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ABSTRACT
For the $GI/G/1$ queueing model with heavy-tailed service- and arrival time distributions and traffic $a < 1$ the limiting distribution of the contracted actual waiting time $\Delta(a)\mathbf{w}$ has been derived for $\Delta(a) \downarrow 0$ for $a \uparrow 1$, see [2]. In the present study we consider the workload process $\{\mathbf{v}_t, t > 0\}$, when properly scaled, i.e. $\Delta(a)\mathrm{v}_{\tau/\Delta_1(a)}$ for $a \uparrow 1$ with $\Delta_1(a) = \Delta(a)(1-a)$. We further consider the noise traffic $\mathbf{n}_t = \mathbf{k}_t - at$ and the virtual backlog $\mathbf{h}_t = \mathbf{k}_t - t$, with $\mathbf{k}_t$ the traffic generated in $[0, t)$. It is shown that $\mathbf{n}_t$ and $\mathbf{h}_t$, when scaled similarly as $\mathbf{v}_t$, have a limiting distribution for $a \uparrow 1$. We further consider the $M/G_R/1$ model. It is a model with instantaneous workload reduction. The arrival process is a Poisson process and the service time distribution and that of the workload reduction are both heavy-tailed. Of this model two variants have to be considered. The $M/G_R/1$ model is for the present purpose, the more interesting one, and for this model the properly scaled workload-, noise traffic- and virtual backlog process are shown to converge weakly when the scaling parameters tend to zero as a function of the traffic b for $b \uparrow 1$. The limiting processes of the noise traffic and virtual backlog (properly scaled) appear to be ν-stable Lévy motions for $1 < \nu < 2$, ν being the index of the heavy tails. The LSTs of these limiting distributions are derived. They have the same structure as those for the $GI/G/1$ model.

The results so far obtained lead to the introduction of the $\mathcal{L}_{v_1}/\mathcal{L}_{v_2}/1$ model. For $\boldsymbol{\nu}_1 = \nu_2 = \nu$, $1 < \nu < 2$, this is a buffer storage model of which the virtual backlog process is a Lévy motion with a negative drift -c. It is shown that for $0 < c < 1$ the workload or buffer content process $\{\mathbf{v}_t, t \geq 0\}$ possesses a stationary distribution and its LST has been derived.

The results of the present study are new and lead to a better understanding of the stochastics of queueing models of which the modelling distributions have heavy-tails of a type $t^{-\mathrm{v}}\mathcal{S}(t)$ for $t \to \infty$, $1 < \nu \leq 2$ and $\mathcal{S}(t)$ a slowly varying function at infinity.

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1. Introduction

In author's previous study [2] the $GI/G/1$ queue has been analysed for the case with heavy-tailed server- and interarrival time distributions, i.e. for $t \to \infty$, $1 < \nu_j \leq 2$,

$$1 - B(t) \sim t^{-\nu_2} S_2(t) \quad \text{and} \quad 1 - A(t) \sim t^{-\nu_1} S_1(t), \tag{1.1}$$

with $S_j(t)$ a slowly varying function at infinity.

The main result in [2] concerns the stationary actual waiting time $\mathbf{w}$ of the stable $GI/G/1$ queue with traffic $a \in (0, 1)$. It was shown that a unique scaling factor $\Delta(a)$, the so-called contraction factor, exists such that $\Delta(a)\mathbf{w}$ converges in distribution for $a \uparrow 1$. This contraction factor $\Delta(a)$ is defined as that root of the contraction equation which tends to zero for $a \uparrow 1$. The limiting distribution of $\Delta(a)\mathbf{w}$ is a proper probability distribution. It has beend derived in [2] for the three cases $\nu_1 < \nu_2$, $\nu_1 = \nu_2$ and $\nu_1 > \nu_2$.

The present study is an extension of [2]. Concerning $\mathbf{k}_t$, the total traffic generated by the arrivals in $[0, t)$, the question arrises whether a proper scaling of $\mathbf{k}_t$ exists which converges in distribution for $a \uparrow 1$. In [5] it has been shown, that for the $M/G/1$ model with $1 - B(t)$, for $t \to \infty$ as given in (1.1), such a scaling exists indeed. In the present study it will be shown that this result can be extended to the $GI/G/1$ model. For $\mathbf{n}_t$, the so-called noise traffic, i.e., for $t \geq 0$,

$$\mathbf{n}_t := \mathbf{k}_t - at, \tag{1.2}$$

it is shown that, with $\Delta_1(a) = (1 - a)\Delta(a)$,

$$\mathbf{N}(\tau; a) := \Delta(a)\mathbf{n}_{\tau/\Delta_1(a)} \tag{1.3}$$

converges in distribution for $a \uparrow 1$. With $\mathbf{N}(\tau)$ a stochastic variable with distribution this limiting distribution of $\mathbf{N}(\tau; a)$, it appears that the process $\{\mathbf{N}(\tau), \tau \geq 0\}$ is a ν-stable Lévy motion for $\nu := \min(\nu_2, \nu_2) < 2$; for $\nu = 2$ it is the Brownian motion.

To obtain more insight between the limiting structure of $\mathbf{N}(\tau; a)$ and that of $\Delta(a)\mathbf{w}$ for $a \uparrow 1$, we analyse in this study the queueing model with load removal. This model, to be called the $M/G_R/1$ model, is most easily described in terms of an infinite buffer model. In [7] the relation between this model and the $GI/G/1$ model has been exposed. The $M/G_R/1$ model is described at follows.

At epochs $\mathbf{t}_n$ of a Poisson point process with rate λ the buffer content is increased by an amount $\boldsymbol{\tau}_n^+$ with probability p^+, whereas with probability p^- it is decreased by the minimum of the buffer content at t_n and $\boldsymbol{\tau}_n^-$. Here $\{\boldsymbol{\tau}_n^+, n = 0, 1, \dots\}$ and $\{\boldsymbol{\tau}_n^-, n = 0, 1, 2, \dots\}$ are independent families of i.i.d. nonnegative stochastic variables, these families are also independent of the $\mathbf{t}_n$-sequence. This process is studied for the case that $\boldsymbol{\tau}_n^+$ and $\boldsymbol{\tau}_n^-$ both have a heavy tail. With $\mathbf{m}_t$ the number of jump epochs $\mathbf{t}_n$ in $(0, t)$ and $\boldsymbol{\tau}_n = \boldsymbol{\tau}_n^\pm$ with probability $p^\pm$, the 'traffic load' $\mathbf{k}_t$ generated in $[0, t)$ is given by

$$\mathbf{k}_t = \sum_{n=0}^{\mathbf{m}_t} \boldsymbol{\tau}_n. \tag{1.4}$$

This $M/G_R/1$ process may be considered as a buffer model with unit decrease per unit of time. With

$$\beta := p^+\mathrm{E}\{\boldsymbol{\tau}_n^+\} - p^-\mathrm{E}\{\boldsymbol{\tau}_n^-\}, \tag{1.5}$$

it may be shown by using a standard technique that

$$b := \lambda\beta < 1, \tag{1.6}$$

is the stability condition. E.g. with $\mathbf{w}_n$ the buffer content immediately before the n-th jump epoch, it can be shown that $\mathbf{w}_n$ converges in distribution for $n \to \infty$. From our analysis it turns out that for the heavy traffic analysis of this $M/G_R/1$ model we have to distinguish two cases, viz. $\beta > 0$ and $\beta < 0$.

Let us first consider the case with $\beta > 0$, i.e.

$$0 < b < 1. \tag{1.7}$$

As for the $GI/G/1$ model we consider here also the noise traffic

$$\mathbf{n}_t := \mathbf{k}_t - bt. \tag{1.8}$$

The $\{\mathbf{n}_t, t \geq 0\}$ process is obviously a process with independent stationary increments. In the present study it is shown that a contraction factor $\nabla(b)$ exists such that, with $\nabla_1(b) = \nabla(b)(1 - b)$,

$$\mathbf{N}(\tau; b) := \nabla(b)\mathbf{n}_{\tau/\nabla_1(b)}, \tag{1.9}$$

converges for $\nabla(b) \downarrow 0$, i.e. for $b \uparrow 1$, in distribution. With $\mathbf{N}(\tau)$ a stochastic variable with distribution the limiting distribution of $\mathbf{N}(\tau; b)$, it appears that $\{\mathbf{N}(\tau), \tau \geq 0\}$ is again a ν-stable Lévy motion for $\min(\nu^-, \nu^+) < 2$ and a Brownian motion when this minimum is equal to 2; here $\nu^\pm$ is the index of the tail of the distribution of $\boldsymbol{\tau}_n^\pm$.

Let $\mathbf{v}_t$ be the buffer contents at time t and $\mathbf{v}$ and $\mathbf{w}$ stochastic variables with distribution the limiting distribution of $\mathbf{v}_t$ for $t \to \infty$ and $\mathbf{w}_n$ for $n \to \infty$, respectively. It is shown that $\nabla(b)\mathbf{v}$ and $\nabla(b)\mathbf{w}$ both converge in distribution. They have the same limiting distribution and its structure is the same as that of the analogous distributions of the $GI/G/1$ model.

Whenever

$$\lambda\beta < 0, \tag{1.10}$$

i.e. $\beta < 0$, it appears that for finite λ, a scaling of $\mathbf{n}_t$ like that in (1.9) does not exist. However for $\lambda = \infty$ a scaling does exists. It is then shown that the noise traffic when property scaled again converges in distribution for $\beta \uparrow 0$. For $\lambda = \infty$ the $\mathbf{w}_n$-sequence for the $M/G_R/1$ model is defined by

$$\mathbf{w}_{n+1} = [\mathbf{w}_n + \boldsymbol{\tau}_n]^+, \qquad n = 0, 1, \ldots . \tag{1.11}$$

The condition $\beta < 0$ implies that $\mathbf{w}_n$ converges in distribution for $n \to \infty$. With $\mathbf{w}$ as defined above, it is shown that a scaling of $\mathbf{w}$ exists which converges in distribution for $\beta \uparrow 0$. The relevant limiting distribution has again the same structure as the analogous one for the $GI/G/1$ model.

We conclude this introduction with an overview of the various sections.

In Section 2 we describe the $GI/G/1$ model with heavy-tailed service- and interarrival time distributions $B(t)$ and $A(t)$. The index of the tail of $A(t)$ is indicated by ν_1 that of $B(t)$ by ν_2. The cases $\nu_1 = \nu_2$, $\nu_2 < \nu_1$ and $\nu_1 < \nu_2$ with $\nu_j \leq 2$ should be distinguished.

In Section 3 the LSTs of the generated traffic $\mathbf{k}_t$ and of the noise traffice $\mathbf{n}_t$ for the $GI/G/1$ model are derived. Section 4 starts with the formulation of the contraction equation for the $GI/G/1$ model. This equation resulted from the analysis in [2], and it defines the contraction coefficient. With this contraction coefficient the contracted noise traffic $\mathbf{N}(\tau; a)$ is defined for $a < 1$, here a is the traffic load of the $GI/G/1$ model. Theorem 4.1 states that the contracted noise traffic converges in distribution for $a \uparrow 1$, and it describes the LST of the limiting distribution. With $\mathbf{N}(\tau)$ a stochastic variable with this limiting distribution, it is seen in Section 5 that the process $\{\mathbf{N}(\tau), \tau \geq 0\}$ is a ν-stable Lévy motion for $1 < \nu < 2$, and a Brownian motion for $\nu = 2$. For $\nu < 2$, the probabilities of the up- and downward jumps of the Lévy motion are expressed in terms of the characteristics of the tails of the service- and interarrival times distributions $B(t)$ and $A(t)$.

In Section 6, where $\mathbf{v}$ and $\mathbf{w}$ are variables with distributions the stationary distributions of the workload and the waiting time of the $GI/G/1$ model, it is shown that the contracted variables $\Delta(a)\mathbf{v}$ and $\Delta(a)\mathbf{w}$ converge in distribution for $a \uparrow 1$; they have the same limiting distribution. The LST transform of this distribution, already derived in [2], is described in (6.5).

In Section 7 the $M/G_R/1$ model is described. Here the heavy tailed distributions of $\boldsymbol{\tau}_n^\pm$ are defined. In Section 8 the noise traffic for the $M/G_R/1$ model is defined. It turns out that for the heavy traffic analysis two cases have to be distinguished, see (8.9). For both cases it is seen that similar heavy traffic results can be derived for the noise traffic and the waiting times (or buffer content, for the latter) see Section 9. The analysis proceeds along similar times as that for the $GI/G/1$ model. The essential point is here the construction of the contraction equation. Once this has been obtained the determination of the limiting distribution of the contracted waiting time requires the derivation of the solution of a Wiener-Hopf problem of a type studied in [2]. This limiting distribution has the same structure as that for the $GI/G/1$ model.

In Section 10 the limiting distributions for the $GI/G/1$ model and the two variants of the $M/G_R/1$ model are compared. It leads to an inisght concerning the influence of the tails of the various heavy-tailed distributions on the coefficients of the heavy-traffic limiting distributions.

In Section 11 we investigate the question whether the functional relation between the contracted workload and the contracted virtual backlog for the $M/G_R^{(1)}/1$ model with $0 < b < 1$ also holds (in distribution) for $b \uparrow 1$. It appears that this is indeed the case.

In Section 12 the buffer model $\mathcal{L}_\nu/\mathcal{L}_\nu/1$ is introduced. The buffer has an infinite capacity and its virtual backlog process is described by a ν-stable Lévy motion. By using the results of Section 10 it is shown that, whenever this Lévy motion has a negative drift, then the workload process of this buffer model possesses a stationary distribution. The LSF of this distribution is obtained. The model $\mathcal{L}_{\nu_1}/\mathcal{L}_{\nu_2}/1$ with $\nu_1 \neq \nu_2$ is defined as a special case of the $\mathcal{L}_\nu/\mathcal{L}_\nu/1$ model.

In Section 13 we consider the traffic discription of a buffer model with heavy tailed distributions as developed in [10]. Here the authors introduce a scaling of the noise traffic and show that this scaled noise traffic process converges when the scaling factor $T \to \infty$. It is illustrated that this limiting process can be used only for the analysis of the workload process of the buffer if T is a specified function of the traffic load a with $T \to \infty$ for $a \uparrow 1$. The relation between a and T is provided by the contraction equation.

2. The $GI/G/1$ model

We consider the $GI/G/1$ model with interarrival time distribution $A(t)$ and service time distribution $B(t)$.

$$\alpha := \int_0^\infty t\mathrm{d}A(t) < \infty, \qquad \beta := \int_0^\infty t\mathrm{d}B(t), \tag{2.1}$$

$$a := \beta/\alpha < 1,$$

with interarrival- and service time distributions of the following type. For a finite $T > 0$ it is assumed that: for $t > T$,

$$1 - A(t) = G_{11}(t) + G_{12}(t), \tag{2.2}$$

$$1 - B(t) = G_{21}(t) + G_{22}(t),$$

with

$$|\int_T^\infty \mathrm{e}^{-\rho t} G_{j1}(t)\mathrm{d}t| < \infty \text{ for } \mathrm{Re}\,\rho > -\delta, \qquad j = 1, 2,$$

for a $\delta > 0$, and where $G_{j2}(t)$ characterises for $t \to \infty$ the dominant terms of the tails. For instance

$$G_{j2}(t) = t^{-\nu} S_j(t), \quad t \to \infty, \quad 1 < \mathrm{v} \leq 2,$$

with $S_j(t)$ a slowly varying function at infinity, is an example of a heavy-tailed distribution.

Denote by $\alpha(\rho)$ and $\beta(\rho)$ the Laplace-Stieltjes transform (LST) of $A(t)$ and $B(t)$, respectively; for $\mathrm{Re}\,\rho \geq 0$,

$$\alpha(\rho) = \int_{0-}^\infty \mathrm{e}^{-\rho t}\mathrm{d}A(t), \qquad \beta(\rho) := \int_{0-}^\infty \mathrm{e}^{-\rho t}\mathrm{d}B(t). \tag{2.3}$$

For many distributions of the type (2.2) the LSTs can be characterized by: for $\mathrm{Re}\,\rho \geq 0$,

$$1 - \frac{1 - \alpha(\rho)}{\rho\alpha} = g_1(\gamma\rho) + C_1(\gamma\rho)^{\nu_1 - 1} L_1(\gamma\rho), \tag{2.4}$$

$$1 - \frac{1 - \beta(\rho)}{\rho\beta} = g_2(\gamma\rho) + C_2(\gamma\rho)^{\nu_2 - 1} L_2(\gamma\rho),$$

with: for $j = 1, 2$,

i. $C_j \geq 0$ and $C_1 + C_2 > 0$, (2.5)

$\gamma > 0,$ here γ stands for the unit of time,

ii. $1 < \nu_j \leq 2$,

iii. $g_j(\gamma\rho)$ is regular for $\mathrm{Re}\,\gamma > -\delta$ for a $\delta > 0$, $g_j(0) = 0$,

iv. $L_j(\gamma\rho)$ is regular for $\mathrm{Re}\,\rho > 0$, continuous for $\mathrm{Re}\,\rho \geq 0$, except possibly at $\rho = 0$, with $L_j(\gamma\rho) \to b_j > 0$ for $|\rho| \to 0$, $\mathrm{Re}\,\rho \geq 0$, and

$$b_j \leq \infty \quad \text{for} \quad 1 < \nu_j \leq 2,$$

$$b_j = \infty \quad \text{for} \quad \nu_j = 2,$$

$$\lim_{x \downarrow 0} \frac{L_j(x\gamma\rho)}{L_j(x)} = 1 \text{ for every } \rho \text{ with } \mathrm{Re}\,\rho \geq 0, \rho \neq 0.$$

It is further assumed that the following limit exists: for $C_1 > 0$, $C_2 > 0$,

$$f := \lim_{x \downarrow 0} \frac{L_2(x)}{L_1(x)} \text{ and } 0 < f < \infty; \text{ for } C_1 > 0,\ C_2 > 0. \tag{2.6}$$

Note that we exclude the cases $f = 0$ and $f = \infty$, actually $f = \infty$ leads to the same result as $C_1 = 0, C_2 > 0$, similarly $f = 0$ to those with $C_1 > 0$, $C_2 = 0$.

Further, we define:

$$c_1 := \alpha/\gamma, \qquad c_2 := \beta/\gamma, \qquad a = c_2/c_1. \tag{2.7}$$

We make the following remarks concerning the assumptions introduced above.

Remark2.1. Many valued functions such as ρ^{ν_i}, $\log\rho$, arctanρ are defined by their principal values, so $\rho^{\nu_i} > 0$ for $\rho > 0$, $\log\rho$ is real for $\rho > 0$ and $|\mathrm{arctan}\rho| < \frac{1}{2}\pi$ for $-\infty < \rho < \infty$. □

Remark 2.2. The class of distributions $A(t)$ and $B(t)$ satisfying (2.4) is actually a subclass of those characterised by (2.2) see [1], [3], for examples.

Remark 2.3. Because the lefthand sides in (2.4) are positive for $\rho > 0$ it follows that $L_j(\gamma\rho)$ is real for $\rho > 0$, and since these lefthand sides are zero for $\rho = 0$ it is seen that $x^{\nu_j - 1}L_j(x) \to 0$ for $x \to 0$, $\mathrm{Re}\,x \geq 0$. □

Remark 2.4. The case for which the limit in (2.6) does not exist, will not be considered in this study. □

Remark 2.5. We have excluded the case $C_1 = 0$ and $C_2 = 0$, note $C_1 = 0$ corresponds to the case that $A(t)$ has a negative exponential tail because $\delta > 0$, analogously if $C_2 = 0$.

Further we define:

$$\begin{array}{llllll} \text{whenever} & \nu_1 = \nu_2 & \text{and} & C_1 > 0,\ C_2 > 0 & \text{then} & \nu := \nu_1 = \nu_2, \\ \text{whenever} & \nu_2 < \nu_1 & \text{or} & C_1 = 0, & \text{then} & \nu := \nu_2, \\ \text{whenever} & \nu_1 < \nu_2 & \text{or} & C_2 = 0 & \text{then} & \nu := \nu_1. \end{array} \tag{2.8}$$

□

3. On the noise traffic

Let $\boldsymbol{\omega}_t$ be the number of arrivals in $[0,t)$ and $\mathbf{k}_t$ the total traffic load generated in $[0,t)$ by these arrivals, $\boldsymbol{\omega}_{0+} = \mathbf{k}_{0+} = 0$. The virtual backlog $\mathbf{h}(t)$ and the noise traffic $\mathbf{n}(t)$ are defined by: for $t \geq 0$,

$$\begin{aligned} \mathbf{h}_t &:= \mathbf{k}_t - t, \\ \mathbf{n}_t &:= \mathbf{k}_t - at. \end{aligned} \tag{3.1}$$

Remark 3.1. A more appropriate definition of noise traffic would be

$$n_t := \mathbf{k}_t - \mathrm{E}\{\mathbf{k}_t\},$$

because it has a zero expectation for all $t \geq 0$. However, the definition in (3.1) is technically some what easier to handle; moreover, we are mainly concerned with the asymptotics for $t \to \infty$ and $\mathrm{E}\{\mathbf{n}_t\} \to 0$ for $t \to \infty$. $\square$

Because

$$\Pr\{\boldsymbol{\omega}_t = n\} = A^{n*}(t) - A^{(n+1)*}(t), \qquad n = 0,1,2,\dots, \tag{3.2}$$

we have for $\mathrm{Re}\,\rho \geq 0$, $t \geq 0$,

$$\mathrm{E}\{\mathrm{e}^{-\rho \mathbf{k}_t}\} = \sum_{n=0}^{\infty} \beta^n(\rho)\{A^{n*}(t) - A^{(n+1)*}(t)\}. \tag{3.3}$$

Hence: for $\mathrm{Re}\, s > 0$, $\mathrm{Re}\,\rho \geq 0$,

$$\int_0^\infty \mathrm{e}^{-st}\mathrm{E}\{\mathrm{e}^{-\rho \mathbf{n}_t}\}\mathrm{d}t = \int_0^\infty \mathrm{e}^{-st}\mathrm{E}\{\mathrm{e}^{-\rho(\mathbf{k}_t - at)}\}\mathrm{d}t = \frac{1-\alpha(s-a\rho)}{s-a\rho}\,\frac{1}{1-\beta(\rho)\alpha(s-a\rho)} \tag{3.4}$$

$$= \left[s - \frac{a\rho}{\frac{1-\alpha(s-a\rho)}{s-a\rho}}\left[\frac{1-\alpha(s-a\rho)}{(s-a\rho)\alpha} - \alpha(s-a\rho) + \alpha(s-a\rho)\{1 - \frac{1-\beta(\rho)}{\rho\beta}\}\right]\right]^{-1},$$

here the last equality follows via some simple algebra, we need it in the next section.

4. The contracted traffic for the $GI/G/1$ model

In [2] Sections 3, 6 and 7 we have introduced the contraction equations for the three cases $\nu_1 = \nu_2$, $\nu_1 > \nu_2$ and $\nu_1 < \nu_2$. In the definition given here we combine these three cases. The contraction equation reads: for $0 < a < 1$.

$$1 - a = x^{\nu-1}|L(x)|, \quad x > 0, \tag{4.1}$$

with

$$\begin{aligned} &\text{i.} \quad L(x) &&= aC_2L_2(x) + (-1)^\nu C_1L_1(x) && \text{for} \quad \nu = \nu_1 = \nu_2 \ \text{and} \ C_1 > 0,\ C_2 > 0, \\ &\text{ii.} &&= aC_2L_2(x) && \text{for} \quad \nu = \nu_2 < \nu_1 \quad \text{or} \quad C_1 = 0, \\ &\text{iii.} &&= (-1)^\nu C_1L_1(x) && \text{for} \quad \nu = \nu_1 < \nu_2 \quad \text{or} \quad C_2 = 0, \end{aligned} \tag{4.2}$$

and

$$c_1 = 1, \qquad c_2 = a. \tag{4.3}$$

REMARK 4.1. Only the cases mentioned in (4.3) are of interest for our analysis. For another characterisation of these cases. see Remark 4.3. □

The contraction coefficient $\Delta(a)$ is defined to be that root of (4.1) which tends to zero for $a \uparrow 1$.

REMARK 4.2. For a sufficiently close to one, i.e. $0 < 1 - a << 1$ the root $\Delta(a)$ of (4.1) always exists and its multiplicity is one, so it will always be assumed that

$$0 < 1 - a << 1.$$

For a proof of an analogous statement see Appendix A. □

Next we introduce the following scaling:

$$\begin{aligned} \rho = &\ r\Delta(a), \quad \operatorname{Re} r \geq 0, \\ t = &\ \tau/\Delta_1(a), \quad \tau \geq 0, \\ s = &\ \sigma\Delta_1(a), \quad \operatorname{Re} \sigma \geq 0, \end{aligned} \tag{4.4}$$

with

$$\Delta_1(a) := (1 - a)\Delta(a). \tag{4.5}$$

The contracted noise traffic $\mathbf{N}(\tau; a)$ and the contracted virtual backlog $\mathbf{H}(\tau; a)$ are defined as follows: for $0 < 1 - a << 1$,

$$\begin{aligned} \mathbf{N}(\tau; a) &:= \Delta(a)\mathbf{n}_{\tau/\Delta_1(a)}, \\ \mathbf{H}(\tau; a) &:= \Delta(a)\mathbf{h}_{\tau/\Delta_1(a)} = \mathbf{N}(\tau; a) - \tau. \end{aligned} \tag{4.6}$$

cf. (3.1) and (4.4), i.e traffic is scaled by $\Delta(a)$ and time by $\Delta_1(a)$.

REMARK 4.3. In our analysis we shall frequently write

$$\begin{aligned} &\Delta \quad \text{instead of } \Delta(a), \\ &\Delta_1 \quad \text{instead of } \Delta_1(a). \end{aligned}$$

□

From (3.4), (4.4) and (4.6) we obtain: for $0 < 1 - a << 1$, $\operatorname{Re} r = 0$ and $\operatorname{Re} \sigma > 0$,

$$\int_0^\infty \mathrm{e}^{-\sigma\tau}\mathrm{E}\{\mathrm{e}^{-r\mathbf{N}(\tau;a)}\}\mathrm{d}\tau = \tag{4.7}$$

$$\Big[\sigma - \frac{ar}{1-a}\left[\hat{\alpha}(\sigma\Delta_1 - ar\Delta)\right]^{-1}\{\hat{\alpha}(\sigma\Delta_1 - ar\Delta) - \alpha(\sigma\Delta_1 - ar\Delta) + \alpha(\sigma\Delta_1 - ar\Delta)(1 - \frac{1-\beta(r\Delta)}{\beta r\Delta})\}\Big]^{-1}.$$

with

$$\hat{\alpha}(s) := \frac{1 - \alpha(s)}{\alpha s}, \quad \operatorname{Re} s \geq 0. \tag{4.8}$$

From (1.4) we have: for Re $r \geq 0$, Re $\sigma \geq 0$,

$$\begin{aligned} \frac{1-\alpha(\sigma\Delta_1 - a\Delta r)}{(\sigma\Delta_1 - a\Delta r)\alpha} &= 1 - g_1((\sigma(1-a) - ar)\Delta\gamma) \\ &- (-1)^{\nu_1 - 1} C_1[-(\sigma(1-a)+ar)\gamma]^{\nu_1 - 1}\Delta^{\nu_1 - 1} L_1((\sigma(1-a) - ar)\Delta\gamma), \end{aligned} \tag{4.9}$$

$$1 - \frac{1-\beta(r\Delta)}{\beta r\Delta} = g_2(\gamma r\Delta) + C_2(\gamma r)^{\nu_2 - 1}\Delta^{\nu_2 - 1} L_2(\gamma r\Delta)),$$

and: for Re $r \geq 0$, Re $\sigma > 0$, $c_1 = 1$,

$$\begin{aligned} \alpha(\sigma\Delta_1 - ar\Delta) &= 1 - (\sigma(1-a) - ar)\gamma\Delta[1 - g_1((\sigma(1-a) - ar)\gamma\Delta)] \\ &+ (-1)^{\nu_1} C_1[(-\sigma(1-a)+ar)\gamma]^{\nu_1}\Delta^{\nu_1} L_1((\sigma(1-a) - ar)\Delta\gamma). \end{aligned} \tag{4.10}$$

From (2.5)iv and (4.1) it follows readily, since $1 < \nu_j \leq 2$, that

$$\frac{\Delta(a)}{1-a} = \Delta^{2-\nu_j}(a)\big|L(\Delta(a))\big|^{-1} \to 0 \quad \text{for} \quad a \uparrow 1. \tag{4.11}$$

Hence from (2.5)iii: for $a \uparrow 1$, Re $r = 0$, Re $\sigma \geq 0$,

$$\begin{aligned} \frac{1}{1-a} g_1((\sigma(1-a) - ar)\Delta\gamma) &\to 0, \\ \frac{1}{1-a} g_2(\gamma r\Delta) &\to 0, \\ \frac{1}{1-a}(\sigma(1-a) - ar)\gamma\Delta[1 - g_1((\sigma(1-a) - ar)\gamma\Delta)] &\to 0. \end{aligned} \tag{4.12}$$

From (4.1) and (2.5)iv for: $C_1 > 0$, $a \uparrow 1$, Re $r = 0$, Re $\sigma \geq 0$,

$$\begin{aligned} &[(ar - \sigma(1-a)\gamma]^{\nu_1}\Delta^{\nu_1} L_1((\sigma(1-a) - ar)\Delta\gamma) = \\ &[(ar - \sigma(1-a)\gamma]^{\nu_1}\frac{\Delta^{\nu_1 - 1}}{1-a}|L(\Delta)|\frac{L_1((\sigma(1-a) - ar)\Delta\gamma)}{|L(\Delta)|}\Delta C_1 \to 0, \end{aligned} \tag{4.13}$$

because $\Delta \to 0$ and, cf. (2.6) and (4.2): for $\Delta \downarrow 0$,

$$\frac{L_1((\sigma(1-a) - ar)\Delta\gamma)}{|L_1(\Delta)|} \to 1, \quad \frac{L_1(\Delta)}{|L(\Delta)|} \to \frac{1}{(-1)^\nu C_1 + C_2 f}, \quad C_1 > 0.$$

Put for Re $r = 0$, Re $\sigma > 0$.

$$\begin{aligned} I(\Delta) := (-1)^{\nu_1} C_1[(-\sigma(1-a)+ar)\gamma]^{\nu_1 - 1}\frac{L_1((\sigma(1-a) - ar)\Delta\gamma)}{L_1(\Delta)}\ \frac{L_1(\Delta)}{|L(\Delta)|}\ \frac{\Delta^{\nu_1 - 1}}{1-a}\ |L(\Delta)| + \\ C_2(\gamma r)^{\nu_2 - 1}\frac{L_2(\gamma r\Delta)}{L_2(\Delta)}\ \frac{L_2(\Delta)}{|L(\Delta)|}\ \frac{\Delta^{\nu_2 - 1}}{1-a}|L(\Delta)|, \end{aligned} \tag{4.14}$$

and, cf. (2.6) and [2], formula (3.12),

$$d_j := C_j \lim_{x \downarrow 0} \frac{L_j(x)}{|L(x)|}, \qquad j = 1, 2, \ldots . \tag{4.15}$$

By using (2.5)iv, (4.1) and (4.3) we obtain from (4.14) and (4.15): for $\mathrm{Re}\, r = 0$, $\mathrm{Re}\, \sigma > 0$,

$$\begin{aligned} \lim_{a \uparrow 1} I(\Delta(a)) &= (\gamma r)^{\nu-1}[d_2 + (-1)^\nu d_1] \quad \text{for } \nu = \nu_1 = \nu_2 \text{ and } C_1 > 0,\ C_2 > 0, \\ &= (\gamma r)^{\nu_2-1} d_2 \quad \text{for } \nu_2 < \nu_1 \quad \text{or} \quad C_1 = 0, \\ &= (\gamma r)^{\nu_1-1}(-1)^{\nu_1} d_1 \quad \text{for } \nu_1 < \nu_2 \quad \text{or} \quad C_2 = 0. \end{aligned} \tag{4.16}$$

Next, we insert the expressions (4.9) and (4.10) into the righthand side of (4.7) and consider the resulting expression for $a \uparrow 1$, so $\Delta(a) \downarrow 0$. By using the relations (4.11), ..., (4.16) it is not difficult to show, although it is rather laborious, that the following limit exists and that: for $\mathrm{Re}\, r = 0$, $\mathrm{Re}\, \sigma > 0$, and $a \uparrow 1$,

$$\begin{aligned} \int_0^\infty \mathrm{e}^{-\sigma\tau} \mathrm{E}\{\mathrm{e}^{-r\mathbf{N}(\tau;a)}\} \mathrm{d}\tau &\to \left[\sigma - r[d_2(\gamma r)^{\nu-1} - d_1(\gamma \bar{r})^{\nu-1}]\right]^{-1} \quad \text{for } \nu = \nu_1 = \nu_2, \quad C_1 > 0,\ C_2 > 0, \\ &\to \left[\sigma - r d_2(\gamma r)^{\nu-1}\right]^{-1} \quad \text{for } \nu = \nu_2 < \nu_1 \quad \text{or} \quad C_1 = 0, \\ &\to \left[\sigma + r d_1(\gamma \bar{r})^{\nu-1}\right]^{-1} \quad \text{for } \nu = \nu_1 < \mathrm{v}_2 \quad \text{or} \quad C_2 = 0. \end{aligned} \tag{4.17}$$

The results obtained above lead to the following theorem.

Theorem 4.1.

i. The stochastic variables $\mathbf{N}(\tau; a)$ and $\mathbf{H}(\tau; a)$ converge in distribution for $a \uparrow 1$;
ii. Let $\mathbf{N}(\tau)$ and $\mathbf{H}(\tau)$ be stochastic variables with the limiting distribution of $\mathbf{N}(\tau; a)$ and $\mathbf{H}(\tau; a)$, respectively, then for $\tau \geq 0$, $\mathrm{Re}\, r = 0$,

$$\begin{aligned} \mathrm{E}\{\mathrm{e}^{-r\mathbf{N}(\tau)/\gamma}\} &= \mathrm{e}^{[d_2 r^\nu + d_1 \bar{r}^\nu]\tau/\gamma} \quad \text{for } \nu = \nu_1 = \nu_2, \quad C_1 > 0,\ C_2 > 0, \\ &= \mathrm{e}^{d_2 r^\nu \tau/\gamma} \quad \text{for } \nu = \nu_2 < \nu_1 \quad \text{or} \quad C_1 = 0, \\ &= \mathrm{e}^{d_1 \bar{r}^\nu \tau/\gamma} \quad \text{for } \nu = \nu_1 < \nu_2 \quad \text{or} \quad C_2 = 0, \\ \mathrm{E}\{\mathrm{e}^{-r\mathbf{H}(\tau)/\gamma}\} &= \mathrm{e}^{r\tau/\gamma} \mathrm{E}\{\mathrm{e}^{-r\mathbf{N}(\tau)/\gamma}\}. \end{aligned} \tag{4.18}$$

Proof. Denote by $J(\sigma, r; a)$ for $0 < 1 - a << 1$, $\mathrm{Re}\, r = 0$, $\mathrm{Re}\, \sigma > 0$, $\tau \geq 0$, the righthand side of the expression (4.7). We may then write by using the inversion integral for the Laplace transform: for $0 < 1 - a << 1$, $\mathrm{Re}\, r = 0$, $\tau \geq 0$,

$$\mathrm{E}\{\mathrm{e}^{-r\mathbf{N}(\tau;a)}\} = \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty+\varepsilon}^{\mathrm{i}\infty+\varepsilon} \mathrm{e}^{\sigma\tau} J(\sigma, r; a) \mathrm{d}\sigma, \quad \text{for} \quad \varepsilon > 0. \tag{4.19}$$

The expectation in the lefthand side of (4.7) is bounded in absolute value by one, so that $|J(\sigma, r; a) < \sigma^{-1}$. As above it follows that: for $\mathrm{Re}\, r = 0$, $\tau \geq 0$, $\sigma = \varepsilon^{-1}$, and $\nu = \nu_1 = \nu_2$, $C_1 > 0$, $C_2 > 0$,

$$\lim_{a \uparrow 1} J(\sigma, r; a) = \left[\sigma - r[d_2(\gamma r)^{\nu-1} - d_1(\gamma \bar{r}^{\nu-1}]]\right]^{-1}. \tag{4.20}$$

Because $e^{\sigma\tau}$, $\operatorname{Re}\sigma = \varepsilon > 0$ is bounded for every finite τ, dominated convergence shows that the integral in (4.19) converges for $a \uparrow 1$, $\operatorname{Re} r = 0$, $\tau \geq 0$, i.e.

$$
\begin{aligned}
\mathrm{E}\{\mathrm{e}^{-r\mathbf{N}(\tau;a)}\} &= \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty+\varepsilon}^{\mathrm{i}\infty+\varepsilon} \frac{e^{\sigma\tau}\mathrm{d}\tau}{\sigma - r[d_2(\gamma r)^{\nu-1} - d_1(\gamma\bar{r})^{\nu-1}]} \qquad (4.21)\\
&= \mathrm{e}^{\tau[d_2(\gamma r)^{\nu}+d_1(\gamma\bar{r})^{\nu}]}, \quad \nu = \nu_1 = \nu_2, \quad C_1 > 0,\ C_2 > 0.
\end{aligned}
$$

Obviously, the righthand side of (4.21) is continuous at $r = 0$, $\operatorname{Re} r = 0$, and by the continuity theorem for characteristic functions it follows that $\mathbf{N}(\tau;a)$, converges in distribution for $a \uparrow 1$. Hence the statements concerning $\mathbf{N}(\tau;a)$ and $\mathbf{N}(\tau)$ have been proved for the case $\nu = \nu_1 = \nu_2$, $C_1 > 0$, $C_2 > 0$. Those for the other cases are proved similarly. By using (4.6) the statements concerning $\mathbf{H}(\tau;a)$ and $\mathbf{H}(\tau)$ follow immediately from those for $\mathbf{N}(\tau;a)$. □

Remark 4.3. From (4.6) it is seen that we may take $\mathbf{H}(\tau)$ in such a way that

$$\mathbf{H}(\tau) - \tau = \mathbf{N}(\tau), \quad \tau \geq 0. \qquad (4.22)$$

Remark 4.4. From the definition of d_j, cf. (4.15), it is readily seen that, see also [2], formula (3.13),

$$d_1 = \frac{C_1}{D} \geq 0, \qquad d_2 = \frac{C_2 f}{D} \geq 0,$$

with

$$D := \left[C_1^2 + f^2 C_2^2 + 2(\cos\nu\pi) f C_1 C_2\right]^{1/2},$$

and f as defined in (2.6). In this definition f is nonzero and finite. However, the cases with $f = 0$ or ∞ also make sense. Whenever f is infinite then $d_1 = 0$, and $d_2 = 0$ for $f = 0$. Hence it is seen that the three cases in (4.18) of Theorem 4.1 may be also characterized by

$$
\begin{aligned}
&\nu = \nu_1 = \nu_2, \qquad d_1 > 0,\ d_2 > 0, \qquad (4.23)\\
&\nu = \nu_2 < \nu_1 \ \text{ or } \ d_1 = 0,\ d_2 > 0,\\
&\nu = \nu_1 < \nu_2 \ \text{ or } \ d_1 > 0,\ d_2 = 0,
\end{aligned}
$$

whenever $f \geq 0$, i.e. in the assumption (2.6) $0 < f < \infty$ is replaced by $f \geq 0$. □

5. The stable Lévy noise traffic

The distribution of the stochastic variable $\mathbf{N}(\tau)$, $\tau \geq 0$ is a ν-stable distribution. We characterise the distribution of $\mathbf{N}(\tau)$ in terms of the notation used in [4], see [4], page 5. From (4.18) we have: for $\operatorname{Re} r = 0$, $\tau > 0$, $1 < \nu \leq 2$,

$$
\begin{aligned}
\log \mathrm{E}\{\mathrm{e}^{r\mathbf{N}(\tau)/\gamma}\} &= (\tau/\gamma)[d_2 r^{\nu} + d_1 \bar{r}^{\nu}] \qquad (5.1)\\
&= |r|^{\nu}[(d_2 + d_1)\cos\frac{\nu}{2}\pi - (d_2 - d_1)\mathrm{i}\frac{r}{|r|}\sin\frac{\nu}{2}\,\pi](\tau/\gamma)\\
&= -|r|^{\nu}[-(d_2 + d_1)\cos\frac{\nu}{2}\,\pi]\frac{\tau}{\gamma}\Big[1 - \mathrm{i}\frac{d_2 - d_1}{d_2 + d_1}\,\frac{r}{|r|}\,\tan\frac{\nu}{2}\,\pi\Big].
\end{aligned}
$$

With the notation used in [4] the righthand side is the logarithm of the characteristic function of the ν-stable distribution $S_\nu(\sigma, d, \mu)$ with $\sigma \geq 0$, $-1 \leq d \leq 1$, $-\infty < \mu < \infty$, where (note $\cos \frac{\nu}{2}\pi \leq 0$ for $1 < \nu \leq 2$).

$$\sigma = \Big[-(d_2 + d_1)\cos\frac{\nu}{2}\pi\Big]^{1/\nu}\Big[\tau/\gamma\Big]^{1/\nu}, \tag{5.2}$$

$$d = \frac{d_2 - d_1}{d_2 + d_1} \quad \text{and} \quad \mu = 0.$$

Similarly, it is seen that $\mathbf{H}(\tau)$ has a stable distribution with the same σ and d but $\mu = -\tau/\gamma$.

The process $\{\mathbf{N}(\tau),\ \tau \geq 0\}$ is obviously a process with independent increments and it is self-similar with index $1/\nu$, i.e. $\mathbf{N}(b\tau)$ and $b^{1/\nu}\mathbf{N}(\tau)$, $b > 0$, have the same distribution for every $\tau > 0$. In [4] this process $\{\mathbf{N}(\tau),\ \tau \geq 0\}$ with $1 < \nu < 2$ is called a ν-stable Lévy motion with independent self-similar stationary increments, for $\nu = 2$ it is the Brownian motion.

In the context of Queueing Theory the $\{\mathbf{N}_t,\ \tau \geq 0\}$ process will be called the ν-stable Lévy noise traffic, the process $\{\mathbf{H}(\tau),\ \tau \geq 0\}$ will be referred to as the ν-stable virtual backlog process. Note that: for $\tau > 0$,

$$\mathrm{E}\{\mathbf{N}(\tau)\} = 0, \qquad \mathrm{E}\{\mathbf{H}(\tau)\} = -\tau, \tag{5.3}$$

since the properly scaled noise traffic $\mathbf{n}_t$ and the virtual backlog $\mathbf{h}_t$, when properly scaled, i.e. $\mathbf{N}(\tau; a)$ and $\mathbf{H}(\tau; a)$, converge in distribution to $\mathbf{N}(\tau)$ and $\mathbf{H}(\tau)$, respectively. We shall not investigate here whenever the finite dimensional distributions for the $\mathbf{N}(\tau; a)$ process converge for $a \uparrow 1$ to those of $\mathbf{N}(\tau)$. For the $M/G/1$ process this is readily verified, since for this case the $\{\mathbf{h}_t,\ t \geq 0\}$ process has stationary independent increments, see [5].

In [4], see Section 3.12, a sample function description of a ν-stable Lévy motion $1 < \nu < 2$ is given which is appropriate for the description of the traffic process of a queueing model. The sample functions of the ν-stable $\mathbf{N}(\tau)$ motion are here, see [4], p. 311, characterized as pure jump processes. The probability of an upward jump being $(1 + d)/2 = d_2/(d_1 + d_2)$, that for a downward jump is $(1 - d)/2 = d_1/(d_1 + d_2)$, see (5.2). The jump epochs of jump $\mathbf{y}$ with $\mathbf{y} \in (y, y + \mathrm{d}y)$ in an interval $(0, t)$ form a Poisson process with intensity depending on y. Actually the expectation of the number of jumps $\mathbf{y}$ with $\mathbf{y} \in (y, y + \mathrm{d}y)$ in $t \div \mathrm{d}t$ is given by:

$$\frac{d_2}{d_1 + d_2}\,|y|^{-(\nu+1)}\mathrm{d}y\mathrm{d}t \quad \text{for} \quad y > 0,$$

$$\frac{d_1}{\mathrm{d}_1 + d_2}\,|y|^{-(\nu+1)}\mathrm{d}y\mathrm{d}t \quad \text{for} \quad y < 0.$$

Note that the expectations are unbounded.

Remark 5.1. From the definition of d_1 and d_2, cf. (4.15) and Remark 4.4, it is seen that d_2/d_1 is independent of ν and so the probabilities of an upward and of a downward jump are independent of ν. □

6. The contracted workload process of the $GI/G/1$ model

Denote for the $GI/G/1$ queue by $\mathbf{v}_t$, $t \geq 0$, the workload at time t. Then $\mathbf{v}_t$, is given by Reich's formula; see [6], p. 170. With $\mathbf{v}_0 = \mathrm{v}_0 \geq 0$, and $t \geq 0$,

$$\mathbf{v}_t := \max[\mathrm{v}_0 + \mathbf{h}_t, \sup_{0<u<t}(\mathbf{h}_t - \mathbf{h}_u)]. \tag{6.1}$$

In this section we consider the contracted workload $\Delta(a)\mathbf{v}_{\tau/\Delta_1(a)}$ for $a \uparrow 1$, cf. (4.4).

It is well-known that the actual waiting time process $\{\mathbf{w}_n,\ n=1,2,\dots\}$ and the workload process $\{\mathbf{v}_t,\ t\geq 0\}$ both possess a stationary distribution if $a<1$. Denote by $\mathbf{w}$ and $\mathbf{v}$ stochastic variables with distribution the stationary distribution of the $\mathbf{w}_n$-process and the $\mathbf{v}_t$-process, respectively. Further, it is known, cf. [6], p. 296, formula (5.109) or p. 373, formula (6.110), that for $\mathrm{Re}\,\rho\geq 0$, $a<1$,

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{v}}\}=1-a+a\frac{1-\beta(\rho)}{\rho\beta}\,\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}. \tag{6.2}$$

In [2] it has been shown that the contracted waiting $\Delta(a)\mathbf{w}$ converges in distribution for $a\uparrow 1$. Replace in (6.2) ρ by $r\Delta(a)$, $\mathrm{Re}\,r\geq 0$, so that for $\mathrm{Re}\,r\geq 0$,

$$\mathrm{E}\{\mathrm{e}^{-r\Delta(a)\mathbf{v}}\}=1-a+a\frac{1-\beta(r\Delta(a))}{\beta r\Delta(a)}\,\mathrm{E}\{\mathrm{e}^{-r\Delta(a)\mathbf{w}}\}. \tag{6.3}$$

From (2.4), (2.5) and (4.1) it is readily seen that

$$\lim_{a\uparrow 1}\frac{1-\beta(r\Delta(a))}{\beta r\Delta(a)}=1\quad\text{for}\quad\mathrm{Re}\,r\geq 0. \tag{6.4}$$

Because $\Delta(a)\mathbf{w}$ converges in distribution for $a\uparrow 1$ it is seen from (6.3) and (6.4) that $\Delta(a)\mathbf{v}$ converges in distribution and that it has the same limiting distribution as $\Delta(a)\mathbf{w}$ for $a\uparrow 1$. Let $\mathbf{V}$ be a stochastic variable with distribution the limiting distribution of $\Delta(a)\mathbf{v}$.

In [2] the LST of the distribution of $\Delta(a)\mathbf{w}$ has been derived, and so this leads immediately to the LST of the distribution of $\mathbf{V}$. We quote here some results from [2].

For the case with $\nu=\nu_1=\nu_2$, $1<\nu<2$ and $C_1>0$, $C_2>0$, see Theorem 4.1 of [2],

$$\mathrm{E}\{\mathrm{e}^{-r\mathbf{V}/\gamma}\}=\lim_{a\uparrow 1}\mathrm{E}\{\mathrm{e}^{-r\Delta(a)\mathbf{v}/\gamma}\}=\mathrm{e}^{\Phi(r)}, \tag{6.5}$$

with

$$\begin{aligned}
\Phi(r)&=\frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}\infty}^{-\mathrm{i}\infty}[\log[1+d_2\xi^{\nu-1}-d_1\bar{\xi}^{\nu-1}]]\frac{r}{\xi-r}\,\frac{\mathrm{d}\xi}{\xi}, &&\mathrm{Re}\,r>0,\\
&=-\frac{1}{\pi}\int_0^\infty\{\arctan\frac{A(rs)^{\nu-1}}{1+B(rs)^{\nu-1}}\,\}\frac{1}{1+s^2}\,\frac{\mathrm{d}s}{s}\\
&-\ \frac{1}{2\pi}\int_0^\infty\log\{1+2B(rs)^{\nu-1}+C(rs)^{2(\nu-1)}\}\frac{\mathrm{d}s}{1+s^2}\ , &&r\geq 0;
\end{aligned}$$

here

$$\begin{aligned}
A&:=(d_2+d_1)\sin\frac{\nu-1}{2}\pi,\quad B=(d_2-d_1)\cos\frac{\nu-1}{2}\,\pi,\\
C&:=(d_1-d_2)^2+4d_1d_2\sin^2\frac{\nu-1}{2}\,\pi;
\end{aligned} \tag{6.6}$$

and for $\nu=\nu_1=\nu_2=2$, $C_1>0,\ C_2>0$, see [2], Theorem 5.1.,

$$\mathrm{E}\{\mathrm{e}^{-r\mathbf{V}/\gamma}\}=\frac{1}{1+r},\quad\mathrm{Re}\,r\geq 0. \tag{6.7}$$

For the case with $\nu=\nu_2<\nu_1$ or $C_1=0$, see [2], Theorem 6.1: for $\mathrm{Re}\,r\geq 0$, $\nu_2\leq 2$,

$$\mathrm{E}\{\mathrm{e}^{-r\mathbf{V}/\gamma}\}=\frac{1}{1+r^{\nu_2-1}}\ , \tag{6.8}$$

and for the case with $\nu = \nu_1 < \nu_2$ or $C_2 = 0$, see [2], Theorem 7.1: for Re $r \geq 0$,

$$\mathrm{E}\{\mathrm{e}^{-r\mathbf{V}/\gamma}\} = \frac{1}{1+r}\ . \tag{6.9}$$

A further result proved in [2] concerns the idle time $\boldsymbol{i}$ of the $GI/G/1$ model. For $\hat{\boldsymbol{i}}$ a stochastic variable with

$$\mathrm{E}\{\mathrm{e}^{-\rho\hat{\boldsymbol{i}}}\} = \frac{1 - \mathrm{E}\{\mathrm{e}^{-\rho\boldsymbol{i}}\}}{\rho\mathrm{E}\{\boldsymbol{i}\}}\ , \quad \mathrm{Re}\ \rho = 0,$$

it has been shown in [2] that: for Re $r < 0$,

$$\lim_{\Delta(a)\downarrow 0} \mathrm{E}\{\mathrm{e}^{-r\Delta(a)\hat{\boldsymbol{i}}}\} = \exp\Big[\frac{1}{2\pi\mathrm{i}} \int_{\xi=-\mathrm{i}\infty}^{\mathrm{i}\infty} [\log\{1 + d_2\xi^{\nu-1} - d_1\bar{\xi}^{\nu-1}\}]\frac{r}{\xi - r}\,\frac{\mathrm{d}\xi}{\xi}\Big]\ . \tag{6.10}$$

From the description of the limiting noise process $\{\mathbf{N}(\tau),\ \tau \geq 0\}$ it is seen that the characteristics ν, d_1 and d_2 of this Lévy motion, cf. also Remark 4.1, determine completely the limiting distribution of $\Delta(a)\mathbf{v}$ for $a \uparrow 1$. It is further noted that these characteristics depends only on the tails of $A(t)$ and $B(t)$. For the asymptotic behaviour of the tail of the distribution of $\mathbf{V}$ the reader is referred to [2].

7. The $M/G_R/1$ model, workload reduction

In order to obtain a better insight in the limiting results obtained in the preceding sections we consider in this and the next sections a variant of the $M/G_R/1$ model. This model will be characterized by the symbol $M/G_R/1$ and it is described as follows.

Let $\mathbf{t}_n,\ n = 0, 1, 2, \ldots$ with $\mathbf{t}_0 = t_0 > 0$, be a Poisson point process with arrival rate λ. Put

$$\boldsymbol{\sigma}_{n+1} := \mathbf{t}_{n+1} - \mathbf{t}_n, \quad n = 0, 1, \ldots\ , \tag{7.1}$$

so that $\boldsymbol{\sigma}_n$, $n = 1, 2, \ldots$, is a sequence of i.i.d. stochastic variables with

$$\Pr\{\boldsymbol{\sigma}_n < \sigma\} = 1 - \mathrm{e}^{-\lambda\sigma}, \quad \sigma \geq 0. \tag{7.2}$$

We further introduce two sequences of i.i.d. nonnegative stochastic variables, viz. $\boldsymbol{\tau}_n^+$, $n = 0, 1, 2, \ldots$, and $\boldsymbol{\tau}_n^-$, $n = 0, 1, 2, \ldots$, with $B^+(\tau)$ and $B^-(\tau)$ the distribution of $\boldsymbol{\tau}_n^+$ and $\boldsymbol{\tau}_n^-$, respectively. It is assumed that the three families $\{\boldsymbol{\sigma}_n,\ n = 1, 2, 3, \ldots\}$, $\{\boldsymbol{\tau}_n^+,\ n = 0, 1, 2, \ldots\}$ and $\{\boldsymbol{\tau}_n^-,\ n = 0, 1, 2, \ldots\}$ are independent families.

With

$$0 < p^{\pm} < 1 \quad \text{and} \quad p^+ + p^- = 1, \tag{7.3}$$

the sequence $\mathbf{w}_n$, $n = 0, 1, 2, \ldots$, is recursively defined by: for $n = 1, 2, \ldots$,

$$\begin{aligned} \mathbf{w}_{n+1} &:= \Big[\mathbf{w}_n + \boldsymbol{\tau}_n^+ - \boldsymbol{\sigma}_{n+1}\Big]^+ \quad \text{with prob. } p^+, \\ &:= \Big[\mathbf{w}_n - \boldsymbol{\tau}_n^- - \boldsymbol{\sigma}_{n+1}\Big]^+ \quad \text{with prob. } p^-, \\ \mathbf{w}_0 &= w_0 \geq 0. \end{aligned} \tag{7.4}$$

The stochastic variable $\mathbf{v}_t, t > 0$, is defined as follows. For $\mathbf{t}_n < t < \mathbf{t}_{n+1}$, $n = 0, 1, 2, \ldots$, and

$$\begin{aligned} \boldsymbol{\tau}_n &:= \quad \boldsymbol{\tau}_n^+ \quad \text{with prob. } p^+, \\ &:= -\boldsymbol{\tau}_n^- \quad \text{with prob. } p^-, \end{aligned} \tag{7.5}$$

$\mathbf{v}_t$, $t > 0$, is defined by

$$\mathbf{v}_t := \Big[\big[\mathbf{w}_n + \boldsymbol{\tau}_n\big]^+ - (t - \mathbf{t}_n)\Big]^+, \tag{7.6}$$

$$\mathbf{v}_{t_n+} = \Big[\mathbf{w}_n + \boldsymbol{\tau}_n\Big]^+, \qquad \mathbf{v}_{t_{n+1}-} = \mathbf{w}_{n+1}.$$

Obvious the $M/G_R/1$ model may be considered as an $M/G/1$ model where at an arrival epoch $\mathbf{t}_n$ the workload is increased with probability p^+ by $\boldsymbol{\tau}_n^+$ or is reduced with probability p^- by $\boldsymbol{\tau}_n^-$. This model has been considered in [7] as an $M/G/1$ model with *negative* customers or workload removal.

We shall investigate this $M/G_R/1$ model for the case that $B^+(t)$ and $B^-(t)$ have heavy tails.

Set

$$\begin{aligned}
\beta^{\pm}(\rho) &:= \int_{0-}^{\infty} \mathrm{e}^{-\rho t}\mathrm{d}B^{\pm}(t), \qquad \mathrm{Re}\ \rho \geq 0, \qquad (7.7)\\
\beta(\rho) &:= p^+\beta^+(\rho) + p^-\beta^-(\bar{\rho}), \qquad \mathrm{Re}\ \rho = 0,\\
\beta^{\pm} &:= \int_{0}^{\infty} t\mathrm{d}B^{\pm}(t),\\
\beta &:= p^+\beta^+ - p^-\beta^-,\\
b^{\pm} &:= \lambda p^{\pm}\beta^{\pm}, \qquad b = b^+ - b^- = \lambda\beta.
\end{aligned}$$

It is assumed that: for $\mathrm{Re}\ \rho \geq 0$,

$$1 - \frac{1-\beta^{\pm}(\rho)}{\rho\beta^{\pm}} = g^{\pm}(\gamma\rho) + C^{\pm}(\gamma\rho)^{\nu^{\pm}-1}L^{\pm}(\rho), \tag{7.8}$$

$$f^{(0)} := \lim_{x\downarrow 0}\frac{L^+(x)}{L^-(x)} \quad \text{with} \quad f^{(0)} \geq 0 \ \text{ for } \ C^{\pm} > 0.$$

Here $g^{\pm}(\,\cdot\,)$, $C^{\pm}$, $\nu^{\pm}$ and $L^{\pm}(\,\cdot\,)$ have the same properties as the analogous functions and constants in (2.5).

As in Remark (3.5) we introduce here the following convention:

$$\begin{aligned}
\nu := \nu^+ &\quad \text{whenever} \quad \nu^+ = \nu^- \quad \text{and} \quad C^{\pm} > 0, \qquad (7.9)\\
:= \nu^+ &\quad \text{whenever} \quad \nu^+ < \nu^- \quad \text{or} \quad C^- = 0,\\
:= \nu^- &\quad \text{whenever} \quad \nu^- = \nu^+ \quad \text{or} \quad C^+ = 0.
\end{aligned}$$

8. On the noise traffic for the $M/G_R/1$ model

In the description of the traffic of the $M/G_R/1$ model we shall use in this and the following sections the same symbols as in the preceding sections for the $GI/G/1$ model; so here the symbols are defined for the $M/G_R/1$ model.

Denote by $\mathbf{k}_t$ the traffic generated in $[0, t)$. Since the arrival process $\mathbf{t}_n$, $n = 0, 1, \ldots,$ is a Poisson point process with rate λ, we have: for Re $\rho = 0$, $t \geq 0$,

$$\begin{aligned}\mathrm{E}\{\mathrm{e}^{-\rho \mathbf{k}_t}\} &= \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \mathrm{e}^{-\lambda t}\big[p^+\beta^+(\rho) + p^-\beta^-(\bar{\rho})\big]^n \\ &= \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \mathrm{e}^{-\lambda t}\beta^n(\rho) = \mathrm{e}^{-\lambda t\{1-\beta(\rho)\}}.\end{aligned} \tag{8.1}$$

It follows that

$$\mathrm{E}\{\mathbf{k}_t\} = bt. \tag{8.2}$$

Set, for $t \geq 0$,

$$\begin{aligned}\mathbf{n}_t &:= \mathbf{k}_t - bt, \\ \mathbf{h}_t &:= \mathbf{k}_t - t = \mathbf{n}_t - (1-b)t.\end{aligned} \tag{8.3}$$

A simple calculation shows that: for Re $\rho = 0$,

$$\mathrm{E}\{\mathrm{e}^{-\rho \mathbf{n}_t}\} = \mathrm{e}^{\rho t[b^+\{1-\frac{1-\beta^+(\rho)}{\rho\beta^+}\} - b^-\{1-\frac{1-\beta^-(\bar{\rho})}{\bar{\rho}\beta^-}\}]}, \qquad t \geq 0. \tag{8.4}$$

With (7.8) we obtain from (8.4): for Re $\rho = 0$,

$$\begin{aligned}&\log \mathrm{E}\{\mathrm{e}^{-\rho \mathbf{n}_t}\} = \rho t[b^+g^+(\gamma\rho) - b^-g^-(\gamma\bar{\rho}) + b^+C^+(\gamma\rho)^{\nu-1}L^+(\gamma\rho) - b^-C^-(\gamma\bar{\rho})^{\nu-1}L^-(\gamma\bar{\rho})] = \\ &\quad \lambda\rho t[p^+\beta^+g^+(\gamma\rho) - p^-\beta^-g^-(\gamma\bar{\rho}) + p^+\beta^+C^+(\gamma\rho)^{\nu-1}L^+(\gamma\rho) - p^-\beta^-C^-(\gamma\bar{\rho})^{\nu-1}L^-(\gamma\bar{\rho})].\end{aligned} \tag{8.5}$$

We introduce again a scaling of ρ and t, viz. for Re $r \geq 0$, $\tau \geq 0$,

$$\rho = r\nabla \quad \text{and} \quad t = \tau/\nabla_1, \quad \nabla_1 = \nabla(1-\varepsilon), \quad \nabla > 0, \quad 1-\varepsilon > 0, \tag{8.6}$$

where ∇ and ε will be specified below. Put

$$\mathbf{N}(\tau;\varepsilon) := \nabla \mathbf{n}_{\tau/\nabla_1}, \tag{8.7}$$

then from (8.5), (8.6) and (8.7): for Re $r = 0$,

$$\begin{aligned}\log \mathrm{E}\{\mathrm{e}^{-r\mathbf{N}(\tau;\varepsilon)}\} &= \frac{\lambda r\tau}{1-\varepsilon}\Big[p^+\beta^+g^+(\gamma r\nabla) - p^-\beta^-g^-(\gamma\bar{r}\nabla) \\ &+ \{p^+\beta^+C^+(\gamma r)^{\nu-1}L^+(\gamma r\nabla) - p^-\beta^-C^-(\gamma\bar{r})^{\nu-1}L^-(\gamma\bar{r}\nabla)\}\nabla^{\nu-1}\}\Big].\end{aligned} \tag{8.8}$$

For the specification of ∇ and ε we distinguish two cases, viz.

$$\begin{aligned}&\text{i.} \qquad \beta = \; p^+\beta^+ - p^-\beta^- > 0 \quad \text{and} \quad b = \lambda\beta < 1, \\ &\text{ii.} \qquad \beta = \; p^+\beta^+ - p^-\beta^- < 0.\end{aligned} \tag{8.9}$$

In the next section it will be seen that $\mathbf{w}_n$ as defined in (7.4) for the $M/G_R/1$ model converges in distribution for $n \to \infty$ for each of the two cases distinguished in (8.9), however the limiting distributions are different for the non-heavy traffic case.

We consider first the case

$$\beta > 0, \qquad b = \lambda\beta < 1. \tag{8.10}$$

In this case we take

$$\varepsilon := b = \lambda\beta < 1, \tag{8.11}$$

so that (8.8) may be written as: for Re $r = 0$,

$$\log \mathrm{E}\{\mathrm{e}^{-r\mathbf{N}(\tau;b)}\} = \frac{r\tau}{1-\lambda\beta}[b^+ g^+(\gamma r\nabla) - b^- g^-(\gamma\bar{r}\nabla)] + \tag{8.12}$$

$$\frac{r\tau\nabla^{\nu-1}}{1-b}\{b^+C^+(\gamma r)^{\nu-1}L^+(\gamma r\nabla) - b^-C^-(\gamma\bar{r})^{\nu-1}L^-(\gamma\bar{r}\nabla)\}.$$

For the present case we define ∇ as that root $\nabla(b)$ of the contraction equation

$$\frac{x^{\nu-1}}{1-b}\left|K_1(x)\right| = 1, \qquad x > 0, \tag{8.13}$$

with

$$K_1(x) = b^+C^+L^+(x) + (-1)^\nu b^-C^-L^-(x), \qquad x > 0, \qquad b = b^+ - b^-, \tag{8.14}$$

which tends to zero for $b \uparrow 1$. As in Appendix A it is shown that $\nabla(b)$ is uniquely defined and is a simple root for $0 < 1 - b << 1$.

As in Section 4 it is shown that the following theorem holds.

Theorem 8.1. For $\beta > 0$ and $\varepsilon = b = \lambda\beta < 1$,

i. $\mathbf{N}(\tau, b)$ converges in distribution for $\varepsilon \uparrow 1$,

ii. for $\mathbf{N}_1(\tau)$, $\tau \geq 0$, a stochastic variable with distribution the limiting distribution of $\mathbf{N}(\tau, b)$ holds: for Re $r = 0$,

$$\mathrm{E}\{\mathrm{e}^{-r\mathbf{N}_1(\tau)/\gamma}\} = \mathrm{e}^{[d_1^+ r^\nu + d_1^- \bar{r}^\nu]\tau/\gamma}, \tag{8.15}$$

with

$$d_1^+ := \frac{b^+C^+f^{(0)}}{F_1} \geq 0, \qquad d_1^- := \frac{b^-C^-}{F_1} \geq 0, \qquad b^+ - b^- = 1, \tag{8.16}$$

$$F_1 := \Big[(b^-C^-)^2 + (b^+f^{(0)}C^+)^2 + 2(\cos\nu\pi)b^-b^+f^{(0)}C^+C^-\Big]^{1/2}.$$

Remark 8.1. The limit $\varepsilon \uparrow 1$ in Theorem 8.1 should be interpreted as $\lambda \to \beta^{-1}$ with β fixed. □

Remark 8.2. From (8.16) it follows that

$$1 = \Big[(d_1^-)^2 + (d_2^+)^2 + 2(\cos\nu\pi)d_1^- d_2^+\Big]^{1/2}. \qquad \square$$

Next we consider the case, cf. (8.9),

$$\beta < 0. \tag{8.17}$$

In (8.8) we take again $\varepsilon = \lambda\beta$ so that $\varepsilon < 0$ and the first factor in the righthand side of (8.8) becomes

$$\frac{\lambda}{1-\lambda\beta}\ .$$

Because $\beta < 0$ this factor cannot tend to infinity for finite $\lambda > 0$. We, therefore, consider the case that $\lambda \to \infty$. To do this rewrite (8.4) by using (8.6) as: for $\mathrm{Re}\, r = 0$,

$$\mathrm{E}\{\mathrm{e}^{-r\nabla \mathbf{n}_{\tau/\nabla_1}}\} = \exp\Big[\frac{r\lambda\tau}{1-\lambda\beta}[p^+\beta^+\{1-\frac{1-\beta^+(r\nabla)}{r\nabla\beta^+}\ \} - p^-\beta^-\{1-\frac{1-\beta^-(\bar{r}\nabla)}{\bar{r}\nabla\beta^-}\ \}]\Big].$$

Obviously the righthand has a limit for $\lambda \to \infty$ and it is readily seen that this limit is a characteristic function of a proper probability distribution, since this characteristic function is continuous in $r = 0$, $\mathrm{Re}\, r = 0$. Hence $\nabla \mathbf{n}_{\tau/\nabla_1}$, $\tau > 0$ converges in distribution for $\lambda \to \infty$. By $\mathbf{N}_2(\tau;\varepsilon)$, $\tau \geq 0$, we shall denote a stochastic variable with distribution the limiting distribution of $\nabla \mathbf{n}_{\tau/\nabla_1}$ with $\nabla > 0$. Therefore, we obtain from (8.17): for $\mathrm{Re}\, r = 0$, $\nabla > 0$ and every $\tau \geq 0$,

$$\mathrm{E}\{\mathrm{e}^{-r\mathbf{N}(\tau;\varepsilon)}\} = \exp\Big[\frac{r\tau}{|\beta|}\{p^+\beta^+g^+(\gamma r\nabla) - p^-\beta^-g^-(\gamma\bar{r}\nabla)\} + \tag{8.18}$$

$$\frac{r\tau}{|\beta|}\{p^+\beta^+C^+(\gamma r)^{\nu-1}L^+(\gamma r\nabla) - p^-\beta^-C^-(\gamma\bar{r})^{\nu-1}L^-(\gamma\bar{r}\nabla)\}\nabla^{\nu-1}\Big].$$

For the present case, cf. (8.17), we define ∇ as that root $\tilde{\nabla}(\beta)$ of the contraction equation

$$\frac{p^-\beta^-}{|\beta|}x^{\nu-1}|K_2(x)| = 1, \qquad x > 0, \tag{8.19}$$

where

$$K_2(x) := \frac{p^+\beta^+}{p^-\beta^-}C^+L^+(x) + (-1)^\nu C^-L^-(x), \qquad x > 0, \tag{8.20}$$

which tends to zero for $|\beta| \downarrow 0$. As in Appendix A it is shown $\tilde{\nabla}(\beta)$ is uniquely defined and is a simple root for $0 < |\beta| << 1$.

As in Section 4 it is shown that the following theorem holds.

Theorem 8.2. For $\beta < 0$, and $1 - \varepsilon = |\beta|$,

i. $\mathbf{N}(\tau;\varepsilon)$ converges in distribution for $\varepsilon \to 1$, (8.21)

ii. with $\mathbf{N}_2(\tau)$, $\tau \geq 0$, a stochastic variable with distribution the limiting distribution of $\mathbf{N}(\tau;\varepsilon)$ holds: for $\mathrm{Re}\, r = 0$,

$$\mathrm{E}\{\mathrm{e}^{-r\mathbf{N}_2(\tau)/\gamma}\} = \mathrm{e}^{[d_2^+r^\nu + d_2^-\bar{r}^\nu]\tau/\gamma}, \tag{8.22}$$

with

$$d_2^+ \ := \ \frac{C^+f^{(0)}}{F_2} \geq 0, \qquad d_2^- := \frac{C^-}{F_2} \geq 0, \tag{8.23}$$

$$F_2 \ := \ \Big[(C^-)^2 + (f^{(0)}C^+)^2 + 2(\cos\nu\pi)f^{(0)}C^+C^-\Big]^{1/2}.$$

Remark 8.3. The limit $\varepsilon \to 0$, i.e. $\beta \uparrow 0$, should be understood as $p^+/p^- \to \beta^-/\beta^+$ with $p^+ + p^- = 1$. □

9. THE $\mathbf{w}_n$-PROCESS FOR $M/G_R/1$

In this section we consider the $\{\mathbf{w}_n,\ n = 0,1,2,\dots\}$ process for the $M/G_R/1$ model; it is recursively defined in (7.4). We shall further consider the sequence $\mathbf{i}_n$, $n = 0,1,2,\dots$, defined by: for $n = 0,1,2,\dots$,

$$\begin{aligned} \mathbf{i}_n \quad &:= \quad -\Big[\mathbf{w}_n + \boldsymbol{\tau}_n^+ - \boldsymbol{\sigma}_{n+1}\Big]^- \qquad \text{with prob. } p^+, \\ &:= \quad -\Big[\mathbf{w}_n - \boldsymbol{\tau}_n^- - \boldsymbol{\sigma}_{n+1}\Big]^- \qquad \text{with prob. } p^+; \end{aligned} \tag{9.1}$$

obviously, we have

$$\mathbf{i}_n \geq 0, \qquad n = 0,1,2,\dots . \tag{9.2}$$

Write

$$\tilde{\alpha}(\rho) := \mathrm{E}\{\mathrm{e}^{-\rho\boldsymbol{\sigma}_n}\}, \qquad \mathrm{Re}\,\rho > -\lambda.$$

It follows from (7.1) that

$$\tilde{\alpha}(\rho) = \frac{\lambda}{\lambda+\rho}\,, \qquad \mathrm{Re}\,\rho > -\lambda. \tag{9.3}$$

To derive a relation for $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}_n}\}$ we use the following identity: for every complex ρ and real x,

$$\mathrm{e}^{-\rho x} = \mathrm{e}^{-\rho[x]^+} + \mathrm{e}^{-\rho[x]^-} - 1. \tag{9.4}$$

Again we take $w_0 = 0$, cf. (7.8), and we shall surpress the condition $\mathbf{w}_0 = w_0 = 0$, in expressions containing a mathematical expectation, i.e. we shall write

$$\mathrm{E}\{\dots\} \quad \text{for} \quad \mathrm{E}\{\dots|\mathbf{w}_0 = 0\},$$

since we are mainly interested in results for $n \to \infty$, which are in general independent of $\mathbf{w}_0$.

Starting from (9.4) we obtain from (7.3) and (9.1): for $\mathrm{Re}\,\rho = 0$, $n = 0,1,2,\dots$,

$$\begin{aligned} \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}_{n+1}}\} = p^+\mathrm{E}\{\mathrm{e}^{-\rho[\mathbf{w}_n+\boldsymbol{\tau}_n^+-\boldsymbol{\sigma}_{n+1}]^+}\} + p^-\mathrm{E}\{\mathrm{e}^{-\rho[\mathbf{w}_n-\boldsymbol{\tau}_n^--\boldsymbol{\sigma}_{n+1}]^-}\} = \\ p^+\mathrm{E}\{e^{-\rho(\mathbf{w}_n+\boldsymbol{\tau}_n^+-\boldsymbol{\sigma}_{n+1})}\} + p^-\mathrm{E}\{\mathrm{e}^{-\rho(\mathbf{w}_n-\boldsymbol{\tau}_n^--\boldsymbol{\sigma}_{n+1})}\} \\ +1 - p^+\mathrm{E}\{\mathrm{e}^{-\rho[\mathbf{w}_n+\boldsymbol{\tau}_n^+-\boldsymbol{\sigma}_{n+1}]^-} - p^-\mathrm{E}\{\mathrm{e}^{-\rho[\mathbf{w}_n-\boldsymbol{\tau}_n^--\boldsymbol{\sigma}_{n+1}]^-}\} = \\ \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}_n}\}[p^+\beta^+(\rho) + p^-\beta^-(\bar{\rho})]\tilde{\alpha}(\bar{\rho}) + 1 - \mathrm{E}\{\mathrm{e}^{\rho\mathbf{i}_n}\}. \end{aligned} \tag{9.5}$$

Hence with $\beta(\rho)$ as defined in (7.7) we have: for $\mathbf{w}_0 = 0$, $\mathrm{Re}\,\rho = 0$, $n = 0,1,\dots$,

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}_{n+1}}\} = \beta(\rho)\tilde{\alpha}(\bar{\rho})\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}_n}\} + 1 - \mathrm{E}\{\mathrm{e}^{\rho\mathbf{i}_n}\}. \tag{9.6}$$

Introduction of the generating functions

$$\begin{aligned} &\sum_{n=0}^{\infty} r^n\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}_n}\}, \quad \mathrm{Re}\,\rho \geq 0, \\ &\sum_{n=0}^{\infty} r^n\mathrm{E}\{\mathrm{e}^{\rho\mathbf{i}_n}\}, \quad \mathrm{Re}\,\rho \leq 0, \end{aligned} \tag{9.7}$$

with $|r|<1$ leads to: for Re $\rho=0$, $|r|<1$,

$$[1-r\beta(\rho)\tilde{\alpha}(-\rho)]\sum_{n=0}^{\infty}r^n\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}_n}\}=1+r\sum_{n=0}^{\infty}r^n\{1-\mathrm{E}\{\mathrm{e}^{\rho\mathbf{i}_n}\}\}. \tag{9.8}$$

Relations of this type have been frequently studied in Queueing Theory, see e.g. [6], see Sections II.5.2, II.6.7, or [8]. The relations (9.8) formulates a Wiener-Hopf problem for the two generating functions in (9.7), and as such its solution can be constructed. From the solution so obtained it may be shown that $\mathbf{w}_n$ and $\mathbf{i}_n$ both converge in distribution for $n\to\infty$, and their limiting distributions are proper probability distributions if $b<1$, cf. (7.7). We omit the proof, see also (9.1) below.

With $\mathbf{w}$ and $\mathbf{i}$ stochastic variables with distributions these limiting distributions, respectively, it may be shown that for $0<b<1$, and Re $\rho\geq 0$,

$$\begin{aligned}\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\} &= \lim_{r\uparrow 1}(1-r)\sum_{n=0}^{\infty}r^n\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}_n}\},\\ \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{i}}\} &= \lim_{r\uparrow 1}(1-r)\sum_{n=0}^{\infty}r^n\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{i}_n}\}.\end{aligned} \tag{9.9}$$

We omit also the proofs of these statements, and refer herefor to [6], [8] and [9].

Multiplying (9.8) by $1-r$ and letting $r\uparrow 1$ yields, by using (9.9), that: for Re $\rho=0$,

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}=\frac{1-\mathrm{E}\}\mathrm{e}^{\rho\mathbf{i}}\}}{1-\beta(\rho)\tilde{\alpha}(\bar{\rho})}=\frac{(1-\mathrm{E}\{\mathrm{e}^{\rho\mathbf{i}}\}(\lambda-\rho)}{\lambda-\rho-\lambda\beta(\rho)}\ . \tag{9.10}$$

Because

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}\Big|_{\rho=0}=\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{i}}\}\Big|_{\rho=0}=1,$$

it follows from (7.11) and (9.3) that

$$\mathrm{E}\{\mathbf{i}\}=(1-b)/\lambda>0 \text{ for } b<1. \tag{9.11}$$

Let $\hat{\mathbf{i}}$ be a stochastic variable with distribution defined by: for Re $\rho\geq 0$,

$$\mathrm{E}\{\mathrm{e}^{-\rho\hat{\mathbf{i}}}\}=\frac{1-\mathrm{E}\}\mathrm{e}^{-\rho\mathbf{i}}\}}{\rho\mathrm{E}\{\mathbf{i}\}}\ , \tag{9.12}$$

and put: for Re $\rho=0$,

$$\psi(\rho):=\frac{-\rho\mathrm{E}\{\mathbf{i}\}}{1-\beta(\rho)\alpha(\bar{\rho})}\ . \tag{9.13}$$

Further, from (9.10) and (9.13): for Re $\rho=0$,

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}=\mathrm{e}^{-\log\psi(\rho)}\mathrm{E}\{\mathrm{e}^{\rho\hat{\mathbf{i}}}\}, \tag{9.14}$$

and

i. $\quad \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}$ and $\mathrm{E}\{\mathrm{e}^{-\rho\hat{\mathbf{i}}}\}$ are regular for Re $\rho>0$, continuous for Re $\rho\geq 0$, (9.15)

ii. $\quad \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}=\mathrm{E}\{\mathrm{e}^{-\rho\hat{\mathbf{i}}}\}=1$ for $\rho=0$.

It is seen that (9.14) and (9.15) formulate a Riemann Boundary value problem with the line Re $\rho = 0$ as the line of discontinuity. A boundary value problem with $\beta(\rho)$ and $\alpha(\rho)$, described by (7.7) and (9.3) has been discussed in [9], see also [2]. We may use the solution derived in [9], since it can be shown that (7.8) and (9.3) imply that the conditions for the validity of the solution in [9] are fulfilled. Using the solution derived in [9] it is seen that

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\} = \exp\Big[\frac{1}{2\pi\mathrm{i}}\int\limits_{\xi=-\mathrm{i}\infty}^{\mathrm{i}\infty}[-\log\psi(\xi)]\frac{\rho}{\xi-\rho}\frac{\mathrm{d}\xi}{\xi}\Big], \quad \mathrm{Re}\,\rho > 0, \tag{9.16}$$

$$\mathrm{E}\{\mathrm{e}^{\rho\hat{\mathbf{i}}}\} = \exp\Big[\frac{1}{2\pi\mathrm{i}}\int\limits_{\xi=-\mathrm{i}\infty}^{\mathrm{i}\infty}[-\log\psi(\xi)]\frac{\rho}{\xi-\rho}\frac{\mathrm{d}\xi}{\xi}\Big], \quad \mathrm{Re}\,\rho < 0,$$

and by using the Plemelj-Sokhotski formulas it is seen that: for Re $\rho = 0$,

$$\begin{aligned}\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\} &= \mathrm{e}^{\frac{1}{2}\log\psi(\rho)+\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}[-\log\psi(\xi)]\frac{\rho}{\xi-\rho}\frac{\mathrm{d}\xi}{\xi}}, \\ \mathrm{E}\{\mathrm{e}^{\rho\hat{\mathbf{i}}}\} &= \mathrm{e}^{-\frac{1}{2}\log\psi(\rho)+\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}[-\log\psi(\xi)]\frac{\rho}{\xi-\rho}\frac{\mathrm{d}\xi}{\xi}}.\end{aligned} \tag{9.17}$$

Next we consider the expression $\psi(\rho)$ in some detail. From (7.2), (7.7), (9.12) and (9.13) we have: for Re $\rho = 0$,

$$\begin{aligned}\psi^{-1}(\rho) &= \frac{1-\lambda\{1-\beta(\rho)\}/\rho}{(\lambda-\rho)(1-b)/\lambda} = \\ &\frac{1}{1-\rho/\lambda}\left[1+\frac{\lambda}{1-\lambda\beta}\{p^+\beta^+\{1-\frac{1-\beta^+(\rho)}{\beta^+\rho}\}-p^-\beta^-\{1-\frac{1-\beta^-(\bar\rho)}{\beta^-\bar\rho}\}\}\right].\end{aligned} \tag{9.18}$$

In (9.18) we insert the expressions (7.8) for the LSTs of the heavy-tailed service time distribution $B^{\pm}(t)$, and we again distinguish the two cases in (8.9).

i. The case with

$$\beta > 0, \qquad \lambda\beta < 1.$$

With $\rho = r\nabla(b)$ and $\nabla(b)$ that zero of the contraction equation (8.13) which tends to zero for $b \uparrow 1$, we obtain from (9.18) (deleting the algebra): for Re $r = 0$,

$$\phi(\gamma r) := \lim_{b\uparrow 1}[\psi(r\nabla(b))]^{-1} = 1 + d_1^+(\gamma r)^{\nu-1} - d_1^-(\gamma\bar r)^{\nu-1}. \tag{9.19}$$

From (9.16) we have: for Re $r = 0$,

$$\mathrm{E}\{\mathrm{e}^{-r\nabla(b)\mathbf{w}}\} = \exp\Big[\frac{1}{2\pi\mathrm{i}}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty}[-\log\psi(\xi\nabla(b))]\frac{r}{\xi-r}\frac{\mathrm{d}\xi}{\xi}\Big], \qquad \mathrm{Re}\,r > 0.$$

Because of dominated convergence of the integral in (9.19) for $\nabla(b) \downarrow 0$, it follows that

$$\lim_{b\uparrow 1} \mathrm{E}\{\mathrm{e}^{-r\nabla(b)\mathbf{w}/\gamma}\} = \mathrm{e}^{\frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\mathrm{i}\infty}[\log\phi(\xi)]\frac{r}{\xi-r}\frac{\mathrm{d}\xi}{\xi}}, \qquad \mathrm{Re}\, r > 0, \tag{9.20}$$

$$= \mathrm{e}^{-\frac{1}{2}\log\phi(r)+\frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\mathrm{i}\infty}[\log\phi(\xi)]\frac{r}{\xi-r}\frac{\mathrm{d}\xi}{\xi}}, \quad \mathrm{Re}\, r = 0,$$

and, similarly,

$$\lim_{b\uparrow 1} \mathrm{E}\{\mathrm{e}^{r\nabla(b)\hat{\mathbf{i}}/\gamma}\} = \mathrm{e}^{\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}[\log\phi(\xi)]\frac{r}{\xi-r}\frac{\mathrm{d}\xi}{\xi}}, \qquad \mathrm{Re}\, r < 0, \tag{9.21}$$

$$= \mathrm{e}^{\frac{1}{2}\log\phi(r)+\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}[\log\phi(\xi)]\frac{r}{\xi-r}\frac{\mathrm{d}\xi}{\xi}}, \quad \mathrm{Re}\, r = 0.$$

The analysis above leads to the following theorem.

THEOREM 9.1. *For the $M/G_R/1$ model with $0 < b = \lambda\beta < 1$, and LSTs $\beta^{\pm}(\rho)$ of $B^{\pm}(t)$ as given in (7.8) the contracted variables $\nabla(b)\mathbf{w}$ and $\nabla(b)\hat{\mathbf{i}}$ both converge in distribution for $b \uparrow 1$. The LSTs of their limiting distributions are given by (9.20) and (9.21) with: for* $\mathrm{Re}\, r = 0$,

$$\begin{aligned}
\phi(r) &= 1 + d_1^+ r^{\nu-1} - d_1^- \bar{r}^{\nu-1} && \text{for} \quad \nu = \nu^+ = \nu^-, \quad C^{\pm} > 0,\\
&= 1 + d_1^+ r^{\nu-1} && \text{for} \quad \nu = \nu^+ < \nu^- \text{ or } C^- = 0,\\
&= 1 - d_1^- \bar{r}^{\nu-1} && \text{for} \quad \nu = \nu^- < \nu^+ \text{ or } C^+ = 0,
\end{aligned}$$

and, cf. (8.16),

$$d_1^+ = \frac{b^+ c^+ f^{(0)}}{F_1}, \qquad d_1^- = \frac{b^- C^-}{F_1}, \qquad b^+ - b^- = 1,$$

$$F_1 = \Big[(b^- C^-)^2 + (b^+ f^{(0)} C^+)^2 + 2(\cos\nu\pi) b^- b^+ f^{(0)} C^+ C^-\Big]^{\frac{1}{2}},$$

here $\nabla(b)$ is that root of the contraction equation (8.13) which tends to zero for $b \uparrow 1$, cf. (8.13).

PROOF. The solution of the boundary value problem (9.14), (9.15) is unique and given by (9.16) and (9.17). Because for $b < 1$, $\mathbf{w}_n$ converges in distribution for $n \to \infty$ and the LST of its limiting distribution, which is given by (9.16) and (9.17), is continuous in $r = 0$, $\mathrm{Re}\, r \geq 0$, its follows from the continuity theorem for characteristic functions and (9.28) that $\nabla(b)\mathbf{w}$ converges for $b \uparrow 1$ and that its limiting distribution is a proper distribution of which the LST is given by the righthand side of (9.28). Analogously for $\nabla(b)\hat{\mathbf{i}}$. □

COROLLOARY 9.1. *For the $M/G_R/1$ model with $b < 1$ the stochastic variable $\mathbf{v}_t$ converges in distribution for $t \to \infty$. For the stochastic variable $\mathbf{v}$ with distribution that limiting distribution of $\mathbf{v}_t$ for $t \to \infty$ holds that $\nabla(b)\mathbf{v}$ converges in distribution for $b \uparrow 1$ and its limiting distribution is the same of that of $\nabla(b)\mathbf{w}$ for $b \uparrow 1$.*

PROOF. The proof that $\mathbf{v}_t$ converges in distribution for $t \to \infty$ for $b < 1$ is analogous to that for the stable $M/G/1$ of $GI/G/1$ model, see [6], and therefore omitted. Because the jump epochs of the

$M/G_R/1$ model form a Poisson point process the PASTA property can be applied here, and so it results that $\mathbf{w}_n$ for $n \to \infty$ and $\mathbf{v}_t$ for $t \to \infty$ have the same limiting distribution, i.e. $\mathbf{v}$ and $\mathbf{w}$ have the same distribution. So the convergence in probability of $\nabla(b)\mathbf{w}$ for $b \uparrow 1$ implies that of $\nabla(b)\mathbf{v}$ for $b \uparrow 1$. □

Next we consider the second case in (8.9), i.e.

$$\beta = p^+\beta^+ - p^-\beta^- < 0. \tag{9.22}$$

In order not to complicate the analysis we assume that $\boldsymbol{\tau}_n$ is not a lattice variable.

From the analysis in the preceding section, cf. the discussion after (8.17), it turned out that the noise traffic $\mathbf{n}_t$ when properly scaled can only converge in probability if first $\lambda \to \infty$. Actually this implies that we have to take

$$\tilde{\alpha}(\rho) = 1 \text{ for } \mathrm{Re}\ \rho \geq 0. \tag{9.23}$$

This implies that (9.10) has to be replaced by: for Re $\rho = 0$,

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\} = \frac{1 - \mathrm{E}\{\mathrm{e}^{\rho\mathbf{i}}\}}{1-\beta(\rho)}. \tag{9.24}$$

Since

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}\Big|_{\rho=0} = \mathrm{E}\{\mathrm{e}^{\rho\mathbf{i}}\}\Big|_{\rho=0} = 1,$$

we have

$$\mathrm{E}\{\mathbf{i}\} = -\beta > 0,$$

as it follows from (9.22). Again with $\hat{\mathbf{i}}$ a stochastic variable with distribution defined by: for Re $\rho \geq 0$,

$$\mathrm{E}\{\mathrm{e}^{-\rho\hat{\mathbf{i}}}\} = \frac{1 - \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{i}}\}}{\rho\mathrm{E}\{\mathbf{i}\}}, \tag{9.25}$$

we obtain from (9.24): for Re $\rho = 0$,

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\} = \frac{-\rho\beta}{1-\beta(\rho)}\mathrm{E}\{\mathrm{e}^{\rho\hat{\mathbf{i}}}\}. \tag{9.26}$$

and

i. $\quad \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\},\ \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{i}}\}$ are regular for Re $\rho \geq 0$, continuous for Re $\rho \geq 0$, (9.27)

ii. $\quad \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}\Big|_{\rho=0} = \mathrm{E}\{\mathrm{e}^{-\rho\hat{\mathbf{i}}}\}\Big|_{\rho=0} = 1.$

Again it can be shown that the boundary value problem formulated by (9.26) and (9.27) has a unique solution for $\beta(\rho)$ as given in (7.7) and (7.8), cf [9]. It follows further, cf. [9], that this solution is given by (9.16) and (9.17) with: for Re $\rho = 0$,

$$\psi^{-1}(\rho) = \frac{1-\beta(\rho)}{-\rho\beta} = 1 + \frac{1}{|\beta|}\Big[p^+\beta^+\{1 - \frac{1-\beta^+(\rho)}{\rho\beta^+}\} - p^-\beta^-\{1 - \frac{1-\beta^-(\bar{\rho})}{\bar{\rho}\beta^-}\}\Big]. \tag{9.28}$$

With $\tilde{\nabla}(\beta)$ that root of (8.19) which tends to zero for $\beta \to 0$, we obtain from (9.28): for Re $r = 0$,

$$\phi(\gamma r) := \lim_{|\beta|\to 0} \psi^{-1}(r\tilde{\nabla}/\beta) = 1 + d_2^+(\rho r)^{\nu-1} - d_2^-(\rho\bar{r})^{\mathrm{v}-1}, \tag{9.29}$$

with $d_2^{\pm}$ as defined in (8.23). Analogously to Theorem 9.1 the following theorem is proved.

Theorem 9.2. *For the $M/G_R/1$ model with $\beta < 0$, $\alpha(\rho) = 1$ for $\mathrm{Re}\,\rho \geq 0$, and the LSTs of $B^{\pm}(t)$ as given in (7.8) the contracted variables $\tilde{\nabla}(\beta)\mathbf{w}$ and $\tilde{\nabla}(\beta)\hat{\mathbf{i}}$ both converge in distribution for $\beta \to 0$. The LSTs of their limiting distributions are given by*

$$\lim_{\beta\to 0} \mathrm{E}\{\mathrm{e}^{-r\tilde{\nabla}(\beta)\mathbf{w}/\gamma}\} = \mathrm{e}^{\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \log\phi(\xi)\frac{r}{\xi-r}\frac{\mathrm{d}\xi}{\xi}}, \qquad \mathrm{Re}\,r > 0, \tag{9.30}$$

$$= \mathrm{e}^{-\frac{1}{2}\log\phi(r)+\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \log\phi(\xi)\frac{r}{\xi-r}\frac{\mathrm{d}\xi}{\xi}}, \quad \mathrm{Re}\,r = 0,$$

$$\lim_{\beta\to 0} \mathrm{E}\{\mathrm{e}^{r\tilde{\nabla}(\beta)\hat{\mathbf{i}}/\gamma}\} = \mathrm{e}^{\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \log\phi(\xi)\frac{r}{\xi-r}\frac{\mathrm{d}\xi}{\xi}}, \qquad \mathrm{Re}\,r < 0, \tag{9.31}$$

$$= \mathrm{e}^{\frac{1}{2}\log\phi(r)+\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} [\log\phi(\xi)]\frac{r}{\xi-r}\frac{\mathrm{d}\xi}{\xi}}, \quad \mathrm{Re}\,r = 0,$$

with: for $\mathrm{Re}\,r = 0$,

$$\begin{aligned}
\phi(\xi) &= 1 + d_2^+ r^{\nu-1} - d_2^- \bar{r}^{\nu-1} && \textit{for} \quad \nu = \nu^+ = \nu^-, \quad C^{\pm} > 0,\\
&= 1 + d_2^+ r^{\nu-1} && \textit{for} \quad \nu = \nu^+ < \nu^- \quad \textit{or} \quad C^- = 0,\\
&= 1 - d_2^- \bar{r}^{\nu-1} && \textit{for} \quad \nu = \nu^- < \nu^+ \quad \textit{or} \quad C^+ = 0,
\end{aligned}$$

and, cf. (8.23),

$$\begin{aligned}
d_2^+ &= \frac{C^+ f^{(0)}}{F_2} \geq 0, \qquad d_2^- = \frac{C^-}{F_2},\\
F_2 &= \Big[(C^-)^2 + (C^+ f^{(0)})^2 + 2(\cos\nu\pi) f^{(0)} C^+ C^-\Big]^{\frac{1}{2}},
\end{aligned}$$

and $\tilde{\nabla}(\beta)$ that root of the contraction equation (8.19) which tends to zero for $\beta \to 0$.

Remark 9.1. The complex integrals in the expressions (9.20) and (9.30) have the same structure as those in (6.5), and so as in section 6 they can be transformed into real integrals, see also [2] for details. □

10. Comparison of the $GI/G/1$ and the $M/G_R/1$ model

In this section we compare the noise traffic $\mathbf{N}(\tau)$ for the $GI/G/1$ model, see Sections 4 and 5, and that for the $M/G_R/1$ model, see section 8. In Section 8 we have discussed two variants of the $M/G_R/1$ model. That with $0 < \lambda\beta < 1$ will be indicated by $M/G_R^{(1)}/1$, that with $\lambda = \infty$, $\beta < 0$ by $M/G_R^{(2)}/1$. Further we shall restrict the discussion to the case that $\nu = \nu_1 = \nu_2$, $C_1 > 0$, $C_2 > 0$, cf. (2.4), and $\nu = \nu^+ = \nu^-$, $C^{\pm} > 0$, cf. (7.8), since for the other cases, cf. (2.8) and (7.9), the comparison is not essentially different. These $\{\mathbf{N}(\tau),\ \tau \geq 0\}$ processes are ν-stable Lévy motions for $1 < \nu < 2$, and

Brownian motions for $\nu = 2$. The Lévy motions are pure jump processes, but Brownian motion is not. We therefore, consider only the case with

$$1 < \nu < 2. \tag{10.1}$$

From Theorems 4.1, 8.1 and 8.2, it is seen that the distribution of $\mathbf{N}(\tau)$ is the limiting distribution of the properly scaled noise traffic $\mathbf{n}_t$ when the scaling factor ∇ goes to zero. Comparison of these distributions for the $GI/G/1$- and the $M/G_R^{(2)}/1$ model shows that their limiting distributions are the same, cf. (4.15), (4.18) and (8.22) and (8.23), whenever

$$C_2 = C^+, \quad C_1 = C^-, \quad f = f^{(0)}, \tag{10.2}$$

i.e. whenever $B(t)$ and $B^+(t)$, and also $A(t)$ and $B^-(t)$ have the same tails, cf. (2.4) and (7.8). From (10.2) it then follows that

$$d_2 = d^+ \quad \text{and} \quad d_1 = d^-, \tag{10.3}$$

nothe that d_1/d_2 is independent of ν.

Because $d_2/(d_1 + d_2)$ and $d_1/(d_1 + d_2)$ are the probabilities of an up- and a downward jump in the $\mathbf{N}(\tau)$-process of the $GI/G/1$ model, it is seen from (10.3) that these probabilities are equal to those for the $\mathbf{N}(\tau)$-process of the $M/G_R^{(2)}/1$ model. This comparison of the limiting noise traffics for the $GI/G/1$ and the $M/G_R^{(2)}/1$ model elucidates the effect of the distribution $A(t)$ (its tail) on the limiting noise traffic of the $GI/G/1$ model.

Comparison of the noise traffics for the $GI/G/1$ and the $M/G_R^{(1)}/1$ model shows again that the limiting noise traffics for these models have the same structure, cf.(4.18) and (8.15). However, from (4.21) and (8.16), it is seen that (10.2) does not imply that (10.3) holds. Obviously the component $\tilde{\alpha}(\rho) = \lambda/(\lambda + \rho)$, Re $\rho \geq 0$ with $0 < \lambda\beta < 1$, cf. (8.9)i, has its own influence on the probabilities of the up- and downward jumps of the limiting noise traffic of the $M/G_R^{(1)}/1$ model.

Next we compare, again for the case $1 < \nu < 2$ the contracted actual waiting $\Delta(a)\mathbf{w}$ for $a \uparrow 1$ of the $GI/G/1$ model to the analogous characteristics for the $M/G_R/1$ model.

Noticing that the actual waiting time $\mathbf{w}$ and the virtual waiting $\mathbf{v}$ have the same (stationary) distribution, it follows readily from (6.5) and Theorem 9.1 that the limiting distributions for the contracted waiting times $\Delta(a)\mathbf{w}$ and $\nabla(b)\mathbf{w}$ for the $GI/G/1$- and the $M/G_R^{(1)}/1$ model have the same structure. However, they are not identical whenever (10.2) holds, viz. for the same reasons as above for the limiting noise traffics.

Comparison of the limiting distributions of the contracted actual stationary waiting times of the $GI/G/1$ and of the $M/G_R^{(2)}/1$ model shows that they have also the same structure and are equal whenever (10.2) holds, cf. (6.5) and Theorem 9.2.

11. On weak convergence of the noise traffic

In this section we consider the weak convergence of the noise traffic $\{\mathbf{N}(\tau; b),\ \tau \geq 0\}$ of the $M/G_R^{(1)}/1$ mode; with, cf. (8.9),

$$0 < b = \lambda\beta < 1. \tag{11.1}$$

The noise traffic $\mathbf{n}_t$ and the virtual backlog $\mathbf{h}_t$ are given by, cf. (8.3),

$$\mathbf{n}_t = \mathbf{k}_t - bt, \tag{11.2}$$

$$\mathbf{h}_t = \mathbf{n}_t - (1 - b)t.$$

With the scaling factor $\nabla(b)$ and $\nabla_1(b) = \nabla(b)(1-b)$, cf. Section 8, we consider the contracted noise traffic and the contracted virtual backlog: for $\tau \geq 0$,

$$\begin{aligned} \mathbf{N}(\tau; b) &:= \nabla(b)\mathbf{n}_{\tau/\nabla_1(b)}, \\ \mathbf{H}(\tau; b) &:= \nabla(b)\mathbf{h}_{\tau/\nabla_1(b)} = \mathbf{N}(\tau; b) - \tau, \end{aligned} \tag{11.3}$$

here the contraction factor $\nabla(b)$ is that root of (8.13) which tends to zero for $b \uparrow 1$.

From Theorem 8.1 it is known that $\mathbf{N}(\tau; b),\ \tau \geq 0$, converges in distribution for $b \uparrow 1$ and so (11.3) implies that this holds also for $\mathbf{H}(\tau; b)$. Let $\mathbf{N}_1(\tau)$ and $\mathbf{H}_1(\tau)$, $\tau \geq 0$, be variables with distributions the limiting distributions for $b \uparrow 1$ of $\mathbf{N}(\tau; b)$ and $\mathbf{H}(\tau; b)$, respectively. The characteristic function of $\mathbf{H}(\tau; b)$ follows from (11.3) and for that of $\mathbf{N}(\tau; b)$, see (8.12) with $\varepsilon = b$.

The characteristic function of the distribution of $\mathbf{N}_1(\tau)$ is given in (8.15), that of $\mathbf{H}_1(\tau)$ follows from (8.15) and (11.3), i.e. for $\operatorname{Re} r = 0$, $\tau \geq 0$,

$$\mathrm{E}\{\mathrm{e}^{-r\mathbf{H}_1(\tau)/\gamma}\} = \mathrm{e}^{r[1+d_1^+ r^{\nu-1} - d_1^- \bar{r}^{\nu-1}]\tau/\gamma}. \tag{11.4}$$

From the definition of $\mathbf{n}_t$ and $\mathbf{h}_t$ for the $M/G_R^{(1)}/1$ model it is readily seen that the processes $\{\mathbf{n}_t, t \geq 0\}$ and $\{\mathbf{h}_t, t \geq\}$ may be so defined that their sample functions are continuous from the right with limits from the left, similarly for the processes $\{\mathbf{N}(\tau; b), \tau \geq 0\}$ and $\{\mathbf{H}(\tau; b), \tau \geq 0\}$, with $0 < 1 - b << 1$. All these just mentioned processes are processes with independent increments. The characteristic functions of $\mathbf{N}_1(\tau)$, $\tau \geq 0$ and $\mathbf{H}_1(\tau),\ \tau \geq 0$, cf. (8.15), (11.4), show that the processes $\{\mathbf{N}_1(\tau),\ \tau \geq 0\}$ and $\{\mathbf{H}_1(\tau), l\tau \geq 0\}$ both have independent increments and $\mathbf{N}_1(0) = \mathbf{H}_1(0) = 0$, a.s. These processes are both ν-stable Lévy motions with zero and nonzero, negative drift, respectively, cf. [14], when $\nu < 2$, for $\nu = 2$ they are Brownian motion. The sample functions of these processes may again be so defined that they are continuous from the left with righthand limits whenever $1 < \nu < 2$. For $\nu = 2$, i.e. Brownian motion, the sample functions are continuous.

For the workload $\mathbf{v}_t$ of the $M/G_R^{(1)}/1$ model we have cf. (6.1) or (7.6) with $\mathbf{v}_0 = 0$, for $t \geq 0$,

$$\mathbf{v}_t = \max[\mathbf{h}_t, \sup_{0<u<t} (\mathbf{h}_t - \mathbf{h}_u)]. \tag{11.5}$$

With

$$\mathbf{V}(\tau; b) := \nabla(b)\mathbf{v}_{\tau/\nabla_1(b)}, \qquad \tau \geq 0, \tag{11.6}$$

we obtain from (11.5); for $\tau \geq 0$, $0 < 1 - b << 1$,

$$\mathbf{V}(\tau; b) = \max[\mathbf{H}(\tau; b), \sup_{0<u<t} (\mathbf{H}(\tau; b) - \mathbf{H}(\tau; u)]. \tag{11.7}$$

From the characteristic function of $\mathbf{N}(\tau; b)$ and of $\mathbf{N}_1(\tau)$, cf. (8.12) and Theorem 8.1 it follows that the finite dimensional distributions of the $\{\mathbf{N}(\tau; b),\ \tau \geq 0\}$ process converge for $b \uparrow 1$ to those of the $\{\mathbf{N}_1(\tau),\ \tau \geq 0\}$ process, and similarly for the $\{\mathbf{H}(\tau; b),\ \tau \geq 0\}$ and the $\{\mathbf{H}_1(\tau),\ \tau \geq 0\}$ process.

Define: for $\tau \geq 0$,

$$\mathbf{V}_1(\tau) := \max[\mathbf{V}_1(0) + \mathbf{H}_1(\tau), \sup_{0<u<\tau} (\mathbf{H}_1(\tau) - \mathbf{H}_1(u))], \tag{11.8}$$

and take presently $\mathbf{V}_1(0) = 0$.

We now apply a result from [11] concerning the convergence in the Skorokhod topology of a functional of a sequence of stochastic processes with independent increments to the same functional of the limiting stochastic process. The theorem in [2], Section 3.2 can be applied here for the sequence $\mathbf{H}(\tau; b_n)$ with $b_n \uparrow 1$ and the functional of $\mathbf{H}(\tau; b)$ in the righthand side of (11.7). It then follows that the righthand side of (11.7) converges in distribution for $b \uparrow 1$, the limiting distribution being that of the righthand side of (11.8). Consequently we have:

$$\mathbf{V}(\tau;b) \textit{ converges in distribution for } b \uparrow 1 \textit{ with limiting distribution the distribution of } \mathbf{V}_1(\tau) \textit{ for every } \tau > 0. \tag{11.9}$$

We consider again the workload process of the $M/G_R^{(1)}/1$ model but now with a stochastic initial workload $\mathbf{v}_0$: for $t \geq 0$,

$$\mathbf{v}_t = \max[\mathbf{v}_0 + \mathbf{h}_t, \sup_{0<u<t} (\mathbf{h}_t - \mathbf{h}_u)]. \tag{11.10}$$

Since $0 < \lambda\beta < 1$ we know that $\mathbf{v}_t$ converges in distribution for $t \to \infty$. Let again $\mathbf{v}$ be a stochastic variable with limiting distribution that of $\mathbf{v}_t$ for $t \to \infty$, actually, this distribution is independent of that of $\mathbf{v}_0$.

For the distribution of $\mathbf{v}_0$ we take that of $\mathbf{v}$, the stationary distribution of the Markov process $\{\mathbf{v}_t,\ t \geq 0\}$. It then follows that

$$\mathbf{v} \stackrel{\text{d}}{=} \max[\mathbf{v} + \mathbf{h}_t, \sup_{0<u<t} (\mathbf{h}_t - \mathbf{h}_u)], \qquad t \geq 0, \tag{11.11}$$

where $\stackrel{\text{d}}{=}$ indicates that the left- and righthand side of (11.11) have the same distribution. From (11.11) we obtain by scaling that: for $\tau \geq 0$,

$$\nabla(b)\mathbf{v} \stackrel{\text{d}}{=} \max[\nabla(b)\mathbf{v} + \mathbf{H}(\tau;b), \sup_{0<u<\tau} (\mathbf{H}(\tau;b) - \mathbf{H}(u;b))], \tag{11.12}$$

here $\nabla(b)\mathbf{v}$ is the stationary distribution of the Markov process $\nabla(b)\mathbf{v}_{t/\nabla_1(b)}$, $0 < 1 - b << 1$.

From Corollary 9.1 it is known that $\nabla(b)\mathbf{v}$ converges in distribution for $b \uparrow 1$. Let $\mathbf{V}_1$ be a stochastic variable with distribution this limiting distribution of $\nabla(b)\mathbf{v}$ for $b \uparrow 1$.

Using again Section 3.2 of [11] it follows by letting $b \uparrow 1$ in (11.12) that: for $\tau \geq 0$,

$$\mathbf{V}_1 \stackrel{\text{d}}{=} \max[\mathbf{V}_1 + \mathbf{H}_1(\tau), \sup_{0<u<\tau} (\mathbf{H}_1(\tau) - \mathbf{H}_1(u))]. \tag{11.13}$$

Take in (11.8) $\mathbf{V}_1(0) \stackrel{\text{d}}{=} \mathbf{V}_1$ it then follows from (11.8) and (11.13) that

$$\textit{the distribution of } \mathbf{V}_1 \textit{ is a stationary distribution of the Markov process } \{\mathbf{V}_1(\tau),\ \tau \geq 0\}. \tag{11.14}$$

From Theorem 9.1 and Corollary 9.1 it now follows that the LST of the distribution of $\mathbf{V}_1$ is given by

$$\begin{aligned} \mathrm{E}\{\mathrm{e}^{-r\mathbf{V}_1/\gamma}\} &= \mathrm{e}^{\frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \{\log\phi(\xi)\} \frac{r}{\xi - r} \frac{\mathrm{d}\xi}{\xi}} && \text{for Re } r > 0, \\ &= \mathrm{e}^{-\frac{1}{2}\log\phi(r) + \frac{1}{2\pi\mathrm{i}} \int_{-i\infty}^{i\infty} \{\log\phi(\xi)\} \frac{r}{\xi - r} \frac{\mathrm{d}\xi}{\xi}} && \text{for Re } r = 0, \end{aligned} \tag{11.15}$$

with, cf. (9.19),

$$\phi(r) := 1 + d_1^+ r^{\nu-1} - d_1^- \bar{r}^{\nu-1}, \qquad \text{Re } r = 0, \tag{11.16}$$

and $d_1^\pm$ as defined in (8.16).

REMARK 11.1. Above we have proved only the weak convergence of the process $\{\mathbf{V}(\tau;b),\ \tau \geq 0\}$ with $b \uparrow 1$ for the $M/G_R^{(1)}/1$ model; that for the $M/G_R^{(2)}/1$ model is analogous. For the $GI/G/1$ model a somewhat more refined analysis is required, and the approach as used in [13] may then be used. □

12. The Lévy queueing model $\mathcal{L}_\nu/\mathcal{L}_\nu/1$

Let $\mathbf{L}_\nu(t)$, $t \geq 0$, $1 < \nu < 2$, be a ν-stable Lévy motion with characteristic function given by: for $\operatorname{Re} r = 0$, $t \geq 0$,

$$\mathrm{E}\{\mathrm{e}^{-r\mathbf{L}_\nu(t)/\gamma}\} = \mathrm{e}^{[cr + a^+ r^\nu + a^- \bar{r}^\nu]t/\gamma}, \tag{12.1}$$

with

$$c > 0, \qquad a^\pm > 0, \qquad \gamma > 0.$$

Define the process $\{\mathbf{v}_t,\ t \geq 0\}$ by

$$\mathbf{v}_t := \max[\mathbf{v}_0 + \mathbf{L}_\nu(t), \sup_{0<u<t} (\mathbf{L}_\nu(t) - \mathbf{L}_\nu(u))], \tag{12.2}$$

with $\mathbf{v}_0$ the initial value of $\mathbf{v}_t$.

This $\mathbf{v}_t$-process will be called the workload process of a $\mathcal{L}_\nu/\mathcal{L}_\nu/1$ buffer model, see also Remark 12.1 below. The input process is described by the noise traffic $\mathbf{n}_t$ and the virtual backlog $\mathbf{h}_t$.

$$\begin{aligned} \mathbf{n}_t &= \mathbf{L}_\nu(t) + tc, \quad \mathrm{E}\{\mathbf{n}_t\} = 0, \\ \mathbf{h}_t &= \mathbf{L}_\nu(t), \qquad\quad \mathrm{E}\{\mathbf{h}_t\} = -ct. \end{aligned} \tag{12.3}$$

Obviously, $\mathbf{v}_t$ is the buffer content at time t.

The analysis of this section concerns the question wether the $\mathbf{v}_t$-process possesses a stationary distribution and if so to derive an expression for this distribution.

We restricted here the analysis to the case $1 < \nu < 2$ and $a^\pm > 0$; that with $\nu = 2$ and/or $a^+ + a^- > 0$, $a^\pm \geq 0$ is analogous and simpler.

To investigate this problem we try to construct an $M/G_R^{(1)}/1$ buffer model for which the workload process, when properly scaled, possesses a limiting process for the scaling factors approaching limits, see the analysis in Sections 8, 9 and 10. It is then tried to identify the limiting workload process with that described by (12.2).

With $\mathbf{k}_t$ the traffic generated in $[0, t)$ it follows from the definition of $\mathbf{h}_t$ and from (12.3) for $t \geq 0$,

$$\begin{aligned} \mathbf{h}_t &= \mathbf{k}_t - t, \\ \mathrm{E}\{\mathbf{k}_t\} &= (1-c)t. \end{aligned} \tag{12.4}$$

In order that the present model makes sense as a buffer model it is obviously necessary that, cf. (12.3) and (12.4),

$$0 < c < 1, \tag{12.5}$$

and in the following it will be assumed that (12.5) applies.

With

$$\hat{\Delta} := \left[\frac{c}{|a^+ + (-1)^\nu a^-|}\right]^{\frac{1}{\nu-1}} > 0, \tag{12.6}$$

put

$$\begin{aligned} \text{i.} \qquad \hat{\gamma} &:= \frac{\gamma}{\hat{\Delta} c}, \\ \text{ii.} \qquad \alpha^\pm &:= \frac{a^\pm}{|a^+ + (-1)^\nu a^-|}, \end{aligned} \tag{12.7}$$

and replace in (12.1) r by $r\hat{\Delta}$ then: for Re $r = 0$,

$$\mathrm{E}\{\mathrm{e}^{-r\mathbf{L}_\nu(t)/(\hat{\gamma}c)}\} = e^{[r+\alpha^+ r^\nu + \alpha^- \bar{r}^\nu]t/\hat{\gamma}}. \tag{12.8}$$

Define

$$\hat{\mathbf{n}}_t := \frac{1}{c}\mathbf{L}_\nu(t) + t, \qquad \hat{\mathbf{h}}_t = \frac{1}{c}\mathbf{L}_\nu(t), \tag{12.9}$$

so that

$$\mathrm{E}\{\hat{\mathbf{n}}_t\} = 0,\ \mathrm{E}\{\hat{\mathbf{h}}_t\} = -t. \tag{12.10}$$

Put

$$\hat{\mathbf{v}}_t := \max[\tilde{\mathbf{v}}_0 + \hat{\mathbf{h}}_t, \sup_{0<u<t}(\hat{\mathbf{h}}_t - \hat{\mathbf{h}}_u)], \qquad t \geq 0. \tag{12.11}$$

Obviously, the relations (11.4) and (12.8) have the same structure when comparing $\mathbf{H}_1(\tau)/\gamma$ and $\frac{1}{c}\mathbf{L}_\nu(t)/\hat{\gamma}$ since $d_1^\pm \geq 0$ and $|d_1^+ + (-1)^\nu d_1^-| = 1$, cf. Theorem 9.1; also (11.8) and (12.11) have the same structure. Next, we investigate whether the Lévy motion $\hat{\mathbf{h}}_t = \frac{1}{c}\mathbf{L}_\nu(t)$ can be the limiting virtual backlog of a properly scaled $M/G_R^{(1)}/1$ model, cf. the analysis which has led to the definition of $\mathbf{H}_1(\tau)$ in Section 11.

Choose positive number $p^\pm$, $\beta^\pm$ with

$$p^+ + p^- = 1, \qquad p^+\beta^+ - p^-\beta^- > 0. \tag{12.12}$$

For these $p^\pm$, $\beta^\pm$ choose a $\tilde{\lambda} > 0$ such that

$$0 < \hat{\lambda}p^+\beta^+ - \hat{\lambda}p^-\beta^- < 1, \tag{12.13}$$

and define λ by

$$\lambda(p^+\beta^+ - p^-\beta^-) = 1. \tag{12.14}$$

Consider the following equations

$$\alpha^+ = \frac{\lambda p^+\beta^+ C^+}{D} f^{(0)}, \qquad \alpha^- = \frac{\lambda p^-\beta^- C^-}{D}, \tag{12.15}$$

for $C^\pm > 0$, $0 < f^{(0)} < \infty$, and with

$$D = |\lambda p^+\beta^+ C^+ f^{(0)} + (-1)^\nu \lambda p^-\beta^- C^-|. \tag{12.16}$$

It is seen that the equations (12.15) are compatible with (12.7)ii, so that

$$C^+ f^0 = \alpha^+ \frac{D}{\lambda p^+\beta^+}, \qquad C^- = \alpha^- \frac{D}{\lambda p^-\beta^-}, \tag{12.17}$$

for every $D > 0$ and every $0 < f^{(0)} < 1$.

Next we construct two probability distributions $B^\pm(t)$ such that, for the $\beta^\pm$, $p^\pm, ; C^\pm$ introduced above

i. $$\beta^{\pm} = \int_0^{\infty} t\mathrm{d}B^{\pm}(t), \tag{12.18}$$

ii. $$1 - B^{\pm}(t) = 1 - B_1^{\pm}(t) - \frac{\beta^{\pm}C^{\pm}}{\Gamma(1-\nu)}(t/\tilde{\gamma})^{\nu}L^{\pm}\Big(\frac{\hat{\gamma}}{t}\Big), \qquad t \geq T,$$

for a $T > 0$ with $L^{\pm}\Big(\frac{\hat{\gamma}}{t}\Big)$ slowly varying functions at infinity and

$$\lim_{t\to\infty} \frac{L^{+}(\hat{\gamma}/t)}{L^{-}(\hat{\gamma}/t)} = f^{(0)}, \qquad 0 < f^{(0)} < \infty,$$

and for a $\delta > 0$,

$$|\int_T^{\infty} e^{-\rho t}\{1 - B_1^{\pm}(t)\}\mathrm{d}t| < \infty \text{ for } \mathrm{Re}\,\rho > -\delta.$$

It is not difficult to show that for given $\beta^{\pm} > 0$, $C^{\pm} > 0$, $f^{(0)}$ and ν with $0 < f^{(0)} < \infty$, $1 < \nu < 2$, a $T > 0$, a $\delta > 0$ and functions $B_1^{\pm}(t) > 0$, $L^{\pm}(\hat{\gamma}/t)$ can always been constructed such that $B^{\pm}(t)$, $t \geq 0$, are probability distributions for which (12.18) i and ii hold. Note there is still some room for the choice of $B^{\pm}(t)$ for $0 \leq t \leq T$.

We may now formulate and prove the following theorem.

THEOREM 12.1. *For the $\mathcal{L}_\nu/\mathcal{L}_\nu/1$ buffer model with virtual backlog $\mathbf{h}_t = \mathbf{L}_\nu(t)$, $1 < \nu < 2$, given by*

$$\mathrm{E}\{\mathrm{e}^{-r\mathbf{h}_t}\} = \mathrm{E}\{\mathrm{e}^{-r\mathbf{L}_\nu(t)}\} = \mathrm{e}^{[cr + a^{+}r^{\nu} + a^{-}\bar{r}^{\nu}]t/\gamma}, \qquad \mathrm{Re}\, r = 0,$$

with

$$0 < c < 1, \tag{12.19}$$

and the workload process $\{\mathbf{v}_0,\ t \geq 0\}$ defined by

$$\mathbf{v}_t = \max[\mathbf{v}_0 + \mathbf{h}_t, \sup_{0<\nu<t}(\mathbf{h}_t - \mathbf{h}_u)], \tag{12.20}$$

holds:

i. *$\mathbf{v}_t$ converges in distribution for $t \to \infty$, its limiting distribution is the stationary distribution of the $\hat{\mathbf{v}}_t$-process;*

ii. *for $\mathbf{V}$ a stochastic variable with distribution this limiting distribution its LST is given by:*

$$\begin{aligned} \mathrm{E}\{\mathrm{e}^{-r\mathbf{V}/\gamma}\} &= \mathrm{e}^{\Phi(r)} && \mathit{for} \quad \mathrm{Re}\, r > 0, \\ &= \mathrm{e}^{-\frac{1}{2}\log\omega(r)+\Phi(r)} && \mathit{for} \quad \mathrm{Re}\, r = 0, \end{aligned} \tag{12.21}$$

where

$$\Phi(r) \quad = \quad \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \{\log \omega(\xi)\} \frac{r}{\xi - r} \frac{\mathrm{d}\xi}{\xi},$$

$$\omega(r) \quad = \quad 1 + \frac{a^+}{c} r^{\nu-1} - \frac{a^-}{c} \bar{r}^{\nu-1}, \qquad \mathrm{Re}\, r = 0.$$

PROOF. For c satisfying the condition of the theorem, cf. (12.19), we consider the $M/G_R^{(1)}/1$ buffer model with arrival rate $\hat{\lambda}$ satisfying (12.14) and distributions$B^{\pm}(t)$ as constructed above, see (12.18). From the results in Section 9, cf. Theorem 9.1 and Corollary 9.1, and Section 11 it is seen that the workload process $\tilde{\mathbf{v}}_t$ of this $M/G_R^{(1)}/1$ model when scaled by $\nabla(b)$ and $\nabla_1(b) = \nabla(b)(1-b)$ with $b = \hat{\lambda}(p^+\beta^+ + p^-\beta^-)$, i.e. $\nabla(b)\tilde{\mathbf{v}}_{\tau/\nabla_1(b)}$, converges weakly for $b \uparrow 1$ to a process $\{\hat{\mathbf{v}}_t, \tau \geq 0\}$ with

$$\hat{\mathbf{v}}_t = \max[\hat{\mathbf{h}}_t, \sup_{0<u<\tau} (\hat{\mathbf{h}}_\tau - \hat{\mathbf{h}}_u)], \qquad \tau \geq 0, \tag{12.22}$$

where, cf. (12.8): for Re $r = 0$,

$$\mathrm{E}\{\mathrm{e}^{-r\mathbf{i}\hat{\mathbf{h}}_\tau/\hat{\gamma}}\} = \mathrm{E}\{\mathrm{e}^{-\frac{r}{c}\mathbf{L}_\nu(\tau)/\hat{\gamma}}\} = \mathrm{e}^{[r+\alpha^+ r^\nu + \alpha^- \bar{r}^\nu]\tau/\hat{\gamma}}. \tag{12.23}$$

Obviously (12.22) may be rewritten as

$$\hat{\mathbf{v}}_\tau = \max\left[\frac{1}{c}\mathbf{L}_\nu(\tau), \sup_{0<u<\tau} (\frac{1}{c}\mathbf{L}_\nu(t) - \frac{1}{c}\mathbf{L}_\nu(u))\right]. \tag{12.24}$$

Moreover it follows that $\hat{\mathbf{v}}_t$ converges in distribution for $\tau \to \infty$, see Corollary 9.1, and so with $\hat{\mathbf{V}}$ a stochastic variable with distribution the limiting distribution of $\hat{\mathbf{v}}_\tau$ for $t \to \infty$ it follows from (11.15) that

$$\mathrm{E}\{\mathrm{e}^{-r\hat{\mathbf{V}}/\tilde{\gamma}}\} \quad = \mathrm{e}^{\frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \{\log \omega(\xi)\} \frac{r}{\xi-r} \frac{\mathrm{d}\xi}{\xi}}, \qquad \mathrm{Re}\, r > 0, \tag{12.25}$$

$$= e^{-\frac{1}{2}\log \omega(r) + \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \{\log \omega(\xi)\} \frac{r}{\xi-r} \frac{\mathrm{d}\xi}{\xi}}, \quad \mathrm{Re}\, r = 0,$$

with

$$\hat{\omega}(r) := 1 + \alpha^+ r^{\nu-1} - \alpha^- \bar{r}^{\nu-1}, \qquad \mathrm{Re}\, r = 0.$$

Multiplication of (12.25) by c shows that $c\hat{\mathbf{V}}$ has as distribution the limiting distribution of the process $\{\mathbf{v}_t, t \geq 0\}$ as defined in (12.2), so that $c\hat{\mathbf{V}} \overset{\mathrm{d}}{=} \mathbf{V}$. It follows from Corollary 9.1 and (11.15) that

$$\mathrm{E}\{\mathrm{e}^{-\frac{r}{c}\mathbf{V}/\tilde{\gamma}}\} = \mathrm{e}^{\frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \{\log \hat{\omega}(\xi)\} \frac{r}{\xi-r} \frac{\mathrm{d}\xi}{\xi}}, \qquad \mathrm{Re}\, r > 0 \tag{12.26}$$

with

$$\hat{\omega}(r) = 1 + \alpha^+ r^{\nu-1} - \alpha^- \bar{r}^{\nu-1} \quad \text{for} \quad \mathrm{Re}\, r = 0.$$

Since, cf. (1.7), $\hat{\gamma} = \gamma/\hat{\Delta}c$, we have

$$\frac{1}{c}\hat{\mathbf{V}}/\gamma = \hat{\Delta}\mathbf{V}/\gamma.$$

so that by replacing $r\Delta$ by r in (12.26) we obtain (12.21). The proof is complete. $\Box$

REMARK 12.1. As in (6.5) a real integral expression for $\mathrm{E}\{\mathrm{e}^{-r\mathbf{V}/\gamma}\}$ can be derived, also asymptotics for the tail of the distribution of $\mathbf{V}$ can be derived. This asymptotic analysis is similar to that given in [2].

REMARK 12.2. Above we have introduced the $\mathcal{L}_\nu/\mathcal{L}_\nu/1$ model for the workload process of a buffer with virtual backlog $L_\nu(t)$. The model $\mathcal{L}_{\nu_1}/\mathcal{L}_{\nu_2}/1$ is defined as follows. For $\nu_2 < \nu_1$, $1 < \nu_2 < 2$ it is the above model with $\nu = \nu_2$ and $a^- = 0$, whereas for $1 < \nu_1 < \nu_2$, $\nu_1 < 2$ it is the $\mathcal{L}_\nu/\mathcal{L}_\nu/1$ model with $\nu = \nu_1$ and $a^+ = 0$. The $\mathcal{L}_{\nu_1}/\mathcal{L}_{\nu_2}/1$ model should be compared with the heavy tailed $GI/G/1$ model of Section 2, in particular see Remark 2.5. It is not difficult to show that Theorem 12.1 also applies for the $\mathcal{L}_{\nu_1}/\mathcal{L}_{\nu_2}/1$ model with $\nu_1 \neq \nu_2$. The integral in (12.16) is then evaluated as the similar one in (6.5), cf. (6.8) and (6.19). $\Box$

13. ON SELF-SIMILAR TRAFFIC MODELING FOR FLUID QUEUEING MODELS

In [10] the authors discuss the modelling of self-similar traffic for a fluid buffer model. Traffic measurements suggested that self-similar processes could be adequate for the analysis of buffer models used in ATM systems. In this section we compare the approach developed in [10] with that developed in the preceding sections.

When considering the models discussed in the preceding sections as buffer models then an essential characteristic is the model assumption that a message ariving at an arrival epoch is instantaneously fed into the buffer, so that the buffer content then experiences a jump. In a fluid model the packets of an arriving message are successively fed into the buffer. During ON-periods traffic is fed into the buffer, no traffic is generated during OFF-periods; ON- and OFF-periods alternate. Their durations are independent stochastic variables and those of the ON-periods are i.i.d., similarly those of the OFF-periods. In [10] the authors develop a limit theorem for the scaled traffic generated in a time interval $[0, t)$ for the case that the ON- and OFF-periods have heavy-tailed distributions.

The approach in [10] is not restricted to a fluid model. It may applied as well to the $M/G/I$ buffer model with instantaneous input. For the comparison of the approach in [10] and that of the present study if suffices to consider the simple $M/G/1$ model with heavy tailed service time distribution $B(t)$ and arrival rate λ. For the sake of simplicity we take for the LSF transform $\beta(\rho)$ of $B(t)$, cf. (2.4): for Re $\rho \geq 0$,

$$1 - \frac{1-\beta(\rho)}{\rho\beta} = g(\gamma\rho) + C(\gamma\rho)^{\nu-1}, \tag{13.1}$$

with

i. $\quad \beta = \int\limits_0^\infty t\mathrm{d}B(t), \qquad 0 < a = \lambda\beta < 1, \qquad (13.2)$

ii. $\quad 1 < \nu < 2, \qquad C > 0,$

iii. $\quad g(\gamma\rho)$ is regular for Re $\rho > -\delta$ for a $\delta > 0$, $\quad g(0) = 0$.

To keep the comparison as simple as possible we take $1 < \nu < 2$ and in (3.4) $L(\rho) \equiv 1$.

For $\mathbf{k}_t$ the traffic generated in $[0, t)$ the noise traffic $\mathbf{n}_t$ and the virtual backlog $\mathbf{h}_t$ are given by

$$\begin{aligned} \mathbf{n}_t &= \mathbf{k}_t - at, \qquad (13.3) \\ \mathbf{h}_t &= \mathbf{k}_t - t = \mathbf{n}_t - (1-a)t. \end{aligned}$$

As in (8.4) it follows for the present case that: for Re $\rho = 0$,

$$\mathrm{E}\{\mathrm{e}^{-\rho \mathbf{n}_t}\} = e^{a\rho t\{1-\frac{1-\beta(\rho)}{\rho\beta}\}}. \tag{13.4}$$

For a $T > 0$ the scaling in [10] reads,

$$t = T\tau \text{ and } \rho = rT^{-k}, \quad \mathrm{Re}\, r \geq 0, \quad k > 0. \tag{13.5}$$

It then follows from (13.1), (13.2) and (13.5) that: for Re $r = 0$,

$$\log \mathrm{E}\{\mathrm{e}^{-r\mathbf{n}_{T\tau}/T^k}\} = ar\tau T^{1-k}\{g(\gamma r T^{-k}) + C(\gamma r)^{\nu-1}T^{-k(\nu-1)}\}. \tag{13.6}$$

Take

$$k = 1/\nu, \tag{13.7}$$

so that $2k > 1$, cf. (13.2)ii. It then follows from (13.2)iii that for $T >> 1$, Re $r = 0$,

$$\mathrm{E}\{\mathrm{e}^{-r\mathbf{n}_{T\tau}/T^{1/\nu}}\} = \mathrm{e}^{aC(\gamma r)^{\nu}(\tau/\gamma)}\{1 + ar\tau \mathrm{O}(T^{1-2/\nu})\}. \tag{13.8}$$

From (13.8) it follows readily that the stochastic variable $\mathbf{n}_{\tau T}/T^{1/\nu}$ converges in distribution for $T \to \infty$ and every $a > 0$. The limiting distribution is actually a ν-stable distribution. As in Section 11 it is shown that $\{\hat{\mathbf{N}}_\tau,\ \tau \geq 0\}$ is a ν-stable Lévy motion, here $\hat{\mathbf{N}}_\tau$, $\tau \geq 0$ is a stochastic variable with distribution this limiting distribution.

The weak convergence of the scaled noise traffic $T^{-1/\mathrm{v}}\mathbf{n}_{\tau T}$ for $T \to \infty$ for the heavy tailed $M/G/1$ model is a similar limiting result as given in Theorem 3 of [11] for the fluid model with heavy tailed ON- and OFF-periods.

Next we consider the virtual backlog $\mathbf{h}_t$. We obtain from (13.3) for Re $\rho = 0$, $t \geq 0$,

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{h}_t}\} = \mathrm{e}^{(1-a)\rho t}\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{n}_t}\}. \tag{13.9}$$

Hence: for Re $r = 0$, $T >> 1$,

$$\mathrm{E}\{\mathrm{e}^{-r\mathbf{h}_{\tau T}/T^{1/\nu}}\} = \mathrm{e}^{(1-a)r\tau T^{t-1/\nu}}\mathrm{e}^{aC(\gamma r)^{\nu}(\tau/\gamma)}\{1 + ar\tau)\mathrm{O}(T^{1-2/\nu})\}. \tag{13.10}$$

Because $1-1/\nu > 0$ for $1 < \nu < 2$ it is seen from (13.10) that for $0 < a < 1$ the characteristic function of the scaled virtual backlog $\mathbf{h}_{\tau T}/T^{1/\nu}$ does not converge for $T \to \infty$.

The workload $\mathbf{v}_t$ of the $M/G/1$ model with virtual backlog $\mathbf{h}_t$ is described by

$$\mathbf{v}_t := \max[\mathbf{v}_0 + \mathbf{h}_t, \sup_{0<u<t}(\mathbf{h}_t - \mathbf{h}_u)], \qquad t \geq 0. \tag{13.11}$$

For the $M/G/1$ model with the scaled noise traffic $\mathbf{n}_{\tau T}/T^{1/\nu}$ the workload $\hat{\mathbf{v}}_{\tau T} = \mathbf{v}_{\tau T}/T^{1/\nu}$ is given by

$$\hat{\mathbf{v}}_{\tau T} := \max[\hat{\mathbf{v}}_0 + \mathbf{h}_{\tau T}/T^{1/\nu}, \sup_{0<u<\tau}(\mathbf{h}_{\tau T}/T^{1/\nu} - \mathbf{h}_{uT}/T^{1/\nu})]. \tag{13.12}$$

Because the characteristic function of $h_{\tau T}/T^{1/\nu}$ does not converge, it is seen from (13.12) that the function of $\hat{\mathbf{v}}_{\tau T}$ cannot have a true limiting distribution for $T \to \infty$. Consequently it is seen that the scaling of time by $t = \tau T$ and that of $\mathbf{n}_{\tau T}$ by $T^{-1/\nu}$ is for $T \to \infty$ not an appropriate scaling of the noise traffic $\mathbf{n}_t$ since $\mathbf{v}_{\tau T}/T^{1/\nu}$ does not converge in distribution for $T \to \infty$.

However when taking

$$T = [\Delta(a)(1-a)]^{-1}, \tag{13.13}$$

with $\Delta(a)$ that zero of the contraction equation: for $0 < a < 1$,

$$aCx^{\nu-1} = 1 - a, \qquad x > 0. \tag{13.14}$$

which goes to zero for $a \uparrow 1$, it follows that

$$(1-a)T^{1-\frac{1}{\nu}} = aC, \qquad T^{-1/\nu} = \Delta(a), \tag{13.15}$$

$$T \to \infty \quad \text{for} \quad a \uparrow 1.$$

and so we obtain from (13.10): for $\operatorname{Re} r = 0$.

$$\mathrm{E}\{\mathrm{e}^{-r\Delta(a)h_{\tau/\Delta_1(a)}}\} = \mathrm{e}^{[\gamma r + \gamma r^{\nu}]aC\tau/\gamma}\{1 + ar\tau\mathrm{O}(aC\frac{\Delta(a)}{1-a})\}, \tag{13.16}$$

with $\Delta_1(a) = \Delta(a)(1-a)$. Since $\Delta_1(a)/(1-a) \downarrow 0$ for $a \uparrow 1$ it is seen from (3.16) that $\Delta(a)\mathbf{h}_{\tau/\Delta_1(a)}$ converges in distribution for $a \uparrow 1$ Considering again the $\mathbf{v}_t$-process it is seen as in Section 10 that $\Delta(a)\mathbf{v}_{\tau/\Delta_1(a)}$ converges in distribution for $a \uparrow 1$, and every $\tau > 0$.

The analysis above illustrates that the scaling of the noise traffic as in [10] is not adequate for the analysis of the workload process $\mathbf{v}_t$ for large t, unless the scaling depends on the traffic parameter. This dependency is determined by the contraction equation. The ultimate result is then a heavy traffic theorem, see the preceding sections.

Appendix A

In this appendix we show that the equation (8.1),

$$1 - b = x^{\mathrm{v}-1}|K(x)|, \quad x > 0, \tag{a.1}$$

with

$$K(x) = b^+C^+L^+(x) + (-1)^{\nu}b^-C^-L^-(x), \quad x > 0, \tag{a.2}$$

has for $b < 1$ and b sufficiently close to one, i.e. $0 < 1 - b << 1$, a single zero $\nabla(b)$ with the property that $\nabla(b) \downarrow 0$ for $b \uparrow 1$. We consider here only the case that $\nu = \nu_1 = \nu_2$, $C^{\pm} > 0$, the other cases are proven similarly.

From (7.16) with $\rho > 0$ we obtain, cf. also Remark 2.3,

$$\rho^{\nu-1}L^{\pm}(\rho) \to 0 \quad \text{for} \quad \rho \downarrow 0, \tag{a.3}$$

since

$$g^{\pm}(\rho) = \mathrm{O}(\rho) \quad \text{and} \quad 1 < \nu \leq 2.$$

From (a.2) we have for $x > 0$,

$$|K(x)| = [\{b^+C^+L^+(x)\}^2 + \{b^-C^-L^-(x)\}^2 + 2(\cos\nu\pi)b^+b^-C^+C^-L^+(x)L^-(x)]^{\frac{1}{2}}. \tag{a.4}$$

Hence from (a.3) (and Remark 2.1),

$$x^{\nu-1}|K(x)| \downarrow 0. \tag{a.5}$$

From (a.5) it follows that (a.1) has for $0 < 1 - b << 1$ a single zero $\nabla(b) > 0$ which tends to zero for $b \uparrow 1$, note that $|K(x)| \neq 0$ for $0 < x << 1$.

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