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D.J.N. van Eijck

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CWI
P.O. Box 94079
1090 GB Amsterdam
The Netherlands

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P.O. Box 94079, 1090 GB Amsterdam (NL)
Kruislaan 413, 1098 SJ Amsterdam (NL)
Telephone +31 20 592 9333
Telefax +31 20 592 4199

Axiomatising Dynamic Logics for Anaphora

Jan van Eijck

CWI, Amsterdam, ILLC, Amsterdam, Uil-OTS, Utrecht

EMAIL: jve@cwi.nl

CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

ABSTRACT

A new task put on the agenda of philosophical logic by the recent dynamic turn, the account for the dynamics of anaphoric context shifts in reasoning, is taken up. The paper gives a sequent axiomatisation of some well known dynamic anaphora logics. The calculi are shown to be sound and complete. They differ from earlier calculi for the dynamics of anaphoric context in that they take the notion of anaphoric context change from premiss to conclusion as starting point of the analysis, and they do not rely on an implicit translation to first order predicate logic.

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1. Pronominal Reference in Reasoning

The recent 'dynamic shift' in logic has led to an increased awareness for the process character of aspects of reasoning. One of the new topics on the logical agenda is anaphoric processing: the way in which possible antecedents for pronominal reference are set up in discourse. As always, awareness of new facts gives occasion for new challenges. Instead of just giving an account of valid reasoning, why not include an account of the way in which premisses can set up an 'anaphoric context' for use by pronouns in the conclusion:

1 *Every man is mortal. Suppose Socrates is a man.* $\models$ *Then* he *is mortal.*

2 *If a man owns a house, then he owns a garden. Suppose Socrates is a man who owns a house.* $\models$ *Then* he *owns a garden.*

3 *If a man owns a house, then he owns a garden. If a man owns a garden, then he sprinkles it.* $\models$ *If a man owns a house, then there is a thing that he sprinkles.*

The task for a suitably dynamicized version of logic is to account for the validity of the above inferences, *and* for the pronoun-antecedent linkings in the discourses that make up the reasoning.

The most streamlined version of recent proposals for a logical representation language for anaphoric linking is Dynamic Predicate Logic or DPL (Groenendijk and Stokhof [7]), the first order version of the representation language proposed in Barwise [1]. This dynamic variation on predicate logic was discovered more than once or twice, witness the existence of a German version, virtually identical to [7]: see Staudacher [11].

The dynamic logic of Barwise [1], Staudacher [11] and Groenendijk and Stokhof [7] essentially represents introduction of new antecedents by means of random assignment to a variable. The meaning of $\exists x$ becomes the relation between variable states s, r with the property that s and r differ at most in their x value:

$$s[\![\exists x]\!]r \text{ iff } s \sim_x r.$$

The language L of DPL is like that of predicate logic, but with the connective $\wedge$ replaced with a connective ; for sequential composition. Let a set of constants C and a set of variables V be given, and assume that c ranges over C and v over V. Also, assume a set of predicate constants P^i_j, with i indicating arity, and let P range over this set. We will omit the arity superscripts, and rely on context to indicate the arity of a predicate.

Definition 1 (L)

$$\begin{array}{rcl} t & ::= & c \mid v \\ A & ::= & \bot \mid \exists v \mid Pt_1 \cdots t_n \\ \phi & ::= & A \mid A;\phi \mid \neg(\phi) \mid \neg(\phi_1);\phi_2. \end{array}$$

Note that we have built into the language that ; creates a flat list structure. (Alternatively, we could allow $\phi;\psi;\chi$ to be ambiguous between $(\phi;\psi);\chi$ and $\phi;(\psi;\chi)$, but then we would have to build associativity of ; into the calculus below.)

We will omit unnecessary parentheses, writing $\neg(Pt_1 \cdots t_n)$ as $\neg Pt_1 \cdots t_n$, etcetera, and we will use the following abbreviations:

$$\begin{array}{rcl} \top & := & \neg\bot \\ \phi_1 \rightarrow \phi_2 & := & \neg(\phi_1;\neg\phi_2) \\ \forall v\phi & := & \neg(\exists v;\neg\phi). \end{array}$$

The key feature of dynamic predicate logic is the ability of the existential quantifier to bind variables outside its proper scope. Assuming for a moment that ; creates structured lists, consider the DPL text $(\exists x; Px); (\exists y; Qy); Rxy$. Here the x and y of Rxy are bound, by $\exists x$ and $\exists y$ respectively, outside of the 'proper scope' of the quantifiers, as indicated by the bracketings. It follows that variables can be viewed as anaphoric elements linked to a preceding existential quantifier that introduces a referent. It is precisely because of the feature that $(\exists x; Px); Qx$ and $\exists x; (Px; Qx)$ are semantically equivalent that we can get away with considering ; as a flat list constructor.

The semantics of DPL is given as a relation between input and output assignments, given a first order model $\mathcal{M}$. We use $s \sim_v r$ in case s and r differ at most in the value they assign to v, and $s \sim_X r$ $(X \subseteq V)$ in case s and r differ at most in the values they assign to the variables in X. We use sI_Xr if $\forall x \in X : s(x) = r(x)$. Note that sI_Xr iff $s \sim_{(V-X)} r$.

Definition 2 (Term evaluation for L)

$$\begin{array}{rcl} [\![c]\!]^{\mathcal{M}}_s & = & c^{\mathcal{M}} \\ [\![v]\!]^{\mathcal{M}}_s & = & s(v). \end{array}$$

Definition 3 (Semantics of L)

$s[\![\bot]\!]^{\mathcal{M}}r$		*never*
$s[\![\exists v]\!]^{\mathcal{M}}r$	*iff*	$s \sim_v r$
$s[\![Pt_1 \cdots t_n]\!]^{\mathcal{M}}r$	*iff*	$s = r$ *and* $\langle [\![t_1]\!]^{\mathcal{M}}_s, \ldots, [\![t_n]\!]^{\mathcal{M}}_s \rangle \in P^{\mathcal{M}}$
$s[\![\neg(\phi)]\!]^{\mathcal{M}}r$	*iff*	$s = r$ *and there is no* s' *with* $s[\![\phi]\!]^{\mathcal{M}}s'$
$s[\![A;\phi]\!]^{\mathcal{M}}r$	*iff*	*there is an* s' *with* $s[\![A]\!]^{\mathcal{M}}s'$ *and* $s'[\![\phi]\!]^{\mathcal{M}}r$
$s[\![\neg(\phi_1);\phi_2]\!]^{\mathcal{M}}r$	*iff*	$s[\![\neg\phi_1]\!]^{\mathcal{M}}s$ *and* $s[\![\phi_2]\!]^{\mathcal{M}}r$.

Note that in this semantics for DPL, a repeated assignment to a single variable by means of a repeated use of the same existential quantifier-variable combination blocks off the individual introduced by the first use of the quantifier from further anaphoric reference. After $\exists xPx; \exists xQx$, the variable x will refer to the individual introduced by $\exists xQx$, and the individual introduced by $\exists xPx$ has become inaccessible.

A suitable consequence relation for this language is the following:

Definition 4 $\phi \models \psi :\Longleftrightarrow$ *for all* $\mathcal{M}, s, t$*: if* $s[\![\phi]\!]^{\mathcal{M}}t$ *then there is an* r *with* $t[\![\psi]\!]^{\mathcal{M}}r$.

Note that this consequence relation allows carrying anaphoric links from premiss to conclusion. For example: from 'a man walks and he talks' it follows that 'he talks':

$$\exists x; Mx; Wx; Tx \models Tx.$$

A DPL formula ϕ succeeds in a state s for model $\mathcal{M}$ in case for some r, $s[\![\phi]\!]^{\mathcal{M}}r$. The weakest precondition for success of a DPL formula ϕ gives what can be called the truth condition for that formula; it can be described as $\langle\phi\rangle\top$, where $\langle\phi\rangle$ is a 'modal' operator interpreted with accessibility relation $[\![\phi]\!]$.

The following translation procedure in a language of classical predicate logic with DPL modalities constitutes the full precondition calculus for DPL:

$$\begin{array}{rcl}
\langle\bot\rangle\phi & \leftrightarrow & \bot \\
\langle\exists v\rangle\phi & \leftrightarrow & \exists v\phi \\
\langle Pt_1 \cdots t_n\rangle\phi & \leftrightarrow & Pt_1 \cdots t_n \wedge \phi \\
\langle\neg\psi\rangle\phi & \leftrightarrow & [\psi]\bot \wedge \phi \\
\langle\psi_1;\psi_2\rangle\phi & \leftrightarrow & \langle\psi_1\rangle\langle\psi_2\rangle\phi.
\end{array}$$

This reduction to classical predicate logic provides a kind of translation 'in the context of a given postcondition'. Without the postcondition context, the translation procedure would break down for the existential quantifier case, for $\exists v$ *per se* has no counterpart in classical predicate logic, of course. The translation is in the spirit of systems of precondition reasoning for imperative programs. See Van Eijck and De Vries [4] for further discussion of this connection. Such a calculus has its use for reasoning about DPL, but although it has been sold to the general public as an axiomatisation of DPL (in Van Eijck [3]), it is closer to a translation procedure than to a real axiomatisation.

2. Substitution in DPL

In the calculus we are about to present, we need some notions about freedom and bondage. First of all, we need the notion of being freely substitutable. For this, we first define a function *var* on the terms and formulas.

Definition 5 **(**$var(t), var(\phi)$**)**

$$
\begin{array}{lcl}
var(c) & := & \emptyset \\
var(v) & := & \{v\} \\
var(\bot) & := & \emptyset \\
var(\exists v) & := & \{v\} \\
var(Pt_1 \cdots t_n) & := & \bigcup_{1 \leq i \leq n} var(t_i) \\
var(\neg \phi) & := & var(\phi) \\
var(\bot ; \phi) & := & var(\phi) \\
var(\exists v; \phi) & := & \{v\} \cup var(\phi) \\
var(Pt_1 \cdots t_n; \phi) & := & var(Pt_1 \cdots t_n) \cup var(\phi) \\
var(\neg \phi_1 ; \phi_2) & := & var(\phi_1) \cup var(\phi_2).
\end{array}
$$

Definition 6 **(**t **is free for** v **in** ϕ**)** *1. t is free for v in A.*

2. *t is free for v in $(\neg\psi)$ if t is free for v in ψ.*
3. *t is free for v in $\bot;\psi$ if t is free for v in ψ.*
4. *t is free for v in $\exists v;\psi$.*
5. *t is free for v in $\exists w;\psi$ $(w \neq v)$ if $w \notin var(t)$ and t is free for v in ψ.*
6. *t is free for v in $Pt_1 \cdots t_n;\psi$ if t is free for v in ψ.*
7. *t is free for v in $(\neg\psi_1);\psi_2$ if t is free for v in ψ_1 and t is free for v in ψ_2.*

The set of variables which have a fixed occurrence in a DPL formula is given by a function $\mathit{fix} : L \rightarrow \mathcal{P}V$. The set of variables which are introduced in a formula is given by a function $\mathit{intro} : L \rightarrow \mathcal{P}V$, and the set of variables which have a classically bound occurrence in a formula is given by a function $\mathit{cbnd} : L \rightarrow \mathcal{P}V$.

Definition 7 (fix, intro, cbnd)

- $\mathit{fix}(\exists v) := \emptyset,\ \mathit{intro}(\exists v) := \{v\},\ \mathit{cbnd}(\exists v) := \emptyset.$
- $\mathit{fix}(\bot) := \emptyset,\ \mathit{intro}(\bot) := \emptyset,\ \mathit{cbnd}(\bot) := \emptyset.$
- $\mathit{fix}(Pt_1 \cdots t_n) := var(Pt_1 \cdots t_n),\ \mathit{intro}(Pt_1 \cdots t_n) := \emptyset,\ \mathit{cbnd}(Pt_1 \cdots t_n) := \emptyset.$
- $\mathit{fix}(\neg\phi) := \mathit{fix}(\phi),\ \mathit{intro}(\neg\phi) := \emptyset,\ \mathit{cbnd}(\neg\phi) := \mathit{intro}(\phi) \cup \mathit{cbnd}(\phi).$
- $\mathit{fix}(\phi_1;\phi_2) := \mathit{fix}(\phi_1) \cup (\mathit{fix}(\phi_2) - \mathit{intro}(\phi_1)),$
  $\mathit{intro}(\phi_1;\phi_2) := \mathit{intro}(\phi_1) \cup \mathit{intro}(\phi_2),$
  $\mathit{cbnd}(\phi_1;\phi_2) := \mathit{cbnd}(\phi_1) \cup \mathit{cbnd}(\phi_2).$

We call an occurrence of v in ϕ *free* if it is not introduced and not classically bound, and we use $[t/v]\phi$ for the result of substituting t for all free occurrences of v in ϕ:

Definition 8 **(**$[t/v]\phi$**)**

$$
\begin{array}{lcl}
[t/v]c & := & c \\
[t/v]v & := & t \\
[t/v]w & := & w \\
[t/v]\bot & := & \bot \\
[t/v]\exists v & := & \exists v \\
[t/v]\exists w & := & \exists w \\
[t/v]Pt_1 \cdots t_n & := & P[t/v]t_1 \cdots [t/v]t_n \\
[t/v]\neg\phi & := & \neg[t/v]\phi \\
[t/v](\bot;\phi) & := & \bot;[t/v]\phi \\
[t/v](\exists v;\phi) & := & \exists v;\phi \\
[t/v](\exists w;\phi) & := & \exists w;[t/v]\phi \\
[t/v](Pt_1 \cdots t_n;\phi) & := & P[t/v]t_1 \cdots [t/v]t_n;[t/v]\phi \\
[t/v](\neg\phi_1;\phi_2) & := & \neg([t/v]\phi_1);[t/v]\phi_2.
\end{array}
$$

Note that this substitution takes the dynamic binding force of $\exists v$ over the text that follows into account (cf. the clause for $[t/v](\exists v;\phi)$, where the occurrence of $\exists v$ blocks off the ϕ that follows). Visser [14] calls this substitution 'left' substitution.

We use $s[v := a]$ for the function s' defined by $s'(w) = s(w)$ if $w \neq v$, $s'(w) = a$ if $w = v$.

Lemma 9 (Term Substitution)

$$[\![[t_1/v]t_2]\!]_s^{\mathcal{M}} = [\![t_2]\!]_{s[v:=[\![t_1]\!]_s^{\mathcal{M}}]}^{\mathcal{M}}.$$

Proof. In case t_2 is not a variable, $[t_1/v]t_2$ equals t_2, and the lemma trivially holds.

If $t_2 = w \neq v$, $[t_1/v]t_2$ equals t_2, and the lemma trivially holds.

If $t_2 = v$, $[t_1/v]t_2$ equals t_1, and the lemma holds because $[\![t_1]\!]_s^{\mathcal{M}} = s[v := [\![t_1]\!]_s^{\mathcal{M}}](v)$. □

The following formula substitution lemma makes clear that substitution in dynamic predicate logic behaves differently from substitution in standard predicate logic. When substituting into ϕ, we have to take the context of ϕ (in the form of the set of variables given by *intro*(ϕ)) into account. For further elaboration of this theme we refer to Visser [14].

Lemma 10 (Formula Substitution) *If t is free for v in ϕ and $v \in$ intro(ϕ) then*

$$s[\![[t/v]\phi]\!]^{\mathcal{M}} r \Leftrightarrow s[v := [\![t]\!]_s^{\mathcal{M}}][\![\phi]\!]^{\mathcal{M}} r.$$

If t is free for v in ϕ and $v \notin$ intro(ϕ) then

$$s[\![[t/v]\phi]\!]^{\mathcal{M}} r \Leftrightarrow s[v := [\![t]\!]_s^{\mathcal{M}}][\![\phi]\!]^{\mathcal{M}} r[v := [\![t]\!]_s^{\mathcal{M}}].$$

Proof. Induction on the structure of ϕ.

$\phi = \bot$. OK.

$\phi = \exists v$. Then $v \in$ *intro*(ϕ), and

$$
\begin{array}{ll}
 & s[\![[t/v]\exists v]\!]^{\mathcal{M}} r \\
\Longleftrightarrow & s[\![\exists v]\!]^{\mathcal{M}} r \\
\Longleftrightarrow & s[v := [\![t]\!]_s^{\mathcal{M}}][\![\exists v]\!]^{\mathcal{M}} r.
\end{array}
$$

$\phi = \exists w$, with $w \neq v$. Now $v \notin \mathit{intro}(\phi)$, and

$$\begin{array}{ll} & s[\![[t/v]\exists w]\!]^{\mathcal{M}} r \\ \iff & s[\![\exists w]\!]^{\mathcal{M}} r \\ \iff & s[v := [\![t]\!]_s^{\mathcal{M}}][\![\exists w]\!]^{\mathcal{M}} r[v := [\![t]\!]_s^{\mathcal{M}}]. \end{array}$$

$\phi = Pt_1 \cdots t_n$. Then $v \notin \mathit{intro}(\phi)$, and

$$\begin{array}{ll} & s[\![[t/v]Pt_1 \cdots t_n]\!]^{\mathcal{M}} s \\ \iff & \langle [\![[t/v]t_1]\!]_s^{\mathcal{M}}, \ldots, [\![[t/v]t_n]\!]_s^{\mathcal{M}} \rangle \in P^{\mathcal{M}} \\ \iff & \langle [\![t_1]\!]_{s[v:=[\![t]\!]_s^{\mathcal{M}}]}^{\mathcal{M}}, \ldots, [\![t_n]\!]_{s[v:=[\![t]\!]_s^{\mathcal{M}}]}^{\mathcal{M}} \rangle \in P^{\mathcal{M}} \\ \iff & s[v := [\![t]\!]_s^{\mathcal{M}}][\![Pt_1 \cdots t_n]\!]^{\mathcal{M}} s[v := [\![t]\!]_s^{\mathcal{M}}]. \end{array}$$

$\phi = \neg\psi$. Then $v \notin \mathit{intro}(\phi)$ and

$$\begin{array}{ll} & s[\![[v/t]\neg\psi]\!]^{\mathcal{M}} s \\ \iff & s[\![\neg[v/t]\psi]\!]^{\mathcal{M}} s \\ \iff & \text{there is no } s' \text{ with } s[\![[v/t]\psi]\!]^{\mathcal{M}} s' \\ \iff & \text{there is no } s' \text{ with } s[v := [\![t]\!]_s^{\mathcal{M}}][\![\psi]\!]^{\mathcal{M}} s' \\ \iff & s[v := [\![t]\!]_s^{\mathcal{M}}][\![\neg\psi]\!]^{\mathcal{M}} s[v := [\![t]\!]_s^{\mathcal{M}}]. \end{array}$$

Note that the last step uses the induction hypothesis, and that the appeal to the induction hypothesis works out equally well for the two subcases $v \in \mathit{intro}(\psi)$ and $v \notin \mathit{intro}(\psi)$.

$\phi = A; \psi$.

Subcase 1: $A = \bot$. OK.

Subcase 2: $A = \exists v$. Then $v \in \mathit{intro}(\phi)$, and

$$\begin{array}{ll} & s[\![[t/v]\exists v; \psi]\!]^{\mathcal{M}} r \\ \iff & s[\![\exists v; \psi]\!]^{\mathcal{M}} r \\ \iff & s \sim_v s' \text{ and } s'[\![\psi]\!]^{\mathcal{M}} r \\ \iff & s[v := [\![t]\!]_s^{\mathcal{M}}][\![\exists v; \psi]\!]^{\mathcal{M}} r. \end{array}$$

Subcase 3: $A = \exists w$, with $w \neq v$. Then $v \notin \mathit{intro}(\phi)$ and

$$\begin{array}{ll} & s[\![[t/v](\exists w; \psi)]\!]^{\mathcal{M}} r \\ \iff & s[\![\exists w; [t/v]\psi]\!]^{\mathcal{M}} r \\ \iff & s \sim_w s' \text{ and } s'[\![[t/v]\psi]\!]^{\mathcal{M}} r \\ \iff & s \sim_w s' \text{ and } s'[v := [\![t]\!]_s^{\mathcal{M}}][\![\psi]\!]^{\mathcal{M}} r[v := [\![t]\!]_s^{\mathcal{M}}] \\ \iff & s[v := [\![t]\!]_s^{\mathcal{M}}][\![\exists w; \psi]\!]^{\mathcal{M}} r[v := [\![t]\!]_s^{\mathcal{M}}] \end{array}$$

$\phi = (\neg\psi_1); \psi_2$. Here is the case where $v \in \mathit{intro}(\psi_2)$.

$$\begin{array}{ll} & s[\![[t/v](\neg(\psi_1); \psi_2)]\!]^{\mathcal{M}} r \\ \iff & s[\![\neg([t/v]\psi_1); [t/v]\psi_2)]\!]^{\mathcal{M}} r \\ \iff & s[\![\neg([t/v]\psi_1)]\!]^{\mathcal{M}} s \text{ and } s[\![[t/v]\psi_2)]\!]^{\mathcal{M}} r \\ \iff & s[v := [\![t]\!]_s^{\mathcal{M}}][\![\psi_1]\!] s[v := [\![t]\!]_s^{\mathcal{M}}] \text{ and } s[v := [\![t]\!]_s^{\mathcal{M}}][\![\psi_2]\!]^{\mathcal{M}} r. \end{array}$$

The case where $v \notin \mathit{intro}(\psi_2)$ is similar. □

The following lemmas use the functions *intro* and *fix* (and indeed motivate their definition).

Lemma 11 $s[\![\phi]\!]^{\mathcal{M}} t$ *implies* $s \sim_{\mathit{intro}(\phi)} t$.

Proof. Induction on the structure of ϕ. $\square$

Lemma 12 *If $s[\![\phi]\!]^{\mathcal{M}}s'$ and sI_Xr for $X \supseteq fix(\phi)$ then there is an r' with $r[\![\phi]\!]^{\mathcal{M}}r'$.*

Proof. Induction on the structure of ϕ.

$\bot$: trivial.

$\exists v$. Take $r' = r[v := s'(v)]$.

$Pt_1 \cdots t_n$. Use the fact that $\langle [\![t_1]\!]_s^{\mathcal{M}} \cdots [\![t_n]\!]_s^{\mathcal{M}} \rangle = \langle [\![t_1]\!]_r^{\mathcal{M}} \cdots [\![t_n]\!]_r^{\mathcal{M}} \rangle$.

$(\neg\psi)$. From $\neg\exists s' : s[\![\psi]\!]^{\mathcal{M}}s'$ it follows by induction hypothesis that $\neg\exists r' : r[\![\psi]\!]^{\mathcal{M}}r'$. Therefore $r[\![\neg\psi]\!]^{\mathcal{M}}r$.

$\bot; \psi$. Trivial.

$\exists v; \psi$. Then $s[\![\exists v; \psi]\!]^{\mathcal{M}}s'$, and there is an s'' with $s \sim_v s''$ and $s''[\![\psi]\!]^{\mathcal{M}}s'$. Take $r'' = r[v := s''(v)]$. Then, by the induction hypothesis, there is an r' with $r''[\![\psi]\!]^{\mathcal{M}}r'$, and therefore $r[\![\exists v; \psi]\!]^{\mathcal{M}}r'$.

$Pt_1 \cdots t_n; \psi$. Then we have $s[\![Pt_1 \cdots t_n; \psi]\!]^{\mathcal{M}}s'$ iff $s[\![Pt_1 \cdots t_n]\!]^{\mathcal{M}}s$ and $s[\![\psi]\!]^{\mathcal{M}}s'$ iff $r[\![Pt_1 \cdots t_n]\!]^{\mathcal{M}}r$ (by the fact that $sI_{var(Pt_1 \cdots t_n)}r$) and $r[\![\psi]\!]^{\mathcal{M}}r'$ for some r' (by i.h.) iff $r[\![Pt_1 \cdots t_n; \psi]\!]^{\mathcal{M}}r'$.

$(\neg\phi_1); \phi_2$. Then $s[\![(\neg\psi_1); \psi_2]\!]^{\mathcal{M}}s'$ iff $s[\![\neg\psi_1]\!]^{\mathcal{M}}s$ and $s[\![\psi_2]\!]^{\mathcal{M}}s'$. By the induction hypothesis $r[\![\neg\psi_1]\!]^{\mathcal{M}}r$. and by a second application of the induction hypothesis there is an r' with $r[\![\psi_2]\!]^{\mathcal{M}}r'$. Thus, there is an r' with $r[\![\neg\psi_1; \psi_2]\!]^{\mathcal{M}}r'$. $\square$

3. A Calculus for DPL

In this section, we will give a set of sequent deduction rules for DPL. We will use $\phi \Longrightarrow \psi$, where $\Longrightarrow$ is the sequent separator. Note that $\phi \Longrightarrow \bot$ expresses that ϕ is inconsistent. The calculus defines a relation $\Longrightarrow \subseteq L^2$ by means of: $\phi \Longrightarrow \psi$ iff $\phi \Longrightarrow \psi$ is at the root of a finite tree with sequents at its nodes, such that the sequents at a leaf node are axioms of the calculus, and the sequents at the internal nodes follow by means of a rule of the calculus from the sequent(s) at the daughter node(s) of that internal node.

In the calculus, we use C, with and without subscripts, as a variable over contexts, where a context is a formula or the empty list ϵ. Substitution and evaluation are extended to contexts in the obvious way. If C is a context and ϕ a formula, then we use $C\phi$ for the formula given by: $C\phi := \phi$ if $C = \epsilon$, $C\phi := \psi; \phi$ if $C = \psi$. Similarly for ϕC, and for $C_1\phi C_2$.

It is convenient to extend the definition of substitution to sequents.

Definition 13 **($[t/v]C \Longrightarrow \phi$)** *Induction on the structure of C*

$$[t/v]\epsilon \Longrightarrow \phi \quad := \quad \epsilon \Longrightarrow [t/v]\phi$$

$$[t/v]\psi \Longrightarrow \phi \quad := \quad \begin{cases} ([t/v]\psi) \Longrightarrow \phi & \text{if } v \in intro(\psi) \\ ([t/v]\psi) \Longrightarrow ([t/v]\phi) & \text{otherwise.} \end{cases}$$

Substitution for sequents carries in its wake a notion of being free for a variable in a sequent:

Definition 14 (t is free for v in $C \Longrightarrow \psi$) *1. t is free for v in $\epsilon \Longrightarrow \psi$ if t is free for v in ψ.*

2. t is free for v in $\phi \Longrightarrow \psi$ if t is free for v in ϕ, and either $v \in intro(\phi)$ or t is free for v in ψ.

Figure 1: The Calculus for DPL

test axiom	$\dfrac{}{T \Longrightarrow T}$
transitivity	$\dfrac{\phi \Longrightarrow \psi \qquad \psi \Longrightarrow \chi}{\phi \Longrightarrow \chi}$ $intro(\psi) \cap fix(\chi) = \emptyset$
test contraction	$\dfrac{C_1 T; T C_2 \Longrightarrow \phi}{C_1 T C_2 \Longrightarrow \phi}$ $\dfrac{\phi \Longrightarrow C_1 T; T C_2}{\phi \Longrightarrow C_1 T C_2}$
quantifier contraction	$\dfrac{C_1 \exists v; \exists v\ C_2 \Longrightarrow \phi}{C_1 \exists v\ C_2 \Longrightarrow \phi}$ $\dfrac{\phi \Longrightarrow C_1 \exists v; \exists v\ C_2}{\phi \Longrightarrow C_1 \exists v\ C_2}$
test swap	$\dfrac{C_1 T_1; T_2 C_2 \Longrightarrow \phi}{C_1 T_2; T_1 C_2 \Longrightarrow \phi}$ $\dfrac{\phi \Longrightarrow C_1 T_1; T_2 C_2}{\phi \Longrightarrow C_1 T_2; T_1 C_2}$
quantifier swap	$\dfrac{C_1 \exists v; \exists w C_2 \Longrightarrow \phi}{C_1 \exists w; \exists v C_2 \Longrightarrow \phi}$ $\dfrac{\phi \Longrightarrow C_1 \exists v; \exists w C_2}{\phi \Longrightarrow C_1 \exists w; \exists v C_2}$
test-quantifier swap	$\dfrac{C_1 T; \exists v\ C_2 \Longrightarrow \phi}{C_1 \exists v; T C_2 \Longrightarrow \phi}$ $v \notin fix(T)$ $\dfrac{\phi \Longrightarrow C_1 T; \exists v\ C_2}{\phi \Longrightarrow C_1 \exists v; T C_2}$ $v \notin fix(T)$
quantifier-test swap	$\dfrac{C_1 \exists v; T C_2 \Longrightarrow \phi}{C_1 T; \exists v\ C_2 \Longrightarrow \phi}$ $v \notin fix(T)$ $\dfrac{\phi \Longrightarrow C_1 \exists v; T C_2}{\phi \Longrightarrow C_1 T; \exists v\ C_2}$ $v \notin fix(T)$
quantifier intro	$\dfrac{\phi \Longrightarrow [t/v]\psi}{\phi \Longrightarrow \exists v; \psi}$
var refreshment	$\dfrac{C_1 \exists v\ C_2 \Longrightarrow \phi}{C_1 \exists w [w/v] C_2 \Longrightarrow \phi}$ $w \notin intro(C_1) \cup fix(C_1)$
sequencing	$\dfrac{\psi \Longrightarrow \chi}{\phi; \psi \Longrightarrow \chi}$ $\dfrac{\phi \Longrightarrow \psi \qquad \phi \Longrightarrow \chi}{\phi \Longrightarrow \psi; \chi}$ $intro(\psi) \cap fix(\chi) = \emptyset$
negation	$\dfrac{\phi \Longrightarrow \psi}{\phi; \neg\psi \Longrightarrow \bot}$ $\dfrac{\phi; \psi \Longrightarrow \bot}{\phi \Longrightarrow \neg\psi}$
double negation	$\dfrac{\phi \Longrightarrow \neg\neg\psi}{\phi \Longrightarrow \psi}$ $\dfrac{\phi; \neg\neg\psi \Longrightarrow \bot}{\phi; \psi \Longrightarrow \bot}$

When a rule mentions a substitution $[t/v]\phi$ in the consequent of a sequent then the standard assumption is made hat t is free for v in ϕ. Note that in the rules below, $\Longrightarrow$ binds more strongly than $[t/v]$.When a rule mentions a substitution $C_1[t/v]C_2 \Longrightarrow \phi$ then it is assumed that t is free for v in $C_2 \Longrightarrow \phi$.

In the rules below we will use T as an abbreviation of formulas ϕ with $\mathit{intro}(\phi) = \emptyset$ (T for *Test* formula).

Structural Rules
Test Axiom

$$\frac{}{T \Longrightarrow T}$$

Soundness of Test Axiom If $s[\![T]\!]^{\mathcal{M}}t$ then $s = t$ (because T is a test) and thus there is an r with $t[\![T]\!]^{\mathcal{M}}r$, namely $r = s$. Thus, $T \models T$.

Transitivity Rule

$$\frac{\phi \Longrightarrow \psi \qquad \psi \Longrightarrow \chi}{\phi \Longrightarrow \chi}\ \mathit{intro}(\psi) \cap \mathit{fix}(\chi) = \emptyset$$

In case the side condition is not fulfilled, this can always be remedied by one or more applications of $\exists$ right (see below). Note that in the case where the 'cut' formula is a test the side condition of the rule is always fulfilled, and the rule assumes the familiar format:

$$\frac{\phi \Longrightarrow T \qquad T \Longrightarrow \chi}{\phi \Longrightarrow \chi}$$

Here is an example application:

$$\frac{Rxy \Longrightarrow Ryx \qquad Ryx \Longrightarrow \exists z; Ryz}{Rxy \Longrightarrow \exists z; Ryz}\ \text{trans}$$

Soundness of Transitivity Rule Suppose $\phi \models \psi$ and $\psi \models \chi$. Assume $s[\![\phi]\!]^{\mathcal{M}}t$. Then by $\phi \models \psi$, there is an r with $t[\![\psi]\!]^{\mathcal{M}}r$. By $\psi \models \chi$, there is a u with $r[\![\chi]\!]^{\mathcal{M}}u$. Then by Lemma 11, $t \sim_{\mathit{intro}(\psi)} r$. By the fact that $\mathit{intro}(\psi) \cap \mathit{fix}(\chi) = \emptyset$, $tI_{\mathit{fix}(\chi)}r$. By $tI_{\mathit{fix}(\chi)}r$, $r[\![\chi]\!]^{\mathcal{M}}u$ and Lemma 12 there is an u' with $t[\![\chi]\!]^{\mathcal{M}}u'$. Thus, $\phi \models \chi$.

Weakening Rules Due to the format where ; serves as the concatenation operator for formulas, the rule for ; left does double duty as an antecedent weakening rule. See below. Succedent weakening would be the step from $\phi \Longrightarrow \psi$ to $\phi \Longrightarrow \neg(\neg\psi; \neg\chi)$. This is taken care of by the negation rules. Again: see below.

Test Contraction Rules

$$\frac{C_1T; TC_2 \Longrightarrow \phi}{C_1TC_2 \Longrightarrow \phi} \qquad \frac{\phi \Longrightarrow C_1T; TC_2}{\phi \Longrightarrow C_1TC_2}$$

Note that contraction does not generally hold for formulas which are not tests. For instance,

$$Px; \exists x; Qx; Px; \exists x; Qx$$

demands that the interpretations of P and Q in a model have a non-empty intersection, while $Px; \exists x; Qx$ may be satisfied in models where this is not the case.

Soundness of Test Contraction Rules Immediate from the fact that if $intro(\phi) = \emptyset$ then $s[\![\phi]\!]^{\mathcal{M}}t$ iff $s = t$.

Quantifier Contraction Rules

$$\frac{C_1\exists v; \exists v\ C_2 \Longrightarrow \phi}{C_1\exists v\ C_2 \Longrightarrow \phi} \qquad \frac{\phi \Longrightarrow C_1\exists v; \exists v\ C_2}{\phi \Longrightarrow C_1\exists v\ C_2}$$

Soundness of Quantifier Contraction Immediate from the fact that $\sim_v$ is transitive.

Test Swap Rules

$$\frac{C_1T_1; T_2C_2 \Longrightarrow \phi}{C_1T_2; T_1C_2 \Longrightarrow \phi} \qquad \frac{\phi \Longrightarrow C_1T_1; T_2C_2}{\phi \Longrightarrow C_1T_2; T_1C_2}$$

Soundness of Test Swap Rules Follows from the fact that $s[\![T_1; T_2]\!]^{\mathcal{M}}t$ iff $s[\![T_2; T_1]\!]^{\mathcal{M}}t$.

Quantifier Swap Rules

$$\frac{C_1\exists v; \exists wC_2 \Longrightarrow \phi}{C_1\exists w; \exists vC_2 \Longrightarrow \phi} \qquad \frac{\phi \Longrightarrow C_1\exists v; \exists wC_2}{\phi \Longrightarrow C_1\exists w; \exists vC_2}$$

Soundness of Quantifier Swap Rules Follows from the fact that $s[\![\exists v; \exists w]\!]^{\mathcal{M}}t$ iff $s \sim_{\{v,w\}} t$ iff $s[\![\exists w; \exists v]\!]^{\mathcal{M}}t$.

Quantifier Movement Rules

$$\frac{C_1T; \exists v\ C_2 \Longrightarrow \phi}{C_1\exists v; TC_2 \Longrightarrow \phi}\ v \notin \mathit{fix}(T) \qquad \frac{\phi \Longrightarrow C_1T; \exists v\ C_2}{\phi \Longrightarrow C_1\exists v; TC_2}\ v \notin \mathit{fix}(T)$$

$$\frac{C_1\exists v; TC_2 \Longrightarrow \phi}{C_1T; \exists v\ C_2 \Longrightarrow \phi}\ v \notin \mathit{fix}(T) \qquad \frac{\phi \Longrightarrow C_1\exists v; TC_2}{\phi \Longrightarrow C_1T; \exists v\ C_2}\ v \notin \mathit{fix}(T)$$

These rules allow us to pull $\exists v$ through a test T. To avoid accidental capture of variables, the condition has to be fulfilled that $\exists v$ does not bind anything in T, in other words, that $v \notin \mathit{fix}(T)$.

Soundness of Quantifier Movement Rules The crucial observation is that for a test T with $v \notin \mathit{fix}(T)$ we have that for all models $\mathcal{M}$, $[\![\exists v; T]\!]^{\mathcal{M}} = [\![T; \exists v]\!]^{\mathcal{M}}$.

$\exists$ *Left* This is a special case of ; left. See below.

$\exists$ *Right*

$$\frac{\phi \Longrightarrow [t/v]\psi}{\phi \Longrightarrow \exists v; \psi}$$

Note that it is assumed that t is free for v in ψ. This assumption qis familiar from the Gentzen format of $\exists$-right in standard predicate logic. Here is an example application:

$$\frac{Rxx \Longrightarrow Rxx}{Rxx \Longrightarrow \exists y; Rxy}$$

Rxx equals $[x/y]Rxy$, so this is indeed a correct application of the rule.

Soundness of ∃ Right Assume $s[\![\phi]\!]^{\mathcal{M}}s_1$. Then by the soundness of the premiss there is an s_2 with

$$s_1[\![[t/v]\psi]\!]^{\mathcal{M}}s_2.$$

Assume $v \notin \mathit{intro}(\psi)$. Then by the substitution lemma,

$$s_1[v := [\![t]\!]^{\mathcal{M}}_{s_1}][\![\psi]\!]^{\mathcal{M}}s_2,$$

and thus there is an r with $s_1 \sim_v r$ and $r[\![\psi]\!]^{\mathcal{M}}s_2$. Assume $v \in \mathit{intro}(\psi)$. Then by the substitution lemma,

$$s_1[v := [\![t]\!]^{\mathcal{M}}_{s_1}][\![\psi]\!]^{\mathcal{M}}s_2[v := [\![t]\!]^{\mathcal{M}}_{s_1}],$$

and thus there is an r with $s_1 \sim_v r$ and $r[\![\psi]\!]^{\mathcal{M}}s_2[v := [\![t]\!]^{\mathcal{M}}_{s_1}]$. In both cases there is a t with $s_1[\![\exists v;\psi]\!]^{\mathcal{M}}t$. This establishes $\phi \models \exists v;\psi$.

Variable Refreshment Rule

$$\frac{C_1 \exists v\ C_2 \Longrightarrow \phi}{C_1 \exists w [w/v] C_2 \Longrightarrow \phi}\ w \notin \mathit{intro}(C_1) \cup \mathit{fix}(C_1)$$

Variable refreshment allows the liberation of a captured variable, e.g., of the first two occurrences of x in $\exists x; Px; \exists x; Qx$, by means of replacement by a variable that does not occur as an introduced or active variable in the left context in the given sequent:

$$\frac{\exists x; Px; \exists x; Qx \Longrightarrow Qx}{\exists y; Py; \exists x; Qx \Longrightarrow Qx}$$

It is also possible to change the other occurrences of x in the same example. The following is also a correct application of the rule:

$$\frac{\exists x; Px; \exists x; Qx \Longrightarrow Qx}{\exists x; Px; \exists y; Qy \Longrightarrow Qy}$$

Note that the rule can also be used to recycle a variable:

$$\frac{\exists y; Py; \exists x; Qx \Longrightarrow Qx}{\exists x; Px; \exists x; Qx \Longrightarrow Qx}$$

This application is also correct, for

$$(\exists x; Px; \exists x; Qx \Longrightarrow Qx) = (\exists x; [x/y]Py; \exists x; Qx \Longrightarrow Qx).$$

Soundness of Variable Refreshment Assume $s[\![C_1 \exists w[w/v]C_2]\!]^{\mathcal{M}}r$. Then there are s_1, s_2 with

$$s[\![C_1]\!]^{\mathcal{M}}s_1, s_1 \sim_w s_2, s_2[\![[w/v]C_2]\!]^{\mathcal{M}}r.$$

Let $s' := s[w := s_2(w)]$ and $s'_1 := s_1[w := s_2(w)]$. Because $w \notin \mathit{intro}(C_1)$ and $w \notin \mathit{fix}(C_1)$, we get by Lemmas 11 and 12 that $s'[\![C_1]\!]^{\mathcal{M}}s'_1$. From $s_1 \sim_w s_2$ we get that $s'_1 = s_2$, and therefore $s'_1 \sim_v s_2[v := s_2(w)]$. Suppose $v \in \mathit{intro}(C_2)$. Then by Lemma 10, $s_2[v := s_2(w)][\![C_2]\!]^{\mathcal{M}}r$. Therefore $s'[\![C_1 \exists v\ C_1]\!]^{\mathcal{M}}r$. By the soundness of the premiss it follows from $s'[\![C_1 \exists v C_1]\!]^{\mathcal{M}}r$ that there is a t with $r[\![\phi]\!]^{\mathcal{M}}t$. This establishes $C_1 \exists w; [w/v]C_2 \models \phi$. Suppose on the other hand that $v \notin \mathit{intro}(C_2)$. Then the substitution lemma yields that $s_2[v := s_2(w)][\![C_2]\!]^{\mathcal{M}}r[v := s_2(w)]$. Therefore $s'[\![C_1 \exists v\ C_1]\!]^{\mathcal{M}}r[v := s_2(w)]$. By the soundness of the premiss it follows from $s'[\![C_1 \exists v C_1]\!]^{\mathcal{M}}r[v := s_2(w)]$ that there is a t with $r[v := s_2(w)][\![\phi]\!]^{\mathcal{M}}t$. By another application of the substitution lemma we get from this that there is a t' with $r[\![[w/v]\phi]\!]^{\mathcal{M}}t'$. This establishes $C_1 \exists w; [w/v]C_2 \models [w/v]\phi$.

; Left and Right

$$\frac{\psi \Longrightarrow \chi}{\phi; \psi \Longrightarrow \chi} \qquad\qquad \frac{\phi \Longrightarrow \psi \qquad \phi \Longrightarrow \chi}{\phi \Longrightarrow \psi; \chi}\ \mathit{intro}(\psi) \cap \mathit{fix}(\chi) = \emptyset$$

The first of these does double duty as a left weakening rule. Antecedent weakening is always extension on the lefthand side. This is because extension on the righthand-side might introduce new variable bindings that extend to the consequent. Weakening with a test is valid anywhere in the antecedent; the swap rules account for that.
An example application of the rule for ; right is:

$$\frac{Rxx \Longrightarrow \exists y; Ryx \qquad Rxx \Longrightarrow \exists z; Rxz}{Rxx \Longrightarrow \exists y; Ryx; \exists z; Rxz}\ ;\ \text{right}$$

In case the condition on the rule for ; right is not satisfied, e.g. for the two sequents $\neg Px; \exists x; Px \Longrightarrow \exists x; \neg Px$ and $\neg Px; \exists x; Px \Longrightarrow Px$, this can always be remedied by one or more applications of $\exists$ Right to the second premiss.

Soundness of ; Left Suppose $s[\![\phi;\psi]\!]^{\mathcal{M}}t$. Then there is an s' with $s[\![\phi]\!]^{\mathcal{M}}s'$ and $s'[\![\psi]\!]^{\mathcal{M}}t$. From $s'[\![\psi]\!]^{\mathcal{M}}t$ and the soundness of the premiss it follows that there is an r with $t[\![\chi]\!]^{\mathcal{M}}r$. This proves that $\phi; \psi \models \chi$.

Soundness of ; Right Assume $s[\![\phi]\!]^{\mathcal{M}}t$. By the soundness of the first premiss, there is an r with $t[\![\psi]\!]^{\mathcal{M}}r$. By the soundness of the second premiss, there is an r' with $t[\![\chi]\!]^{\mathcal{M}}r'$.
From $t[\![\psi]\!]^{\mathcal{M}}r$ we get by Lemma 11 that $t \sim_{\mathit{intro}(\psi)} r$ and from the fact that $\mathit{intro}(\psi) \cap \mathit{fix}(\chi) = \emptyset$ we may conclude, by Lemma 12, that there is an r'' with $r[\![\chi]\!]^{\mathcal{M}}r''$, and we have established that $\phi \models \psi; \chi$.

¬ Left and Right

$$\frac{\phi \Longrightarrow \psi}{\phi; \neg\psi \Longrightarrow \bot} \qquad\qquad \frac{\phi; \psi \Longrightarrow \bot}{\phi \Longrightarrow \neg\psi}$$

Soundness of ¬ Left Assume $s[\![\phi;\neg\psi]\!]^{\mathcal{M}}t$. Then $s[\![\phi]\!]^{\mathcal{M}}t$ and there is no r with $t[\![\psi]\!]^{\mathcal{M}}r$. Contradiction with the soundness of the premiss. This establishes $\phi; \neg\psi \models \bot$.

Soundness of ¬ Right Assume $s[\![\phi]\!]^{\mathcal{M}}t$. Then by the soundness of the premiss, there is no r with $t[\![\psi]\!]^{\mathcal{M}}r$. This establishes $\phi \models \neg\psi$.

Double Negation Rules

$$\frac{\phi \Longrightarrow \neg\neg\psi}{\phi \Longrightarrow \psi} \qquad\qquad \frac{\phi; \neg\neg\psi \Longrightarrow \bot}{\phi; \psi \Longrightarrow \bot}$$

Soundness of Double Negation Rules For Double Negation Left, assume $s[\![\phi]\!]^{\mathcal{M}}t$. Then by the soundness of the premiss, there is no r with $t[\![\neg\psi]\!]^{\mathcal{M}}r$. By the fact that $\neg\psi$ is a test, this means that it is not the case that $t[\![\neg\psi]\!]^{\mathcal{M}}t$. But then there is an r with $t[\![\psi]\!]^{\mathcal{M}}r$. This establishes $\phi \models \psi$. The soundness of Double Negation Right is established similarly.
This completes the presentation of the calculus. As we have checked the soundness of every rule as we went along, we have:

Theorem 15 *The Calculus for DPL is sound.*

Proof. Induction on the structure of a proof tree for $\phi \Longrightarrow \psi$. □

4. Derivable Rules for DPL Reasoning

Proposition 16 (Contradiction Rule) *The following rule is derivable:*

$$\frac{\phi; \neg\psi \Longrightarrow \neg\chi \qquad \phi; \neg\psi \Longrightarrow \chi}{\phi \Longrightarrow \psi}$$

Proof. Here is a derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\phi; \neg\psi \Longrightarrow \neg\chi \qquad \phi; \neg\psi \Longrightarrow \chi}{\phi; \neg\psi \Longrightarrow \neg\chi; \chi}\,;\mathrm{r} \qquad \dfrac{\dfrac{\dfrac{}{\neg\chi \Longrightarrow \neg\chi}\,\text{test axiom}}{\neg\chi; \neg\neg\chi \Longrightarrow \bot}\,\neg\mathrm{l}}{\neg\chi; \chi \Longrightarrow \bot}\,\mathrm{dn}}{\phi; \neg\psi \Longrightarrow \bot}\,\mathrm{tr}}{\phi \Longrightarrow \neg\neg\psi}\,\neg\mathrm{r}}{\phi \Longrightarrow \psi}\,\mathrm{dn}$$

□

Proposition 17 (Cases Rule) *The following rule is derivable:*

$$\frac{\phi; \neg\psi \Longrightarrow \chi \qquad \phi; \neg\neg\psi \Longrightarrow \chi}{\phi \Longrightarrow \chi}$$

Proof. Here is a derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\phi; \neg\psi \Longrightarrow \chi}{\phi; \neg\psi; \neg\chi \Longrightarrow \bot}\,\neg\mathrm{l}}{\phi; \neg\chi; \neg\psi \Longrightarrow \bot}\,\mathrm{swap}}{\phi; \neg\chi \Longrightarrow \neg\neg\psi}\,\neg\mathrm{r} \qquad \dfrac{\dfrac{\dfrac{\phi; \neg\neg\psi \Longrightarrow \chi}{\phi; \neg\neg\psi; \neg\chi \Longrightarrow \bot}\,\neg\mathrm{l}}{\phi; \neg\chi; \neg\neg\psi \Longrightarrow \bot}\,\mathrm{swap}}{\phi; \neg\chi \Longrightarrow \neg\neg\neg\psi}\,\neg\mathrm{r}}{\phi \Longrightarrow \chi}\,\mathrm{contrad}$$

□

5. Completeness of the Calculus

To establish the completeness of the calculus, assume that $\phi \not\Longrightarrow \psi$. We will construct a countermodel by a slight modification of the standard Henkin construction for the completeness of classical predicate logic.

Definition 18 $\phi \vdash_\Gamma \psi :\Leftrightarrow$ *there are* $\phi_1, \ldots, \phi_n \in \Gamma$ *with* $\phi; \neg\neg\phi_1; \cdots; \neg\neg\phi_n \Longrightarrow \psi$. *We say that* Γ *is consistent with* ϕ *if there is a* ψ *with* $\phi \not\vdash_\Gamma \psi$.
We call Γ *negation complete with respect to* ϕ *if for every* ψ *either* $\phi \vdash_\Gamma \psi$ *or* $\phi \vdash_\Gamma \neg\psi$.
Call a formula ψ *fixed in* v *if* $v \in fix(\psi)$. Γ *has witnesses for* ϕ *if for every formula* ψ *fixed in* v *such that* $\phi \vdash_\Gamma \exists v; \psi$ *there is a* c *for which* $\neg\neg(\exists v; \psi) \to [c/v]\psi \in \Gamma$.

Note that in the definition of $\phi \vdash_\Gamma \psi$ the extra premisses from Γ do not extend the 'anaphoric context': the context change potential of the premisses from Γ is blocked off by means of double negation signs. This is a key element in the canonical model construction below.

Proposition 19 *If* $\phi \not\vdash_\Gamma \psi$ *then at least one of* $\Gamma \cup \{\psi\}$, $\Gamma \cup \{\neg\psi\}$ *is consistent with* ϕ.

Proof. Use the Cases Rule. □

Let $\chi_1, \ldots$ be a list of all formulas of L that are fixed in v. Let $C_0 := c_1^0, \ldots$ be a list of fresh individual constants. Let L_0 be $L(C_0)$ (the result of adding the constants C_0 to L.

$$\Delta_0 := \{\neg\neg(\exists v; \chi_i) \to [c_i^0/v]\chi_i \mid 1 \leq i\}.$$

Let $\chi_1^m, \ldots$ be a list of all formulas that are fixed in v which occur in L_m. Let $C_{m+1} := c_1^{m+1}, \ldots$ be a list of fresh individual constants. Let $L_{m+1} := L_m(C_{m+1})$.

$$\Delta_{m+1} := \{\neg\neg(\exists v; \chi_i^{m+1}) \rightarrow [c_i^{m+1}/v]\chi_i^{m+1} \mid 1 \leq i\}.$$

Let $C := \bigcup_m C_m$, and let Δ be the set of $L(C)$ formulas given by:

$$\Delta := \bigcup_m \Delta_m.$$

Proposition 20 *If Γ consists of $L(C)$ formulas, and $\Gamma \supseteq \Delta$, then Γ has witnesses for ϕ.*

Proof. Take some ψ fixed in v with $\phi \vdash_\Gamma \exists v; \psi$. Then $\psi \in L_m$ for some m. So there is some $c \in C$ with $\neg\neg(\exists v; \psi) \rightarrow [c/v]\psi \in \Delta_{m+1}$. So $\neg\neg(\exists v; \psi) \rightarrow [c/v]\psi) \in \Delta \subseteq \Gamma$. □

Proposition 21 *If Γ is consistent with ϕ then there is a $\Gamma' \supseteq \Gamma$ which is consistent with ϕ, negation complete with respect to ϕ, and has witnesses for ϕ.*

Proof. Assume Γ consistent with ϕ. Let $\chi_1, \ldots, \chi_i, \ldots$ be an enumeration of all bounded formulas of the language $L(C)$. Extend Γ as follows to a Γ' with the required properties.

$$\Gamma_0 := \Gamma \cup \Delta$$

$$\Gamma_{m+1} := \begin{cases} \Gamma_m \cup \{\chi_m\}) & \text{if } \Gamma_m \cup \{\chi_m\}) \text{ consistent with } \phi, \\ \Gamma_m & \text{otherwise.} \end{cases}$$

$$\Gamma' := \bigcup_m \Gamma_m$$

$\Gamma' \supseteq \Delta$, so by Proposition 20 Γ' has witnesses for ϕ.
Assume Γ' is inconsistent with ϕ. Then some Γ_m has to be inconsistent with ϕ and contradiction with Proposition 19. So Γ' is consistent with ϕ.
Finally, Γ' is negation complete by construction. □

Definition 22 (Canonical Model) *Let Γ be consistent with ϕ, be negation complete with respect to ϕ, and have witnesses for ϕ. Then $\mathcal{M}_\Gamma = (D, I)$ is defined as follows. $D :=$ the set of variables V together with the set of constants C occurring in $\Gamma \cup \{\phi\}$. For all terms of the language, let $I(t) := t$. Let $I(P) := \{\langle t_1, \ldots t_k \rangle \mid \phi \vdash_\Gamma P(t_1, \ldots, t_k)\}$.*

Lemma 23 (Satisfaction Lemma) *Let Γ be consistent with ϕ, be negation complete with respect to ϕ, and have witnesses for ϕ. Then for all ξ: $\phi \vdash_\Gamma \xi$ iff $\exists t$ with $i[\![\xi]\!]^{\mathcal{M}_\Gamma} t$, where i is the identity assignment.*

Proof. Induction on the structure of ξ.
$\phi \not\vdash_\Gamma \bot$ by the fact that ϕ is consistent and Γ is consistent with ϕ.
$\phi \vdash_\Gamma Pt_1 \cdots t_n$ iff $\langle [\![t_1]\!]_i^{\mathcal{M}_\Gamma}, \ldots, [\![t_n]\!]_i^{\mathcal{M}_\Gamma} \rangle \in I(P)$ iff $i[\![Pt_1 \cdots t_n]\!]^{\mathcal{M}_\Gamma} i$.
$\phi \vdash_\Gamma \neg\xi'$ iff (Γ negation complete) $\phi \not\vdash_\Gamma \xi'$ iff (i.h.) there is no t with $i[\![\xi']\!]^{\mathcal{M}_\Gamma} t$ iff (semantic clause for $\neg$) $i[\![\neg\xi']\!]^{\mathcal{M}_\Gamma} i$.
$\phi \not\vdash_\Gamma \bot; \xi'$, for assume the contrary. The following proof tree establishes that $\bot; \xi' \Longrightarrow \bot$:

$$\dfrac{\dfrac{\dfrac{}{\bot \Longrightarrow \bot}\ \text{test axiom}}{\xi'; \bot \Longrightarrow \bot}\ ;\text{left}}{\bot; \xi' \Longrightarrow \bot}\ \text{test swaps}$$

Thus, from $\phi \vdash_\Gamma \bot; \xi'$ and $\bot; \xi' \Longrightarrow \bot$ we get by transitivity that $\phi \vdash_\Gamma \bot$, and contradiction with the facts that ϕ is consistent and that Γ is consistent with ϕ.
$\phi \vdash_\Gamma Pt_1 \cdots t_n; \xi'$ iff $\phi \vdash_\Gamma Pt_1 \cdots t_n$ and $\phi \vdash_\Gamma \xi'$ iff $\langle [\![t_1]\!]_i^{\mathcal{M}_\Gamma}, \dots, [\![t_n]\!]_i^{\mathcal{M}_\Gamma} \rangle \in I(P)$ and $\phi \vdash_\Gamma \xi'$ iff $i[\![Pt_1 \cdots t_n]\!]^{\mathcal{M}_\Gamma} i$ and (i.h.) there is a t with $i[\![\xi']\!]^{\mathcal{M}_\Gamma} t$ iff there is a t with $i[\![Pt_1 \cdots t_n; \xi']\!]^{\mathcal{M}_\Gamma} t$.
$\phi \vdash_\Gamma \exists v; \xi'$ iff (Γ has witnesses) $\phi \vdash_\Gamma [c/v]\xi'$ iff (i.h.) there is a t with $i[\![[c/v]\xi']\!]^{\mathcal{M}_\Gamma} t$ iff $i[\![\exists v; \xi']\!]^{\mathcal{M}_\Gamma} t$.
$\phi \vdash_\Gamma \neg\xi_1; \xi_2$ iff $\phi \vdash_\Gamma \neg\xi_1$ and $\phi \vdash_\Gamma \xi_2$ iff (i.h. twice) $i[\![\neg\xi_1]\!]^{\mathcal{M}_\Gamma} i$ and there is a t with $i[\![\xi_2]\!]^{\mathcal{M}_\Gamma} t$ iff there is a t with $i[\![\neg\xi_1; \xi_2]\!]^{\mathcal{M}_\Gamma} t$.

□

Proposition 24 *Let Γ be consistent with ϕ, be negation complete with respect to ϕ, and have witnesses for ϕ. Then $i[\![\phi]\!]^{\mathcal{M}_\Gamma} i$.*

Proof. Let ϕ_1 be $\exists w_1; \cdots; \exists w_n; \phi$, where $\{w_1, \dots, w_n\} = \mathit{fix}(\phi)$. Next, let ϕ_2 be the result of applying the first variable refreshment rule to ϕ_1 as many times as necessary to ensure that all patterns of the form $\exists v; \psi; \exists v$ in ϕ_1 are replaced by patterns $\exists w; [w/v]\psi; \exists v$. Note that application of the first variable refreshment rule does not affect the set of introduced variables. Finally, let $\exists v_1; \cdots; \exists v_k; \phi'$ be the result of moving all quantifiers of ϕ_2 to the front. This is possible, as the conditions for pulling the quantifiers through tests on their lefthand sides are fulfilled in ϕ_2. Because no introduced variables of ϕ_1 were removed, we have that

$$\{w_1, \dots, w_n\} \subseteq \{v_1, \dots, v_k\},$$

or in other words,

$$\mathit{fix}(\phi) \subseteq \{v_1, \dots, v_k\}.$$

Let $\{u_1, \dots, u_m\}$ be the result of removing the set $\mathit{fix}(\phi)$ from $\{v_1, \dots, v_k\}$.
Then $\exists u_1; \cdots; \exists u_m; \phi'$ is semantically equivalent to ϕ. Moreover, ϕ' is a test, and we have:

$$\dfrac{\dfrac{\dfrac{}{\phi' \Longrightarrow \phi'}\ \text{test axiom}}{\exists u_1; \cdots; u_m; \phi' \Longrightarrow \phi'}\ ;\ \text{left}}{\phi \Longrightarrow \phi'}\ \text{quantifier swaps} \ + \ \text{variable refreshments}$$

Therefore, $\phi \vdash_\Gamma \phi'$, and we get from the satisfaction lemma, plus the fact that ϕ' is a test, that $i[\![\phi']\!]^{\mathcal{M}_\Gamma} i$. Also, by definition of the semantics for $\exists v$, we have that $i[\![\exists u_1; \cdots; u_m]\!]^{\mathcal{M}_\Gamma} i$. By the semantic equivalence of ϕ and $\exists u_1; \cdots; \exists u_m; \phi'$ we get $i[\![\phi]\!]^{\mathcal{M}_\Gamma} i$. □

Theorem 25 (Completeness) *If $\phi \models \psi$ then $\phi \Longrightarrow \psi$.*

Proof. Assume $\phi \not\Longrightarrow \psi$. Then $\neg\psi$ is consistent with ϕ. By proposition 21, there is a $\Gamma \supseteq \{\neg\psi\}$ which is consistent with ϕ, is negation complete with respect to ϕ, and has witnesses for ϕ. Construct the canonical model and apply the satisfaction lemma to get:

$$i[\![\neg\psi]\!]^{\mathcal{M}_\Gamma} i.$$

By the semantic clause for negation we have that for all t:

$$\text{not } i[\![\psi]\!]^{\mathcal{M}_\Gamma} t.$$

By proposition 24:

$$i[\![\phi]\!]^{\mathcal{M}_\Gamma} i.$$

This proves $\phi \not\models \psi$. □

6. Anaphoric Reasoning with Equality

Anaphoric linking makes extensive use of equality. See Van Eijck [2] for an in-depth analysis of the use of equality in anaphoric descriptions. An anaphoric definite description like *the garden* can be treated as a definiteness quantifier followed by a link to a contextually available referent. The translation of *He sprinkles the garden* would then be something like $\iota x : (x \doteq y; Gx); Szx$. Also, the determiner *another* often has an implicit anaphoric element. In such cases, the treatment involves non-identity links to contextually available referents. *He met another woman* gets a translation like $\exists x; x \neq y; Wx; Mzx)$. Below we indicate briefly how to handle equality, while leaving the axiomatisation of definiteness in the present framework for another occasion.

For the treatment of equality, add expressions $t_1 \doteq t_2$ to the language (we assume that $t_1 \neq t_2$ is an abbreviation of $\neg(t_1 \doteq t_2)$). Equalities are tests, or, in other words, $\mathit{intro}(t_1 \doteq t_2) = \emptyset$. The semantics of equality is as you would expect:

$$s[\![t_1 \doteq t_2]\!]^{\mathcal{M}} r \quad \text{iff} \quad s = r \text{ and } [\![t_1]\!]^{\mathcal{M}}_s \text{ equals } [\![t_2]\!]^{\mathcal{M}}_s.$$

The following rules must be added to the calculus to deal with equality statements:

Reflexivity Axiom

$$\frac{}{\phi \Longrightarrow t \doteq t}$$

Soundness of Reflexivity Axiom The axiom expresses that equality is reflexive.

Substitution Rule For this we need the notion $[t_1/t_2]$ of substitution of terms for arbitrary terms. This is defined in the obvious way.

$$\frac{\phi \Longrightarrow [t_1/t_2]\psi}{\phi; t_1 \doteq t_2 \Longrightarrow \psi} \; t_1, t_2 \in C \cup \mathit{intro}(\phi)$$

Soundness of the Substitution Rule Assume $s[\![\phi; t_1 \doteq t_2]\!]^{\mathcal{M}} r$. Then $s[\![\phi]\!]^{\mathcal{M}} r$, and $[\![t_1]\!]^{\mathcal{M}}_r = [\![t_2]\!]^{\mathcal{M}}_r$. By the soundness of the premiss, there is an r' with $r[\![[t_1/t_2]\psi]\!]^{\mathcal{M}} r'$. Therefore, $r[\![\psi]\!]^{\mathcal{M}} r'$. This shows $\phi; t_1 \doteq t_2 \models \psi$.

The completeness of the anaphoric DPL calculus with equality is proved by modifying the Henkin construction in the usual way (taking equivalence classes of terms under provable equality as elements of the canonical model).

7. A Variation on the Theme

To justify the plural in the title of this paper, first note that the approach to axiomatising dynamic predicate logic carries over without a hitch to the analysis of Discourse Representation Theory [8, 5]. The version for DRT looks even more elegant because the shift rules are absorbed by the set theoretic notation for DRSs. This reflects the fact that the syntax of DRT is more abstract than that of DPL. To keep as close as possible to the previous calculus, we consider a version of DRT where DRS negation is primitive and $D_1 \Rightarrow D_2$ is defined in terms of negation. We can view a Discourse Representation Structure or DRS as a triple consisting of a set of fixed referents F, a set of introduced referents I, and a set of conditions $C_1 \cdots C_n$ constrained by the requirement that the free variables in the C_i must be among $F \cup I$.

Concretely, the syntax of DRT looks like this:

Definition 26 (DRT)

$$\begin{array}{lll} t & ::= & c \mid v \\ C & ::= & \bot \mid Pt_1 \cdots t_n \mid \neg D \\ D & ::= & \begin{array}{|c|c|}\hline v_1 \cdots v_n & v_{n+1} \cdots v_m \\ \hline \multicolumn{2}{|c|}{C_1 \cdots C_k} \\ \hline \end{array} \end{array}$$

Figure 2: The Calculus for DRT

test axiom

$$\frac{}{\begin{array}{|c|c|}\hline F & \emptyset \\ \hline \multicolumn{2}{|c|}{C} \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|}\hline F & \emptyset \\ \hline \multicolumn{2}{|c|}{C} \\ \hline \end{array}} \; M(C) \subseteq F$$

transitivity

$$\frac{D \Longrightarrow \begin{array}{|c|c|}\hline F & \emptyset \\ \hline \multicolumn{2}{|c|}{C_1} \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline F & \emptyset \\ \hline \multicolumn{2}{|c|}{C_1} \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C_2} \\ \hline \end{array}}{D \Longrightarrow \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C_2} \\ \hline \end{array}}$$

marker intro

$$\frac{D \Longrightarrow \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{[t/v]\; C} \\ \hline \end{array}}{D \Longrightarrow \begin{array}{|c|c|}\hline F & \{v\} \cup I \\ \hline \multicolumn{2}{|c|}{C} \\ \hline \end{array}} \; v \notin F$$

marker shift

$$\frac{\begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C} \\ \hline \end{array} \Longrightarrow D}{\begin{array}{|c|c|}\hline F - I_0 & I_0 \cup I \\ \hline \multicolumn{2}{|c|}{C} \\ \hline \end{array} \Longrightarrow D}$$

sequencing-l

$$\frac{\begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C} \\ \hline \end{array} \Longrightarrow D}{\begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C_0 \cup C} \\ \hline \end{array} \Longrightarrow D} \; M(C_0) \subseteq F \cup I$$

sequencing-r

$$\frac{D \Longrightarrow \begin{array}{|c|c|}\hline F & I_1 \\ \hline \multicolumn{2}{|c|}{C_1} \\ \hline \end{array} \qquad D \Longrightarrow \begin{array}{|c|c|}\hline F & I_2 \\ \hline \multicolumn{2}{|c|}{C_2} \\ \hline \end{array}}{D \Longrightarrow \begin{array}{|c|c|}\hline F & I_1 \cup I_2 \\ \hline \multicolumn{2}{|c|}{C_1 \cup C_2} \\ \hline \end{array}} \; I_1 \cap I_2 = \emptyset$$

negation

$$\frac{\begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C} \\ \hline \end{array} \Longrightarrow D}{\begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C \cup \{\neg D\}} \\ \hline \end{array} \Longrightarrow \bot} \qquad \frac{\begin{array}{|c|c|}\hline F & I_1 \cup I_2 \\ \hline \multicolumn{2}{|c|}{C_1 \cup C_2} \\ \hline \end{array} \Longrightarrow \bot}{\begin{array}{|c|c|}\hline F & I_1 \\ \hline \multicolumn{2}{|c|}{C_1} \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|}\hline F \cup I_1 & \emptyset \\ \hline \multicolumn{2}{|c|}{\neg \begin{array}{|c|c|}\hline F \cup I_1 & I_2 \\ \hline \multicolumn{2}{|c|}{C_2} \\ \hline \end{array}} \\ \hline \end{array}} \; M(C_1) \cap I_2 = \emptyset$$

d-negation

$$\frac{D \Longrightarrow \begin{array}{|c|c|}\hline F & \emptyset \\ \hline \multicolumn{2}{|c|}{\neg \begin{array}{|c|c|}\hline F & \emptyset \\ \hline \multicolumn{2}{|c|}{\neg \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C} \\ \hline \end{array}} \\ \hline \end{array}} \\ \hline \end{array}}{D \Longrightarrow \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C} \\ \hline \end{array}} \qquad \frac{\begin{array}{|c|c|}\hline F & I_1 \\ \hline \multicolumn{2}{|c|}{C_1 \cup \{\neg \begin{array}{|c|c|}\hline F \cup I_1 & \emptyset \\ \hline \multicolumn{2}{|c|}{\neg \begin{array}{|c|c|}\hline F \cup I_1 & I_2 \\ \hline \multicolumn{2}{|c|}{C_2} \\ \hline \end{array}} \\ \hline \end{array} \}} \\ \hline \end{array} \Longrightarrow \bot}{\begin{array}{|c|c|}\hline F & I_1 \cup I_2 \\ \hline \multicolumn{2}{|c|}{C_1 \cup C_2} \\ \hline \end{array} \Longrightarrow \bot}$$

The (active) markers of a condition C or DRS D are given by:

$$M(v) := \{v\}, M(c) := \emptyset, M(Pt_1 \cdots t_n) := \cup_i M(t_i),$$

$$M(\begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C} \\ \hline\end{array}) := F \cup I, M(\neg \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C} \\ \hline\end{array}) := F.$$

Conditions on the formation rule for a DRS $\begin{array}{|c|c|}\hline v_1 \cdots v_n & v_{n+1} \cdots v_m \\ \hline \multicolumn{2}{|c|}{C_1 \cdots C_k} \\ \hline\end{array}$:

1. $\{v_1 \cdots v_n\} \cap \{v_{n+1} \cdots v_m\} = \emptyset$,
2. $\cup_i M(C_i) \subseteq \{v_1 \cdots v_m\}$.

We can now define the condition

$$\begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C_1 \cdots C_n} \\ \hline\end{array} \Rightarrow D$$

as

$$\neg \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C_1 \cdots C_n, \neg D} \\ \hline\end{array}.$$

Here is a semantics in terms of partial assignments, following the original set-up in [8].

Definition 27 (Semantics of DRT)

$$\begin{array}{lll}
\mathcal{M}, f \models \bot & & \textit{never} \\
\mathcal{M}, f \models Pt_1 \cdots t_n & \textit{iff} & \langle [\![t_1]\!]_f^{\mathcal{M}}, \ldots, [\![t_n]\!]_f^{\mathcal{M}} \rangle \in P^{\mathcal{M}} \\
\mathcal{M}, f \models \neg D & \textit{iff} & \textit{there is no } g \textit{ with } \mathcal{M}, f, g \models D
\end{array}$$

$$\mathcal{M}, f, g \models \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C_1 \cdots C_n} \\ \hline\end{array} \quad \textit{iff} \quad \begin{array}{l} f : F \to dom(\mathcal{M}), \\ g : F \cup I \to dom(\mathcal{M}), \\ M, g \models C_1, \ \ldots \ , M, g \models C_n. \end{array}$$

Definition 28 (DRT Consequence)

$$\begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C_1 \cdots C_n} \\ \hline\end{array} \models \begin{array}{|c|c|}\hline F \cup I & I' \\ \hline \multicolumn{2}{|c|}{C_{n+1} \cdots C_m} \\ \hline\end{array}$$

iff for all $\mathcal{M}, f, g$ *with* $\mathcal{M}, f, g \models \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C_1 \cdots C_n} \\ \hline\end{array}$

there is an h *with* $\mathcal{M}, g, h \models \begin{array}{|c|c|}\hline F \cup I & I' \\ \hline \multicolumn{2}{|c|}{C_{n+1} \cdots C_m} \\ \hline\end{array}$.

A DRT calculus along similar lines as the calculus for DPL above is give in Figure 2 (lists $C_1 \cdots C_k$ abbreviated as C). The calculus uses substitution in constraints and DRSs. This notion is defined by:

$$\begin{array}{rcl}
[t/v] \ Pt_1 \cdots t_n & := & P[t/v]t_1 \cdots [t/v]t_n, \\
[t/v] \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C_1 \cdots C_n} \\ \hline\end{array} & := & \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{[t/v] \ C_1 \cdots [t/v] \ C_n} \\ \hline\end{array}
\end{array}$$

Of course, when a substitution $[t/v]$ is mentioned in a rule, it is assumed that t is free for v in D. It is also assumed that all DRSs mentioned in the rules satisfy the syntactic wellformedness conditions for DRSs.

Note that the counterpart to variable refreshment, marker refreshment, is not among the rules of the DRT calculus. Marker refreshment is the following rule:

$$\frac{\begin{array}{|c|c|}\hline F_1 & \{v\} \cup I_1 \\ \hline \multicolumn{2}{|c|}{C_1} \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|}\hline F_2 & I_2 \\ \hline \multicolumn{2}{|c|}{C_2} \\ \hline \end{array}}{\begin{array}{|c|c|}\hline F_1 & \{w\} \cup I_1 \\ \hline \multicolumn{2}{|c|}{[w/v]C_1} \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|}\hline \{w\} \cup F_2 & I_2 \\ \hline \multicolumn{2}{|c|}{[w/v]\ C_2} \\ \hline \end{array}}$$

It is easy to see that this is an admissible rule of the calculus, in the sense that adding the rule would not change the set of valid DRT sequences. To see why this is so, note that a proof for

$$\begin{array}{|c|c|}\hline F_1 & \{v\} \cup I_1 \\ \hline \multicolumn{2}{|c|}{C_1} \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|}\hline F_2 & I_2 \\ \hline \multicolumn{2}{|c|}{C_2} \\ \hline \end{array}$$

can be transformed into a proof for

$$\begin{array}{|c|c|}\hline F_1 & \{w\} \cup I_1 \\ \hline \multicolumn{2}{|c|}{[w/v]\ C_1} \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|}\hline \{w\} \cup F_2 & I_2 \\ \hline \multicolumn{2}{|c|}{[w/v]\ C_2} \\ \hline \end{array}$$

by simply changing appropriate occurrences of v to w.
Note that marker shift and sequencing left together give the following derived rule:

$$\frac{\begin{array}{|c|c|}\hline F_1 & I_1 \\ \hline \multicolumn{2}{|c|}{C_1} \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|}\hline F_2 & I_2 \\ \hline \multicolumn{2}{|c|}{C_2} \\ \hline \end{array}}{\begin{array}{|c|c|}\hline F_1 - I_0 & I_0 \cup I_1 \\ \hline \multicolumn{2}{|c|}{C_0 \cup C_1} \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|}\hline F_2 & I_2 \\ \hline \multicolumn{2}{|c|}{C_2} \\ \hline \end{array}}$$

Theorem 29 *The calculus for DRT is sound.*

Proof. Induction on the basis that the test axiom is sound and that the rules preserve soundness. Here is one example soundness check, for the rule of marker introduction. Assume $\mathcal{M}, f, g, \models D$. Then by the soundness of the premiss, there is an h with $\mathcal{M}, g, h \models \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{[t/v]\ C} \\ \hline \end{array}$. Thus, $\mathcal{M}, h \models [t/v]\ C$. Since $v \notin F$, h' given by $h'(v) = [\![t]\!]_h^{\mathcal{M}}$, and $h'(w) = h(w)$ for all $w \neq v$ for which h is defined extends g. By (an appropriate DRT version of) the substitution lemma, $\mathcal{M}, g, h' \models \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C} \\ \hline \end{array}$. This proves $D \models \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C} \\ \hline \end{array}$. □

Theorem 30 *The calculus for DRT is complete.*

Proof. Consider the following translation procedure from DRT to DPL:

$$\begin{array}{rcl} \bot^\circ & := & \bot \\ (Pt_1 \cdots t_n)^\circ & := & Pt_1 \cdots t_n \\ (\neg D)^\circ & := & \neg D^\bullet \end{array}$$

$$\begin{array}{|c|c|}\hline v_1 \cdots v_n & v_{n+1} \cdots v_m \\ \hline \multicolumn{2}{|c|}{C_1 \cdots C_k} \\ \hline \end{array}^{\bullet} \quad := \quad \exists v_{n+1}; \cdots ; \exists v_m; C_1^\circ; \cdots ; C_k^\circ$$

Use this procedure to translate a DRT sequent $D_1 \Longrightarrow D_2$ to a DPL sequent $D_1^\bullet \Longrightarrow D_2^\bullet$. Note that the translation maps valid DRT consequences to valid DPL consequences. Assume $D_1 \models D_2$. Then $D_1^\bullet \models D_2^\bullet$. By Theorem 25, $D_1^\bullet \Longrightarrow D_2^\bullet$. Next, transform the proof of $D_1^\bullet \Longrightarrow D_2^\bullet$ into one of $D_1 \Longrightarrow D_2$ making use of the correspondences between the axioms and rules of the two calculi. □

8. A Further Variation

Here is a variation on dynamic predicate logic suggested by Albert Visser in [13]. Visser proposes the following 'prudent' version of DPL. States are pairs (X, s) of a variable set X and an assignment s. The semantics of prudent DPL is given by:

Definition 31 (Semantics of Prudent DPL)

$$
\begin{array}{lll}
(X,s)[\![\bot]\!]^{\mathcal{M}}(Y,r) & & \textit{never} \\
(X,s)[\![\exists v]\!]^{\mathcal{M}}(Y,r) & \textit{iff} & (i)\ v \in X \wedge (X,s) = (Y,r),\ \textit{or} \\
& & (ii)\ v \notin X \wedge X \cup \{v\} = Y\ \textit{and}\ s \sim_v r \\
(X,s)[\![Pt_1 \cdots t_n]\!]^{\mathcal{M}}(Y,r) & \textit{iff} & (X,s) = (Y,r)\ \textit{and}\ \langle [\![t_1]\!]^{\mathcal{M}}_s, \ldots, [\![t_n]\!]^{\mathcal{M}}_s \rangle \in P^{\mathcal{M}} \\
(X,s)[\![\neg(\phi)]\!]^{\mathcal{M}}(Y,r) & \textit{iff} & (X,s) = (Y,r) \\
& & \textit{and there is no}\ (Y',r')\ \textit{with}\ (X,s)[\![\phi]\!]^{\mathcal{M}}(Y',r'). \\
(X,s)[\![A;\phi]\!]^{\mathcal{M}}(Y,r) & \textit{iff} & \textit{there is an}\ (X',s') \\
& & \textit{with}\ (X,s)[\![A]\!]^{\mathcal{M}}(X',s')\ \textit{and}\ (X',s')[\![\phi]\!]^{\mathcal{M}}(Y,r). \\
(X,s)[\![\neg(\phi_1);\phi_2]\!]^{\mathcal{M}}(Y,r) & \textit{iff} & (X,s)[\![\neg\phi_1]\!]^{\mathcal{M}}(X,s) \\
& & \textit{and}\ (X,s)[\![\phi_2]\!]^{\mathcal{M}}(Y,r).
\end{array}
$$

Here the set of 'activated' variables is incorporated in the relational semantics. This set is used in the semantic clause for the existential quantifier. In case variable v has already been activated before, $\exists v$ does nothing.

Semantic consequence for prudent DPL gets defined as $\phi \models \psi$ iff for all models $\mathcal{M}$, all assignments s, r and the variable set X with $(\emptyset, s)[\![\phi]\!]^{\mathcal{M}}(X, r)$ there are (X', r') with $(X, r)[\![\psi]\!]^{\mathcal{M}}(X', r')$.

The semantic change gets reflected in the calculus as follows:

- The restriction $v \notin \mathit{fix}(T)$ on test-quantifier swap (from $T; \exists v$ to $\exists v; T$) is removed.
- The two quantifier contraction rules get replaced by:

$$
\frac{\phi; \exists v\ C \Longrightarrow \psi}{\phi\ C \Longrightarrow \psi}\ v \in \mathit{intro}(\phi) \qquad\qquad \frac{\phi \Longrightarrow C_1 \exists v\ C_2}{\phi \Longrightarrow C_1\ C_2}\ v \in \mathit{intro}(\phi C_1)
$$

- The rule of variable refreshment is removed from the calculus. (This rule is not sound for the prudent semantics.)

It can be proved that this modified calculus is sound and complete for prudent DPL, and the fact that the modified calculus is a simplification of the original can be taken as a point in favour of prudent DPL.

9. Related Work

The present approach to axiomatising dynamic predicate logics derives from a calculus for dynamic reasoning without variables [6] where variables have been replaced by De Bruyn indices (familiar from implementations of lambda calculus), and destructive assignment is replaced by a stack push operation. Such a variable free approach to dynamics has clear advantages over the destructive dynamics of variable assignment if one is interested in incrementality of updates.

An alternative to the present calculus is given in unpublished work by Frank Veltman [12]. Veltman analyses a more baroque version of DPL, where a distinction is made between the 'text connective' ; and the 'sentential connective' $\wedge$. Their semantic interpretations are the same; the difference between them resides in a syntactic restriction on the text connective to the effect that it cannot be outscoped by negation. Also, in Veltman's system free text variables are forbidden on the grounds that these are hard to analyse in terms of partial variable assignments.

Veltman's central notion of substitution is defined for variables that are 'free in ϕ after ψ'. In our terminology, an occurrence of v in ϕ is free in ϕ after ψ if $v \in \mathit{fix}(\phi)$ and $v \notin \mathit{intro}(\psi)$ both hold.

Because Veltman's calculus does not have our division of labour between introduction and elimination rules on one hand and shift rules on the other, the introduction and elimination rules have rather involved side conditions. Veltman suggests that completeness of these rules can be proved by means of a syntactic tour through a variety of equivalent tableaux calculi, but [12] contains only a very sketchy roadmap for carrying this out.
The calculus for DRT given in Section 7 demonstrates how keeping track of both the set of fixed discourse referents and the set of introduced discourse referents of a DRS engenders a considerable simplification of previous calculi for DRT (see e.g. [9, 10]). The approach is very much in the spirit of Visser [13].

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References

1. J. Barwise. Noun phrases, generalized quantifiers and anaphora. In P. Gärdenfors, editor, *Generalized Quantifiers: linguistic and logical approaches*, pages 1–30. Reidel, Dordrecht, 1987.
2. J. van Eijck. The dynamics of description. *Journal of Semantics*, 10:239–267, 1993.
3. J. van Eijck. Axiomatizing dynamic predicate logic with quantified dynamic logic. In J. van Eijck and A. Visser, editors, *Logic and Information Flow*, pages 30–48. MIT Press, Cambridge Mass, 1994.
4. J. van Eijck and F.J. de Vries. Dynamic interpretation and Hoare deduction. *Journal of Logic, Language, and Information*, 1:1–44, 1992.
5. J. van Eijck and H. Kamp. Representing discourse in context. In J. van Benthem and A. ter Meulen, editors, *Handbook of Logic and Language*, pages 179–237. Elsevier, Amsterdam, 1996.
6. Jan van Eijck. Incremental Dynamics. Manuscript, CWI/ILLC, Amsterdam, September 1998.
7. J. Groenendijk and M. Stokhof. Dynamic predicate logic. *Linguistics and Philosophy*, 14:39–100, 1991.
8. H. Kamp. A theory of truth and semantic representation. In J. Groenendijk et al., editors, *Formal Methods in the Study of Language*. Mathematisch Centrum, Amsterdam, 1981.
9. H. Kamp and U. Reyle. A calculus for first order discourse representation structures. *Journal of Logic, Language and Information*, 5(3–4):297–348, 1996.
10. W. Saurer. A natural deduction system for discourse representation theory. *Journal of Philosophical Logic*, 22(3):249–302, 1993.
11. P. Staudacher. Zur Semantik indefiniter Nominalphrasen. In B. Asbach-Schnitker and J. Roggenhofer, editors, *Neue Forschungen zur Wortbildung und Historiographie der Linguistik*, pages 239–258. Gunter Narr Verlag, 1987.
12. F. Veltman. Proof systems for dynamic predicate logic. Handout, November 1997.
13. A. Visser. Digressions on and desiderata for the design of dynamic discourse denotations. Course notes, University of Utrecht, February 1994.
14. A. Visser. Contexts in dynamic predicate logic. *Journal of Logic, Language and Information*, 7(1):21–52, 1998.