sojourn time distribution given by Ott [32]. This transform is derived by representing the sojourn time as a functional on a branching process, cf. [40]. A more general relationship between branching processes and processor sharing queues is investigated in [24]. For an overview on the connections between regular variation and branching processes we refer to Chapter 8.12 of [11] and more in particular to [9, 10].

In order to prove Theorem 4.1, we rewrite the expression for the LST of the sojourn time distribution. Known expressions for the LST of the sojourn time, see [32, 36, 40], all contain contour integrals which are inversion formulas of Laplace transforms. We show how to get rid of these contour integrals and thus obtain a more explicit formula. Using this result, we show how the moments of the sojourn time can be calculated recursively and prove that the k−th moment of the sojourn time is finite iff the k−th moment of the service time is finite.

Apart from the tail behaviour of the sojourn time distribution, we study some properties of the sojourn time in heavy traffic, in particular when the service time distribution satisfies (1.1). We give a new proof of a heavy traffic theorem due to [37, 43] and prove a similar statement for the moments of the sojourn time in heavy traffic. When the service time has a Pareto distribution, it is possible to give an explicit formula for the heavy traffic limiting distribution. More generally, we show that the heavy traffic limiting distribution is regularly varying of index $-\nu$ if the service time distribution is regularly varying of index $-\nu$, $\nu > 1$.

The paper is organised as follows. Preliminary results are given in Section 2. In Section

3, we derive a new expression for the Laplace-Stieltjes transform (LST) of the sojourn time distribution and study the moments of the sojourn time. Section 4 establishes the link between the tail behaviour of the service time and the sojourn time with Theorem 4.1, of which the proof is given in Section 5. The heavy traffic analysis is performed in Section 6. Section 7 contains conclusions and suggestions for further research. A short overview of regular variation is given in the appendix.

2. Preliminaries

For later use, we give in this section a short review of the M/G/1 PS queue. Customers arrive according to a Poisson process with rate $\lambda > 0$. The service time B of a customer has distribution $B(t)$ with $B(0+) = 0$ and first moment $0 < \beta_1 < \infty$. In this study, it is assumed that the workload is less than one, i.e. $\rho := \lambda\beta_1 < 1$. The LST of the service time distribution is given by $\beta(s)$. If the $k-$th moment of the service time exists, it is denoted by β_k.

In the analysis in the next sections, the integrated tail distribution (or excess distribution) of the service time will be useful. The distribution function and LST of the excess service time are respectively given by

$$\widetilde{B}(t) := \frac{1}{\beta_1}\int_0^t (1 - B(x))\mathrm{d}x, \qquad t \geq 0,$$

$$\widetilde{\beta}(s) := \int_0^\infty \mathrm{e}^{-st}\mathrm{d}\widetilde{B}(t) = \frac{1-\beta(s)}{\beta_1 s}, \qquad \mathrm{Re}\ s \geq 0.$$

It is well known, due to Sakata et al. [35] (see also [26]), that the steady state distribution $(P_n)_{n\geq 0}$ of the number of customers in the system is geometrically distributed and only depends on the service time through its mean:

$$P_n = (1-\rho)\rho^n.$$

Determination of the distribution of the *sojourn time* of a customer in steady state, defined by the r.v. V, has turned out to be a more difficult problem. Define the r.v. $V(\tau)$ as the sojourn time of a customer entering the system in steady state having a service time equal to τ. Define the LST of $V(\tau)$ by

$$v(s,\tau) := \mathbb{E}\left[\mathrm{e}^{-sV(\tau)}\right].$$

Yashkov has derived an expression for $v(s,\tau)$ by writing the sojourn time as a functional on a branching process, see Yashkov [40]. Using the structure of the branching process, Yashkov found (and solved) a system of differential equations determining $v(s,\tau)$.

Similar results for $v(s,\tau)$, each obtained by using a different approach, are obtained in [6, 36]. For our purposes, the expression for $v(s,\tau)$ derived in [32] is the most suitable one. It is given by (see also [32], p. 367–368)

$$v(s,\tau) = \frac{1-\rho}{(1-\rho)H_1(s,\tau) + sH_2(s,\tau)}, \tag{2.1}$$

where the functions H_1 and H_2 are given by,

$$\int_0^\infty \mathrm{e}^{-x\tau}\mathrm{d}H_1(s,\tau) = \frac{x-\lambda(1-\beta(x))}{x-s-\lambda(1-\beta(x))}, \qquad \mathrm{Re}\, x > 0, \tag{2.2}$$

$$\int_0^\infty \mathrm{e}^{-x\tau}\mathrm{d}H_2(s,\tau) = \frac{\rho x-\lambda(1-\beta(x))}{x(x-s-\lambda(1-\beta(x)))}, \qquad \mathrm{Re}\, x > 0. \tag{2.3}$$

Define the $k-$th moment of $V(\tau)$ by $v_k(\tau)$. The first moment of $V(\tau)$ is given by, cf. [27], p. 168:

$$v_1(\tau) = \frac{\tau}{1-\rho}. \tag{2.4}$$

Note that $v_1(\tau)$ is linear in τ. An immediate consequence of (2.4) (or of the expression for $(P_n)_{n\geq 0}$ and Little's formula) is that the first moment of the sojourn time $\mathbb{E}[V]$ is finite and equals $\frac{\beta_1}{1-\rho}$ if $\beta_1 < \infty$. In Section 3, we will show that a similar result holds for higher moments of the sojourn time. This property is contrasting with the FCFS service discipline, where finiteness of $\mathbb{E}[V]$ requires $\beta_2 < \infty$. We come back to this in Section 4, where we study the tail behaviour of V. Closing the section, we define the $n-$fold convolution F^{n*} of a distribution function F of a non-negative random variable by, for $x \geq 0$,

$$\begin{aligned} F^{0*}(x) &= 1, \\ F^{n*}(x) &= \int_0^x F^{(n-1)*}(x-u)\mathrm{d}F(u), \qquad n = 1, 2, \ldots . \end{aligned}$$

3. A new expression for the sojourn time distribution

The goal of this section is to provide a novel expression for $v(s,\tau)$ that will be suitable for analysing the tail behaviour of the sojourn time distribution in the next section. In particular, we show that $v(s,\tau)^{-1}$ can be written as a power series in s. It turns out that the expression contains the LST of the waiting time distribution $R(x)$ in the M/G/1 FCFS queue, which is given by the Pollaczek-Khintchine formula, i.e.

$$\omega(s) := \int_0^\infty \mathrm{e}^{-sx}\mathrm{d}R(x) = \frac{1-\rho}{1-\rho\widetilde{\beta}(s)}. \tag{3.1}$$

It can easily be shown by inversion of $\omega(s)^k$ that, for $k \geq 1$ and $x \geq 0$,

$$R^{k*}(x) = (1-\rho)^k \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} \rho^n \widetilde{B}^{n*}(x). \tag{3.2}$$

We introduce some definitions before the main result of this section is presented. Define the coefficients $\alpha_k(\tau)$, with $k \geq 0$ and $\tau \geq 0$, by $\alpha_0(\tau) := 1$, $\alpha_1(\tau) := \frac{\tau}{1-\rho}$, and for $k \geq 2$,

$$\alpha_k(\tau) := \frac{k}{(1-\rho)^k} \int_{x=0}^{\tau} (\tau - x)^{k-1} R^{(k-1)*}(x) \mathrm{d}x. \tag{3.3}$$

Obviously we can write

$$\alpha_k(\tau) = \left(\frac{\tau}{1-\rho}\right)^k - \delta_k(\tau), \tag{3.4}$$

with $\delta_0(\tau) = \delta_1(\tau) := 0$, and

$$\delta_k(\tau) := \frac{k}{(1-\rho)^k} \int_0^{\tau} (\tau - x)^{k-1} (1 - R^{(k-1)*}(x)) \mathrm{d}x, \qquad k = 2, 3, \dots . \tag{3.5}$$

The next theorem expresses $v(s,\tau)^{-1}$ as a power series in s with coefficients $\frac{\alpha_k(\tau)}{k!}$.

Theorem 3.1 *For* $\mathrm{Re}\ s \geq 0, \tau \geq 0$:

$$v(s,\tau) = \left[\sum_{k=0}^{\infty} \frac{s^k}{k!} \alpha_k(\tau)\right]^{-1}. \tag{3.6}$$

The theorem will be proven by analysing the LST of $v(s,\tau)^{-1}$. It is also possible to prove Theorem 3.1 without using transforms, starting from Formula (5.2) in [41]. However, this proof is rather lengthy and therefore omitted. Instead, we give a short proof of Theorem 3.1 with the aid of the following lemma.

Lemma 3.1 *For* $\mathrm{Re}\ s \geq 0$ *and* $\mathrm{Re}\ x > 0$:

$$\int_0^{\infty} \mathrm{e}^{-x\tau} \mathrm{d}v(s,\tau)^{-1} = 1 + \frac{1}{1-\rho} \frac{s}{x} \frac{1}{1 - \frac{1}{1-\rho}\frac{s}{x}\omega(x)}. \tag{3.7}$$

Proof By (2.1)–(2.3) and (3.1) we have for $\mathrm{Re}\ x > 0$:

$$\begin{aligned}
\int_0^{\infty} \mathrm{e}^{-x\tau} \mathrm{d}v(s,\tau)^{-1} &= \frac{x - \lambda(1-\beta(x))}{x - s - \lambda(1-\beta(x))} + \frac{s}{1-\rho} \frac{\rho x - \lambda(1-\beta(x))}{x(x - s - \lambda(1-\beta(x)))} \\
&= 1 + \frac{1}{1-\rho} \frac{s - s\lambda(1-\beta(x))/x}{x - s - \lambda(1-\beta(x))} \\
&= 1 + \frac{1}{1-\rho} \frac{s}{x} \frac{1 - \rho\widetilde{\beta}(x)}{1 - \rho\widetilde{\beta}(x) - \frac{s}{x}} \\
&= 1 + \frac{1}{1-\rho} \frac{s}{x} \frac{1}{1 - \frac{1}{1-\rho}\frac{s}{x}\omega(x)},
\end{aligned}$$

which proves the lemma. ■

Proof of Theorem 3.1 It is sufficient to show that the LST of the power series in the denominator of the right hand side of (3.6) has the same LST as $v(s,\tau)^{-1}$ for Re $x > |s| + \lambda$. It is not difficult to show using the expression for $\omega(s)$ that $\left|\frac{s\omega(x)}{(1-\rho)x}\right| < 1$ if Re $x > |s| + \lambda$. Hence, we have by Lemma 3.1 that

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-x\tau} \mathrm{d}v(s,\tau)^{-1} &= 1 + \frac{1}{1-\rho}\frac{s}{x}\frac{1}{1-\frac{1}{1-\rho}\frac{s}{x}\omega(x)} \\ &= 1 + \sum_{k=1}^{\infty} \left(\frac{1}{1-\rho}\frac{s}{x}\right)^k \omega(x)^{k-1}. \end{aligned} \tag{3.8}$$

On the other hand, we have for $k \geq 1$, cf. (3.3):

$$\int_0^\infty \mathrm{e}^{-x\tau} \mathrm{d}\alpha_k(\tau) = \frac{1}{x^k}\frac{k!}{(1-\rho)^k}\omega(x)^{k-1},$$

which implies (3.5). ■

As a first application of Theorem 3.1 we show how the moments $v_k(\tau)$ can be found recursively. Note that all $v_k(\tau)$ exist and are equal to $(-1)^k\left(\frac{\partial^k}{\partial s^k}v\right)(0,\tau)$, since Theorem 3.1 implies that $v(s,\tau)$ is analytic in $s = 0$. From (3.5) we obtain the identity

$$v(s,\tau)\sum_{n=0}^{\infty}\frac{s^n}{n!}\alpha_n(\tau) = 1.$$

Differentiating both sides k times w.r.t. s and putting $s = 0$, we obtain the following result (with $v_0(\tau) := 1$).

Corollary 3.1 *For $k \geq 1$ and $\tau \geq 0$,*

$$v_k(\tau) = -\sum_{j=1}^{k}\binom{k}{j} v_{k-j}(\tau)\alpha_j(\tau)(-1)^j. \tag{3.9}$$

In particular, the variance of $V(\tau)$ is given by

$$\mathbb{V}\mathrm{ar}(V(\tau)) = \delta_2(\tau), \qquad \tau \geq 0. \tag{3.10}$$

This result is also obtained in [40].

Remark 3.1 Apart from being a tool in the proof of Theorem 3.1, Lemma 3.1 is also useful for the determination of a tractable expression for $v(s,\tau)$. For example, if the service time is exponentially distributed with parameter μ, it is possible to invert the right hand side of (3.7) by partial fraction expansion, which yields the following expression for $v(s,\tau)$:

$$v(s,\tau) = \left[\frac{s}{1-\rho}\frac{\mu+\lambda-x_1(s)}{x_1(s)x_0(s)}\mathrm{e}^{x_1(s)\tau} - \frac{s}{1-\rho}\frac{\mu+\lambda-x_2(s)}{x_2(s)x_0(s)}\mathrm{e}^{x_2(s)\tau} - \frac{2\rho}{1-\rho}\right]^{-1},$$

with $x_0(s) = x_1(s) - x_2(s)$ and

$$x_1(s) = \frac{1}{2}\left[s+\lambda-\mu+\sqrt{(s+\lambda-\mu)^2+4\mu s}\right],$$

$$x_2(s) = \frac{1}{2}\left[s+\lambda-\mu-\sqrt{(s+\lambda-\mu)^2+4\mu s}\right].$$

Equivalence with the result in [16] can be established by noting that $x_1(s) = \lambda\pi(s) - \mu$, where $\pi(s)$ is the LST of the busy period distribution in the M/M/1 queue. We omit the details.

More generally, the right hand side of (3.7) can be inverted when the LST of the service time distribution is a rational function, since then also $\omega(s)$ and the right hand side of (3.7) are rational functions. □

Remark 3.2 A challenging task (and a topic for our further research) is to derive more explicit expressions for the LST and the moments of $V(\tau)$ when the service time distribution has a heavy tail. Recently, an explicit expression for the waiting time distribution in the standard M/G/1 queue with a particular class of heavy-tailed service time distributions has been found by Boxma and Cohen [14], and has been extended by Abate and Whitt [4]. These results can be used to calculate exact expressions for the second moment of $V(\tau)$ (cf. (3.10) and (3.5) for $k = 2$) for a particular class of heavy-tailed service time distribution. It might be possible to use the results in [4, 14] to obtain a tractable expression for $v(s,\tau)$ using Lemma 3.1.

When the LST of the service time distribution is a rational function, it is possible to obtain an explicit expression for the waiting time distribution in the standard M/G/1 queue and to invert the right hand side of (3.7). However, this class of distribution functions does not satisfy (1.1). □

Remark 3.3 If $\beta_2 < \infty$, we have the following two-term asymptotic expansion for $v_k(\tau)$.

$$v_k(\tau) = v_1^k(\tau) + \frac{\beta_2}{2\beta_1}\frac{\rho}{1-\rho}\frac{k(k-1)}{(1-\rho)^k}\tau^{k-1} + \mathrm{o}(\tau^{k-1}), \qquad \tau\to\infty.$$

This result can be derived by analysing the behaviour of $\delta_k(\tau)$ for $\tau\to\infty$ by means of its LST and applying the classical Tauberian theorem (cf. [19], App. 4). Then, apply Corollary 3.1 and induction. We omit the details. The case $k = 2$ is similar to a result in [40]. □

Using Corollary 3.1, it is not difficult to show that the $k-$th moment of the sojourn time is finite iff the $k-$th moment of the service time is finite.

Corollary 3.2 *For integer $k \geq 1$,*

$$\mathbb{E}[V^k] < \infty \qquad \Leftrightarrow \qquad \beta_k < \infty.$$

Proof Since $V \geq B$ for any particular customer, '$\Rightarrow$' is trivial. To prove '$\Leftarrow$', fix $k \geq 1$ and write

$$\mathbb{E}[V^k] = \int_0^\infty v_k(\tau)\mathrm{d}B(\tau). \tag{3.11}$$

Note that, cf. (3.3), for $j \geq 1$,

$$\alpha_j(\tau) \leq \frac{\tau^j}{(1-\rho)^j}.$$

From this and Corollary 3.1, it is easily shown that

$$v_k(\tau) \leq \frac{C_k}{(1-\rho)^k}\tau^k, \tag{3.12}$$

with $C_0 = 1$ and

$$C_k = \sum_{j=0}^{k-1} \binom{k}{j} C_j, \qquad k \geq 1. \tag{3.13}$$

The proof follows from (3.11)–(3.13). ■

Corollary 3.2 indicates that the tail behaviour of the service time distribution and the sojourn time distribution are similar. In the next section, we will study this relation in the case that the service time distribution or the sojourn time distribution has a regularly varying tail of index $-\nu$.

4. Sojourn time asymptotics for a heavy tailed service time distribution

In this section we present Theorem 4.1, the main result in this study. Theorem 4.1 establishes an equivalence between the tail behaviour of the service time distribution and the sojourn time distribution. With L we denote a slowly varying function, cf. the appendix.

Theorem 4.1 *Let $\nu > 1$, ν not an integer. The following are equivalent.*

(i) $\mathbb{P}(B > x) \sim x^{-\nu}L(x), \qquad x \to \infty,$

(ii) $\mathbb{P}(V > x) \sim (1-\rho)^{-\nu}x^{-\nu}L(x), \qquad x \to \infty.$

The proof of Theorem 4.1 is based on Lemma 8.1 in the Appendix and Theorem 3.1 and will be presented in the next section.

Taking Corollary 3.2 into account (see also Remark 4.2 below), we expect the result of Theorem 4.1 to be true for all $\nu > 1$. A proof for integer ν will bring extra difficulties, see Chapter 3 and Theorem 8.1.6 of [11]. We refer to De Meyer and Teugels [30], where a similar statement has been proven for the busy period in the M/G/1 queue. They give a proof for general $\nu > 0$, including integer values. Note that $1 < \nu < 2$, the heavy-tailed case, is the most interesting one, see also Section 1.

Remark 4.1 For the GI/G/1 queue with FCFS service discipline, Cohen [17] has shown that the tail of the waiting time is regularly varying of index $1 - \nu$ iff the service time is regularly varying of index $-\nu$, $\nu > 1$. The same result holds for the sojourn time distribution. Theorem 4.1 shows that, in the M/G/1 queue with processor sharing, the sojourn time is as heavy as the tail of the service time. This reveals a crucial property of processor sharing: Long jobs have a much smaller effect on the delay of other customers than in the case of FCFS, where the heavy-tailed service time distribution gives rise to an even heavier tail of the sojourn time distribution. □

Remark 4.2 Note that both **(i)** and **(ii)** in Theorem 4.1 imply

$$\mathbb{P}(V > x) \sim \mathbb{P}(B > (1-\rho)x), \tag{4.1}$$

for $x \to \infty$. It is possible to give an interpretation of the constant $(1 - \rho)$ appearing in (4.1). When a tagged customer is in the system for a long time, the distribution of the total number of customers is approximately equal to the steady state distribution of the number of customers in a processor sharing queue with one permanent customer. This model is a special case of the M/G/1 *generalised* processor sharing queue, as studied by Cohen [18]. Using the results obtained in [18], it is possible to show that the mean service rate in steady state for the tagged (permanent) customer equals $1 - \rho$. Hence, if a tagged customer has been in the system for x time periods, with x large, one would expect that the amount of service attained is equal to $x(1 - \rho)$.

It must be emphasised that the above heuristics do not apply in general. For example, (4.1) is not true if the service time is exponentially distributed, as can be shown from the expression for $\mathbb{P}(V > x)$ in the M/M/1 PS queue given by Morrison [29]. An explanation for this is that, when the service time distribution is exponential, the tagged customer does not stay in the system long enough to reach the equilibrium situation sketched above. □

Remark 4.3 A result related to the observations in Remark 4.2 is the following. If $\beta_1 < \infty$, then

$$\frac{V(\tau)}{\tau} \to \frac{1}{1-\rho}, \qquad \tau \to \infty,$$

where the convergence is in probability. This follows immediately from $\mathbb{V}\mathrm{ar}(V(\tau)) = \mathrm{o}(\tau^2)$, $\tau \to \infty$, and Chebyshev's inequality (the first argument can be derived from (3.10) and the fact that $R(x)$ is a proper distribution function). We conjecture that the convergence also holds with probability one. $\square$

Remark 4.4 The asymptotics for $v_k(\tau)$ given in Remark 3.3 are not valid if $1 - B(x) = x^{-\nu}L(x)$, $1 < \nu < 2$. However, it is still possible to obtain a two-term expansion for $v_k(\tau)$ in this case. We mention the result for $k = 2$.

$$v_2(\tau) - v_1^2(\tau) = \mathbb{V}\mathrm{ar}(V(\tau)) = \delta_2(\tau) \sim \frac{B(2, 2-\nu)}{(1-\rho)^3}\frac{2\lambda}{\nu - 1}\tau^{3-\nu}L(\tau), \qquad \tau \to \infty, \tag{4.2}$$

where $B(.,.)$ is the Beta-function. This result can be derived from Karamata's theorem (see (8.3) in the appendix) and the asymptotics for $1 - R(x)$ given by Cohen [17]. A similar result holds for $v_k(\tau) - v_1^k(\tau)$, $k > 2$, which can be derived from Corollary 3.1 and asymptotics for $1 - R^{(k-1)*}(x)$ (cf. (8.6)). We omit the details. $\square$

Remark 4.5 Define the *delay time* W of a customer entering the system in steady state as the sojourn time minus the length of the service request. The conditional delay time $W(\tau)$ is given by, cf. [41],

$$W(\tau) = V(\tau) - \tau. \tag{4.3}$$

The LST's of W and $W(\tau)$ are denoted by $w(s)$ and $w(s, \tau)$. Note that

$$\mathbb{E}[W(\tau)] = \frac{\rho\tau}{1-\rho}, \tag{4.4}$$

$$w(s, \tau) = \mathrm{e}^{s\tau}v(s, \tau). \tag{4.5}$$

One can show that the $k-$th moment of W is finite iff the $k-$th moment of the service time is finite. If the latter is the case, this follows from Corollary 3.2 and the fact that $W \leq V$ for any particular customer. If the former holds, use Jensen's inequality and $\rho > 0$:

$$\infty > \mathbb{E}[W^k] = \int_0^\infty \mathbb{E}[W(\tau)^k]\mathrm{d}B(\tau) \geq \left(\frac{\rho}{1-\rho}\right)^k \int_0^\infty \tau^k \mathrm{d}B(\tau) = \left(\frac{\rho}{1-\rho}\right)^k \beta_k.$$

If the service time distribution is regularly varying of index $-\nu$, $1 < \nu < 2$, it is possible to show from (4.3)–(4.5), following a similar analysis as in the proof of Theorem 4.1 in the next section, that for $x \to \infty$,

$$\mathbb{P}(W > x) \sim \mathbb{P}\left(\frac{\rho}{1-\rho}B > x\right).$$

$\square$

5. Proof of Theorem 4.1

First, we will give a proof for the case $1 < \nu < 2$. The proof for the case $\nu > 2$ is somewhat different. In both cases, we will use the relation

$$v(s) = \int_0^\infty v(s,\tau)\mathrm{d}B(\tau),$$

and the expression for $v(s,\tau)$ given by Theorem 3.1. Also, the Tauberian theorem of Bingham and Doney [9] (see Lemma 8.1 in the Appendix) is applied in both cases.

It turns out that the asymptotic analysis of $v(s)$ for $s \to 0$ is rather intricate. In several cases we need to write $v(s) - \beta\left(\frac{s}{1-\rho}\right)$ and related quantities (like the residual term of a Taylor expansion of $v(s) - \beta\left(\frac{s}{1-\rho}\right)$ in the case $\nu > 2$) as

$$v(s) - \beta\left(\frac{s}{1-\rho}\right) = \int_0^{h(s)} \left[v(s,\tau) - \mathrm{e}^{-\frac{s\tau}{1-\rho}}\right]\mathrm{d}B(\tau) + \int_{h(s)}^\infty \left[v(s,\tau) - \mathrm{e}^{-\frac{s\tau}{1-\rho}}\right]\mathrm{d}B(\tau),$$

for some function $h(s)$ like $h(s) = \frac{1}{s}$. This decomposition turns out to be convenient. In most cases, we only need a trivial upper bound for the last integrand, and a non-trivial upper bound for the first integrand, using the inequality $\tau \leq h(s)$.

Case I: $1 < \nu < 2$

By the Tauberian theorem of Bingham and Doney [9] (see Lemma 8.1 in the appendix) and the fact that $\mathbb{E}(V) = \frac{\beta_1}{1-\rho}$, it suffices to show that

$$v(s) - \beta\left(\frac{s}{1-\rho}\right) = \mathrm{o}(s^\nu L(1/s)) \tag{5.1}$$

for $s \downarrow 0$ and s real. First, note that $v(s) - \beta\left(\frac{s}{1-\rho}\right) \geq 0$ for real s by Jensen's inequality and (2.2) (or by using $\delta_k(\tau) \geq 0$ and Theorem 3.1):

$$\begin{aligned} v(s) &= \int_0^\infty v(s,\tau)\mathrm{d}B(\tau) = \int_0^\infty \mathbb{E}\left[\mathrm{e}^{-sV(\tau)}\right]\mathrm{d}B(\tau) \\ &\geq \int_0^\infty \mathrm{e}^{-s\mathbb{E}[V(\tau)]}\mathrm{d}B(\tau) = \int_0^\infty \mathrm{e}^{-\frac{s\tau}{1-\rho}}\mathrm{d}B(\tau) = \beta\left(\frac{s}{1-\rho}\right). \end{aligned}$$

The following representation for $v(s) - \beta\left(\frac{s}{1-\rho}\right)$ is crucial in the remainder of the proof. By our representation in Theorem 3.1 for $v(s,\tau)$ and (3.4) we have:

$$
\begin{aligned}
& v(s) - \beta\left(\frac{s}{1-\rho}\right) \\
= & \int_0^\infty \left(v(s,\tau) - \mathrm{e}^{-\frac{s\tau}{1-\rho}}\right) \mathrm{d}B(\tau) \\
= & \int_0^\infty \left(\frac{1}{\mathrm{e}^{\frac{s\tau}{1-\rho}} - \sum_{k=2}^\infty \frac{s^k}{k!}\delta_k(\tau)} - \mathrm{e}^{-\frac{s\tau}{1-\rho}}\right) \mathrm{d}B(\tau) \\
= & \int_0^\infty \frac{\mathrm{e}^{-\frac{2s\tau}{1-\rho}} \sum_{k=2}^\infty \frac{s^k}{k!}\delta_k(\tau)}{1 - \mathrm{e}^{-\frac{s\tau}{1-\rho}} \sum_{k=2}^\infty \frac{s^k}{k!}\delta_k(\tau)} \mathrm{d}B(\tau).
\end{aligned}
$$

Define

$$
f(s,\tau) := \frac{\mathrm{e}^{-\frac{2s\tau}{1-\rho}} \sum_{k=2}^\infty \frac{s^k}{k!}\delta_k(\tau)}{1 - \mathrm{e}^{-\frac{s\tau}{1-\rho}} \sum_{k=2}^\infty \frac{s^k}{k!}\delta_k(\tau)}.
$$

We assume that statement **(i)** of Theorem 4.1 holds with equality and prove statement **(ii)**. To prove **(ii)**, we have to bound $f(s,\tau)$. It turns out that we only need a non-trivial upper bound for $f(s,\tau)$ if $\tau \le \frac{T_1}{s}$ for some finite constant T_1. Therefore, we write for $0 < T_1 < \infty$:

$$
\int_0^\infty f(s,\tau)\mathrm{d}B(\tau) = \int_0^{\frac{T_1}{s}} f(s,\tau)\mathrm{d}B(\tau) + \int_{\frac{T_1}{s}}^\infty f(s,\tau)\mathrm{d}B(\tau). \tag{5.2}
$$

Bounding the first term in the right hand side of (5.2) will be our main task. The second term can be bounded by using $f(s,\tau) \le v(s,\tau) \le 1$:

$$
\int_{\frac{T_1}{s}}^\infty f(s,\tau)\mathrm{d}B(\tau) \quad \le \quad L(T_1/s)\left(\frac{T_1}{s}\right)^{-\nu}.
$$

Since T_1 can be chosen arbitrarily large, (5.1) and the first part of Theorem 4.1 follow once we have proven that

$$
\int_0^{\frac{T_1}{s}} f(s,\tau)\mathrm{d}B(\tau) = \mathrm{o}(s^\nu L(1/s)). \tag{5.3}
$$

Proof of (5.3) Since $\delta_k(\tau) \le \left(\frac{\tau}{1-\rho}\right)^k$, it follows that

$$
\frac{\mathrm{e}^{-\frac{s\tau}{1-\rho}}}{1 - \mathrm{e}^{-\frac{s\tau}{1-\rho}} \sum_{k=2}^\infty \frac{s^k}{k!}\delta_k(\tau)} \le \frac{e^{-\frac{s\tau}{1-\rho}}}{1 - \mathrm{e}^{-\frac{s\tau}{1-\rho}}\left(e^{\frac{s\tau}{1-\rho}} - \frac{s\tau}{1-\rho} - 1\right)} = \frac{1}{1 + \frac{s\tau}{1-\rho}}.
$$

Hence,

$$f(s,\tau) \leq \frac{1}{1+\frac{s\tau}{1-\rho}} \mathrm{e}^{-\frac{s\tau}{1-\rho}} \sum_{k=2}^{\infty} \frac{s^k}{k!} \delta_k(\tau) \leq \mathrm{e}^{-\frac{s\tau}{1-\rho}} \sum_{k=2}^{\infty} \frac{s^k}{k!} \delta_k(\tau). \tag{5.4}$$

Obtaining an upper bound $\delta_k(\tau)$ will be crucial in the proof of (5.3). In view of (3.5), we need an upper bound for $1-R^{(k-1)*}(x)$. By a result of Cohen [17], we have that $1-R(x)$ is regularly varying of index $1-\nu$. Hence, for each $\varepsilon > 0$ there exists an M such that, for $x \geq 0$,

$$1 - R(x) \leq M(x+1)^{1-\nu+\varepsilon}. \tag{5.5}$$

Let $(W_i)_{i\geq 1}$ be a sequence of i.i.d. random variables (defined on the same probability space) with distribution function $R(x)$. Then, we have

$$1 - R^{(k-1)*}(x) = \mathbb{P}(W_1 + \cdots + W_{k-1} > x) \leq \mathbb{P}(\cup_{i=1}^{k-1}\{W_i > \frac{x}{k-1}\})$$

$$\leq (k-1)\mathbb{P}(W_1 > \frac{x}{k-1}).$$

Hence, by $1 < \nu < 2$ and (5.5) we get for $k \geq 2$:

$$1 - R^{(k-1)*}(x) \leq (k-1)M\left(\frac{x}{k-1}+1\right)^{1-\nu+\varepsilon} \leq (k-1)^2 M(x+1)^{1-\nu+\varepsilon}. \tag{5.6}$$

In the sequel, we choose ε fixed such that $0 < 2\varepsilon < \nu - 1$. By (3.5) and (5.6), we have for some finite constant M' and $k \geq 2$:

$$\delta_k(\tau) \leq \frac{Mk(k-1)^2\tau^{k-1}}{(1-\rho)^k} \int_0^\tau (x+1)^{1-\nu+\varepsilon}\mathrm{d}x \leq M'(k-1)^2 k \frac{\tau^{k+1-\nu+\varepsilon}}{(1-\rho)^k}. \tag{5.7}$$

Define

$$M'' := \frac{M'}{(1-\rho)^2}\left(1 + \frac{T_1}{1-\rho}\right).$$

Since $\tau \leq \frac{T_1}{s}$, it follows from (5.7) after some simple calculations that

$$f(s,\tau) \leq \mathrm{e}^{-\frac{s\tau}{1-\rho}} \sum_{k=2}^{\infty} \frac{s^k}{k!} M'(k-1)^2 k \frac{\tau^{k+1-\nu+\varepsilon}}{(1-\rho)^k}$$

$$= M'\tau^{1-\nu+\varepsilon}\mathrm{e}^{-\frac{s\tau}{1-\rho}} \left(\frac{s\tau}{1-\rho}\right)^2 \sum_{k=0}^{\infty} \frac{k+1}{k!}\left(\frac{s\tau}{1-\rho}\right)^k$$

$$\leq M'\tau^{1-\nu+\varepsilon}\left[\left(\frac{s\tau}{1-\rho}\right)^2+\left(\frac{s\tau}{1-\rho}\right)^3\right] \leq M'\tau^{1-\nu+\varepsilon}\left(\frac{s\tau}{1-\rho}\right)^2\left(1+\frac{T_1}{1-\rho}\right)$$

$$= M''s^2\tau^{3-\nu+\varepsilon} \leq M''s^{\nu+\varepsilon}\tau^{1+2\varepsilon}T_1^{2-\nu+\varepsilon}. \tag{5.8}$$

Since $1+2\varepsilon<\nu$, the right hand side of (5.8) is $B(\tau)$–integrable. This proves (5.3), since $L(1/s)=\mathrm{o}(s^\delta)$, $s\to 0$, for all $\delta>0$, cf. (8.2) in the appendix.

Now, suppose that statement **(ii)** of Theorem 4.1 holds with equality. Again, it suffices to prove (5.1). A simple, but important observation is that for all $x \geq 0$,

$$\mathbb{P}(B>x)\leq \mathbb{P}(V>x). \tag{5.9}$$

First, we show that (5.9) and **(ii)** together imply (5.5). Fix $0<\varepsilon<\frac{1}{2}(\nu-1)$. **(ii)** implies that $\mathbb{E}[V^{\nu-\frac{1}{2}\varepsilon}]<\infty$. Hence, by (5.9), also $\mathbb{E}[B^{\nu-\frac{1}{2}\varepsilon}]<\infty$. It follows that for $s\to 0$, cf. Loève [28] p. 199,

$$\beta(s)=1-\beta_1 s+\mathrm{O}(|s|^{\nu-\frac{1}{2}\varepsilon}),$$

and by (3.1), for $s\to 0$,

$$\omega(s)=1+\mathrm{O}(|s|^{\nu-1-\frac{1}{2}\varepsilon})=1+\mathrm{o}(s^{\nu-1-\varepsilon}).$$

Finally, by Lemma 2.2 in [13] (see also Lemma 8.1 with $C=0$ in the appendix), we have for $x\to\infty$:

$$1-R(x)=\mathrm{o}(x^{1-\nu+\varepsilon}),$$

which implies (5.5). Following the arguments made in the first part of the proof, we obtain (5.8), which in turn implies (5.3). A second application of (5.9) to the second term in (5.2), together with (5.3), yields (5.1).

Case II: $\nu > 2$

The proof in this case is different from case I, because we need the property $\beta_2<\infty$ (which follows from both **(i)** and **(ii)** if $\nu>2$). Again, we will apply Lemma 8.1. Suppose that **(i)** holds with equality for $n<\nu<n+1$. Write for real $s\geq 0$,

$$v(s)-\beta\left(\frac{s}{1-\rho}\right)-\sum_{k=0}^{n}\frac{(-s)^k}{k!}\left(\mathbb{E}[V^k]-\frac{\beta_k}{(1-\rho)^k}\right)=\int_0^\infty R_n(s,\tau)\mathrm{d}B(\tau),$$

with $R_n(s,\tau)$ being the residual term of the $n-$term Taylor expansion of $f(s,\tau)$ in $s=0$, i.e.

$$R_n(s,\tau)=f(s,\tau)-\sum_{k=0}^{n}\frac{s^k}{k!}f^{(k)}(0,\tau), \tag{5.10}$$

with

$$f^{(k)}(0,\tau) := \frac{\partial^k}{\partial s^k} f(0,\tau).$$

Since $f(s,\tau)$ is analytic in $s=0$, we can apply Taylor's theorem, which gives, for s in a neighbourhood of 0,

$$|R_n(s,\tau)| = \left| \int_0^s \frac{(s-u)^n}{n!} f^{(n+1)}(u,\tau)\mathrm{d}u \right| \leq s^n \int_0^s \left| f^{(n+1)}(u,\tau) \right| \mathrm{d}u. \tag{5.11}$$

Note that Taylor's theorem cannot be applied to the $n-$term Taylor expansion of $v(s) - \beta\left(\frac{s}{1-\rho}\right)$, since the latter function is not n+1 times differentiable in $s=0$, because $\beta_{n+1}=\infty$. Hence, one cannot interchange the limit $s \to 0$ and the integration w.r.t. τ (in that case, the proof would be finished since the residual term is $\mathrm{O}(s^{n+1})$ due to the mean value theorem).

It suffices to show that

$$\int_0^\infty |R_n(s,\tau)|\,\mathrm{d}B(\tau) = \mathrm{o}(s^\nu L(1/s)), \qquad s \downarrow 0. \tag{5.12}$$

To prove (5.12), we will construct an upper bound for $|f^{(n+1)}(s,\tau)|$. Using the probabilistic interpretation of $f(s,\tau)$ we get for some finite constant M,

$$\begin{aligned}
\left|f^{(n+1)}(s,\tau)\right| &= \left|\mathbb{E}[V^{n+1}(\tau)\mathrm{e}^{-sV(\tau)}] - \left(\frac{\tau}{1-\rho}\right)^{n+1} \mathrm{e}^{-\frac{s\tau}{1-\rho}}\right| \\
&\leq \left|\mathbb{E}[V^{n+1}(\tau)\mathrm{e}^{-sV(\tau)}] - \mathbb{E}[V^{n+1}(\tau)]\mathbb{E}[\mathrm{e}^{-sV(\tau)}]\right| + \\
&\left|\mathbb{E}[V^{n+1}(\tau)]\mathbb{E}[\mathrm{e}^{-sV(\tau)}] - \left(\frac{\tau}{1-\rho}\right)^{n+1} \mathbb{E}[\mathrm{e}^{-sV(\tau)}]\right| + \\
&\left|\left(\frac{\tau}{1-\rho}\right)^{n+1} \mathbb{E}[\mathrm{e}^{-sV(\tau)}] - \left(\frac{\tau}{1-\rho}\right)^{n+1} \mathrm{e}^{-\frac{s\tau}{1-\rho}}\right| \\
&\leq \left|\mathrm{Cov}\left[V^{n+1}(\tau), \mathrm{e}^{-sV(\tau)}\right]\right| + v_{n+1}(\tau) - \left(\frac{\tau}{1-\rho}\right)^{n+1} + \left(\frac{\tau}{1-\rho}\right)^{n+1} f(s,\tau) \\
&\leq M\tau^{n+\frac{1}{2}}\mathrm{e}^{-s\tau} + v_{n+1}(\tau) - \left(\frac{\tau}{1-\rho}\right)^{n+1} + \left(\frac{\tau}{1-\rho}\right)^{n+1} f(s,\tau).
\end{aligned}$$

The last inequality follows from the inequality of Cauchy-Schwarz, $\sqrt{\mathbb{V}\mathrm{ar}[\mathrm{e}^{-sV(\tau)}]} \leq \mathrm{e}^{-s\tau}$ $(V(\tau) \geq \tau)$, and, cf. Remark 3.3, since $\beta_2 < \infty$,

$$\mathbb{V}\mathrm{ar}[V^{n+1}(\tau)] = v_{2n+2}(\tau) - v_{n+1}^2(\tau) = \mathrm{O}(\tau^{2n+1}), \qquad \tau \to \infty.$$

Hence, from (5.11) it follows that

$$\int_0^\infty |R_n(s,\tau)|\,\mathrm{d}B(\tau) \leq Ms^n \int_0^\infty \tau^{n+\frac{1}{2}} \int_0^s \mathrm{e}^{-u\tau}\mathrm{d}u\mathrm{d}B(\tau) +$$
$$s^{n+1}\int_0^\infty \left[v_{n+1}(\tau) - \left(\frac{\tau}{1-\rho}\right)^{n+1}\right]\mathrm{d}B(\tau) +$$
$$(1-\rho)^{-(n+1)}s^n \int_0^\infty \tau^{n+1}\int_0^s f(u,\tau)\mathrm{d}u\mathrm{d}B(\tau).$$

The second term is integrable, since $\beta_2 < \infty$, cf. Remark 3.3. To bound the other terms, it turns out that it is convenient to split the integrals up in two parts. It suffices to show that

$$s^n \int_0^{s^{-1}} \tau^{n+\frac{1}{2}} \int_0^s \mathrm{e}^{-u\tau}\mathrm{d}u\mathrm{d}B(\tau) = \mathrm{o}(s^\nu L(1/s)), \qquad s \downarrow 0, \tag{5.13}$$

$$s^n \int_{s^{-1}}^\infty \tau^{n+\frac{1}{2}} \int_0^s \mathrm{e}^{-u\tau}\mathrm{d}u\mathrm{d}B(\tau) = \mathrm{o}(s^\nu L(1/s)), \qquad s \downarrow 0. \tag{5.14}$$

$$s^n \int_0^{s^{-\alpha}} \tau^{n+1} \int_0^s f(u,\tau)\mathrm{d}u\mathrm{d}B(\tau) = \mathrm{o}(s^\nu L(1/s)), \qquad s \downarrow 0, \tag{5.15}$$

$$s^n \int_{s^{-\alpha}}^\infty \tau^{n+1} \int_0^s f(u,\tau)\mathrm{d}u\mathrm{d}B(\tau) = \mathrm{o}(s^\nu L(1/s)), \qquad s \downarrow 0. \tag{5.16}$$

In (5.15) and (5.16) $\alpha > 1$ is chosen such that $n + 1 - \frac{1}{\alpha} < \nu$. We prove only (5.15) and (5.16) because the proofs of (5.13) and (5.14) are similar and somewhat easier.

Proof of (5.15) An important observation is that both **(i)** and **(ii)** imply that $\beta_2 < \infty$, since $\nu > 2$. Using the Pollaczek-Khintchine formula for the mean waiting time in the M/G/1 FCFS queue (see e.g. [17], p. 256), we obtain the result

$$\delta_k(\tau) \leq \frac{k\tau^{k-1}}{(1-\rho)^k}\int_0^\infty (1 - R^{(k-1)*}(x))\mathrm{d}x = \frac{\tau^{k-1}}{(1-\rho)^k}k(k-1)\frac{\beta_2}{2\beta_1}\frac{\rho}{1-\rho}. \tag{5.17}$$

It follows easily from the first inequality in (5.4) and (5.17) that, with

$$K := \frac{\beta_2}{2\beta_1}\frac{\rho}{1-\rho},$$

$$f(s,\tau) \le \frac{s^2\tau}{1+s\tau}\frac{K}{(1-\rho)^2} \le \frac{K}{(1-\rho)^2}s. \tag{5.18}$$

The proof of (5.15) follows easily from (5.18). Since $s\tau^{\frac{1}{\alpha}} \le 1$ if $\tau \le s^{-\alpha}$, it follows that

$$s^n \int_0^{s^{-\alpha}} \tau^{n+1} \int_0^s f(u,\tau)\mathrm{d}u\mathrm{d}B(\tau) \le \frac{1}{2}\frac{K}{(1-\rho)^2}s^{n+2}\int_0^{s^{-\alpha}} \tau^{n+1-\frac{1}{\alpha}}\tau^{\frac{1}{\alpha}}\mathrm{d}B(\tau)$$

$$\le \frac{s^{n+1}}{2}\frac{K}{(1-\rho)^2}\int_0^{s^{-\alpha}} \tau^{n+1-\frac{1}{\alpha}}\mathrm{d}B(\tau) \le \frac{s^{n+1}}{2}\frac{K}{(1-\rho)^2}\mathbb{E}[B^{n+1-\frac{1}{\alpha}}] = \mathrm{o}(s^\nu L(1/s)), \qquad s\downarrow 0.$$

To prove (5.13), use the inequality $\mathrm{e}^{-u\tau} \le 1$ and a similar technique as in the last part of the proof of (5.15).

Proof of (5.16) Since $V(\tau) \ge \tau$, we have

$$f(s,\tau) \le v(s,\tau) = \mathbb{E}[\mathrm{e}^{-sV(\tau)}] \le \mathrm{e}^{-s\tau}.$$

It follows for $s \ge 0$,

$$\int_0^s f(u,\tau)\mathrm{d}u \le \int_0^s \mathrm{e}^{-u\tau}\mathrm{d}u = \frac{1}{\tau}(1-\mathrm{e}^{-s\tau}) \le \frac{1}{\tau}.$$

This gives,

$$s^n \int_{s^{-\alpha}}^{\infty} \tau^{n+1}\int_0^s f(u,\tau)\mathrm{d}u\mathrm{d}B(\tau) \le s^n \int_{s^{-\alpha}}^{\infty} \tau^n \mathrm{d}B(\tau). \tag{5.19}$$

Define $\delta := \nu - n > 0$. Fix $\varepsilon > 0$ such that $\alpha\varepsilon < (\alpha-1)\delta$. For s small enough we have that $L(\tau) \le \tau^\varepsilon$ if $\tau \ge s^{-\alpha}$. Using partial integration, we get

$$s^n \int_{s^{-\alpha}}^{\infty} \tau^n \mathrm{d}B(\tau) \le -s^n \int_{s^{-\alpha}}^{\infty} \tau^n \mathrm{d}(1-B(\tau))$$

$$= s^{n(1-\alpha)}(1-B(s^{-\alpha})) + s^n n \int_{s^{-\alpha}}^{\infty} (1-B(\tau))\tau^{n-1}\mathrm{d}\tau$$

$$\leq s^{n+\alpha\delta-\alpha\varepsilon} + ns^n \int_{s^{-\alpha}}^{\infty} \tau^{-1-\delta+\varepsilon} \mathrm{d}\tau = s^{n+\alpha\delta-\alpha\varepsilon} + \frac{n}{\delta-\varepsilon} s^{n+\alpha\delta-\alpha\varepsilon} = \mathrm{o}(s^{\nu} L(1/s)), \qquad s \downarrow 0,$$

since $n + \alpha\delta - \alpha\varepsilon > \nu$. This proves (5.16). The proof of (5.14) is similar to that of (5.16).

We conclude that (5.12) holds, which implies **(ii)**. The proof of the implication **(ii)** $\Rightarrow$ **(i)** follows by the same arguments and (5.9). ■

Remark 6.1 In the proof for the case $\nu < 2$, we constructed an upper bound for $1 - R^{(k-1)*}(x)$. In the part **(i)** $\Rightarrow$ **(ii)**, it is sufficient to use the upper bound provided by (8.7) in the appendix, which is valid for subexponential distribution functions. However, in the converse direction, it is not possible to apply (8.7), since in that case $1 - R(x)$ need not be subexponential. □

6. Heavy traffic and heavy tails

In this section we give a new proof of a heavy traffic theorem, due to [37, 43], based on Theorem 3.1. We will show that the 'contracted' moments of the sojourn times converge to the moments of the limiting distribution. Finally, we give both explicit and asymptotic results for the sojourn time distribution in heavy traffic when the service time distribution has a regularly varying tail.

6.1 General results

We present a new proof for the following result, see [37, 43].

Theorem 6.1 *If $\beta_1 < \infty$, then*

$$\lim_{\rho\to 1} v(s(1-\rho), \tau) = \frac{1}{1+s\tau}, \qquad \mathrm{Re}\, s \geq 0, \quad \tau \geq 0, \tag{6.1}$$

$$\lim_{\rho\to 1} \mathbb{P}((1-\rho)V(\tau) \leq x) = 1 - e^{-\frac{x}{\tau}}, \qquad x \geq 0, \quad \tau \geq 0. \tag{6.2}$$

A heavy traffic theorem for the GI/G/1 PS queue is also known, see Grishechkin [25]. Note that it is only required that the first moment of the service time distribution is finite, which is not the case in the FCFS service discipline, cf. [12, 19].

Proof Note that (6.1) and (6.2) are equivalent. Since

$$v(s(1-\rho), \tau) = \left[1 + s\tau + \sum_{k=2}^{\infty} \frac{s^k}{k!} (1-\rho)^k \alpha_k(\tau)\right]^{-1},$$

it suffices to show that, for $k \geq 2$,

$$\lim_{\rho\to 1} (1-\rho)^k \alpha_k(\tau) = 0. \tag{6.3}$$

This follows immediately from (3.3) and the fact that $\lim_{\rho\to 1} R(x) = 0$ for $x \geq 0$. Indeed, when $\beta_2 < \infty$ this follows from the heavy traffic limit in [19], p. 597. If $\beta_2 = \infty$, then it must hold that $\widetilde{B}(x) < 1$. Hence, since $\widetilde{B}^{n*}(x) \leq \widetilde{B}^{n}(x)$,

$$R(x) = (1-\rho)\sum_{n=0}^{\infty} \rho^n \widetilde{B}^{n*}(x) \leq \frac{1-\rho}{1-\rho\widetilde{B}(x)} \to 0,$$

when $\rho \to 1$. ■

Since $v(s(1-\rho),\tau) \leq 1$ we have by dominated convergence and Theorem 6.1 the following heavy traffic limit for the unconditional sojourn time distribution.

Corollary 6.1 *For* Re $s \geq 0$,

$$\lim_{\rho\to 1} v(s(1-\rho)) = \int_0^\infty \frac{1}{1+s\tau}\mathrm{d}B(\tau), \tag{6.4}$$

and

$$\lim_{\rho\to 1} v(s(1-\rho)) = \int_0^\infty \mathrm{e}^{-x}\beta(sx)\mathrm{d}x. \tag{6.5}$$

Proof (6.4) follows from Theorem 6.1 and Lebesgue's dominated convergence theorem ($v(s(1-\rho),\tau) \leq 1$). (6.5) follows easily from (6.4) since

$$\int_0^\infty \frac{1}{1+s\tau}\mathrm{d}B(\tau) = \int_0^\infty\int_0^\infty \mathrm{e}^{-x-s\tau x}\mathrm{d}x\mathrm{d}B(\tau) = \int_0^\infty \mathrm{e}^{-x}\beta(sx)\mathrm{d}x.$$

■

This result has also been obtained by Sengupta [37]. Note that (6.5) is the LST of a random variable $Y := XB$, where B is equal to the service time and X is exponentially distributed with mean 1 and independent of B. A similar interpretation is given in [37], where it serves as a basis for approximations for the sojourn time distribution in the GI/G/1-PS queue.

We now turn to convergence of the moments of the sojourn time in heavy traffic. It will be shown that the moments of the contracted sojourn times converge to the corresponding moments of the heavy traffic limiting distribution. Instead of using arguments concerning uniform integrability, cf. [8] p. 338, after which Theorem 6.2 below readily follows from (6.2), we follow another approach by using Corollary 3.1.

Theorem 6.2 *If $\beta_1 < \infty$, then*

$$\lim_{\rho\to 1} \mathbb{E}[((1-\rho)V(\tau))^k] = k!\tau^k, \qquad \tau \geq 0, \quad k \geq 1.$$

Proof We apply induction on k. Fix $\tau \geq 0$. By (2.4), the result holds for $k = 1$. Suppose the result is true for $n \leq k-1$. By Corollary 3.1 we have

$$\mathbb{E}[((1-\rho)V(\tau))^k] = (1-\rho)^k v_k(\tau) = -\sum_{j=1}^{k} \binom{k}{j} (1-\rho)^{k-j} v_{k-j}(\tau)(1-\rho)^j \alpha_j(\tau)(-1)^j. \tag{6.6}$$

The result follows after some simple calculations for $n = k$ by the induction hypothesis, (6.3), and $(1-\rho)\alpha_1(\tau) \equiv \tau$. ■

A similar result holds for the unconditional moments of the sojourn time, whenever they exist.

Corollary 6.2 *If $\beta_k < \infty$, $k \geq 1$, then*

$$\lim_{\rho \to 1} \mathbb{E}[((1-\rho)V)^k] = k!\beta_k.$$

Proof. The same idea as in the proof of Corollary 3.2 is used. Write

$$\mathbb{E}[((1-\rho)V)^k] = \int_0^\infty (1-\rho)^k v_k(\tau) \mathrm{d}B(\tau).$$

Note that, cf. (3.3),

$$(1-\rho)^k \alpha_k(\tau) \leq \tau^k. \tag{6.7}$$

From (6.6), (6.7) and by induction on k, it is trivially seen that $(1-\rho)^k v_k(\tau) \leq C_k \tau^k$, with $C_0 = 1$ and

$$C_k = \sum_{j=0}^{k-1} \binom{k}{j} C_j, \qquad k \geq 1.$$

The result follows by dominated convergence and Theorem 6.2. ■

Both Theorem 6.2 and Corollary 6.2 can be used for the approximation of higher moments for the sojourn time distribution. Note that Corollary 6.2 provides a certain robustness, since the heavy traffic behaviour of the k-th moment of the sojourn time is completely determined by the $k-$th moment of the service time. Van den Berg [6], Chapter 4, has proven Theorem 6.2 and Corollary 6.2 in the case $k = 2$. Numerical results in [6] indicate that the heavy traffic approximation for the second moment of the sojourn time performs well.

Remark 6.1 Abate and Whitt [3] perform a heavy traffic analysis for the waiting time in the M/G/1 LIFO system. They prove a heavy traffic theorem for the moments of the waiting time under additional assumptions to meet uniform integrability conditions. The latter concept can also be applied in our case without making any additional assumptions. Note that in our case the $k-$th moment of the heavy traffic limiting distribution is equal to $k!\beta_k$ if $\beta_k < \infty$. □

6.2 An explicit expression for the limiting distribution

Let V_{HT} be a rv with a distribution equal to the heavy traffic limit, i.e.

$$\mathbb{P}(V_{HT} \leq x) := \lim_{\rho \to 1} \mathbb{P}((1-\rho)V \leq x).$$

If the service time has a Pareto-distribution, given by

$$1 - B(\tau) = \left(\frac{r-1}{r}\right)^r \tau^{-r}, \qquad \tau \geq \frac{r-1}{r}, \tag{6.8}$$

($B(\tau) = 0$ otherwise) an explicit expression for $\mathbb{P}(V_{HT} \leq x)$ can be found if r is integer-valued and a multiterm asymptotic expansion is available for $\mathbb{P}(V_{HT} \leq x)$ if r is non-integer. A similar result holds if we consider finite mixtures of (6.8).

To show this, we exploit results of Abate and Whitt [1]. They define the class of Pareto Mixtures of Exponentials (PME) as follows. A distribution function F is a PME if

$$1 - F(x) = \int_0^\infty \mathrm{e}^{-\frac{x}{\tau}} \mathrm{d}B(\tau), \qquad x \geq 0, \tag{6.9}$$

with $B(.)$ given by (6.8). From this definition and (6.2) we can conclude that the heavy-traffic limiting distribution is a PME if the service time distribution is Pareto. We get, cf. [1],

$$\begin{aligned}
\mathbb{P}(V_{HT} > x) &= \int_0^\infty \mathrm{e}^{-\frac{x}{y}} \mathrm{d}B(y) \\
&= \int_{\frac{r-1}{r}}^\infty \mathrm{e}^{-\frac{x}{y}} r \left(\frac{r-1}{r}\right)^r y^{-r-1} \mathrm{d}y \\
&= r \left(\frac{r-1}{r}\right)^r \int_0^{\frac{r}{r-1}} \mathrm{e}^{-yx} y^{r-1} \mathrm{d}y.
\end{aligned}$$

This expression is (up to a multiplicative constant) equal to the incomplete gamma function. Applications of well known results for the incomplete gamma function (see Abramovitz and Stegun [5], (4.2.55) and §6.5)) give the following results. For $r \geq 2$ integer we have,

$$\mathbb{P}(V_{HT} > x) = \left(\frac{r-1}{r}\right)^r \frac{r!}{x^r} \left[1 - \mathrm{e}^{-\frac{rx}{r-1}} \sum_{k=0}^{r-1} \frac{1}{(r-1-k)!} \left(\frac{xr}{r-1}\right)^{r-1-k}\right]. \tag{6.10}$$

And, for non-integer $r > 1$:

$$\mathbb{P}(V_{HT} > x) =$$

$$\left(\frac{r-1}{r}\right)^r \frac{r}{x^r} \left[\Gamma(r) - \left(\frac{rx}{r-1}\right)^{r-1} \mathrm{e}^{-\frac{rx}{r-1}} \left[1 + \frac{r-1}{\frac{rx}{r-1}} + \frac{(r-1)(r-2)}{\left(\frac{rx}{r-1}\right)^2} + \cdots\right]\right]. \tag{6.11}$$

It is not difficult to obtain an explicit expression for $\mathbb{P}(V_{HT} > x)$ when the service time distribution is a mixture of (6.8). In that case, the distribution of V_{HT} is a mixture of PME's, which implies that $\mathbb{P}(V_{HT} > x)$ is given by a mixture of (6.10) and (6.11).

Both (6.10) and (6.11) indicate that a one-term asymptotic expansion for the heavy traffic limiting distribution will behave quite accurately since the residual terms decrease exponentially fast if $x \to \infty$ (cf. the observation in [1] p. 321). Another interesting observation is that the one-term expansion for $\mathbb{P}(V_{HT} > x)$ behaves like $\Gamma(r+1)\mathbb{P}(B > x)$, $x \to \infty$. In the next section, we will show that this property still holds if we only assume that the service time distribution is regularly varying.

6.3 Tail behaviour

In this subsection we study the behaviour of $\mathbb{P}(V_{HT} > x)$ for x large in the case that the service time distribution is regularly varying. In particular, it will be shown that the heavy traffic approximation

$$\mathbb{P}(V > x) \approx \mathbb{P}(V_{HT} > (1-\rho)x)$$

for the sojourn time *over*estimates the true sojourn time distribution for large x.

Theorem 6.3 *If* $1 - B(x) = x^{-\nu}L(x)$ *with* $\nu > 1$, *then*

$$\mathbb{P}(V_{HT} > x) \sim \Gamma(\nu+1)\mathbb{P}(B > x)$$

if $x \to \infty$.

Proof Since $V_{HT} \stackrel{d}{=} YB$, with Y exponentially distributed with mean 1 and B the service time independent of Y (cf. the remark below Corollary 5.1), Theorem 6.3 immediately follows from Proposition 3 in [15], which is stated only for $0 < \nu < 1$, but can easily be extended to $\nu > 0$ (see also [20, 21, 34]). ■

Remark 6.2 It is possible to get more refined asymptotics for $\mathbb{P}(V_{HT} > x)$. Suppose $1-B(x)$ is given by

$$1 - B(x) = \sum_{i=1}^{N} p_i x^{-\nu_i} + \mathrm{o}(x^{-\nu_N}), \qquad x \to \infty, \tag{6.12}$$

with $1 < \nu_1 < \cdots < \nu_N$, and $p_i > 0$. Applying (6.10), (6.11), and Theorem 6.3, we get

$$\mathbb{P}(V_{HT} > x) = \sum_{i=1}^{N} p_i \Gamma(\nu_i + 1) x^{-\nu_i} + \mathrm{o}(x^{-\nu_N}), \qquad x \to \infty. \tag{6.13}$$

This result is useful for numerical purposes. □

Remark 6.3 By Theorem 6.3 and Theorem 4.1, we have the following interesting result:

$$\lim_{x\to\infty}\lim_{\rho\to1} \frac{\mathbb{P}((1-\rho)V > x)}{\mathbb{P}(B > x)} = \Gamma(\nu+1) > 1 = \lim_{\rho\to1}\lim_{x\to\infty} \frac{\mathbb{P}((1-\rho)V > x)}{\mathbb{P}(B > x)}. \tag{6.14}$$

Hence, the heavy traffic approximation for $\mathbb{P}(V > x)$ overestimates the true value when x is large. This indicates that the approximations for the waiting time distribution induced by Theorem 4.1 and Theorem 6.3 will behave differently. We postpone a numerical investigation of this phenomenon to a later study. □

7. Conclusion

In this study, we have investigated asymptotic properties of the sojourn time distribution in the M/G/1 processor sharing queue. Our main result (Theorem 4.1) is that the sojourn time distribution is regularly varying of index $-\nu$ iff the service time distribution is regularly varying of index $-\nu$, with ν not an integer. More precisely, in the case of regular variation, we have $\mathbb{P}(V > x) \sim \mathbb{P}(B > (1-\rho)x)$ for $x \to \infty$.

A generalisation of Theorem 4.1 to integer ν seems difficult. However, it might be possible to apply the techniques used in [30, 31] for a particular class of regularly varying functions. Another extension of Theorem 4.1 is to assume only subexponentiality (cf. the appendix) instead of regular variation. However, proving such an extension seems to be a hard problem, since no characterisation of a subexponential distribution function by its LST is available.

In a future study, we intend to extend Theorem 4.1 in a different way, namely to the M/G/1 processor sharing queue with different types of customers. For this system we intend to show that the sojourn time distribution of a tagged customer is regularly varying of index $-\nu$ iff the service time distribution of the tagged customer is regularly varying of index $-\nu$, even if the tail of the service time distribution of another class of customers is heavier.

Theorem 4.1 is derived from a new expression (Theorem 3.1) for the LST of the sojourn time distribution. That expression has also led us to several other explicit and asymptotic results for the sojourn time distribution. It seems worthwhile to investigate the potential of these results for approximating various characteristics of the sojourn time distribution.

8. Appendix: Regular Variation and Subexponentiality

In this appendix, we give a short overview of regular variation, and show some basic results that are used in this paper. Regular variation is an important concept in probability theory and various other fields. The main reference is the book [11], see also p. 275–284 in [23]. For a more general introduction to (applications of) heavy tailed distributions, the reader is referred to [22]. All the material discussed in this appendix can be found in these books.

A measurable positive function f is called regularly varying with index ϑ if, for all $x > 0$,

$$\lim_{t\to\infty} \frac{f(xt)}{f(t)} = x^{\vartheta}, \tag{8.1}$$

(cf. [11], p. 18). The class of regularly varying functions with index ϑ is called $\mathcal{R}_\vartheta$. A random variable, or its distribution function F, is said to be regularly varying if $1-F$ is a regularly varying function. When $L \in \mathcal{R}_0$, we call L a *slowly varying* function. In this paper, a slowly varying function is denoted by L. The following basic property for slowly varying functions is often used without mention. Let L be a slowly varying function. Then, for all $\varepsilon > 0$, there exists a T such that, if $x > T$,

$$x^{-\varepsilon} \le L(x) \le x^{\varepsilon}. \tag{8.2}$$

see e.g. Feller [23], p. 271. The next result is part of Karamata's theorem ([11], p. 26) and is useful to determine the tail behaviour of excess distributions, cf. Section 2. Let $\vartheta > -1$, let $T \geq 0$ and let L be locally bounded in $\{x : x \geq T\}$. Then for, $t \to \infty$,

$$\frac{1}{t^{\vartheta+1}L(t)} \int_T^t L(x)x^\vartheta \mathrm{d}x \to \frac{1}{\vartheta+1}. \tag{8.3}$$

An important result due to Bingham and Doney [9] (see also Theorem 8.1.6 in [11]) characterises the tail behaviour of a regularly varying distribution function uniquely by the behaviour around the origin of its LST. Let X be a random variable with distribution function F, LST $\phi(s)$, and finite first n moments $\mu_1, ..., \mu_n$ (and $\mu_0 = 1$). Define

$$\phi_n(s) := (-1)^{n+1} \left[\phi(s) - \sum_{j=0}^n \mu_j \frac{(-s)^j}{j!} \right].$$

We have the following lemma from [9, 13].

Lemma 8.1 *Let $n < \nu < n+1$, $C \geq 0$. The following are equivalent.*

$$\phi_n(s) \sim (C + o(1))s^\nu L(1/s), \quad s \downarrow 0, \quad s \text{ real}, \tag{8.4}$$

$$1 - F(t) \sim (C + o(1)) \frac{(-1)^n}{\Gamma(1-\nu)} t^{-\nu} L(t), \quad t \to \infty. \tag{8.5}$$

The case $C > 0$ is due to [9]. The case $C = 0$ is treated in [13], Lemma 2.2. For the more complicated case if ν is integer, we refer to Theorem 8.1.6 and Chapter 3 of [11].

If a distribution function F is regularly varying, then (see e.g. [23])

$$\lim_{x\to\infty} \frac{1 - F^{2*}(x)}{1 - F(x)} = 2. \tag{8.6}$$

A random variable of which the distribution function F satisfies (8.6) is called *subexponential*. The following property for subexponential distributions is often useful, see also Remark 4.1. Let F satisfy (8.6). Then, for all $\varepsilon > 0$ there exists a $K(\varepsilon) < \infty$ such that for all $n \geq 1$ and all $x \geq 0$,

$$\frac{1 - F^{n*}(x)}{1 - F(x)} \leq K(\varepsilon)(1+\varepsilon)^n. \tag{8.7}$$

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References

1. J. Abate, G.L. Choudhury, and W. Whitt. Waiting-time tail probabilities in queues with long-tail service-time distributions. *Queueing Systems,* 16:311-338, 1994.
2. J. Abate, G.L. Choudhury, and W. Whitt. Exponential approximations for tail probabilities in queues. I. Waiting times. *Operations Research,* 43:885-901, 1995.
3. J. Abate and W. Whitt. Limits and approximations for the M/G/1 LIFO waiting-time distribution. *Operations Research Letters,* 20:199-206, 1997.
4. J. Abate and W. Whitt. Explicit M/G/1 waiting-time distributions for a class of long-tail service-time distributions, preprint, AT&T Labs, 1998.
5. M. Abramovitz and I.A. Stegun. *Handbook of Mathematical Functions.* Dover, New York, 1965.
6. J.L. van den Berg. *Sojourn Times in Feedback and Processor Sharing Queues.* PhD Thesis, Utrecht, 1990.
7. J. Beran, R. Sherman, M.S. Taqqu, and W. Willinger. Long-range dependence in variable-bit-rate video. *IEEE Transactions on Communications*, 43:1566-1579, 1995.
8. P. Billingsley, *Probability and Measure.* Third edition, Wiley, New York, 1996.
9. N.H. Bingham and R.A. Doney. Asymptotic properties of supercritical branching processes I: The Galton-Watson process. *Advances in Applied Probability*, 6:711-731, 1974.
10. N.H. Bingham and R.A. Doney. Asymptotic properties of supercritical branching processes II: Crump-Mode and Jirina processes. *Advances in Applied Probability,* 7:66-82, 1975.
11. N.H. Bingham, C.M. Goldie, and J.L. Teugels. *Regular Variation.* Cambridge University Press, Cambridge, 1987.
12. O.J. Boxma and J.W. Cohen. Heavy-traffic analysis for the GI/G/1 queue with heavy-tailed distributions. Report PNA-R9710, CWI, Amsterdam, 1997.
13. O.J. Boxma and V. Dumas. The busy period in the fluid queue. Report PNA-R9718,

CWI, Amsterdam, 1997.

14. O.J. Boxma and J.W. Cohen. The M/G/1 queue with heavy-tailed service time distribution. To appear in *IEEE Journal on Selected Areas in Communications*, 1998.

15. L. Breiman. On some limit theorems similar to the arc-sin law. *Theory of Probability and its Applications,* 10:323-331, 1965.

16. E.G. Coffman, R.R. Muntz, and H. Trotter. Waiting time distributions for processor-sharing systems. *Journal of ACM,* 17:123-130, 1970.

17. J.W. Cohen. Some results on regular variation in queueing and fluctuation theory. *Journal of Applied Probability,* 10:343-353, 1973.

18. J.W. Cohen. The multiple phase service network with generalized processor sharing. *Acta Informatica,* 12:245-284, 1979.

19. J.W. Cohen. *The Single Server Queue.* Second edition, North Holland, Amsterdam, 1982.

20. R.A. Davis and S.I. Resnick. Limit theory for bilinear processes with heavy-tailed noise. *Annals of Applied Probability,* 6:1191-1210, 1996.

21. P. Embrechts and C. Goldie. On closure and factorization properties of subexponential and related distributions. *Journal of the Australian Mathematical Society A*, 29:243-256, 1980.

22. P. Embrechts, C. Klüppelberg, and T. Mikosch. *Modelling Extremal Events.* Springer Verlag, Berlin, 1997.

23. W. Feller. *An Introduction to Probability Theory and its Applications,* Volume II. Second edition, Wiley, New York, 1971.

24. S. Grishechkin. On a relationship between processor-sharing queues and Crump-Mode-Jagers branching processes. *Advances in Applied Probability,* 24:653-698, 1992.

25. S. Grishechkin. GI/G/1 processor sharing queue in heavy traffic. *Advances in Applied Probability,* 26:539-555, 1994.

26. F.P. Kelly. *Reversibility and Stochastic Networks.* Wiley, New York, 1979.

27. L. Kleinrock. *Queueing Systems, Volume II: Computer Applications.* Wiley, New York, 1976.

28. M. Loève. *Probability Theory.* Second edition, New York, 1960.

29. J.A. Morrison. Response-time distribution for a processor sharing system. *SIAM Journal of Applied Mathematics* 45:152-167, 1985.

30. A. de Meyer and J.L. Teugels. On the asymptotic behaviour of the distributions of the busy period and service time in M/G/1. *Journal of Applied Probability,* 17:802-813, 1980.

31. A. de Meyer. On a theorem of Bingham and Doney. *Journal of Applied Probability,* 19:217-220, 1982.

32. T.J. Ott. The sojourn-time distribution in the M/G/1 queue with processor sharing. *Journal of Applied Probability,* 21:360-378, 1984.

33. V. Paxson and S. Floyd. Wide area traffic: The failure of Possion modeling. *IEEE/ACM Transactions on Networking*, 3:226-244, 1995.

34. S. Resnick. Point processes, regular variation and weak convergence. *Advances in Applied Probability,* 18:66-138, 1986.

35. M. Sakata, S. Noguchi, and J. Oizumi. Analysis of a processor shared queueing model for time sharing systems. *Proceedings of the second Hawaii International Conference on System Sciences,* 625-628, 1969.

36. R. Schassberger. A new approach to the M/G/1 processor sharing queue. *Advances in Applied Probability,* 16:802-813, 1984.

37. B. Sengupta. An approximation for the sojourn-time distribution for the GI/G/1 processor-sharing queue. *Communications in Statistics - Stochastic Models*, 8:35-57, 1992.

38. H.C. Tijms. *Stochastic Models.* Wiley, Chichester, 1994.

39. W. Willinger, M.S. Taqqu, W.E. Leland, and D.V. Wilson. Self-similarity in high-speed packet traffic: Analysis and modeling of ethernet traffic measurements. *Statistical Science*, 10:67-85, 1995.

40. S.F. Yashkov. A derivation of response time distribution for a M/G/1 processor sharing queue. *Problems of Control and Information Theory* 12:133-148, 1983.

41. S.F. Yashkov. Processor-sharing queues: Some progress in analysis. *Queueing Systems,* 2:1-17, 1987.

42. S.F. Yashkov. Mathematical problems in the theory of shared-processor systems. *Journal of Soviet Mathematics,* 58:101-147, 1992.

43. S.F. Yashkov. On a heavy traffic limit theorem for the M/G/1 processor-sharing queue. *Communications in Statistics - Stochastic Models,* 9:467-471, 1994.