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ABSTRACT
Redistribution of oil and water in a long horizontal column, including oil entrapment by water, is described by a nonlinear diffusion problem with a spatially varying diffusion coefficient. This problem admits a similarity solution that was found previously for redistribution of water with capillary hysteresis. The distributions of both the free and the trapped oil saturations are computed and additionally the effect of initially trapped oil on the solution is demonstrated.

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1. Introduction

Both in oil reservoir and environmental engineering hysteresis due to oil entrapment by water is an important process. Quantitative descriptions of multi-phase flow involving entrapment are, however, complicated by the hysteretic nature of fluid entrapment. As a result, only few analytical solutions for this type of problems are available, see e.g. [4, 8].

Philip [6] developed a similarity solution for horizontal redistribution of water with capillary hysteresis, starting from uniform but different saturations in the two halves of a long homogenous soil column. Van Duijn and De Neef [12] investigated a similar problem, i.e. two-phase redistribution in a horizontal soil column with different entry pressures and absolute permeabilities in the two halves of the column. Both problems lead to a nonlinear diffusion equation with different diffusion functions on each side of the interface that separates the two halves of the column. At the interface continuity of flux and a condition on the capillary pressure are imposed.

The present study shows that the above similarity solution is applicable to horizontal redistribution of two immiscible phases (say oil and water) taking into account capillary entrapment of the nonwetting phase, according to the two-phase constitutive relations of Parker and Lenhard [2, 5] and the trapping model of Kaluarachchi and Parker [1]. In this model the trapped oil saturation is an increasing function of the water saturation. Hysteresis in the constitutive relations is assumed to be caused by entrapment only [3].

The corresponding diffusion function involves both a hysteretic capillary pressure and a hysteretic oil relative permeability function, contrary to [6, 12]. We show that the solution is also applicable to situations where initially trapped nonwetting fluid is present at uniform but different saturations in the two halves of the column, similar to the history-dependent initial conditions for the water redistribution problem [7]. We indicate how the initially trapped fluid affects the behaviour of the solution.

2. Model

Following [12], we use the water mass balance equation for the effective water saturation S_w

$$\frac{\partial S_w}{\partial t} + \frac{\partial F_w}{\partial x} = 0 \qquad \text{for} \quad t > 0, \; -\infty < x < \infty, \tag{2.1}$$

to describe, in dimensionless variables, the water and oil redistribution in an infinitely long horizontal soil column. Since the total flow rate of oil and water is equal to zero, the water flux F_w is specified as

$$F_w = \lambda(S_w)\,\frac{\partial p_c(S_w)}{\partial x}, \tag{2.2}$$

where p_c denotes the capillary pressure and λ the mobility function, which is given by

$$\lambda = \frac{k_{ro}\,k_{rw}}{k_{ro} + M\,k_{rw}}. \tag{2.3}$$

In (2.3) M is the mobility ratio, i.e. the oil-water viscosity ratio, and k_{ro} and k_{rw} are the relative permeabilities of oil and water, respectively.

We use the constitutive relations [2, 5]

$$p_c(S_w) = \bar{p}_c(S_{wa}(S_w)) = (S_{wa}^{-\frac{1}{m}} - 1)^{1-m} \tag{2.4}$$

$$k_{rw}(S_w) = S_w^{\frac{1}{2}}(1 - (1 - S_w^{\frac{1}{m}})^m)^2 \tag{2.5}$$

$$k_{ro}(S_w) = \bar{k}_{ro}(S_{wa}(S_w)) = (1 - S_{wa})^{\frac{1}{2}}(1 - S_{wa}^{\frac{1}{m}})^{2m}, \tag{2.6}$$

with $0 < m < 1$. Observe that no explicit entry pressure for the nonwetting phase appears in relation (2.4). Hence, p_c can be taken continuously everywhere [12]. Distinguishing a (effective) free oil saturation S_{of} and a (effective) trapped oil saturation S_{ot}, with $S_w + S_{of} + S_{ot} = 1$, relations (2.4-2.6) depend on S_{ot} through the apparent water saturation S_{wa}, which is defined by

$$S_{wa} = S_w + S_{ot}. \tag{2.7}$$

Defining S_m as the historic minimum of S_w, indicating the latest reversal from drainage to imbibition of water at a given location, and S_{or}^{max} as the maximum residual saturation, according to [1] the trapped oil saturation is given by

$$S_{ot} = \begin{cases} \dfrac{1 - S_m}{1 + R(1 - S_m)} - \dfrac{1 - S_{wa}}{1 + R(1 - S_{wa})} & \text{for } S_{wa} > S_m \\ 0 & \text{for } S_{wa} \leq S_m, \end{cases} \tag{2.8}$$

where $R = 1\,/\,S_{or}^{max} - 1$ is Land's factor. After combination of (2.7) and (2.8) S_w is related to S_{wa} by

$$S_w = \begin{cases} S_{wa} - \dfrac{1 - S_m}{1 + R(1 - S_m)} + \dfrac{1 - S_{wa}}{1 + R(1 - S_{wa})} & \text{for } S_{wa} > S_m \\ S_{wa} & \text{for } S_{wa} \leq S_m, \end{cases} \tag{2.9}$$

which reveals a simplified but consistent description of the hysteretic relation $p_c(S_w)$ due to entrapment.

At the start of the redistribution $t = 0$, we impose the initial conditions

$$S_w(x) = \begin{cases} S_{w,l} & \text{for } x < 0 \\ S_{w,r} & \text{for } x > 0, \end{cases} \quad \text{and} \quad S_{ot}(x) = \begin{cases} S_{ot,l} & \text{for } x < 0 \\ S_{ot,r} & \text{for } x > 0, \end{cases} \tag{2.10}$$

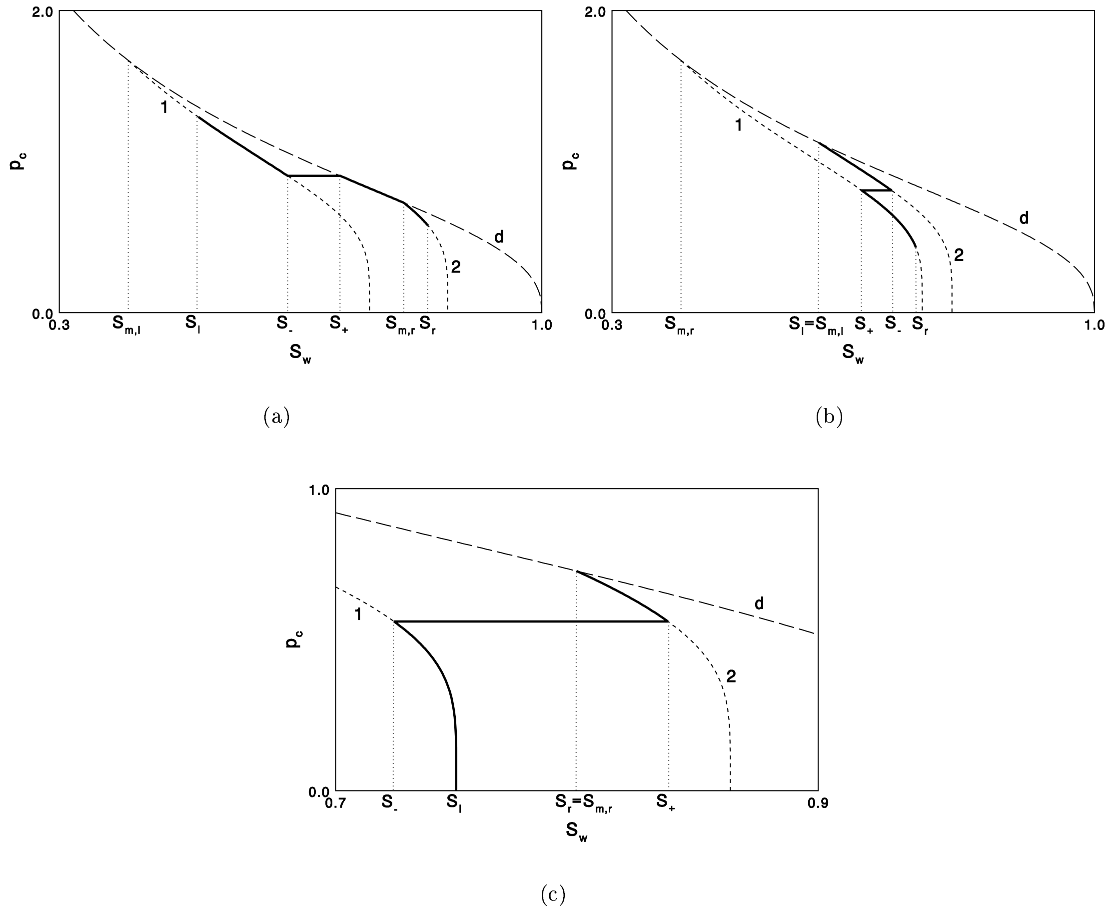

Figure 1: Hysteretic capillary pressure-saturation relations with parameters $m = 0.6$ and $S_{or}^{max} = 0.3$. Flow paths for (a) $S_{m,l} < S_{m,r}$ and $p_c(S_l) > p_c(S_r)$, (b) $S_{m,l} > S_{m,r}$ and $p_c(S_l) > p_c(S_r)$, (c) $S_{m,l} < S_{m,r}$ and $p_c(S_l) < p_c(S_r)$.

where the constants $S_{w,l}$ and $S_{w,r}$ satisfy $0 \leq S_{w,l} \leq S_{w,r} \leq 1$. The constants $S_{ot,\alpha}$ ($\alpha = l, r$) are chosen such that they correspond through the first of relations (2.8), with $S_{wa} = S_{ot,\alpha} + S_\alpha$, to values $S_m = S_{m,\alpha}$ that satisfy $0 \leq S_{m,\alpha} \leq S_\alpha$. Assuming that during the redistribution process no reversal takes place, $S_{m,l}$ and $S_{m,r}$ are fixed for all $t > 0$.

For all $t > 0$, we require continuity of water flux and capillary pressure throughout the domain, whereas the latter implies continuity of S_{wa} (2.4), especially at $x = 0$ [6, 11], i.e.

$$S_{wa}(S_-) = S_{wa}(S_+), \tag{2.11}$$

with $S_-(t) = \lim_{x\uparrow 0} S_w(x,t)$ and $S_+(t) = \lim_{x\downarrow 0} S_w(x,t)$.

In Figure 1 we show possible flow paths for the redistribution process in the (p_c, S_w) plane. We have drawn the main drainage curve (d), which reflects the nonhysteretic function $\bar{p}_c(S_{wa})$ (2.7), and the scanning curves 1 and 2. The latter reflect the relation $p_c(S_w)$, according to relations (2.7) and (2.9), for $S_w > S_{m,\alpha}$, $\alpha = l, r$. For $S_w \leq S_{m,\alpha}$, relation (2.9) reveals that $p_c(S_w) = \bar{p}_c(S_{wa})$. Observe that the ordering of the $S_{m,\alpha}$ determines the ordering of the scanning curves. The flow paths connect $(p_c(S_l), S_l)$ to $(p_c(S_r), S_r)$, such that p_c varies monotonically. In Figure 1.a we have $S_{m,l} < S_{m,r}$,

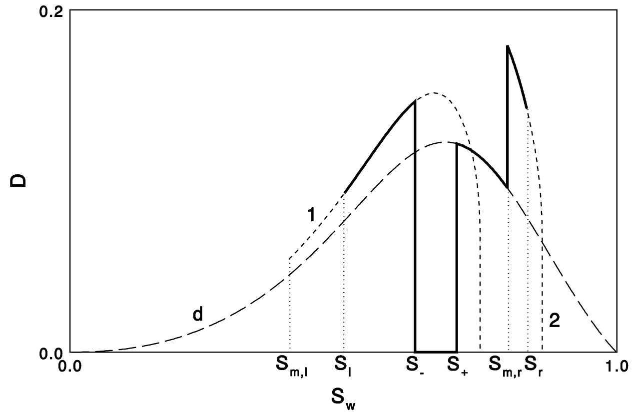

Figure 2: Operative diffusion function with parameters $m = 0.6$, $M = 2.0$ and $S_{or}^{max} = 0.3$, which corresponds to the flow path of Figure 1.a.

hence curve 1 below curve 2. At $x = 0$, where the path jumps from the curve corresponding to S_l to the curve corresponding to S_r, the saturations S_- and S_+ satisfy $S_- < S_+$. Furthermore, since the trapped oil saturation S_{ot} satisfies $S_{ot} = S_{wa} - S_w$ (2.7), the change of the flow path from curve 2 to the main drainage curve at $S_w = S_{m,r}$ indicates release of all trapped oil in part of the right half of the domain. In Figure 1.b the situation for $S_{m,l} > S_{m,r}$ is shown, leading to $S_- > S_+$. Since water flows in the direction of increasing p_c, see relation (2.2), Figure 1.b shows that accross the saturation jump water flows unconventionally from a higher to a lower water saturation. In Figure 1.c we find the same ordering as in Figure 1.a except that p_c increases from S_l to S_r, leading to completely unconventional water flow from the lower S_l to the higher S_r [7]. It follows that the hysteresis in the capillary pressure-saturation relation that corresponds to the presence of trapped oil, may lead to water flow in the direction of increasing water saturations.

To analyze the redistribution process, we combine equations (2.1) and (2.2), leading to a diffusion equation for S_w, i.e.

$$\frac{\partial S_w}{\partial t} = \frac{\partial}{\partial x}\left(D(S_w)\,\frac{\partial S_w}{\partial x}\right). \tag{2.12}$$

with the diffusion function

$$D(S_w) = -\lambda(S_w)\,\frac{d p_c}{d S_w}(S_w). \tag{2.13}$$

In Figure 2 the operative diffusion function corresponding to the flow path of Figure 1.a is shown. D is equal to zero for $S_- < S_w < S_+$, where p_c is constant, and also discontinuous at $S_w = S_{m,r}$ due to the discontinuity of dp_c/dS_w. Note that for the unconventional cases of Figures 1.b and 1.c D can not be represented as a unique function of S_w.

3. The similarity solution

We solve (2.12) by means of the similarity transformation [6, 12]

$$f(\eta) = S_w(x,t) \quad \text{with } \eta = \frac{x}{\sqrt{t}}, \tag{3.1}$$

yielding for $f(\eta)$ the ordinary differential equation

$$\frac{1}{2}\eta f' + (D(f)f')' = 0 \quad \text{for } -\infty < \eta < \infty. \tag{3.2}$$

The initial condition (2.10) requires the boundary conditions

$$f(-\infty) = S_l \quad \text{and} \quad f(\infty) = S_r. \tag{3.3}$$

At $\eta = 0$ we impose continuity of $f_a(f)$ and of $D(f)f'$, where $f_a(f) = S_{wa}(f)$. f_a is defined by the inverse of relation (2.9), which can be obtained explicitly.

Following [12] we apply the transformation

$$y = -D(f)\frac{d\,f}{d\,\eta}, \tag{3.4}$$

where y represents the water flux (2.2). Combining (3.2) and (3.4) leads to

$$\frac{d\,y}{d\,f} = \frac{1}{2}\eta, \tag{3.5}$$

whenever $f' \neq 0$. Differentiating (3.5) with respect to f and using (3.4), yields for $y(f)$ the equation

$$-y\,y'' = \frac{D(f)}{2}, \tag{3.6}$$

where primes $'$ now denote differentiation with respect to f. Analogous to the definitions of S_- and S_r we introduce the constants $f_- = \lim_{\eta\uparrow 0} f(\eta)$ and $f_+ = \lim_{\eta\downarrow 0} f(\eta)$. Hence, equation (3.6) is solved on the two intervals (S_l, f_-) and (f_+, S_r) with solutions y_l and y_r respectively, due to the conditions

$$y_l(S_l) = y_r(S_r) = 0 \quad \text{and} \quad y_l'(f_-) = y_r'(f_+) = 0, \tag{3.7}$$

following from the assumptions of vanishing fluxes at $\eta = \pm\infty$ and from relation (3.5) at $\eta = 0$ respectively. Although the pairs $(f_-, y_l(f_-))$ and $(f_+, y_r(f_+))$ are unknown a priori, they are linked by $f_a(f_-) = f_a(f_+)$ and $y_l(f_-) = y_r(f_+)$. The details of numerically solving (3.6) for y_l and y_r and the values of f_- and f_+ are described in [12].

Before computing the similarity profile $f(\eta)$, we analyze the behaviour of f near the end values S_l and S_r. If the diffusion coefficient $D(f)$ degenerates for $f = S_r$ or $f = S_l$, finite numbers $\eta_l < 0$ or $\eta_r > 0$ (free boundaries) exist, satisfying $f(\eta_l) = S_l$ and $f(\eta_r) = S_r$, and the solution of (3.2) for $\eta \leq \eta_l$ is given by $f(\eta) \equiv S_l$ or for $\eta \geq \eta_r$ by $f(\eta) \equiv S_r$, respectively. When f decreases towards S_α, with $\alpha = l, r$, degeneration occurs if $D(f)/(f - S_\alpha)$ is integrable in a neighbourhood of $f = S_\alpha$. When f increases towards S_α, degeneration occurs if $D(f)/(S_\alpha - f)$ is integrable [9]. If this condition is not met, $f(\eta)$ tends asymptotically to S_α for $\eta \to \pm\infty$.

It follows easily that degeneration may only occur if $D(S_\alpha) = 0$ and Figure 2 shows that this happens only if $S_\alpha = 0$ or if $f_a(S_\alpha) = 1$. In the latter case we have either $S_\alpha < 1$, representing the endpoint of a scanning curve, or $S_\alpha = 1$, representing the endpoint of the main drainage curve in Figure 1. When $S_\alpha = 0$, the behaviour of $D(f)$ for $f \downarrow 0$ is given by [12]

$$D(f) \sim m\,(1-m)\,f^{\frac{1}{m}+\frac{1}{2}}, \tag{3.8}$$

which yields degeneration for all $0 < m < 1$. When $f_a(S_\alpha) = 1$, we find

$$D(f) \sim \begin{cases} \dfrac{1-m}{M\,m^{m+1}}\,(1-f)^{m+\frac{1}{2}} & \text{if } S_\alpha = 1 \\ \dfrac{1-m}{2\,R^{\frac{m}{2}+\frac{3}{4}}\,M\,\bar{k}\,m^{m+1}}\,(S_\alpha - f)^{\frac{m}{2}-\frac{1}{4}} & \text{if } S_\alpha < 1 \end{cases} \tag{3.9}$$

for $f \uparrow S_\alpha$, with $\bar{k} = k_{rw}(1 - (1 - S_{m,\alpha})/(1 + R(1 - S_{m,\alpha})))$. Consequently, if $S_\alpha = 1$, degeneration occurs for all $0 < m < 1$, but if $S_\alpha < 1$, degeneration occurs only for $1/2 < m < 1$.

Furthermore, for the degenerate cases relation (3.5) yields the positions of the free boundaries $\eta_\alpha = 2\,\frac{dy}{df}(S_\alpha)$. The behaviour of f for $\eta \to \eta_\alpha$ is given by [9]

$$\frac{df}{d\eta} \sim \begin{cases} -\frac{\eta}{2}\,\frac{f}{D(f)} & \text{if } f \downarrow 0 \\ \frac{\eta}{2}\,\frac{S_\alpha - f}{D(f)} & \text{if } f \uparrow S_\alpha. \end{cases} \tag{3.10}$$

Hence, for all possible degenerate cases we find with (3.8) and (3.9)

$$\frac{df}{d\eta} \to \begin{cases} \infty & \text{for } 0 < m < 1 & \text{if } \eta \downarrow \eta_l \quad (S_l = 0) \\ 0 & \text{for } \frac{1}{2} < m < 1 & \text{if } \eta \uparrow \eta_l \quad (S_l < 1) \\ 0 & \text{for } \frac{1}{2} < m < 1 & \text{if } \eta \uparrow \eta_r \quad (S_r < 1) \\ \left.\begin{array}{ll} 0 & \text{for } 0 < m < \frac{1}{2} \\ \frac{\eta_r M}{2\sqrt{2}} & \text{for } m = \frac{1}{2} \\ \infty & \text{for } \frac{1}{2} < m < 1 \end{array}\right\} & & \text{if } \eta \uparrow \eta_r \quad (S_r = 1). \end{cases} \tag{3.11}$$

Computation of the similarity profile $f(\eta)$ is possible by inverting relation (3.5), but for both the nondegenerate cases and for the degenerate situations with zero slope at the free boundary much higher accuracy is obtained by solving equation (3.2) as two initial value problems [10]. For $\eta < 0$ the solution for $y(f)$ yields the conditions $f(0) = f_-$ and $(D(f)\,f')(0) = y_l(f_-)$, and for $\eta > 0$ we obtain $f(0) = f_+$ and $(D(f)\,f')(0) = y_r(f_+)$, with $y_l(f_-) \equiv y_r(f_+)$ by continuity of flux.

In Figure 3 we show the similarity profiles of the water saturation and the water flux for some typical situations. The similarity solution for the trapped oil saturation is found according to (2.7) as the difference $f_a - f$. Figure 3.a shows the situation with initially no trapped oil, leading to entrapment only for $\eta < 0$. Furthermore, at the left a free boundary exists, where f has an infinite slope, and at the right a free boundary exists, where f has a finite slope, because $m = 0.5$. Water flow is everywhere from right to left, yielding negative values of y (3.4). Figure 3.b shows the situation with initially trapped oil present at both sides of $\eta = 0$. At the right side the water saturation decreases and trapped oil is completely released in part of this half of the column. Although we have $f_a(S_r) = 1$, no degeneracy occurs at the right, because $m = 0.3$. Figure 3.c shows the situation with $S_{m,l} > S_{m,r}$ of Figure 1.b, yielding at $\eta = 0$ unconventional water flow from the lower (right) saturation to the higher (left) saturation, which corresponds to the multi-valued $y(f)$ relation. At the right a free boundary exists, where f has zero slope. Figure 3.d shows the situation with $p_c(S_l) > p_c(S_r)$ of Figure 1.c, leading to completely unconventional water flow from left to right, which is reflected by the nonnegative values of y for all f.

Figure 3 indicates that contrary to the water saturation profiles the apparent water saturation profiles are always continuous and monotone. This follows, according to relation (2.4), from the continuity and monotonicity of capillary pressure. As a result, water always flows in the direction of decreasing f_a.

4. Conclusions

This study shows how the similarity solution for hysteretic water redistribution can be applied to redistribution of two phases taking into account capillary entrapment of the nonwetting phase. The entrapment leads to hysteresis of both the capillary pressure and the relative permeability functions. The similarity solution applies also to situations where initially trapped nonwetting fluid is present. The latter may lead to unconventional flow in the direction of increasing phase saturations. Furthermore, for the present choice of the constitutive relations initially trapped fluid may avoid the occurrence of free boundaries and affects the corresponding saturation profiles. By means of a simple

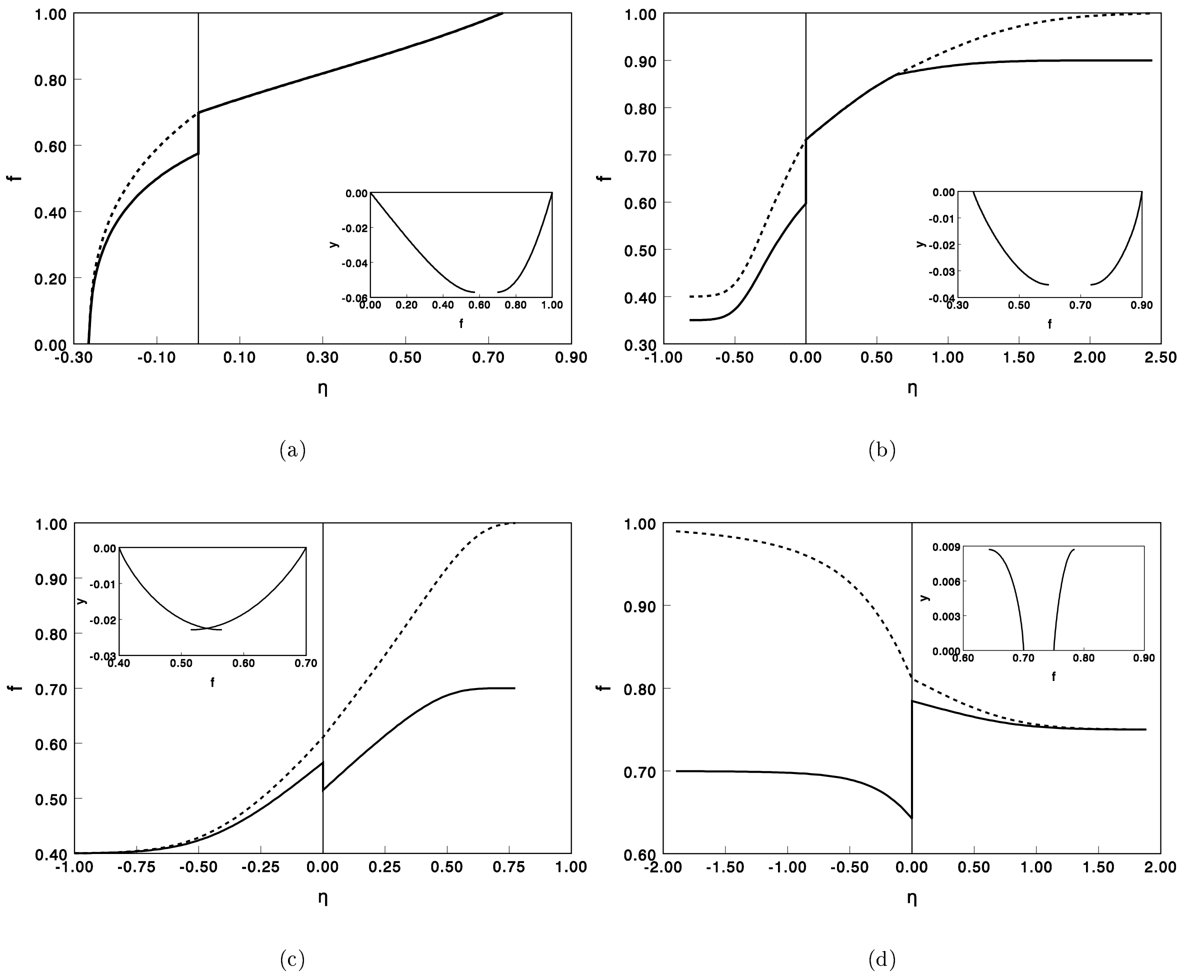

Figure 3: Similarity profiles of the water saturation f (solid lines) and apparent water saturation f_a (dashed lines, whenever different from f) and the water flux y as a function of water saturation f (insets), with (a) initially no trapped oil and degeneration at both endpoints ($m = 0.5$), (b) complete release of initially trapped oil in part of the domain ($m = 0.3$), (c) unconventional flow at $\eta = 0$ and degeneration at the right endpoint ($m = 0.8$), (d) completely unconventional flow ($m = 0.3$). Furthermore, $M = 2.0$ and $S_{or}^{max} = 0.3$ for all cases.

computation procedure the similarity solution provides a quantitative description of the two-phase flow process including nonwetting phase entrapment, which can be used to test numerical codes for hysteretic multi-phase flow.

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