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On Interpolation Series Related to the Abel-Goncharov Problem, with Applications to Arithmetic-Geometric Mean Relationship and Hellinger Integrals

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ABSTRACT

In this paper a convergence class is characterized for special series associated with Gelfond's interpolation problem (a generalization of the Abel-Goncharov problem) when the interpolation nodes are equidistantly distributed within the interval $[0, 1]$. As a result, an expansion is derived of the arithmetic-geometric mean difference in terms of certain central moments. Another result concerns an expansion of the Hellinger integral.

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1. Introduction

1.1 Interpolation problem

In [3] and [9], section 5, the investigation of a special interpolation series has been initiated, with a view to applications within certain probabilistic context. The present paper improves upon the previous results; concerning the applications of the new results, see the concluding subsection 1.3 of this introduction.

Our interpolation problem (see the formulation at the end of this subsection) belongs to a class of problems introduced in Gelfond's book [6]. In section 1.5 of this book a triangular array of interpolation nodes is considered

$$
\begin{array}{llll}
x_{00} & & & \\
x_{10} & x_{11} & & \\
\cdots & & & \\
x_{n0} & x_{n1} & \cdots & x_{nn} \\
\cdots & & &
\end{array}
$$

that determines for each $n = 0, 1, \ldots$ the divided difference $[x_{n0}\, x_{n1} \cdots x_{nn}]f$ of a certain analytic function f of a complex variable $z \in \mathbb{C}$. It is required to construct for each $n = 0, 1, \ldots$ a polynomial $p_n(\cdot; f)$ of degree n so that

$$[x_{k0}\, x_{k1} \cdots x_{kk}]p_n = [x_{k0}\, x_{k1} \cdots x_{kk}]f \quad \text{for} \quad k = 0, 1, \ldots, n. \tag{1.1}$$

Recall that for a set of arguments $\{x_k\}_{k=0,1,\dots,n}$ the *devided deference* $[x_0\, x_1\, \cdots\, x_n]$ is a functional that acts on a function f as follows: $[x_n]f = f(x_n)$ for $n \geq 0$ and

$$[x_0\, x_1\, \cdots\, x_n]f = \frac{[x_0\, x_1\, \cdots\, x_{n-1}]f - [x_1\, x_2\, \cdots\, x_n]f}{x_0 - x_n} \quad \text{for } n \geq 1, \tag{1.2}$$

see e.g. [6], section 1.1, [14], section 2.11, or [13], chapter I. If f is assumed to be an analytic function on $\mathbb{C}$ and C is a closed rectifiable contour in the complex plane which contains all the points $x_0, \dots, x_n$, then we have the following contour integral representation:

$$[x_0\, x_1\, \cdots\, x_n]f = \frac{1}{2\pi i} \int\limits_C \frac{f(z)\, dz}{(z - x_0) \cdots (z - x_n)}, \tag{1.3}$$

see [6], section 1.4.3, or [13], section 1.7. It is easily seen by taking into consideration the recurrent definition (1.2) of the devided differences, that Gelfond's class of problems includes the classical interpolation problem (with the well-known solutions by Lagrange, Newton or Hermite: see, e.g. [6], section 1.1, [14], vol. II, section 2.11, or [13], chapter I) of constructing a polynomial $p_n(\cdot; f)$ that coincides with the function f in question at certain fixed interpolation nodes, say $x_0, \dots, x_n$. Indeed, this requirement is a special case of (1.1) when the above triangular array is constructed by starting from $x_{00} = x_0$ and by adding in each consecutive row a new node to the previous nodes, say x_n in the n^{th} row. The latter row then consists of the entries $\{x_{nk} = x_k\}_{k=0,1,\dots,n}$. In another extreme case when each row consists of equal entries x_n say (so that in the n^{th} row $\{x_{nk} = x_n\}_{k=0,1,\dots,n}$), the conditions (1.1) turn into

$$p_n^{(k)}(x_k; f) = f^{(k)}(x_k) \quad \text{for} \quad k = 0, 1, \dots, n,$$

due to the integral representation (1.3) and Cauchy's integral formula for the n^{th} derivative $f^{(n)}$ of a function f (see [14], vol. I, chapter 14). In this special case we thus deal with the so-called *Abel-Goncharov* interpolation problem (see [9], section 5, and the references therein, e.g. [1], [5], [6] or [17]).

As was already mentioned, the main subject of our interest is a quite different special case of the above triangular scheme – the case when each row consists of the nodes spread equidistantly over the interval $[0, 1]$ so that the n^{th} row, e.g., consists of $\{x_{nk} = \frac{k}{n}\}_{k=0,1,\dots,n}$. In this special case lemma 2.1 below tells us that the n^{th} divided difference, denoted throughout by $\Delta_n \equiv [0\, \frac{1}{n}\, \cdots\, 1]$ for convenience, may be presented as follows:

$$\Delta_n f = \frac{n^n}{n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f\left(\frac{k}{n}\right). \tag{1.4}$$

For instance

$$\Delta_0 f = f(0),\ \Delta_1 f = f(1) - f(0),\ \Delta_2 f = 2[f(1) - 2f(\frac{1}{2}) + f(0)], \dots$$

Let the first $n + 1$ of these numbers be given. Our interpolation problem is then defined as follows.

Problem: Construct the interpolation polynomial $p_n(\cdot; f)$ of degree n so that

$$\Delta_k p_n = \Delta_k f \quad \text{for} \quad k = 0, 1, \dots, n. \tag{1.5}$$

For the solution to this problem see section 3.2, proposition 3.2.

1.2 Interpolation series

The main task of the interpolation theory is to show the consistency of the interpolation polynomial as $n \to \infty$, i.e. to show that for a function f under consideration the remainder term

$$r_n(\cdot; f) \equiv f(\cdot) - p_n(\cdot; f) \tag{1.6}$$

converges absolutely to zero, pointwise as evaluated at a certain point, or even uniformly within a certain range of the argument. Gelfond's general results in this direction are presented in [6], sections 1.5.2 and 1.5.3. These two results are direct extensions of Goncharov's original results, see e.g. [1], theorem 9.11.1, and [17], theorem 7, respectively. Only the first of these results is applicable to the case of our interest, i.e. to the interpolation problem (1.5). This leads in [3] (cf also [9], chapter 5) to certain results on the convergence of the associated series. The present paper improves upon these results. In order to get the idea on this improvement, consider the example of the exponential function $f : x \mapsto e^{ax}$. In [3] the following expansion has been obtained: at each fixed $x \in [0, 1]$

$$e^{ax} = 1 + \sum_{n=1}^{\infty} \frac{n^n}{n!} \, c_n(x) \, (e^{a/n} - 1)^n \tag{1.7}$$

where $c_0 \equiv 1, c_1, c_2, \ldots$ is a sequence of certain basic polynomials; see section 3.1 for details. The range of the parameter a has been $|a| < \log 2$. It was shown that this range cannot be essentially enlarged by arguments based on Goncharov's results mentioned above. To avoid this drawback we will pursue in section 3 a different, direct approach and obtain theorem 3.3 concerning expansion (3.8) of type (1.7) for any analytic function f of an exponential type, cf condition (3.10). The class of such functions includes the exponentials $f : x \mapsto e^{ax}$ for every $a \in \mathbb{R}$, which means that the earlier restriction $|a| < \log 2$ is superfluous and the expansion (1.7) holds for any real valued parameter a. See section 3.3 for further comments. Note, in conclusion, that the essential improvement is achieved by sharper estimation of the basic polynomials $c_0 \equiv 1, c_1, c_2, \ldots$ in proposition 3.1 (this might be of some interest within the theory of Stirling numbers, via the connection discussed in section 2.2, example 3).

1.3 Applications

In section 3.4 we consider a binary random variable X which takes on either the value e^a with probability x or the value 1 with probability $1 - x$, where a is a non-zero parameter. Note that any binary random variable X can be presented in this form by suitable normalization, if needed. The geometric mean $\mathbb{G}X \equiv e^{\mathbb{E}\log X}$ and the arithmetic mean $\mathbb{E}X$ of X equal e^{ax} and $1 + x(e^a - 1)$ respectively. We will show in theorem 3.4 that (1.7) implies the following expansion of the difference between the geometric mean and the arithmetic mean of X:

$$\mathbb{G}X - \mathbb{E}X = \sum_{n=2}^{\infty} \sum_{m=0}^{[\frac{n}{2}]} \frac{n^n}{n!} \, a_{nm} \, \mathbb{E}_{nm} \, \mathbb{E}_{n\, n-m} \tag{1.8}$$

with the central moments $\mathbb{E}_{nm} \equiv \mathbb{E}(X^{\frac{1}{n}} - \mathbb{E}X^{\frac{1}{n}})^m$. The coefficients a_{nm}, uniquely determined in theorem 3.4, are independent of the distribution of X, i.e. independent of x and a. The arithmetic-geometric mean relationship is a classical subject (see [2] or [7]), however the expansion (1.8) seems new. The fact that the arithmetic mean always exceeds geometric mean is very well-known. Less known are verious refinements of this inequality, see [2], section II.3, where the first term $2\mathbb{E}_{22}$ of the series in (1.8) occurs to characterize the difference. Cp (3.16) where the explicit expression of the next few terms is displayed. Finally we want to stress the significance of expansion (1.8) in applications where only fractional moments of X make sense (as in the context of [4] or [8], say) and the simple results like (3.14) below are unapplicable.

In section 3.5 we will give another application of the expansion (1.7). Let f and g be positive probability density functions on $\mathbb{R}$. For $n \in \mathbb{N}^+$ define

$$h_n(f,g) = \int_{-\infty}^{\infty} \left(f(t)^{\frac{1}{n}} - g(t)^{\frac{1}{n}} \right)^n \, dt. \tag{1.9}$$

In case n is even $h_n(f,g)^{\frac{1}{n}}$ is called the *Hellinger distance* of order n between f and g. By Newton's binomial formula

$$h_n(f,g) = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} H_{m/n}(f,g) \tag{1.10}$$

where $H_x(f,g)$ is the *Hellinger integral* of order $x \in [0,1]$ defined by

$$H_x(f,g) = \int_{-\infty}^{\infty} f(t)^x g(t)^{1-x} \, dt, \tag{1.11}$$

see e.g. [12], section 3.2, or [16], chapter 1. By using expansion (1.7) we show that the following relation (inverse to (1.10)) holds:

$$H_x(f,g) = 1 + \sum_{m=1}^{\infty} \frac{m^m}{m!} \, c_m(x) \, h_m(f,g). \tag{1.12}$$

The expansion (1.12) might be useful in the setup of [4] or [8]. The expansion up to the second term, for instance, plays a central rôle in proving the important functional central limit theorem in [8], p 554.

2. Divided differences

2.1 Application to polynomials

In the sequel we will need to apply the functional $[x_0\, x_1\, \cdots\, x_n]$ to polynomials of various degree. If p_k is a certain monic polynomial of degree $k \le n$, i.e. a polynomial with unit leading coefficient, then

$$[x_0\, x_1\, \cdots\, x_n] p_k = \delta_{nk}. \tag{2.1}$$

This is easily seen due to the following mean value representation

$$[x_0\, x_1\, \cdots\, x_n] f = \frac{f^{(n)}(\xi)}{n!}$$

where ξ is a certain point from the smallest segment covering all the points $x_0, x_1, \dots, x_n$ (see [6], section 1.4.1 or [13], p 6). Take now a polynomial of degree exceeding n, say a monic polynomial

$$p(z; y_0, \dots, y_{n+m}) \equiv (z-y_0)\cdots(z-y_{n+m})$$

of degree $n+m+1$, where $m \geq 0$. Then the n^{th} divided difference may be calculated by means of the following formula:

$$\frac{1}{2\pi i}\int_C \frac{p(z; y_0, \dots, y_{n+m})}{p(z; x_0, \dots, x_n)} dz = \sum_{0\leq k_0\leq\cdots\leq k_m\leq n}\prod_{j=0}^{m}\left(x_{k_j} - y_{k_j+j}\right). \tag{2.2}$$

This formula is easily obtained by induction, using repeatedly the identity

$$\begin{aligned}\int_C \frac{p(z; y_0, \dots, y_{n+m})}{p(z; x_0, \dots, x_n)}\, dz \quad &= \quad (x_0 - y_0)\int_C \frac{p(z; y_1, \dots, y_{n+m})}{p(z; x_0, \dots, x_n)}\, dz \\ &+ \int_C \frac{p(z; y_1, \dots, y_{n+m})}{p(z; x_1, \dots, x_n)}\, dz.\end{aligned}$$

Since the forthcoming sections are restricted to the particular case of the equidistant nodes $\{\frac{k}{n}\}_{k=0,1,\dots,n}$, it will be convenient to use throughout the special notation $\gamma_{n+1}(\cdot) \equiv p(\cdot; 0, \frac{1}{n}, \dots, 1)$ for the monic polynomial with zeros at the nodes $\{\frac{k}{n}\}_{k=0,1,\dots,n}$, that is

$$\gamma_{n+1}(z) = \prod_{m=0}^{n}\left(z - \frac{m}{n}\right). \tag{2.3}$$

2.2 Equidistant nodes

The following lemma has been already mentioned in the introduction. Assertion (i) is an easy consequence of the following explicit expression for the n^{th} divided difference:

$$[x_0\, x_1\, \cdots\, x_n]f = \sum_{k=0}^{n}\frac{f(x_k)}{\prod\limits_{\substack{m=0\\ m\neq k}}^{n}(x_k - x_m)} \tag{2.4}$$

(see [6], section 1.4.2, or [10], section 1.9). Assertion (ii) gives yet another expression (2.5) for $\Delta_n f$.

Lemma 2.1. *(i) In the special case of the equidistant nodes $\{\frac{k}{n}\}_{k=0,1,\dots,n}$ the divided differences take the form (1.4).*

(ii) Put $u_0 \equiv 0$ and for $n \geq 1$ let $u_n = (u_{n1} + \cdots + u_{nn})/n$ be the arithmetic mean of independent random variables $u_{n1}, \dots, u_{nn}$, which are uniformly distributed in the interval $[0,1]$. If for some $n \in \mathbb{N}$ the function f possesses an integrable n^{th} derivative, denoted by $f^{(n)}$, then

$$\Delta_n f = \frac{1}{n!}\,\mathbb{E}f^{(n)}(u_n). \tag{2.5}$$

Proof. *(i)* Note that in the present special case the product in the denominator on the right hand side of (2.4) equals to

$$\frac{1}{n^n}\prod_{\substack{m=0\\m\neq k}}^{n}(k-m)=(-1)^{n-k}\frac{n!}{n^n}\binom{n}{k}^{-1}$$

so that (2.4) reduces to

$$[0\,\frac{1}{n}\,\cdots\,1]f=\frac{n^n}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f\left(\frac{k}{n}\right).$$

In view of (1.4) the proof of assertion (i) is complete.
(ii) Clearly the desired equality (2.5) is true for $n=0$. For $n\geq 1$, it may be written in the following integral form

$$\Delta_n f=\frac{1}{n!}\int_0^1\cdots\int_0^1 f^{(n)}\left(\frac{x_1+\cdots+x_n}{n}\right)dx_1\cdots dx_n.$$

But this is equivalent to

$$\Delta_n f=\frac{n^n}{n!}\int_0^{\frac{1}{n}}\int_{t_1}^{t_1+\frac{1}{n}}\cdots\int_{t_{n-1}}^{t_{n-1}+\frac{1}{n}} f^{(n)}(t_n)\,dt_n\cdots dt_1$$

which we get from (1.4) by the Newton-Leibniz formula, see [15], p 165. □

Let us consider several applications.

Example 1
Let f be the exponential function $f:x\mapsto e^{ax}$, with a certain constant a, cf (1.7). Obviously, $\Delta_0 f=1$. As $m>0$, apply (1.4) to get

$$\Delta_m f=\frac{m^m}{m!}\,(e^{\frac{a}{m}}-1)^m. \tag{2.6}$$

Example 2
Consider the kernel $f:x\mapsto\frac{1}{a-x}$, $|a|>1$. Obviously, $\Delta_0 f=\frac{1}{a}$ and for $m>0$

$$\Delta_m f=\frac{1}{2\pi i}\int_C\frac{1}{a-z}\,\frac{1}{\gamma_{m+1}(z)}\,dz=\frac{1}{\gamma_{m+1}(a)} \tag{2.7}$$

with γ_{m+1} given by (2.3). This is easily obtained from the integral representation (1.3) by applying Cauchy's integral formula (see [14], vol. I, chapter 14).

Example 3
For a nonnegative integer n, let e_n denote a monomial, i.e. a special function $e_n:x\mapsto x^n$. In this section a relationship will be discussed between the numbers

$$d_{nm}=\Delta_m e_n \tag{2.8}$$

and the *Stirling numbers of the second kind.* As is well-known (see for example [10], p 175-176) these Stirling numbers $\mathcal{S}_{nm}$ are usually defined either by the generating function

$$\frac{1}{m!}\,(e^t-1)^m = \sum_{n=m}^{\infty} \mathcal{S}_{nm}\,\frac{t^n}{n!} \tag{2.9}$$

or

$$\frac{t^m}{(1-t)(1-2t)\cdots(1-mt)} = \sum_{n=m}^{\infty} \mathcal{S}_{nm}\,t^n. \tag{2.10}$$

In order to obtain the generating function for the numbers (2.8), apply the functional Δ_m to two absolutely convergent Taylor series (of the same functions of $x \in [0,1]$ as in the previous examples)

$$e^{ax} = \sum_{n=0}^{\infty} \frac{a^n x^n}{n!} \quad \text{and} \quad \frac{1}{a-x} = \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{x}{a}\right)^n$$

with a parameter $a \in \mathbb{R}$ in the first case and $|a| > 1$ in the second case. The absolute convergence allows for the term by term application of Δ_m, so that in the first case we get for $m > 0$

$$\frac{m^m}{m!}\,(e^{\frac{a}{m}}-1)^m = \sum_{n=m}^{\infty} d_{nm}\,\frac{a^n}{n!} \tag{2.11}$$

by using (2.6), and in the second case

$$\frac{1}{\gamma_{m+1}(a)} = \sum_{n=m}^{\infty} \frac{d_{nm}}{a^{n+1}} \tag{2.12}$$

by using (2.7). Compare (2.9) with (2.11) and (2.10) with (2.12). It is seen that $\mathcal{S}_{nm} = m^{n-m} d_{nm}$ (see [9], sections 5.2 and 5.3, for more details concerning the numbers (2.8)). We only mention here the following simple estimate:

$$d_{nm} \le \binom{n}{m} \quad \text{as } m \le n, \tag{2.13}$$

which is easily obtained by applying lemma 2.1, assertion (ii), to the monomial and by taking into consideration definition (2.8). Indeed, we get

$$d_{nm} = \binom{n}{m}\,\mathbb{E}u_m^{n-m}$$

which in turn yields (2.13), since the arithmetic means u_n are confined to $[0,1]$. Clearly, $d_{nm} = 0$ as $m > n$ due to (2.1).

Example 4

Let $\{\gamma_n\}_{n=1,2,\ldots}$ be the sequence of the monic polynomials γ_n of degree n defined in (2.3). Put $\gamma_0 \equiv 1$, for completeness. We associate with this sequence of polynomials a triangular scheme of numbers $\{b_{nk}\}_{k=0,1,\dots,n}$ with the entries $b_{nk} = \Delta_k \gamma_n$ in the n^{th} row, $n = 0, 1, \dots,$ given by

$$b_{nk} = \frac{1}{2\pi i} \int_C \frac{\gamma_n(z)}{\gamma_{k+1}(z)} dz, \tag{2.14}$$

cf (1.3). Obviously, $b_{n0} = \gamma_n(0) = \delta_{n0}$ and $b_{n1} = \gamma_n(1) - \gamma_n(0) = \delta_{n1}$. Moreover, examining the entries diagonally we observe 1's on the main diagonal $\{b_{nn}\}_{n=0,1,\dots}$ (apply (2.1) that also allows us to view our triangular scheme as a lower triangular matrix, if needed) and 0's on the first diagonal $\{b_{n+1\,n}\}_{n=0,1,\dots}$. The latter is directly seen from (2.2) which gives at once $b_{kn+1\,n} = 0$ for $k = 1, 2, \dots$. Furthermore, for an integer $m > 0$ consider the $m+1^{th}$ diagonal with the entries $\{b_{n+m+1\,n}\}_{n=0,1,\dots}$ and apply again (2.2). Along with $b_{m+1\,0} = 0$ and $b_{m+2\,1} = 0$, we obtain for $n > 1$ that

$$b_{n+m+1\,n} = \left(\frac{m}{n+m}\right)^{m+1} \sum_{0<k_0\le\cdots\le k_m<n} \gamma\left(\frac{k_0}{n}, \dots, \frac{k_m}{n}\right) \tag{2.15}$$

where γ is a function of $m+1$ variables given by

$$\gamma(z_0, \dots, z_m) = \prod_{j=0}^{m} \left(z_j - \frac{j}{m}\right). \tag{2.16}$$

Note that $\gamma(z, \dots, z) = \gamma_{m+1}(z)$, cf (2.3). So, in case $n = 2$, for instance, the right hand side of (2.15) consists of a single term that yields

$$b_{m+2\,2} = \left(\frac{m-1}{m+1}\right)^m \gamma_m\left(\frac{1}{2}\right) = \begin{cases} 0 & \text{if } m \text{ is odd} \\ \frac{(-1)^{m/2}}{\pi} \left(\frac{1}{m+1}\right)^m \Gamma^2\left(\frac{m+1}{2}\right) & \text{if } m \text{ is even} \end{cases}$$

(this is not hard to see by evaluating the gamma function as in [11], formula (1.2.6), but we do not enter into details).

We will show in proposition 2.2 that all the odd diagonals consist of 0's, not only the first one as above. As for the even diagonals, their entries will be estimated by the inequality (2.17), that is all we need for the present purposes. For comparison, however, observe that the sum on the right hand side of (2.15) is an approximation to the following integral evaluated at $z = 1$:

$$I_{m+1}(z) = m^{m+1} \int_0^z \int_0^{z_m} \cdots \int_0^{z_1} \gamma(z_0, \dots, z_m) dz_0 \cdots dz_m.$$

For example $I_2(1) = -\frac{1}{4!}$, $I_4(1) = -\frac{3^2\,7}{8!}$, $I_6(1) = -\frac{5^2\,11\,13\,15}{12!}$, and the exact values of $\{b_{n+m\,n}\}_{m=2,4,6}$ may be obtained from

$$b_{n+m\,n} = I_m(1) \prod_{j=1}^{m-1} \left(1 - \frac{c_{mj}}{n(n+m-1)}\right)$$

where $c_{21} = 2$, $\{c_{41}, c_{42}, c_{43}\} = \{18, 4, -2\}$ and $\{c_{61}, \dots, c_{65}\} = \{\frac{200}{13}, 50, 6, -4, -6\}$.

Proposition 2.2. *Let $\{b_{nk}\}_{k=0,1,\dots,n}$ with $n = 0, 1, \dots$ be the triangular scheme defined by (2.14). Then for $m = 1, 3, \dots$ we have*

(i) $b_{n+m\,n} = 0$, i.e. all the odd diagonals consist of 0's;

(ii) the estimates to the entries in the even diagonals are provided by the inequality

$$|b_{n+m+1\,n}| \leq \frac{1}{m+1} \prod_{j=1}^{m+1} \left(1 - \frac{j}{n+m}\right). \tag{2.17}$$

The proof of this proposition is rather technical. Therefore we postpone it untill section 4.2.

3. Main results

3.1 Basic polynomials

The basis for the solution to our interpolation problem (to be formulated below in proposition 3.2) is provided by a sequence of certain monic polynomials $\{c_n\}_{n=0,1,\dots}$ that will be described next.

For each $m \in \{0, 1, \dots, n\}$ define a polynomial c_m of degree m, subject to the following conditions:

$$\Delta_k c_m = \begin{cases} 0 & \text{if } k \neq m \\ 1 & \text{if } k = m. \end{cases} \tag{3.1}$$

It is easily verified that in this manner polynomials $\{c_n\}_{n=0,1,\dots}$ are uniquely defined and may be presented in the following form:

$$c_n = \begin{vmatrix} d_{00} & & \bigcirc & e_0 \\ \vdots & \ddots & & \vdots \\ d_{n-1\,0} & \cdots & d_{n-1\,n-1} & e_{n-1} \\ d_{n0} & \cdots & d_{n\,n-1} & e_n \end{vmatrix} \tag{3.2}$$

where the notations are used of *example 3* in section 2.2. Indeed, apply the linear functionals $\{\Delta_k\}_{k=0,1,\dots,n}$ to both sides of (3.2) by taking into consideration that on the right we have a linear combination of the monomials $\{e_k\}_{k=0,1,\dots,n}$. Then (2.8) yields (3.1). Equally easy to verify that for all nonnegative integers n

$$e_n = \sum_{m=0}^{n} d_{nm}\, c_m. \tag{3.3}$$

Thus starting from $c_0 \equiv 1$, we have the following recurrence relationship

$$c_n = e_n - \sum_{m=0}^{n-1} d_{nm}\, c_m$$

since $d_{nn} = 1$. Note that the polynomial c_n is monic. For more details on these polynomials we refer to [3], or [9], section 5.

Obviously, the monomials in (3.2) may be substituted by any other monic polynomials of equal degree. For instance, with the notations of *example 4* in section 2.2

$$c_n = \begin{vmatrix} b_{00} & & \bigcirc & \gamma_0 \\ \vdots & \ddots & & \vdots \\ b_{n-1\,0} & \cdots & b_{n-1\,n-1} & \gamma_{n-1} \\ b_{n0} & \cdots & b_{n\,n-1} & \gamma_n \end{vmatrix} \tag{3.4}$$

so that analogously to (3.3) we have $\gamma_0 = c_0$, $\gamma_1 = c_1$ and

$$\gamma_{2n} = \sum_{m=1}^{n} b_{2n\,2m}\, c_{2m}, \quad \gamma_{2n+1} = \sum_{m=1}^{n} b_{2n+1\,2m+1}\, c_{2m+1}$$

as $n = 1, 2, \ldots$ (since certain coefficients vanish according to proposition 2.2, assertion (i)).

As is seen in [3] or [9], section 5, the first few polynomials c_n are of simple form due to easily localized zeros. Apart from $c_0 \equiv 1$, first few instances are

$$\begin{aligned} c_1(x) &= x \\ c_2(x) &= x(x-1) \\ c_3(x) &= x(x-\frac{1}{2})(x-1) \\ c_4(x) &= x(x-\frac{1}{2})^2(x-1) \\ c_5(x) &= x(x-\frac{1}{3})(x-\frac{1}{2})(x-\frac{2}{3})(x-1) \\ c_6(x) &= x(x-\frac{1}{4})(x-\frac{1}{2})^2(x-\frac{3}{4})(x-1). \end{aligned}$$

But after awhile they take on quite complicated form and their estimation becomes a complicated task. For instance

$$\begin{aligned} c_7(x) &= c_5(x)(x^2-x+\frac{8}{45}) \\ c_8(x) &= c_6(x)(x^2-x+\frac{25}{144}) \\ c_9(x) &= c_5(x)(x^4-2x^3+\frac{5\cdot 17}{3^2\cdot 7}x^2-\frac{2\cdot 11}{3^2\cdot 7}x+\frac{2^3\cdot 1301}{3^4\cdot 5^4\cdot 7}) \\ c_{10}(x) &= c_6(x)(x^4-2x^3+\frac{43}{2^5}x^2-\frac{11}{2^5}x+\frac{5\cdot 7\cdot 11}{2^9\cdot 3^3}). \end{aligned}$$

Note the symmetry: for all $n \geq 2$

$$c_n(x) = (-1)^n c_n(1-x), \tag{3.5}$$

see [9], proposition 5.6. In [9], proposition 5.22, a uniform (over $x \in [0,1]$) upper bound for c_n has been obtained (and in the concluding section 5.12 the possibilities for further refinement have been discussed; cf [3], section 8). In the next proposition an improvement upon this bound is provided.

Proposition 3.1. *For a positive integer n*

$$\sup_{x\in[0,1]} |c_n(x)| \le \left(\frac{3}{2}\right)^{[n/2]}. \tag{3.6}$$

The proof is provided in section 4.4. As usual [ ] denotes the integral part so that $[\frac{n}{2}] = \frac{n}{2}$ if n is even and $[\frac{n}{2}] = \frac{n-1}{2}$ if n is odd.

3.2 Convergence of the interpolation series
Consider the interpolation problem formulated at the end of section 1.1. The corresponding interpolation polynomials $p_n(\cdot; f)$ are constructed as follows:

Proposition 3.2. *For each fixed integer n the polynomial of degree n*

$$p_n(\cdot; f) = \sum_{m=0}^{n} c_m(\cdot)\Delta_m f \tag{3.7}$$

with the basic polynomials $\{c_m\}_{m=0,1,\dots,n}$ introduced in section 3.1, satisfies the conditions (1.5).

Proof. Apply the linear functional Δ_k to both sides of (3.7) by taking into consideration (3.1). □

Evaluate (3.7) at a certain point $x \in [0, 1]$ and let $n \to \infty$. In the next theorem a convergence class of functions is characterized, i.e. a class of functions f for which the remainder term (1.6) evaluated at a certain point $x \in [0, 1]$, vanishes as $n \to \infty$ and the following expansion holds

$$f(x) = \sum_{n=0}^{\infty} c_n(x)\,\Delta_n f. \tag{3.8}$$

Theorem 3.3. *Let f be an analytic function whose power series expansion at $x \in [0, 1]$*

$$f(x) = \sum_{n=0}^{\infty} a_n\, x^n \tag{3.9}$$

is such that for a certain integer $k_0 \in \mathbb{N}$ the coefficients satisfy the inequality

$$|a_n| \le \frac{A^n}{n!} \quad \text{if} \quad n \ge k_0, \tag{3.10}$$

where $A > 0$ is a certain positive constant. Then f can be developed in series (3.8).

Proof. Since the power series (3.9) converges absolutely, we can apply the functional Δ_m term by term. By (2.8) we get

$$\Delta_m f = \sum_{n=m}^{\infty} a_n\, d_{nm}. \tag{3.11}$$

Now, substitute (3.3) in (3.9) and interchange the summation order to get

$$f = \sum_{n=0}^{\infty} a_n \sum_{m=0}^{n} d_{nm}\, c_m = \sum_{m=0}^{\infty} c_m \sum_{n=m}^{\infty} a_n\, d_{nm},$$

which by (3.11) implies the desired expansion (3.8). This interchange is justified, provided

$$\sum_{m=0}^{\infty} |c_m| \sum_{n=m}^{\infty} |a_n|\, d_{nm} < \infty. \tag{3.12}$$

It remains thus to prove (3.12). In view of the inequalities $|c_m| \le \left(\frac{3}{2}\right)^m$ and $d_{nm} \le \binom{n}{m}$ (cf (3.6) and (2.13)), it suffices to show that

$$\sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{3}{2}\right)^m \sum_{n=m}^{\infty} \frac{n!\,|a_n|}{(n-m)!} < \infty.$$

But under the present conditions the latter inequality is certainly true due to inequality (3.10). For instance

$$\sum_{m=k_0}^{\infty} \frac{1}{m!} \left(\frac{3}{2}\right)^m \sum_{n=m}^{\infty} \frac{n!\,|a_n|}{(n-m)!} \le \sum_{m=k_0}^{\infty} \frac{A^m}{m!} \left(\frac{3}{2}\right)^m \sum_{n=m}^{\infty} \frac{A^{n-m}}{(n-m)!} \le e^{\frac{5A}{2}}.$$

The proof is complete. □

3.3 Remarks

The convergence in (3.8) is in fact uniform in the sense that under the conditions of theorem 3.3

$$\sup_{x\in[0,1]} |r_n(x,f)| \to 0 \quad \text{as} \quad n \to \infty, \tag{3.13}$$

cf (1.6). Indeed, in view of theorem 3.3 the n^{th} remainder evaluated at fixed $x \in [0,1]$ may be written in the form

$$r_n(x,f) = \sum_{m=n+1}^{\infty} c_m(x)\Delta_m f = \sum_{m=n+1}^{\infty} c_m(x) \sum_{k=m}^{\infty} a_k\, d_{km}$$

by (3.11). Hence if n is sufficiently large, exceeding k_0 of condition (3.10), then

$$\sup_{x\in[0,1]} |r_n(x,f)| \le \sum_{m=n+1}^{\infty} \frac{A^m}{m!} (\frac{3}{2})^m \sum_{k=m}^{\infty} \frac{A^{k-m}}{(k-m)!} = e^A \sum_{m=n+1}^{\infty} \frac{A^m}{m!} (\frac{3}{2})^m$$

which yields (3.13).

As was mentioned in the introduction, in the special case of section 2.2, *example 1*, the conditions of theorem 3.3 are satisfied and the expansion (1.7) is valid for each $a \in \mathbb{R}$. But the case of *example 2* is not covered, hence the question remains open whether the corresponding n^{th} remainder term evaluated at $x \in [0,1]$ so that

$$r_n(x,f) = \frac{1}{a-x} - \sum_{m=0}^{n} \frac{c_m(x)}{\gamma_{m+1}(a)}$$

(cf (1.6) and (2.7)) vanishes as $n \to \infty$. Cf [9], section 5.5 (where only a certain "weak convergence" is established for an appropriate range of the parameter a).

3.4 Arithmetic-geometric mean relation
Let X be a binary random variable with the probability distribution described at the beginning of section 1.3. It is easy to characterize the arithmetic-geometric mean ratio in terms of the central moments of a Bernoulli random variable η that takes on either the value 1 with probability x or the value 0 with probability $1-x$. Indeed, in view of the explicit expressions for $\mathbb{E}X$ and $\mathbb{G}X$ given in section 1.3, we have

$$\frac{\mathbb{E}X}{\mathbb{G}X} = \mathbb{E}e^{a(\eta-\mathbb{E}\eta)} = \sum_{n=0}^{\infty} \frac{a^n}{n!}\chi_n(x) \tag{3.14}$$

where

$$\chi_n(x) \equiv \mathbb{E}(\eta-\mathbb{E}\eta)^n = x(1-x)^n + (-1)^n(1-x)x^n. \tag{3.15}$$

Obviously, $\chi_1 \equiv 0$. In many applications like in [4] or [8], however, the above development is unsuitable, instead a development in terms of fractional moments of X is required. In the next theorem such an expansion (1.8) is proved and the coefficients a_{nm} are determined. It will be seen, in particular, that the first few terms are

$$\begin{aligned} \mathbb{G}X = \mathbb{E}X &- \frac{2^2}{2!}\mathbb{E}_{22} + \frac{3^3}{3!}\frac{1}{2}\mathbb{E}_{33} - \frac{4^4}{4!}\left(\frac{1}{2}\right)^2\left[\mathbb{E}_{44} - \mathbb{E}_{42}^2\right] \\ &+ \frac{5^5}{5!}\frac{1}{2}\frac{1}{3}\frac{2}{3}\left[\mathbb{E}_{55} - \frac{5}{2}\mathbb{E}_{52}\,\mathbb{E}_{53}\right] \\ &- \frac{6^6}{6!}\left(\frac{1}{2}\right)^2\frac{1}{4}\frac{3}{4}\left[\mathbb{E}_{66} - \mathbb{E}_{62}\,\mathbb{E}_{64} - \frac{10}{3}\mathbb{E}_{63}^2\right] + \ldots \end{aligned} \tag{3.16}$$

with the same $\mathbb{E}_{nm} = \mathbb{E}[(X^{\frac{1}{n}} - \mathbb{E}X^{\frac{1}{n}})^m]$ as in (1.8) that indeed involves only fractional moments of X.

Theorem 3.4. *Let the random variable X take on either the value e^a with probability x or the value 1 with probability $1-x$. Then the expansion (1.8) holds with the constants a_{nm} that are expressed in terms of the coefficients of the polynomials c_n as follows: for each $n \geq 2$, write c_n in the form*

$$c_n(x) = \sum_{k=1}^{n-1} p_{nk}(x-1)^k x^{n-k}. \tag{3.17}$$

Then the numbers a_{nm} are completely determined by the following identities: For $n \geq 2$

$$(-1)^{n+1}a_{n0} = p_{n1} \tag{3.18}$$

and for $n \geq 4$

$$(-1)^{n+1}a_{nm} = \begin{cases} p_{n\,m-1} - 2p_{nm} + p_{n\,m+1} & \text{if } 2 \leq m \leq [\frac{n}{2}]-1 \\ p_{n\,[\frac{n}{2}]-1} - 2p_{n\,[\frac{n}{2}]} + p_{n1} & \text{if } m = [\frac{n}{2}] \end{cases} \tag{3.19}$$

Note that for $n \geq 2$ each polynomial c_n can be written in the form (3.17) (which is sometimes called Bernstein's expansion), with the coefficients p_{nk} so that $p_{nk} = p_{n\,n-k}$ due to the symmetry property (3.5) and to the fact that both 0 and 1 are zeros of c_n. Then the polynomial $c_n^*(x) = x^n c_n(x^{-1})$, reciprocal to c_n, takes the form

$$c_n^*(x) = \sum_{k=1}^{n-1} p_{nk}(1-x)^k. \tag{3.20}$$

Proof of theorem 3.4. Since $c_0(x) = 1$ and $c_1(x) = x$, as we already know, the expressions for the arithmetic and geometric mean and expansion (1.7) allow us to equate the difference $\mathbb{G}X - \mathbb{E}X$ to the series $\sum_{n=2}^{\infty} \frac{n^n}{n!} c_n(x)\,(e^{\frac{a}{n}} - 1)^n$. Compare the result with (1.8) term by term. It is then seen that the following equality has to be satisfied for $n \geq 2$

$$\sum_{\substack{m=0 \\ m\neq 1}}^{[\frac{n}{2}]} a_{nm}\mathbb{E}_{nm}\,\mathbb{E}_{n\,n-m} = c_n(x)\,(e^{\frac{a}{n}} - 1)^n \tag{3.21}$$

with a suitable choice of the constants a_{nm}. For $n = 2$, for instance, the left hand side equals $a_{20}\,\mathbb{E}_{22}$ with $\mathbb{E}_{22} = x(1-x)(e^{\frac{a}{2}} - 1)^2$, so that $a_{20} = -1$ since $c_2(x) = x(x-1)$.

In general, all the central moments for $n \geq 2$ involved in (3.21) are easily determined: $\mathbb{E}_{nm} = \chi_m(x)(e^{\frac{a}{n}} - 1)^m$, with 1 on both sides if $m = 0$, cf (3.15). Hence the equality (3.21) is reduced to

$$c_n(x) = \sum_{m=0}^{[\frac{n}{2}]} a_{nm}\,\chi_{nm}(x) \quad \text{as} \quad n \geq 2, \tag{3.22}$$

with the polynomials $\chi_{n0} \equiv \chi_n$ and $\chi_{nm} \equiv \chi_n\chi_{n-m}$ for $2 \leq m \leq [\frac{n}{2}]$. Note that for each $m = 0, 2, \ldots, [\frac{n}{2}]$ the polynomials χ_{nm} in the expansion (3.22) are of degree n and they possess the same symmetry property (3.5) as c_n. Their reciprocals $\chi_{nm}^*(x) = x^n\chi_{nm}(x^{-1})$ are

$$\chi_{nm}^*(x) = \frac{(-1)^n}{x^2}[(1-x)^m - (1-x)][(1-x)^{n-m} - (1-x)].$$

Hence, (3.22) rewritten in terms of reciprocals, yields

$$\begin{aligned}
(-1)^n c_n^*(x) &= \frac{1}{x^2}\sum_{0}^{[\frac{n}{2}]} a_{nm}[(1-x)^m - (1-x)][(1-x)^{n-m} - (1-x)] \\
&= -a_{n0}\sum_{k=1}^{n-1}(1-x)^k + \sum_{m=2}^{[\frac{n}{2}]} a_{nm}\sum_{k=1}^{m-1}\sum_{l=1}^{n-m-1}(1-x)^{k+l}.
\end{aligned}$$

Compare the coefficients in this expression with that of (3.20). We get (3.18), as well as the following matrix identity $P = (-1)^n ALL^\top$ for $n \geq 4$, where the square lower triangular matrix L has the entries $L_{ij} = 1$ if $i \geq j$ and 0 elsewhere, $L^\top$ is its transpose, while P and A are rows with entries $\{p_{n2} - p_{n1}, \ldots, p_{n\,[\frac{n}{2}]} - p_{n1}\}$ and $\{a_{n2}, \ldots, a_{n\,[\frac{n}{2}]}\}$, respectively. This identity yields (3.19), since the inverse L^{-1} is the lower triangular matrix with only non-zero entries $L_{ii} = 1$ and $L_{i+1\,i} = -1$. □

We have shown that expansion (1.8) with the constants a_{nm} determined by (3.22) is identical to series (1.7) and hence, according to the remark in secton 3.3 concerning this series, the convergence of the remainder term in (1.8) is uniform in x and a.

3.5 Hellinger integrals

In this section we will give another application of series (1.7), namely we will prove expansion (1.12) of the Hellinger integral (1.11).

Theorem 3.5. *For two positive probability density functions f and g the n^{th} remainder term in expansion (1.12)*

$$R_n(x) = H_x(f,g) - 1 - \sum_{m=1}^{n} \frac{m^m}{m!}\, c_m(x)\, h_m(f,g)$$

vanishes as $n \to \infty$, in the sense that $\sup_{x\in[0,1]} |R_n(x)| \to 0$.

Proof. Substitute $a = \log f(t) - \log g(t)$ in (1.7) to conclude by uniform convergence of this series (cf section 3.3) that

$$\sup_{x\in[0,1],\, t\in\mathbb{R}} |u(t,x) - u_n(t,x)| \to 0 \quad \text{as} \quad n \to \infty \tag{3.23}$$

with $u(t,x) = [f(t)/g(t)]^x$ and $u_n(t,x) = 1 + \sum\limits_{m=1}^{n} \frac{m^m}{m!}\, c_m(x) \left(u(t,\frac{1}{m})^{\frac{1}{m}} - 1\right)^m$. This yields the desired result, since by (1.9) and (1.11)

$$\sup_{x\in[0,1]} |R_n(x)| = \sup_{x\in[0,1]} \left| \int_{-\infty}^{\infty} [u(t,x) - u_n(t,x)]\; g(t)\; dt \right|$$

does not exceed the left hand side of (3.23) and thus tends to 0. □

4. Some estimates

4.1 Basic lemma

The proof of proposition 2.2 to be presented in section 4.2 is based on the following lemma concerning an absolute upper bound for the summands in (2.15).

Lemma 4.1. *For a positive integer m, let γ be a function of $m+1$ variables given by (2.16). Then*

$$\sup_{0<z_0\le\cdots\le z_m<1} |\gamma(z_0,\ldots,z_m)| \le \frac{m!}{m^{m+1}}. \tag{4.1}$$

Proof. For a notational convenience, let the interval $[0,1]$ be divided into m subintervals $L_1,\ldots,L_m$ of equal width, that is $L_k = [\frac{k-1}{m}, \frac{k}{m}]$. Fix a particular distribution of successive points $0 < z_0 \le \cdots \le z_m < 1$ over these subintervals that is determined in the following manner. Select non-empty subintervals, say $L_{i_1}, \dots, L_{i_\ell}$ with $i_1 < \cdots < i_\ell$, $\ell \le m$, and assume each individual subinterval L_{i_k} to cover ν_k of the points. Clearly $\nu_1 + \cdots + \nu_\ell = m+1$. The points thus covered by L_{i_k} satisfy

$$\frac{i_k - 1}{m} < z_{n_{k-1}} \le \cdots \le z_{n_k - 1} < \frac{i_k}{m}, \tag{4.2}$$

where $n_0 = 0$ and $n_k = \nu_1 + \cdots + \nu_k$ for $0 < k \le \ell$ so that $n_\ell = m+1$.

In order to estimate

$$|\gamma(z_0, \dots, z_m)| = \prod_{j=0}^{m} |z_j - \frac{j}{m}| = \prod_{k=1}^{\ell} \prod_{z_j \in L_{i_k}} |z_j - \frac{j}{m}|, \tag{4.3}$$

we first obtain the inequality

$$m^{m+1} |\gamma(z_0, \dots, z_m)| \le \prod_{k=1}^{\ell} \prod_{n_{k-1} \le j < n_k} \left\{ (j - i_k + 1) I_{\{i_k \le j\}} + (i_k - j) I_{\{i_k > j\}} \right\} \tag{4.4}$$

by taking into consideration that due to the inequalities (4.2) the inner product on the right hand side of (4.3) (over the set of points $\{z_j \in L_{i_k}\}$, i.e. over the set of integers $\in [n_{k-1}, n_k)$) satisfies

$$m^{\nu_k} \prod_{n_{k-1} \le j < n_k} |z_j - \frac{j}{m}| \le \prod_{n_{k-1} \le j < n_k} \left\{ (j - i_k + 1) I_{\{i_k \le j\}} + (i_k - j) I_{\{i_k > j\}} \right\}.$$

We either have $\{i_k < n_{k-1}\}$ or $\{n_{k-1} \le i_k < n_k\}$ or $\{i_k \ge n_k\}$. Consider all these three cases separately. In the first case the product on the right hand side of the latter inequality equals $\nu_k! \binom{n_k - i_k}{\nu_k}$, in the second case $\nu_k! / \binom{\nu_k}{n_k - i_k}$, in the third case $\nu_k! \binom{i_k - n_{k-1}}{\nu_k}$. Thus, the inequality (4.4) reduces to

$$\frac{m^{m+1}}{m!} |\gamma(z_0, \dots, z_m)| \le \frac{\nu_1! \cdots \nu_\ell!}{m!} \Pi^{i_1 \cdots i_\ell}_{\nu_1 \cdots \nu_\ell} \tag{4.5}$$

with

$$\Pi^{i_1 \cdots i_\ell}_{\nu_1 \cdots \nu_\ell} = \prod_{k=1}^{\ell} \left\{ \binom{n_k - i_k}{\nu_k} I_{\{i_k < n_{k-1}\}} + \frac{I_{\{n_{k-1} \le i_k < n_k\}}}{\binom{\nu_k}{n_k - i_k}} + \binom{i_k - n_{k-1}}{\nu_k} I_{\{i_k \ge n_k\}} \right\}.$$

Compare this with the desired inequality (4.1). As is easily seen, it remains to prove that the product on the right hand side of (4.5) is less then 1, i.e.

$$\Pi^{i_1 \cdots i_\ell}_{\nu_1 \cdots \nu_\ell} \le \frac{m!}{\nu_1! \cdots \nu_\ell!} = \frac{1}{m+1} \binom{m+1}{\nu_1 \, \cdots \, \nu_\ell}. \tag{4.6}$$

As is shown below, the equality in (4.6) is attained only in the special case of $\ell = 1$, when all the $\nu_1 = m+1$ points are covered by the subinterval L_{i_1} with some $i_1 \in \{1, \dots, m\}$, i.e. $\frac{i_1 - 1}{m} < z_0 \le \cdots \le z_m < \frac{i_1}{m}$. In this case $\Pi^{i_1}_{\nu_1} = 1 / \binom{\nu_1}{i_1}$ and inequality (4.6) is equivalent to $\binom{m+1}{i_1} \ge m+1$. Clearly, this holds for all $i_1 \in \{1, \dots, m\}$, with the equality only when $i_1 = 1$ or $i_1 = m$.

To make our arguments transparent, we treat in details the next special case $\ell = 2$ when all the points are distributed over two subintervals L_{i_1} and L_{i_2} with some $i_1 < i_2$ from $\{1, \dots, m\}$ and $n_2 = m+1$. In this case $\Pi^{i_1\, i_2}_{\nu_1\, \nu_2}$ is the product of $\frac{I_{\{i_1<\nu_1\}}}{\binom{\nu_1}{\nu_1-i_1}} + \binom{i_1}{\nu_1} I_{\{i_1 \geq \nu_1\}}$ and $\binom{n_2-i_2}{\nu_2} I_{\{i_2<\nu_1\}} + \frac{I_{\{\nu_1\leq i_2\}}}{\binom{\nu_2}{n_2-i_2}}$, hence it takes on three possible values: either $\binom{m+1-i_2}{\nu_2}/\binom{\nu_1}{i_1}$ as $\nu_1 > i_2$, or $1/\binom{\nu_1}{i_1}\binom{\nu_2}{i_2-\nu_1}$ as $i_1 < \nu_1 \leq i_2$, or $\binom{i_1}{\nu_1}/\binom{\nu_2}{i_2-\nu_1}$ as $\nu_1 \leq i_1$. Note the symmetry $\Pi^{i_1\, i_2}_{\nu_1\, \nu_2} = \Pi^{j_2\, j_1}_{\nu_2\, \nu_1}$ with $j_1 = m+1-i_1$ and $j_2 = m+1-i_2$, which means that on looking for an upper bound of $\Pi^{i_1\, i_2}_{\nu_1\, \nu_2}$ one has to take into account only the first possibility (for the third possibility would give the same result and the second one is irrelevant as the value taken on is always less then 1). In the present case the desired inequality

$$\Pi^{i_1\, i_2}_{\nu_1\, \nu_2} \leq \frac{1}{m+1}\binom{m+1}{\nu_1} \tag{4.7}$$

(cf (4.6)) is thus proved by verifying that if $\nu_1 > i_2$, then the ratio

$$\frac{m+1}{\binom{\nu_1}{i_1}} \frac{\binom{m+1-i_2}{\nu_2}}{\binom{m+1}{\nu_2}} = \frac{\nu_1 - i_2 + 1}{\binom{\nu_1}{i_1}} \frac{(m-i_2+1)_{\nu_2-1}}{(m)_{\nu_2-1}} < 1 \tag{4.8}$$

(with $(\cdot)_r$ the falling r-factorial, cf [10], §16). We hence see that in (4.7) the equality is excluded and the ratio is largest for $\Pi^{1\,2}_{m\,1} = 1 - \frac{1}{m} < 1$.

The arguments used above extend beyond the cases $\ell = 1, 2$ as follows. Fix a certain $\ell \in \{3, \dots, m\}$. Firstly, it suffices to deal only with the situation in which $i_2 < n_1, i_3 < n_2, \dots, i_\ell < n_{\ell-1}$, and the product $\Pi^{i_1\, \cdots\, i_\ell}_{\nu_1\, \cdots\, \nu_\ell}$ takes on the value

$$\Pi^{i_1\, \cdots\, i_\ell}_{\nu_1\, \cdots\, \nu_\ell} = \frac{1}{\binom{\nu_1}{i_1}} \prod_{k=2}^{\ell} \binom{n_k - i_k}{\nu_k}. \tag{4.9}$$

Secondly, it is verified that this value satisfies the inequality

$$\Pi^{i_1\, \cdots\, i_\ell}_{\nu_1\, \cdots\, \nu_\ell} \leq \frac{1}{m+1} \prod_{k=2}^{\ell} \binom{n_k}{\nu_k} = \frac{1}{m+1}\binom{m+1}{\nu_1\ \cdots\ \nu_\ell} \tag{4.10}$$

(cf (4.6)). To this end, we follow the same arguments as in (4.8): taking the ratio of the right hand sides in (4.9) and (4.10), we get the inequality

$$\frac{m+1}{\binom{\nu_1}{i_1}} \prod_{k=2}^{\ell} \frac{\binom{n_k-i_k}{\nu_k}}{\binom{n_k}{\nu_k}} = \frac{\nu_1 - i_2 + 1}{\binom{\nu_1}{i_1}} \prod_{k=3}^{\ell} \frac{n_{k-1} - i_k + 1}{n_{k-1}} \prod_{k=2}^{\ell} \frac{(n_k - i_k)_{\nu_k - 1}}{(n_k - 1)_{\nu_k - 1}} < 1.$$

Finally, note that the ratio is largest for $\{i_1, i_2, \dots, i_\ell\} = \{1, 2, \dots, \ell\}$ and $\{\nu_1, \nu_2, \dots, \nu_\ell\} = \{m+2-\ell, 1, \dots, 1\}$, when the inequality (4.10) reduces to

$$\Pi^{i_1\, \cdots\, i_\ell}_{\nu_1\, \cdots\, \nu_\ell} = \frac{(m+1-\ell)^{\ell-1}}{m+2-\ell} < (m)_{\ell-2}.$$

We have thus seen that the inequality (4.6) holds for all $\ell \in \{1, \dots, m\}$. The proof is complete. □

4.2 Proof of proposition 2.2

Proof of assertion (i). By definition (2.14), the case $m = 1$ is trivial. For $m = 3, 5, \ldots$ the assertion is verified by taking into consideration that

$$\gamma(z_0, \ldots, z_m) = (-1)^{m+1}\gamma(1 - z_m, \ldots, 1 - z_0).$$

Indeed, on the right hand side of (2.15) the sum consists of an equal number of positive and negative terms cancelling each other. Notice that the number of non-zero terms is always even, while the total number equals to $\binom{n+m-1}{m+1}$. This is calculated as follows:

$$\sum_{0<k_0\leq\cdots\leq k_m<n} 1 = \binom{n+m-1}{m+1} \tag{4.11}$$

by applying repeatedly the combinatorial identity

$$\sum_{k=0}^{m} \binom{n+k}{n} = \binom{n+m+1}{n+1}. \tag{4.12}$$

Proof of assertion (ii). Taking into consideration the number (4.11) of terms on the right hand side of (2.15), we get by lemma 4.1 that

$$|b_{n+m+1\,n}| \leq \left(\frac{m}{n+m}\right)^{m+1} \binom{n+m-1}{m+1} \frac{m!}{m^{m+1}}$$

which is equivalent to (2.17). □

4.3 Estimation of the polynomials (2.3)

We conclude this section by the assertion that the right hand side of (4.1) may serve as an upper bound for γ_{m+1}, as well.

Lemma 4.2. *For a positive integer m, let γ_{m+1} be the monic polynomial of degree $m+1$ defined in (2.3). Then*

$$\sup_{z\in[0,1]} |\gamma_{m+1}(z)| \leq \frac{m!}{m^{m+1}}.$$

Proof. It is directly verified that

$$\sup_{z\in[0,\frac{1}{m}]} |\gamma_{m+1}(z)| = \sup_{z\in[0,\frac{1}{m}]} z\prod_{j=1}^{m}\left(\frac{j}{m} - z\right) \leq \frac{m!}{m^{m+1}}.$$

Thus it suffices to show that the supremum we are looking for is located in $[0, \frac{1}{m}]$. But this follows from the fact that for any $z \in [0, \frac{1}{m}]$ and any $k \in \{1, \ldots, [m/2]\}$

$$\frac{|\gamma_{m+1}(\frac{k}{m} - z)|}{|\gamma_{m+1}(\frac{k}{m} + z)|} = \prod_{j=k+1}^{m-k}\left(1 + \frac{2z}{\frac{k}{m} - z}\right) \geq 1.$$

The proof is complete. □

4.4 Proof of proposition 3.1
For $2n > 0$ formula (3.4) yields

$$c_{2n} = \sum_{m=1}^{n} (-1)^m \beta_{nm} \gamma_{2m} \tag{4.13}$$

with the coefficients

$$\beta_{nm} = \begin{vmatrix} b_{2(m+1)\,2m} & b_{2(m+1)\,2(m+1)} & & \bigcirc \\ \vdots & \vdots & \ddots & \\ b_{2(n-1)\,2m} & b_{2(n-1)\,2(m+1)} & \cdots & b_{2(n-1)\,2(n-1)} \\ b_{2n\,2m} & b_{2n\,2(m+1)} & \cdots & b_{2n\,2(n-1)} \end{vmatrix} \tag{4.14}$$

which are estimated as follows:

Lemma 4.3. *For positive integers n and m, the determinant (4.14) is estimated as follows:*

$$|\beta_{n+m\,n}| \leq \frac{1}{2}\left(\frac{3}{2}\right)^{m-1}. \tag{4.15}$$

Proof. By developing the determinant (4.14) along the lowest row we get

$$\sum_{k=m}^{n} (-1)^k \, b_{2n\,2k} \, \beta_{km} = 0. \tag{4.16}$$

For each fixed $n > 0$ we have $\beta_{nn} = 1$ and $\beta_{n+1\,n} = b_{2(n+1)\,2n}$. Further, (4.16) yields for any $m > 1$

$$(-1)^{m-1}\beta_{n+m\,n} = b_{2(n+m)\,2n} + \sum_{k=1}^{m-1} (-1)^k \, b_{2(n+m)\,2(n+k)} \, \beta_{n+k\,n}. \tag{4.17}$$

For instance, substitute $m = 2$ to get

$$\beta_{n+2\,n} = -b_{2(n+2)\,2n} + b_{2(n+2)\,2(n+1)} \, b_{2(n+1)\,2n}.$$

We may use (4.17) iteratively so that for $m > 2$ the first step yields

$$\begin{aligned} (-1)^{m-1}\beta_{n+m\,n} = {} & b_{2(n+m)\,2n} - \sum_{k=1}^{m-1} b_{2(n+m)\,2(n+k)} \, b_{2(n+k)\,2n} \\ & - \sum_{k_2=2}^{m-1} \sum_{k_1=1}^{k_2-1} (-1)^{k_1} \, b_{2(n+m)\,2(n+k_2)} \, b_{2(n+k_2)\,2(n+k_1)} \, \beta_{n+k_1\,n}. \end{aligned}$$

For any $m > 1$, the final result of this procedure may be written in the following form:

$$\begin{aligned} (-1)^{m-1}\beta_{n+m\,n} = {} & b_{2(n+m)\,2n} \\ & + \sum_{0<j<m} (-1)^j \sum_{k_0<\cdots<k_{j+1}} \prod_{i=0}^{j} b_{2(n+k_{i+1})\,2(n+k_i)} \end{aligned} \tag{4.18}$$

where $k_0 = 0$ and $k_{j+1} = m$ so that the summation in $\sum_{k_0<\cdots<k_{j+1}}$ is taken j times, first $\sum_{k_1=1}^{k_2-1}$, then $\sum_{k_2=2}^{k_3-1}$ and so on, at last $\sum_{k_j=j}^{m-1}$. The number of summands equals

$$\sum_{k_0<\cdots<k_{j+1}} 1 = \sum_{k_j=j}^{m-1} \sum_{k_{j-1}=j-1}^{k_j-1} \cdots \sum_{k_1=1}^{k_2-1} 1 = \binom{m-1}{j} \tag{4.19}$$

which is calculated by applying repeatedly the combinatorial identity (4.12). Due to (4.19) and the inequality

$$|b_{2(n+k_{i+1})\,2(n+k_i)}| \leq \frac{1}{2}$$

(cf (2.17)), we obtain from (4.18) that

$$|\beta_{n+m\,n}| \leq \frac{1}{2} + \sum_{j=1}^{m-1} \left(\frac{1}{2}\right)^{j+1} \binom{m-1}{j}.$$

By the binomial formula the right hand side equals to that of (4.15). □

Proof of (3.6). Consider the case of an even index in (3.6). For $2n > 0$ we have (4.13) with the coefficients β_{nm} that satisfy the inequalities (4.15). Besides, $|\gamma_{2m}| \leq 1$ according to lemma 4.2. So, (4.13) implies

$$|c_{2n}| \leq \frac{1}{2} \sum_{k=0}^{n-1} \left(\frac{3}{2}\right)^k \leq \left(\frac{3}{2}\right)^n.$$

The case of odd indices in (3.6) is handled analogously. □

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