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ABSTRACT

Suppose we cover the set of vertices of a graph G by non-overlapping monomers (singleton sets) and dimers (pairs of vertices corresponding to an edge). Each way to do this is called a monomer-dimer configuration. If G is finite and $\lambda > 0$, we define the monomer-dimer distribution for G (with parameter λ) as the probability distribution which assigns to each monomer-dimer configuration a probability proportional to $\lambda^{|\text{dimers}|}$, where $|\text{dimers}|$ is the number of dimers in that configuration. If the graph is infinite, monomer-dimer distributions can be constructed in the standard way, by taking weak limits.

We are particularly interested in the monomer-dimer model on (subgraphs of) the d-dimensional cubic lattice. Heilmann and Lieb (1972) prove absence of phase transition, in terms of smoothness properties of certain thermodynamic functions. They do this by studying the location in the complex plane of the zeros of the partition function.

We present a different approach and show, by probabilistic arguments, that boundary effects become negligible as the distance to the boundary goes to ∞. This gives absence of phase transition in a related, but generally not equivalent sense as above. However, the decay of boundary effects appears to occur in such a strong way that, by results on general Gibbs systems of Dobrushin and Shlosman (1987) and Dobrushin and Warstat (1990), smoothness properties of thermodynamic functions follow. More precisely we show that, in their terminology, the model is $completely\ analytic$.

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1. Introduction

One of the purposes of this paper is to give a clear example of the use of 'disagreement percolation' arguments in the study of spatial dependencies in Markov fields. These arguments were introduced in [2], and further developed in [3] - [5] and [10]. There are no totally new ideas in this paper, so in some sense it is a review-like paper. However, although the *proof* of the main result, Theorem 1, is (at least in spirit) 'almost' in [2] and [5] and in [13], the result itself has not been explicitly observed before. This, plus the fact that this result and its proof method may shed some new light on the classical Heilmann-Lieb results, are the main motivation for this paper.

I have tried to write the paper in such a way that it has something interesting for experts as well as beginners in the field of Gibbs measures and phase transitions. As to the latter group of readers, although the paper is not self-contained and part of it assumes some general knowledge about Gibbs measures, I hope that it gives at least an idea of the kind of problems and methods in this field, and that it is entertaining enough to stimulate further study. To avoid confusion: this paper is *not* about the *pure dimer* model (in which monomers are not allowed), which has in some sense a more interesting behaviour (see [14]).

The monomer-dimer model originates from statistical physics: dimers represent molecules which occupy two adjacent vertices of a graph (crystal), and monomers molecules which occupy a single vertex (see the introduction of [11] for further background and references). Molecules are not allowed to overlap. The energy contribution of a dimer may be different from that of a monomer. By standard statistical-physics considerations it is then natural to assign to each monomer-dimer configuration a probability proportional to $\lambda_1^{|\text{monomers}|} \times \lambda_2^{|\text{dimers}|}$, where |monomers| and |dimers| denote the number of monomers and the number of dimers in the configuration respectively. Since the number of monomers plus twice the number of dimers equals the total number of vertices (hence does not depend on the configuration), the above is (assuming $\lambda_1 > 0$) exactly the same as assigning to each configuration a probability proportional to $\lambda^{|\text{dimers}|}$, where $\lambda = \lambda_2/\lambda_1^2$. The case $\lambda_1 = 0$ gives the pure dimer model which, as we mentioned before, will not be studied in this paper.

The monomer-dimer model also arises in operations research, combinatorial optimization and theoretical computer science. For instance, some vertices may represent tasks and the others persons, and the aim may be to match persons with tasks (not more than one task per person and vice versa). The edges indicate which person-task pairs are allowed. Since, for large λ, 'typical' configurations have many dimers, it makes sense to sample randomly from the monomer-dimer distribution with large λ in the hope to get a close to optimal matching. For rigorous results on how to do this 'efficiently' and related problems, see [12]. See also Remark 1 in Section 4.

More formally, the monomer-dimer model is described as follows: Let G be a finite graph. Denote the set of vertices of G by $V(G)$ and its set of edges by $E(G)$. We will often write just V and E respectively. If two vertices i and j are adjacent (i.e. share an edge), we write $i \sim j$. If e and e' are edges, $e \sim e'$ means that e and e' have a common endpoint. The *state space* Ω is given by $\Omega = \{0,1\}^E$. Elements of Ω are called configurations and will typically be denoted by $\omega (= (\omega_i, i \in E))$, α etc. To denote a randomly chosen element from Ω we will mostly use the notation σ. A configuration ω is called *feasible* if there are no edges e, e' with $e \sim e'$ and $\omega_e = \omega_{e'} = 1$.
Now consider the following probability distribution on Ω:

$$\mu(\omega) = \frac{\lambda^{|\omega|} I(\omega \text{ is feasible})}{Z}, \tag{1.1}$$

where $\lambda > 0$ is the parameter of the model, $|\omega|$ denotes $\sum_{e \in E} \omega_e$, and Z (which depends of course on λ) the normalizing constant (partition function). We call this the monomer-dimer distribution for G

with parameter λ.

It should be clear that this corresponds with the more informal description given before. For instance, if $\omega_e = 1$, this means that the pair of endpoints of e is covered by a dimer. If, for a vertex i, each edge e having i as one of its endpoints has $\omega_e = 0$, then i is covered by a monomer. Feasibility means that no molecules overlap, $|\omega|$ is the number of dimers, etc.

Before we go on we need some more notation. If $\Lambda \subset E$, Ω_Λ denotes $\{0,1\}^\Lambda$. If $\omega \in \Omega$, ω_Λ is the 'restriction' of ω to Λ, i.e. $\omega_\Lambda = (\omega_i, i \in \Lambda)$. Further, $\partial\Lambda$ denotes the boundary of Λ, that is, the set of all $e \in E \setminus \Lambda$ with $e \sim e'$ for some $e' \in \Lambda$. (If Λ has only one element, say i, we usually write ∂i instead of $\partial\{i\}$). If U and Λ are disjoint subsets of E, $\alpha \in \Omega_U$ and $\beta \in \Omega_\Lambda$, then $\alpha\beta$ denotes the 'concatenation' of α and β, i.e. the element of $\Omega_{U\cup\Lambda}$ for which $(\alpha\beta)_e$ equals α_e for all $e \in U$ and β_e for all $e \in \Lambda$. Finally, if A and B are sets, $A \subset\subset B$ means that A is a finite subset of B.

Now let G be an infinite, connected locally finite graph. Monomer-dimer distributions on G are defined in the standard way, as follows. Let $\lambda > 0$. Define, for each $\Lambda \subset E$ and each $\alpha \in \Omega_{\partial\Lambda}$ the monomer-dimer distribution for Λ with *boundary condition* α by

$$\mu_\Lambda^\alpha(\omega) = \frac{\lambda^{|\omega|}\, I(\omega \text{ is feasible w.r.t. } \alpha)}{Z_\Lambda}, \tag{1.2}$$

where the expression in the indicator function means that the configuration ω is feasible in the sense given before, and that, moreover, there are no $e \in \Lambda$ and $e' \in \partial\Lambda$ with $e \sim e'$ and $\omega_e = \alpha_e = 1$. A distribution μ on Ω is now called an (infinite-volume) monomer-dimer distribution for G (or infinite-volume Gibbs measure for the monomer-dimer model on G) if, for all $\Lambda \subset\subset E$, and all $\alpha \in \Omega_\Lambda$,

$$\mu(\sigma_\Lambda = \alpha \mid \sigma_e, e \in \Lambda^c) = \mu_\Lambda^{\sigma_{\partial\Lambda}}(\alpha),\ \mu - \text{a.s.}$$

Such distributions can be constructed in the standard way, as follows: take a nested sequence $\Lambda_1, \Lambda_2, \cdots$ of finite subsets of E with union E, and a sequence of boundary conditions $\alpha(i) \in \Omega_{\partial\Lambda_i}$ $i = 1, 2, \cdots$. Then take weak limits of subsequences of the sequence $\mu_1, \mu_2, \cdots$ on Ω, defined by

$$\mu_i(\sigma_{\Lambda_i} = \beta) = \mu_{\Lambda_i}^{\alpha(i)}(\beta),\ \beta \in \Omega_{\Lambda_i},\ i = 1, 2, \cdots, \text{ and}$$

$$\mu_i(\sigma_{\Lambda_i^c} \equiv 0) = 1,\ i = 1, 2, \cdots$$

A central question is whether there is a unique infinite-volume Gibbs measure. The above suggests that this is the case if, roughly speaking, the influence of the boundary condition becomes negligible as the boundary moves to infinity. This can indeed be stated in a precise way and is not difficult to prove. If the infinite-volume Gibbs measure is not unique, we say that there is a phase transition. An alternative notion of (absence of) phase transition is given in terms of smoothness properties of so-called thermodynamic functions, in particular the free energy function $\lim_{n\to\infty}(\log Z_{\Lambda_n})/|\Lambda_n|$ (this limit exists for sufficiently nice graphs and appropriate sequences (Λ_n)). For further study of these matters (for general Gibbs measures) see [9] and [8].

We will now concentrate on the d-dimensional cubic lattice, although many of the results hold in much more generality. This is the graph whose vertices are the elements of $\mathbf{Z}^d$ and were two vertices have an edge if their euclidian distance equals 1. With some abuse of notation we denote this graph simply by $\mathbf{Z}^d$.

Heilmann and Lieb (1972) proved in [11] (which has become a classical paper on the subject) absence of phase transition in terms of smoothness properties mentioned above. They did this by studying the zeros of the partition function (as a function on the complex plane). We follow a different, more probabilistic approach, and study the influence of boundary conditions. This is done by applying quite general percolation-like arguments introduced in [2]. In fact, the monomer-dimer model offers one of the nicest examples to illustrate these arguments: we will show that for every λ the above mentioned decay of boundary effects is so strong that one of the so-called *complete analyticity conditions* (namely

condition III_c in [7]) of Dobrushin and Warstat (1990) holds. To state this condition we need a little more terminology and notation. If $\Lambda \subset\subset E$, $\Delta \subset \Lambda$ and $\alpha \in \Omega_{\partial\Lambda}$, $\mu^{\alpha}_{\Lambda,\Delta}$ denotes the marginal distribution on Ω_Δ of μ^{α}_{Λ}. Recall that if μ and ν are two probability distributions on a finite set A, then the *variational distance* between μ and ν is defined by $\mathrm{Var}(\mu,\nu) = 1/2 \sum_{i\in A} |\mu(i) - \nu(i)|$. This is also equal to $\max_{B\subset A} |\mu(B) - \nu(B)|$. The complete analyticity condition mentioned above can then be formulated as follows:

Dobrushin-Warstat condition III_c *There exist K and $\kappa > 0$ such that for all $\Lambda \subset\subset E$, $\Delta \subset \Lambda$, $e \in \partial\Lambda$, and all $\alpha, \beta \in \Omega_{\partial\Lambda}$ with $\alpha_s = \beta_s$, $s \neq e$, the following holds:*

$$Var(\mu^{\alpha}_{\Lambda,\Delta}, \mu^{\beta}_{\Lambda,\Delta}) \leq K \exp(-\kappa\, \rho(e,\Delta)), \tag{1.3}$$

where $\rho(e,\Delta)$ is the minimal length of a path which starts in e and ends in Δ.

Remarks:
(i) The above is an obvious translation of condition III_c of Dobrushin and Warstat ([7]). In the original formulation the randomness is attached with the vertices instead of the edges, but this is immaterial.
(ii) It is easy to see that this condition implies uniqueness of the infinite-volume Gibbs measure. In fact, it is much stronger.
(iii) Dobrushin and Warstat state nine other conditions and prove that all ten conditions are 'essentially' equivalent (see the end of this remark). Some of those conditions are concerned with the zeros of the partition function, (and of the partition functions corresponding with small perturbations of the interactions) and imply that the free energy and related functions have analytic continuations (which partly explains the name complete analyticity).

The word 'essentially' in the beginning of this remark needs some explanation: In Section 2 of their paper Dobrushin and Warstat define a natural metric on the space of interactions (of bounded range). For each subset of interactions, the *main component* of that set is defined as its maximal connected open subset containing the 0 interaction. More precisely now, their main result is that for each of the ten conditions, the set of interactions satisfying that condition has the same main component. This common component is then called the class of completely analytic interactions.

(iv) The work of Dobrushin and Warstat is an extension of the results by Dobrushin and Shlosman in [6]. The Dobrushin-Warstat results are applicable to a larger class of interactions, including the monomer-dimer model. Our main result is Theorem 1 below:

Theorem 1 *The monomer-dimer model on $\mathbf{Z}^d$ is completely analytic.*

Remark:
As the remarks after (1.3) indicate, complete analyticity is a very strong property, and, as far as I understand, does not follow from the results of Heilmann and Lieb (1972). In particular, the perturbations mentioned in Remark iii above are much more general than those only involving the 'ordinary' model parameters. To give one example, it also includes the possibility of adding an extra factor $x^{|\mathrm{pairs}|}$ to (1.1), with the 'extra' parameter x sufficiently close to 1, where |pairs| denotes the number of pairs of distinct edges $\{e, e'\}$ s.t. $\omega_e = \omega'_e = 1$ and for which there is an edge e^* with $e \sim e^* \sim e'$. On the other hand, although the analyticity notion in the Heilmann-Lieb paper is restricted to the 'ordinary' model parameters they have several detailed results which do not follow from ours.

In Section 2 we review the earlier mentioned percolation-like method for getting upper bounds for boundary effects. The main result in that section, Lemma 1, holds for all so-called Markov fields. Although it is, in spirit, already in [2], we state it here because the argument in [2] deals immediately

with infinite-volume Gibbs measures, while here we need a more explicit expression for the boundary effects for finite graphs. In Section 3 we apply the Lemma to the special case of the monomer-dimer model, to prove Theorem 1. Section 4 gives a brief discussion and also mentions some other results in the literature.

2. Disagreement percolation and Markov fields

For those not familiar with the topic we explain the concept of Markov fields (on finite graphs): Let G be a finite graph, S a finite set and the state space $\Omega = S^V$, where $V = V(G)$ is the set of vertices of G. Note that the state space here involves, as is standard in the literature, the vertices instead of the edges, while in the formulation of the monomer-dimer model it was the other way round. We come back to this soon. We use the obvious analogs and generalizations of the notation in Section 1. For instance, if $\Lambda \subset V$, then $\partial\Lambda$ denotes $\{i \in \Lambda^c : \exists j \in \Lambda \text{ with } j \sim i\}$. And, if μ is a probability distribution on Ω, $\Lambda \subset V$ and $\alpha \in S^{\partial\Lambda}$ with $\mu(\sigma_{\partial\Lambda} = \alpha) > 0$, then μ^α_Λ denotes the conditional distribution of σ_Λ given $\sigma_{\partial\Lambda} = \alpha$. A probability distribution μ on Ω is called a Markov field if for all $\Lambda \subset V$ the conditional distribution of σ_Λ given σ_{Λ^c} depends only on $\sigma_{\partial\Lambda}$. Well-known examples in the literature are the Ising model and the hard-core lattice gas model. In the latter model we take $S = \{0, 1\}$ and

$$\mu(\omega) = \frac{\lambda^{|\omega|} I(\omega \text{ is feasible })}{Z},$$

where $|\omega| = \sum_{i \in V} \omega_i$ and 'ω is feasible' means that there are no vertices i, j with $i \sim j$ and $\omega_i = \omega_j = 1$. Comparison with (1.1) and a few moments reflection show that the monomer-dimer model on a graph G corresponds to the hard-core model on the so-called line graph of G. (The line graph of G is the graph whose vertices correspond to the edges of G, and in which two vertices have an edge if the corresponding two edges in G have a common endpoint). More about this correspondence below. It is easy to check that the hard-core model is indeed a Markov field. In particular we have the following observation: For each $i \in V$, the conditional probability that $\sigma_i = 1$ given the $\sigma-$values of all the other vertices, is $\lambda/(1 + \lambda)$ if $\sigma_{\partial i} \equiv 0$, and 0 otherwise. This will be used in Section 3.

As we said before, we want to give an upper bound for boundary effects. So let $\Delta \subset \Lambda \subset V$, and $\alpha, \beta \in S^{\partial\Lambda}$. We want to estimate the variational distance between the marginal distribution on Ω_Δ for boundary condition α and that for boundary condition β. We now introduce the notion 'path of disagreement': If $\omega, \omega' \in S^\Lambda$ and $j \in \Lambda$, then we say that the pair (ω, ω') has a path of disagreement from j to Δ if there is a sequence $j = i_1 \sim i_2, \cdots \sim i_n$, of vertices in Λ with $i_n \in \Delta$ and $\omega_{i_k} \neq \omega'_{i_k}, k = 1, \cdots, n-1$. Further, we say that (ω, ω') has a path of disagreement from $\partial\Lambda$ to Δ if there is a $j \in \partial\Lambda$ with $\alpha_j \neq \beta_j$, an $i \in \Lambda$ with $i \sim j$ and a path of disagreement from i to Δ. The following lemma is essentially in [2], but not explicitly stated there.

Lemma 1

$$\mathrm{Var}(\mu^\alpha_{\Lambda,\Delta}, \mu^\beta_{\Lambda,\Delta}) \leq (\mu^\alpha_\Lambda \times \mu^\beta_\Lambda)(\exists \text{ a path of disagreement from } \partial\Lambda \text{ to } \Delta). \tag{2.1}$$

Remark: In words, Lemma 1 says that the variational distance between the 'restrictions to Δ' of the two distributions μ^α_Λ and μ^β_Λ is smaller than or equal to the probability that, if we randomly and independently choose two realizations in Ω_Λ, one with the first distribution above, the other with the second distribution, this pair of realizations has a path of disagreement from $\partial\Lambda$ to Δ. The r.h.s. of (2.1) involves the product of μ^α_Λ and μ^β_Λ which can be considered as a trivial coupling of these distributions. A more clever coupling is given in [4]. However, it appears that for hard-core models these couplings are essentially the same.

Proof of Lemma 1 Let $A \subset S^\Lambda \times S^\Lambda$ be the complement of the event in the r.h.s. of the inequality in the lemma. Define the "cluster of disagreement of Δ (w.r.t. the pair (ω, ω')", notation $C(\Delta)$, by

$$C(\Delta) = \Delta \cup \{i \in \Lambda : (\omega, \omega') \text{ has a path of disagreement from } i \text{ to } \Delta\}.$$

Let $T : \Omega_\Lambda \times \Omega_\Lambda \to \Omega_\Lambda \times \Omega_\Lambda$ be the map which exchanges ω and ω' on $C(\Delta)$. More precisely,

$$T(\omega, \omega') = (\omega'_{C(\Delta)}\omega_{\Lambda\setminus C(\Delta)},\, \omega_{C(\Delta)}\omega'_{\Lambda\setminus C(\Delta)}),$$

where we have used the notation for concatenation given in Section 1. It is clear that T is a 1-1 map and that $(\omega, \omega') \in A \Leftrightarrow T((\omega, \omega')) \in A$. Moreover, it follows immediately from the definition of A that if $(\omega, \omega') \in A$ then $(\omega\alpha)_{\partial(C(\Delta))} = (\omega'\alpha)_{\partial(C(\Delta))}$. Combining these observations with the Markov property, we see that T preserves the measure $\mu^\alpha_\Lambda \times \mu^\beta_\Lambda$ on A. Now let $B \subset S^\Delta$. We have the following trivial identity:

$$\begin{aligned}
\mu^\alpha_{\Lambda,\Delta}(B) &= \mu^\alpha_\Lambda(\sigma_\Delta \in B) \\
&= (\mu^\alpha_\Lambda \times \mu^\beta_\Lambda)(\{\omega, \omega') \in \Omega_\Lambda \times \Omega_\Lambda : \omega_\Delta \in B\}) \\
&= (\mu^\alpha_\Lambda \times \mu^\beta_\Lambda)(\{(\omega, \omega') \in \Omega_\Lambda \times \Omega_\Lambda : \omega_\Delta \in B,\, (\omega, \omega') \in A\}) \\
&+ (\mu^\alpha_\Lambda \times \mu^\beta_\Lambda)(\{(\omega, \omega') \in \Omega_\Lambda \times \Omega_\Lambda : \omega_\Delta \in B,\, (\omega, \omega') \in A^c\}). \qquad (2.2)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mu^\beta_{\Lambda,\Delta}(B) &= (\mu^\alpha_\Lambda \times \mu^\beta_\Lambda)(\{(\omega, \omega') \in \Omega_\Lambda \times \Omega_\Lambda : \omega'_\Delta \in B,\, (\omega, \omega') \in A\}) \\
&+ (\mu^\alpha_\Lambda \times \mu^\beta_\Lambda)(\{(\omega, \omega') \in \Omega_\Lambda \times \Omega_\Lambda : \omega'_\Delta \in B,\, (\omega, \omega') \in A^c\}). \qquad (2.3)
\end{aligned}$$

However, the image under T of the event in the first term in the r.h.s. of (2.2) is exactly the event in the first term of the r.h.s of (2.3). Hence, since T is measure-preserving on A, the absolute value of the difference between the l.h.s. of (2.2) and (2.3) is smaller than or equal to the maximum of the second term in the r.h.s. of (2.2) and that in the r.h.s. of (2.3), which completes the proof of the lemma. □

The next step in [2] and [5] is a stochastic domination argument: Let, for $i \in \Lambda$, p_i be the maximum over all pairs $\alpha, \beta \in S^{\partial i}$ of $(\mu^\alpha_i \times \mu^\beta_i)(\{(s, s') \in S \times S : s \neq s'\})$. (In many models these values can be easily calculated explicitly). The r.h.s of (2.1) is then bounded above by the probability that, for the independent site percolation model on G with parameters $p_i, i \in V$, there is an open path from $\partial\Lambda$ to Δ. In many cases, for instance when G is an $n \times n$ box of the square lattice, known results from percolation can then be used to prove asymptotic properties of the boundary influences.

The hard-core model, described in the beginning of this section, has the special feature that the vertices on a path of disagreement are alternatingly occupied in ω (and hence empty in ω') and vice versa. Using this we can, for that model, refine the argument and replace the above given expression for p_i by the maximum over all $\alpha \in S^{\partial i}$ of $\mu^\alpha_i(1)$, which is simply equal to $\lambda/(1+\lambda)$. This leads then to the result of [5] that if this last expression is smaller than the critical probability for site percolation on an infinite graph G, then the hard-core model on G with parameter λ has a unique Gibbs measure. In fact, by using exponential-decay results for sub-critical percolation, which were first obtained by Kesten in his famous paper [15] for bond percolation on the square lattice, and later in more generality by Aizenman and Barsky ([1]) and Menshikov ([17]), we even get complete analyticity.

In the light of the earlier made observation that the monomer-dimer model corresponds to the hard-core model on the line graph, it is tempting to simply translate the above steps to the monomer-dimer model. This would, for instance, lead to the result that if $\lambda/(1+\lambda)$ is smaller than the critical probability for *bond* percolation on $\mathbf{Z}^d$, the monomer-dimer model on that graph is completely analytic. However, this is not sufficient for our purpose (to prove Theorem 1), because we want this result for *all* λ. This stronger result is obtained by being a little more patient and by observing that, in addition to the above mentioned special feature of the more general hard-core model, the monomer-dimer model has another simple but essential special feature. This will be explained in the next section.

3. Proof of Theorem 1

We now apply the quite general Lemma 1 to the special case of the monomer-dimer model, to prove Theorem 1. So let $\Delta \subset \Lambda \subset\subset \mathbf{E}^d$, $e \in \partial\Lambda$, and $\alpha, \beta \in \Omega_{\partial\Lambda}$ with $\alpha_s = \beta_s$ for all $s \neq e$. We will study for this model the l.h.s. of (1.3) by using Lemma 1. As mentioned before, the monomer-dimer model on a graph corresponds to the hard-core model on its line graph. So, strictly speaking, to apply the lemma we have to translate the monomer-dimer model in terms of the hard-core model, then apply the lemma and then translate the result back to the original monomer-dimer model. One can easily see that this gives an obvious 'edge-analog' of Lemma 1 (recall that Lemma 1 was formulated for models where the randomness involves the vertices rather than the edges). This edge analog of the lemma tells us that the l.h.s. of (1.3) is bounded above by

$$(\mu_\Lambda^\alpha \times \mu_\Lambda^\beta)(\exists \text{ a path of disagreement from } e \text{ to } \Delta), \tag{3.1}$$

where the event above is formally given by

$$\{(\omega, \omega') \in \Omega_\Lambda \times \Omega_\Lambda : \quad \exists \text{ a sequence } e_1 \sim e_2 \cdots \sim e_n \text{ of edges in } \Lambda, \text{ with } e_1 \sim e, e_n \in \Delta \text{ and } \omega_{e_i} \neq \omega'_{e_i}, i = 1, \cdots, n-1\}.$$

We now study the probability of this event. Let x and y be the two endpoints of e. A property which makes the monomer-dimer model so special within the more general hard-core model is that (see also Remark 1 in Section 4), for each integer $k > 0$, there is at most one path of disagreement $e_1, \cdots, e_k$ such that x is an endpoint of e_1. This is so, because of the following. For convenience, we define $e_0 = e_0^* = e$ and $\omega_e = \alpha_e$, $\omega'_e = \beta_e$. Suppose both $e_1, \cdots, e_k$ and $e_1^*, \cdots, e_k^*$ would be such paths. If they are not the same, take the smallest i, $1 \leq i \leq k$ for which $e_i \neq e_i^*$. Since both paths 'start' at the same point (namely x) it is easy to see that the three edges $e_{i-1} (= e_{i-1}^*)$, e_i and e_i^* are distinct and have a common endpoint. We also know that ω differs from ω' on each of these three edges. But then either two of these edges have ω value 1 or two of them have ω' value 1. In both cases we have a conflict with feasibility.

Now suppose $e_1, \cdots, e_k$ is a path of disagreement of length k with x an endpoint of e_1. Without loss of generality we assume that $\alpha_e = 1$. By the uniqueness property above, a path of disagreement of length $k+1$ starting in x (if one exists) must be of the form $e_1, \cdots, e_k, \tilde{e}$, where $\tilde{e}$ is an edge which has a common endpoint with e_k but not with e_{k-1}. There are clearly at most $2d-1$ possible candidates for $\tilde{e}$. Suppose k is odd. It is easy to see (because of $\alpha_e = 1$ and $\beta_e = 0$) that then $\omega_{\tilde{e}}$ must be 1 and $\omega'_{\tilde{e}}$ must be 0. Hence, to have a path of disagreement of length $k+1$ starting at x, at least one of the above mentioned candidates must have ω-value 1. However, by the observation for the hard-core model in Section 2, the conditional probability that all of these have ω-value 0 is at least $(1+\lambda)^{-(2d-1)}$. From these arguments it is easy to conclude that the conditional probability that there is a path of disagreement of length $k+1$ starting at x, given that there is a path of disagreement of length k starting at x is at most $1-(1+\lambda)^{-(2d-1)}$. Since each path from e to Δ has length at least $\rho(e, \Delta)$, we finally see that the probability of having a path of disagreement from e to Δ is at most $2((1-(1+\lambda)^{-(2d-1)})^{\rho(e,\Delta)}$. (The factor 2 comes from the fact that the path could start at y instead of x). This shows that the monomer-dimer interaction satisfies the Dobrushin-Warstat condition III_c. To complete the proof of Theorem 1 we must (according to Remark (iii) after (1.3)) show that it is in the corresponding main component. This follows now by arguments which are quite standard and which we only sketch briefly: First of all, Dobrushin and Warstat have shown that condition III_c also has a so-called *constructive analog* involving boxes Λ of bounded size only. But for a fixed box simple continuity arguments can be applied. In this way it follows that the class of interactions satisfying III_c is open. Hence, by our result that III_c holds for the monomer-dimer model with arbitrary λ, changing λ keeps us in the same open connected component. However, for λ sufficiently small (so that we are in the so-called Dobrushin uniqueness region) it is well-known that the monomer-dimer model is in the main component. This completes the proof of Theorem 1.

4. Further remarks and discussion

1. An essential role in the proof in Section 3 was played by the uniqueness of certain disagreement paths, which made the monomer- dimer model so special. This property can also be formulated as follows: let ω and ω' be two feasible configurations. Then the set $\{e \in \Lambda : \omega_e \neq \omega'_e\}$ consists of pairwise disjoint paths (some of which may be cycles). ('Disjoint' means 'vertex-disjoint' here; so two paths are called disjoint if no edge in the first path has a common endpoint with any edge in the other path). Remarkably, this same property plays an essential role in work (on Markov chain Monte Carlo methods) by Jerrum and Sinclair (see e.g. the interesting review paper [12] and the references given there), which apart from that is very different from ours. They use the property to construct so-called *canonical paths* from a configuration ω to a configuration ω'. (The word 'path' here refers to a sequence in the state space Ω, not in the graph). They then apply a quite general theorem to get an upper bound for the mixing time of a certain dynamics, which has the monomer-dimer distribution as its limit distribution. This dynamics is, roughly speaking, as follows: at each step an edge is chosen uniformly at random, and its value (0 or 1 in this case) is updated according to it conditional distribution given the current values of all the other edges. The mixing time is the number of steps after which the system is 'sufficiently close' to equilibrium.

One of their main results is, again roughly speaking, that the mixing time is polynomially bounded in the input parameters. One may wonder if these results for the '*dynamics*', applied to subgraphs of $\mathbf{Z}^d$, also give, for instance, uniqueness of the Gibbs measure. However, it seems that the bounds on the mixing time they obtain are not strong enough for that purpose. Conversely, one may wonder what our result for *spatial* dependencies yields for the mixing times. Using ideas from the literature (see e.g. [16]) we can use our result to get upper bounds for the mixing time of a certain modified dynamics, a so-called block dynamics. Under this dynamics at each step in the procedure a whole box of edges, instead of a single edge, is updated simultaneously. Unfortunately, by lack of a certain monotonicity, it is not clear what these results tell us about the mixing time for the original (single-edge) dynamics. (Hard-core models on *bipartite* graphs do have the desired monotonicity properties, but the monomer-dimer model corresponds only in trivial cases with such a hard-core model). Moreover, the block size in the block dynamics depends on λ (the larger λ the larger the required block size), and even for moderate block sizes the simultaneous update of the edges within a block involves a huge amount of computations.

2. In the introduction we also mentioned the paper [13] by Kahn and Kayll. Although they too study spatial dependencies (and, e.g. refer to the percolation-like bound for boundary effects in [5]) their goals deviate considerably from ours. Their main motivation comes from combinatorics and graph theory, and the types of assumptions and conclusions in their main theorem (which is quite difficult) are different from those in statistical physics.

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