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The Robust Regulation Problem with Robust Stability

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ABSTRACT
Among the most common purposes of control are the tracking of reference signals and the rejection of disturbance signals in the face of uncertainties. The related design problem is called the 'robust regulation problem'. Here we investigate the trade-off between the robust regulation constraint and the requirement of robust stability. We first formulate the robust regulation problem as an interpolation problem, and derive from this a number of simple necessary conditions for the robust regulation problem to be solvable with a given stability margin. Then we show that these conditions are also sufficient provided the given stability margin is achievable at all.

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1. Introduction

The problem of finding a controller to make the closed-loop system follow or reject signals generated by an 'exosystem' in the presence of small parameter variations is called robust regulation. This problem has been extensively studied from various points of view in the seventies and early eighties, see for instance the books by Wonham [33, Ch.8] and Basile and Marro [7, Ch.6] and the references therein. Another important design objective is robust closed-loop stability. Various formulations are possible; here we shall work with unstructured perturbations, and more specifically we use the model based on normalized coprime factorizations. The problem of designing a controller that optimizes the robustness of closed-loop stability with respect to this perturbation class has been solved by Glover and McFarlane in [19, 24].

The purpose of the present paper is to investigate the relation between the design requirements of robust regulation and robustness of stability. We determine the conditions under which there is a trade-off between these two requirements, and we determine the nature of this trade-off. A similar problem, with the robustness of stability in the sense of normalized right coprime factorizations replaced by a general H_∞ objective, was studied by Abedor *et al.* [1] and Sugie *et al.* [29]. Although the problem studied in this paper can be formulated as an H_∞ optimization problem with certain interpolation constraints, the point here is to show that for the important special case of robust regulation there is a remarkably simple solution, which is not readily apparent from the more general framework (see Thm. 5.2 and Prop. 5.4 below).

The present paper is a continuation of [8, 9] in which the regulator problem is characterized as an interpolation problem on the subspace-valued function associated to controller and the trade-off between the regulation requirement and robust stability is established. The treatment is based on a mixture of classical ideas from the geometric theory of linear systems with rational interpolation theory, via the concepts of subspace-valued functions. The idea of associating to a finite-dimensional linear system with m inputs and p outputs a function from the extended complex plane to subspaces of $(m+p)$-dimensional space can be traced back to Martin and Hermann [23]. Qiu and Davison [26] and the second author [27] have contributed to showing that subspace-valued functions provide a suitable framework for defining a distance measure between linear systems and describing robust stability properties.

Here, we use the same formulation as in our earlier papers [8, 9] to show that the robust regulation requirement is equivalent to an interpolation condition on a subspace-valued function associated to the controller. Finite-dimensional geometry then readily leads to necessary conditions for the solvability of the robust regulation problem when a stability margin γ is imposed. We show that these conditions are also sufficient when another (obvious) condition is also satisfied, namely that the given margin is achievable at all, i. e., without taking the regulation requirement into account.

The paper is organized as follows. A precise problem formulation is given in section 2. As a first step to solving the problem, conditions for robust regulation in interpolation form are given in section 3. The proof of the corresponding result is rather technical and so is relegated to an appendix. Section 4 presents necessary conditions for the problem of robust regulation with robust stability to be solvable. With a constructive proof of the sufficiency side in section 5, we arrive at the main result of the paper. After a discussion of the nonuniqueness of solutions in section 6, a worked example is provided to illustrate the constructive solution procedure. Finally, conclusions follow in section 8.

2. Problem formulation and preliminaries

The standard state-space formulation of the 'regulator problem' or 'servo problem' (cf. for instance [33]) starts with a finite-dimensional linear time-invariant system of the following form:

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) \tag{2.1}$$

$$\dot{x}_2(t) = A_{22}x_2(t) \tag{2.2}$$

$$y(t) = C_1x_1(t) + C_2x_2(t). \tag{2.3}$$

The variable x_1 denotes the state of the plant; x_2 is the state of an 'exosystem' that generates signals which can be disturbances or references. The matrix A_{22} has its eigenvalues on the imaginary axis, allowing the reference/disturbance signals to be steps, ramps, sinusoids, etc. To the plant is connected a linear time-invariant compensator of the form

$$\dot{z}(t) = Fz(t) + Gy(t) \tag{2.4}$$

$$u(t) = Hz(t) + Jy(t). \tag{2.5}$$

The *closed-loop system* takes the form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ z \\ x_2 \end{bmatrix}(t) = A_e \begin{bmatrix} x_1 \\ z \\ x_2 \end{bmatrix}(t) \tag{2.6}$$

$$y(t) = [C_1 \quad 0 \quad C_2] \begin{bmatrix} x_1 \\ z \\ x_2 \end{bmatrix}(t) \tag{2.7}$$

where

$$A_e = \begin{bmatrix} A_{11}+B_1JC_1 & B_1H & A_{12}+B_1JC_2 \\ GC_1 & F & GC_2 \\ 0 & 0 & A_{22} \end{bmatrix}. \tag{2.8}$$

The compensator is said to satisfy the *internal stability requirement* if the closed-loop system is stable when $x_2(t) = 0$, that is, if

$$\sigma\left(\begin{bmatrix} A_{11} + B_1JC_1 & B_1H \\ GC_1 & F \end{bmatrix}\right) \subset \mathbb{C}^- \tag{2.9}$$

where σ denotes spectrum and $\mathbb{C}^-$ is the open left half plane. The compensator is said to satisfy the *regulation requirement* if the regulated output $y(t)$ converges to zero for all possible initial conditions, so if

$$\mathcal{X}_+(A_e) \subset \ker[C_1 \;\; 0 \;\; C_2] \tag{2.10}$$

where $\mathcal{X}_+(A_e)$ denotes the unstable subspace of A_e.

We shall consider the regulation problem under the following standing assumptions.

Assumptions The system (2.1–2.3) satisfies

(A1) the pair (A_{11}, B_1) is stabilizable;

(A2) the pair (C, A) given by

$$C \;=\; [C_1 \;\; C_2], \qquad A \;=\; \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \tag{2.11}$$

is detectable;

(A3) all eigenvalues of A_{22} are on the imaginary axis;

(A4) C_1 has full row rank.

These are fairly usual assumptions for the regulator problem which are either necessary conditions for the solvability or cause no essential loss of generality. For an explanation and justification see for instance [14, 33, 9]. The classical “regulator problem with internal stability” (cf. for instance [33, Thm. 8.1] is the following.

Problem 1 (regulator problem with internal stability: RPIS)

Given the plant and exosystem (2.1–2.3), find a compensator of the form (2.4–2.5) such that both the internal stability requirement and the regulation requirement are satisfied.

In reality the values of the system parameters are not exactly known and it is desired that the properties of internal stability and output regulation are preserved for small variations around the nominal parameters. To discuss robustness issues, assume (as in [33, p. 194]) that A_{22}, C_1, C_2 are fixed (precisely known) while A_{11}, A_{12}, B_1 are subject to uncertainty. Regard (A_{11}, A_{12}, B_1) as a data point in $\mathbf{R}^N$ with $N = n_1^2 + n_1n_2 + n_1m$ where n_1, n_2 and m are the dimensions of the vectors x_1, x_2 and u respectively. The compensator is said to satisfy the *robust regulation requirement* if

$$\mathcal{X}_+(A_e) \subset \ker[C_1 \;\; 0 \;\; C_2] \quad \text{in a neighborhood of } (A_{11}, A_{12}, B_1) \text{ in } \mathbf{R}^N. \tag{2.12}$$

The associated design problem is formulated as follows.

Problem 2 (robust regulation problem with internal stability: RRIS)

Given the plant and exosystem (2.1–2.3), find a compensator of the form (2.4–2.5) such that the internal stability requirement (2.9) and the robust regulation requirement (2.12) are satisfied.

Necessary and sufficient conditions for RPIS and RRIS to be solvable along with synthesis procedures of a suitable compensator are well known. One may refer to [33, Thm. 8.1] for RPIS and [33, Thm. 8.5] for RRIS where this problem is called ‘the regulator problem with structurally stable synthesis’. Now we want to add the element of robust stability. To formulate this, we devote some space to a few definitions and results that will also play a crucial role in the development below.

As in [8, 9], we define some subspace-valued functions as follows. The plant parameters (A_{11}, B_1, C_1) determine the subspace-valued function

$$\mathcal{P}(s) \;=\; \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \mid \exists x \text{ s.t. } \begin{bmatrix} sI - A_{11} & 0 & -B_1 \\ C_1 & -I & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ u \end{bmatrix} = 0 \right\}, \quad \mathcal{P}(\infty) \;=\; \operatorname{im} \begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{2.13}$$

In the same way, the controller determines the subspace-valued function

$$\mathcal{C}(s) \;=\; \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \mid \exists z \text{ s.t. } \begin{bmatrix} sI - F & -G & 0 \\ H & J & -I \end{bmatrix} \begin{bmatrix} z \\ y \\ u \end{bmatrix} = 0 \right\}, \quad \mathcal{C}(\infty) \;=\; \operatorname{im} \begin{bmatrix} I \\ J \end{bmatrix}. \tag{2.14}$$

To the full system (2.1–2.3) we associate

$$\mathcal{M}(s) = \Pi \ \ker \begin{bmatrix} sI - A & 0 & -B \\ C & -I & 0 \end{bmatrix}, \quad \mathcal{M}(\infty) = \operatorname{im} \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{2.15}$$

where $B = [B_1' \ \ 0]'$ and Π denotes the natural projection from $\mathcal{X} \times \mathcal{Y} \times \mathcal{U}$ to $\mathcal{Y} \times \mathcal{U}$. All functions take values in the set of subspaces of the product space $\mathcal{Y} \times \mathcal{U}$, which is an $(m+p)$-dimensional space if m is the number of inputs and p is the number of outputs. The functions above may be considered as functions on the extended complex plane $\mathbb{C} \cup \{\infty\}$, but we shall only need their values on the closed right half plane $\mathbb{C}^+ := \{s \in \mathbb{C} \mid \operatorname{Re} s \geq 0\} \cup \{\infty\}$. The subspace-valued functions were defined in state space terms above. In terms of factorizations over RH_∞, one has the following (cf. for instance [10, Lemma 2.4]).

Lemma 2.1 *Consider a set of state space parameters (A, B, C, D), and assume that (A, B) is stabilizable and that (C, A) is detectable. Let $N(s)D^{-1}(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$ be respectively a right and a left coprime factorization over RH_∞ of the transfer matrix $G(s) = C(sI - A)^{-1}B + D$. Under these conditions, one has*

$$\operatorname{im} \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = \ker[\tilde{D}(s) \ \ -\tilde{N}(s)] = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \mid \exists x \text{ s.t. } \begin{bmatrix} sI - A & 0 & -B \\ C & -I & D \end{bmatrix} \begin{bmatrix} x \\ y \\ u \end{bmatrix} = 0 \right\} \tag{2.16}$$

for all $s \in \mathbf{C}$ with $\operatorname{Re} s \geq 0$, and

$$\operatorname{im} \begin{bmatrix} N(\infty) \\ D(\infty) \end{bmatrix} = \ker[\tilde{D}(\infty) \ \ -\tilde{N}(\infty)] = \operatorname{im} \begin{bmatrix} D \\ I \end{bmatrix}. \tag{2.17}$$

The lemma shows how a subspace-valued function may be given by an *image representation* (corresponding to a right coprime factorization) or by a *kernel representation* (corresponding to a left coprime factorization). As a notational convention, we use script letters for subspace-valued functions and the corresponding roman letters for kernel and image representations, using a tilde to distinguish kernel from image representations (so for instance $\mathcal{P}(s) = \ker \tilde{P}(s) = \operatorname{im} P(s)$).

To measure robustness, we use the concept of the *minimal angle* between two subspaces $\mathcal{Y}$ and $\mathcal{Z}$ of a unitary space $\mathcal{X}$, which is defined as follows (see for instance [20, p. 339]):

$$\sin\phi(\mathcal{Y}, \mathcal{Z}) \;=\; \min\{\|y - z\| \mid y \in \mathcal{Y}, \ \ z \in \mathcal{Z}, \ \ \|y\| = 1\}, \qquad 0 \leq \phi \leq \tfrac{1}{2}\pi. \tag{2.18}$$

This notion may be used to define the following measure of the robustness of closed-loop stability:

$$\sin\phi(\mathcal{P}, \mathcal{C}) \;:=\; \min_{s \in \mathbb{C}^+} \sin\phi(\mathcal{P}(s), \mathcal{C}(s)). \tag{2.19}$$

In various equivalent formulations, the above measure has also been used for instance in [31, 19, 16, 32]. One possible motivation for this particular measure is the observation that the plant-controller combination is stable if and only if

$$\mathcal{C}(s) \oplus \mathcal{P}(s) \;=\; \mathcal{Y} \times \mathcal{U} \quad \text{for all } s \in \mathbb{C}^+. \tag{2.20}$$

as can be proved easily (see for instance [8, Lemma 2.5]). It has been shown in [27] that the minimal angle is the appropriate measure of the robustness of complementarity of two subspaces $\mathcal{Y}$ and $\mathcal{Z}$, in the sense that it gives exactly the distance (in the sense of the gap) of $\mathcal{Y}$ to the set of subspaces that are not complementary to $\mathcal{Z}$. Below we shall also need the following formula for the cosine of the minimal angle (see for instance [27]):

$$\cos\phi(\mathcal{Y}, \mathcal{Z}) \;=\; \|\Pi_{\mathcal{Z}}|_{\mathcal{Y}}\| \tag{2.21}$$

where $\Pi_{\mathcal{Z}}$ is the orthogonal projection onto $\mathcal{Z}$.

After this, we can formulate the *robust stabilization problem* as follows.

Problem 3 (robust stabilization problem with margin γ: RSP(γ))
Given the plant (2.1–2.2) and γ with $0 < \gamma < 1$, find a compensator of the form (2.4–2.5) that satisfies the *robust stability requirement*

$$\min_{s\in\mathbb{C}^+} \sin\phi(\mathcal{P}(s), \mathcal{C}(s)) \;>\; \gamma. \tag{2.22}$$

Necessary and sufficient conditions for the solvability and a parametrization of all solutions of RSP(γ) have been given by Glover and McFarlane [19]. The main problem that will be discussed in this paper is defined as follows.

Problem 4 (robust regulation problem with robust stability margin γ: RRRS(γ))
Given the plant and exosystem (2.1–2.3) and γ with $0 < \gamma < 1$, find a compensator of the form (2.4–2.5) such that the robust regulation requirement (2.12) and the robust stability requirement (2.22) are simultaneously satisfied.

3. Interpolation conditions for robust regulation

It is our purpose in this section to give the robust regulation condition (2.12) a more manageable form. For this we shall follow the interpolation approach of [8]. This requires again the introduction of some definitions. For a given rational matrix function $M(s)$ of size $p \times m$ and a given natural number r, we define a new matrix function of size $rp \times rm$ by

$$M^{[r]}(s) \;=\; \begin{bmatrix} M(s) & 0 & \cdots & \cdots & 0 \\ M'(s) & M(s) & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \frac{1}{(r-1)!}M^{(r-1)}(s) & \cdots & \cdots & M'(s) & M(s) \end{bmatrix}. \tag{3.1}$$

We call $M^{[r]}(s)$ the *r-fold blow-up* of $M(s)$. See also [8] for a more intrinsic definition. We shall sometimes use the notation $[M(s)]^{[r]}$ instead of $M^{[r]}(s)$, in particular when $M(s)$ is a partitioned matrix, and in such cases even write $[M(s)]^{[r]}(\lambda)$ instead of $M^{[r]}(\lambda)$. The following formula for the blow-up of a product readily follows from the definition [8, Lemma 3.1].

Lemma 3.1 *For any matrix functions $T(s) \in \mathbf{R}^{p\times m}(s)$ and $S(s) \in \mathbf{R}^{m\times l}(s)$ and any $r = 1, 2, \ldots$ one has $(TS)^{[r]}(s) = T^{[r]}(s)S^{[r]}(s)$.*

A minor problem is that the blow-up does not commute with matrix partitioning; indeed, if A and B are linear mappings from $\mathcal{X}$ to $\mathcal{Z}$ and from $\mathcal{Y}$ to $\mathcal{Z}$ respectively, then $[A \;\; B]^{[r]}$ is a mapping from $(\mathcal{X} \times \mathcal{Y})^r$ to $\mathcal{Z}^r$, but $[A^{[r]} \;\; B^{[r]}]$ is a mapping from $\mathcal{X}^r \times \mathcal{Y}^r$ to $\mathcal{Z}^r$. To get a proper correspondence we need an operator from $\mathcal{X}^r \times \mathcal{Y}^r$ to $(\mathcal{X} \times \mathcal{Y})^r$ that we call the *mingling* operator. It is defined by

$$Mi : (x_1, \cdots, x_r, y_1, \cdots, y_r) \;\mapsto\; (x_1, y_1, \cdots, x_r, y_r). \tag{3.2}$$

We shall use the mingling operator between various spaces and even use its obvious generalization to products of more than two factors, employing the same symbol Mi every time; this abuse of notation should cause no confusion.

In addition to the blow-ups of matrix functions, we shall also need blown-up versions of the various subspace-valued functions that were introduced above. For the functions $\mathcal{P}(s)$ and $\mathcal{C}(s)$ defined in (2.13) and (2.14) respectively, these can be defined via either image or kernel representations as follows:

$$\mathcal{P}^{[r]}(s) = \ker \tilde{P}^{[r]}(s) = \operatorname{im} P^{[r]}(s) \tag{3.3}$$

$$\mathcal{C}^{[r]}(s) = \ker \tilde{C}^{[r]}(s) = \operatorname{im} C^{[r]}(s). \tag{3.4}$$

It follows from [8, Lemma 3.3, 3.4] that this definition is unambiguous. The subspace-valued function $\mathcal{M}(s)$ is not of constant dimension on the closed right half plane and so we use a more circuitous definition:

$$\mathcal{M}^{[r]}(s) = \Pi^{[r]} \ker \begin{bmatrix} sI - A & 0 & -B \\ C & -I & 0 \end{bmatrix}^{[r]}, \quad \mathcal{M}^{[r]}(\infty) = \operatorname{im} \begin{bmatrix} 0 \\ I \end{bmatrix}^{[r]}. \tag{3.5}$$

For ease of notation, we introduce

$$\mathcal{K} = \{ \begin{bmatrix} y \\ u \end{bmatrix} \mid y = 0 \} \tag{3.6}$$

and denote the natural projection from $\mathcal{Y} \times \mathcal{U}$ to $\mathcal{Y}$ by $\tilde{K} = [I \;\; 0]$, so that

$$\mathcal{K} = \ker \tilde{K} = \operatorname{im} \begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{3.7}$$

Regarding $\tilde{K}$ as a constant matrix-valued function, we can also consider $\tilde{K}^{[r]}$ which is simply a block diagonal matrix with $\tilde{K}$ on the diagonal entries, and $\mathcal{K}^{[r]} = \ker \tilde{K}^{[r]}$. By the *multiplicity* of an eigenvalue of a matrix we mean the length of the longest Jordan chain associated with that eigenvalue.

The main result of [8] can now be formulated as follows.

Theorem 3.2 *A controller of the form (2.4–2.5) is a solution to the regulator problem with internal stability if and only if the associated subspace-valued function $\mathcal{C}(s)$ satisfies the internal stability condition (2.20) and the interpolation conditions*

$$\mathcal{C}^{[r]}(\lambda) \cap \mathcal{M}^{[r]}(\lambda) \;\subset\; \mathcal{K}^{[r]} \qquad \textit{for all } \lambda \textit{ in } \sigma(A_{22}) \textit{ of multiplicity } r. \tag{3.8}$$

We call the conditions (3.8) interpolation conditions since they impose conditions on the subspace-valued function $\mathcal{C}(s)$ at specific points (namely the exosystem poles), although the condition (3.8) of course does not prescribe $\mathcal{C}(\lambda)$ completely. By formulating the conditions for the solvability of the regulator problem in this way, it is evident that the conditions are sensitive to variations in the plant since the subspace-valued function $\mathcal{M}(s)$ depends on the plant. Dropping the intersection leads to a stronger condition which is however no longer dependent on the plant and so is robust with respect to *any* linear plant with the same number of inputs and outputs. In the theorem below, which is the first main result of this paper, we establish that this 'robustified' version is indeed exactly the condition that one needs to solve the robust regulation problem as formulated above. Plausible as the result may be, its proof is unfortunately rather technical and we have relegated it to the appendix.

THEOREM 3.3 *A controller of the form (2.4-2.5) is a solution to the robust regulation problem with internal stability if and only if the associated subspace-valued function $\mathcal{C}(s)$ satisfies the internal stability condition (2.20) and the interpolation conditions*

$$\mathcal{C}^{[r]}(\lambda) \subset \mathcal{K}^{[r]} \quad \text{for all } \lambda \text{ in } \sigma(A_{22}) \text{ of multiplicity } r. \tag{3.9}$$

REMARK **3.4** Condition (3.9) implies that the controller has p poles at λ of multiplicity at least r. This is in accordance with the internal model principle which states that the controller should contain a suitably reduplicated model of the dynamic structure of the exogenous signals for robust regulation [33, 14, 12].

4. Necessary conditions

We first note a result that can be obtained easily by using the interpolation form (3.9) of the robust regulation condition. By the following lemma, we get an upper bound on the achievable robustness of stability (compare (2.20) and (3.9) to (4.1) below).

LEMMA 4.1 *Let $\mathcal{P}, \mathcal{K}$ be subspaces of a unitary space $\mathcal{W}$ satisfying $\mathcal{P}+\mathcal{K} = \mathcal{W}$. Then for any subspace $\mathcal{C}$ satisfying*

$$\mathcal{C} \oplus \mathcal{P} = \mathcal{W} \quad \text{and} \quad \mathcal{C} \subset \mathcal{K} \tag{4.1}$$

we have

$$\phi(\mathcal{P}, \mathcal{C}) \;\leq\; \phi(\mathcal{P}^{\perp}, \mathcal{K}^{\perp}) \tag{4.2}$$

and equality is achieved for $\mathcal{C} = \mathcal{K} \cap (\mathcal{K} \cap \mathcal{P})^{\perp}$.

PROOF From the singular value characterization of the minimal angle (see for instance [27, Prop. 2.4]) we have $\phi(\mathcal{P}, \mathcal{C}) = \phi(\mathcal{P}^{\perp}, \mathcal{C}^{\perp})$. So (4.2) follows from the definition of the minimal angle and the constraint $\mathcal{K}^{\perp} \subset \mathcal{C}^{\perp}$. For $\mathcal{C} = \mathcal{K} \cap (\mathcal{K} \cap \mathcal{P})^{\perp}$ we have $\mathcal{P} \cap \mathcal{C} = \{0\}$, and since $\mathcal{P} + \mathcal{K} = \mathcal{W}$ it follows by counting dimensions that $\mathcal{C} + \mathcal{P} = \mathcal{W}$. Therefore, $\phi(\mathcal{P}, \mathcal{K} \cap (\mathcal{P} \cap \mathcal{K})^{\perp}) = \phi(\mathcal{P}^{\perp}, \mathcal{K}^{\perp} + (\mathcal{P} \cap \mathcal{K})) = \phi(\mathcal{P}^{\perp}, \mathcal{K}^{\perp})$. □

REMARK **4.2** In the important special case of square systems ($m = p$), the only possible choice for a subspace $\mathcal{C}$ satisfying (4.1) is $\mathcal{C} = \mathcal{K}$ and the upper bound (4.2) trivially becomes $\phi(\mathcal{P}, \mathcal{C}) = \phi(\mathcal{P}, \mathcal{K})$. In general this is smaller than $\phi(\mathcal{P}, \mathcal{K} \cap \mathcal{M})$, the upper bound due to regulation constraints as derived in [9]. The difference between $\phi(\mathcal{P}, \mathcal{K})$ and $\phi(\mathcal{P}, \mathcal{K} \cap \mathcal{M})$ represents the price one has to pay, in terms of constraints on the achievable robustness of stability, for insisting on robust regulation as opposed to plain regulation. For single-input-single-output systems, however, this price is zero. Indeed, for such systems the space $\mathcal{Y} \times \mathcal{U}$ is just two-dimensional whereas $\mathcal{P}(\lambda)$ is one-dimensional for all λ, and $\mathcal{M}(\lambda)$ contains $\mathcal{P}(\lambda)$ as a proper subspace for $\lambda \in \sigma(A_{22})$ as a consequence of the definition (2.15) and the detectability assumption (A2). Therefore one must have $\mathcal{M}(\lambda) = \mathcal{Y} \times \mathcal{U}$ for $\lambda \in \sigma(A_{22})$ and consequently $\mathcal{K} \cap \mathcal{M}(\lambda) = \mathcal{K}$. So for SISO systems, one can get robustness of regulation at no extra cost in terms of effects on the robustness of closed-loop stability.

Obviously, a second necessary condition for RRRS(γ) to be solvable is that the robust regulation problem with internal stability is solvable. The necessary and sufficient conditions for this are classical; we state them here in the terminology of subspace-valued functions. Our setting allows a quick proof of these conditions.

THEOREM 4.3 *Under the assumptions (A1-A4), there exists a controller of the form (2.4–2.5) that satisfies both the internal stability requirement (2.20) and the robust regulation requirement (3.9) if and only if*

$$\mathcal{K} + \mathcal{P}(\lambda) = \mathcal{Y} \times \mathcal{U} \quad \text{for all } \lambda \in \sigma(A_{22}). \tag{4.3}$$

PROOF First assume that RRIS is solvable. Let $\mathcal{C}(s)$ be the subspace-valued function of the controller which is a solution of the problem. By Thm. 3.3 $\mathcal{C}(\lambda)\oplus\mathcal{P}(\lambda) = \mathcal{Y}\times\mathcal{U}$ and $\mathcal{C}(\lambda)\subset\mathcal{K}$ for all $\lambda\in\sigma(A_{22})$. It follows that $\mathcal{P}(\lambda)+\mathcal{K} = \mathcal{Y}\times\mathcal{U}$ for all $\lambda\in\sigma(A_{22})$.

To prove the reverse implication, let $P(s)$ be an image representation for the subspace-valued function associated to the plant and $\hat{C}(s)$ be an image representation for a stabilizing controller, i. e. $\operatorname{im} P(s)\oplus\operatorname{im}\hat{C}(s) = \mathcal{Y}\times\mathcal{U}$ for all $s\in\mathbb{C}^+$. For each eigenvalue λ of A_{22} with multiplicity r determine a subspace $\mathcal{C}_\lambda = \operatorname{im} C_\lambda$ satisfying $\mathcal{C}_\lambda\oplus(\mathcal{K}\cap\mathcal{P}(\lambda)) = \mathcal{K}$; then $\mathcal{C}_\lambda\subset\mathcal{K}$ and $\mathcal{C}_\lambda\oplus\mathcal{P}(\lambda) = \mathcal{Y}\times\mathcal{U}$. Because of the latter condition there exist lower triangular matrices $Q_\lambda\in\mathbf{C}^{rm\times rp}$ and $T_\lambda\in\mathbf{C}^{rp\times rp}$ of the form

$$Q_\lambda = \begin{bmatrix} Q^0 & 0 & \cdots & 0 \\ Q^1 & Q^0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ Q^{r-1} & Q^{r-2} & \cdots & Q^0 \end{bmatrix}, \quad T_\lambda = \begin{bmatrix} T^0 & 0 & \cdots & 0 \\ T^1 & T^0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ T^{r-1} & T^{r-2} & \cdots & T^0 \end{bmatrix} \tag{4.4}$$

such that

$$P^{[r]}(\lambda)Q_\lambda + C_\lambda^{[r]}T_\lambda = \hat{C}^{[r]}(\lambda). \tag{4.5}$$

By a simple interpolation of Lagrange-Hermite type (see for instance [2]), one can determine $Q(s)\in RH_\infty^{m\times p}$ satisfying the interpolation conditions $\frac{1}{k!}Q^{(k)}(\lambda) = Q^k$ $(k = 0,\dots,r-1)$. Now let $C(s) = \hat{C}(s) - P(s)Q(s)$. Note that $\operatorname{im} C(\lambda)\oplus\operatorname{im} P(\lambda) = \mathcal{Y}\times\mathcal{U}$ for all $\lambda\in\mathbb{C}^+$; therefore $C(s)$ is an image representation of a stabilizing controller. Moreover, by taking derivatives it follows from Lemma 3.1 and (4.5) that $C^{[r]}(\lambda) = \hat{C}^{[r]}(\lambda) - P^{[r]}(\lambda)Q^{[r]}(\lambda) = C_\lambda^{[r]}T_\lambda$. Thus $\operatorname{im} C^{[r]}(\lambda) = \operatorname{im} C_\lambda^{[r]}\subset\mathcal{K}^{[r]}$ and by Thm. 3.3 $C(s)$ is an image representation for a solution of RRIS. □

REMARK **4.4** By dualizing the proof of [8, Lemma 5.1] it can be shown that the condition (4.3) is equivalent to the requirement that the matrix

$$\begin{bmatrix} \lambda I - A_{11} & B_1 \\ C_1 & 0 \end{bmatrix}$$

has full row rank for all $\lambda\in\sigma(A_{22})$, which is a well known condition for the solvability of the robust regulation problem with internal stability [33]. Another equivalent formulation is

$$\sin\phi(\mathcal{P}^\perp(\lambda),\mathcal{K}^\perp) \;>\; 0 \quad \text{for all } \lambda\in\sigma(A_{22}). \tag{4.6}$$

By Lemma 4.1, the numerical value of the left hand side has a clear meaning: it is an upper bound on the achievable robustness of stability. See also Prop. 5.4 below.

5. Construction of solutions to RRRS

The solution of the robust stabilization problem is given in [19, Thm. 4.1]. Image representations of all controllers which satisfy the robust stability condition (2.22) are obtained by solving the Nehari extension problem

$$\|\,\tilde{P}^*(s) - C(s)\,\|_\infty < \sqrt{1-\gamma^2} \tag{5.1}$$

where $\tilde{P}^*(s)$ denotes $\tilde{P}^T(-s)$, and $\tilde{P}(s)$ is a kernel representation for the subspace-valued function of the plant normalized such that $\tilde{P}(s)\tilde{P}^*(s) = I$. By Nehari's theorem (cf. for instance [13]) the minimum of $\|\tilde{P}^*(s) - C(s)\|_\infty$ is given by $\|\Gamma_{\tilde{P}^*}\|$, the norm of the Hankel operator with symbol $\tilde{P}^*$. So a controller satisfying the robust stability condition (2.22) exists if and only if

$$\gamma < (1 - \|\Gamma_{\tilde{P}^*}\|^2)^{1/2}. \tag{5.2}$$

For $C(s)$ to be a solution of the robust regulation problem with robust stability it should also satisfy the interpolation conditions

$$\operatorname{im} C^{[r]}(\lambda) \subset \ker \tilde{K}^{[r]} \quad \text{for all } \lambda \in \sigma(A_{22}) \text{ of multiplicity } r. \tag{5.3}$$

Therefore our problem is to find $C(s) \in RH_\infty^{(m+p)\times p}$ which satisfies (5.1) and (5.3) simultaneously. The solution of this problem relies on the following lemma from [9] which is about finding a Nehari extension of an RL_∞ function which also satisfies a set of interpolation conditions on the imaginary axis (compare also [21, Thm. 3]).

LEMMA 5.1 *Let $R(s) \in RL_\infty^{q\times p}$, finitely many purely imaginary complex numbers λ_i and matrices $W_{ik} \in \mathbf{C}^{q\times p}$ $(k = 0, 1, \ldots, r_i - 1)$ be given. There exists $W(s) \in RL_\infty^{q\times p}$ such that*

$$\begin{aligned} W(s) - R(s) &\in RH_\infty^{q\times p} && (5.4)\\ \|W(s)\|_\infty &< 1 && (5.5)\\ W^{(k)}(\lambda_i) &= W_{ik} \;\; (k = 0, 1, \ldots, r_i - 1) \quad \text{for all } i && (5.6) \end{aligned}$$

if and only if $\|\Gamma_R\| < 1$ and $\|W_{i0}\| < 1$ for all i.

The necessity of the conditions of Lemma 5.1 is obvious from the norm constraint (5.5) and Nehari's theorem. Sufficiency of the conditions is constructively shown in the proof of Thm. 3.1 in [9]. For the reader's convenience we summarize the construction. First, the solution to the Nehari problem (5.4–5.5) is parametrized as

$$W = (\Theta_{11}G + \Theta_{12})(\Theta_{21}G + \Theta_{22})^{-1} \tag{5.7}$$

where the matrix

$$\Theta(s) = \begin{bmatrix} \Theta_{11}(s) & \Theta_{12}(s) \\ \Theta_{21}(s) & \Theta_{22}(s) \end{bmatrix}$$

can be constructed explicitly from state space data for $R(s)$ [18, 4, 5], and $G(s) \in RH_\infty^{q\times p}$ is a free parameter such that $\|G(s)\|_\infty < 1$. The next step is to translate the interpolation conditions on $W(s)$ into the parameter $G(s)$. For this, define matrices $W_i^{r_i} \in \mathbf{C}^{r_i q\times p}$ and $F^{r_i} \in \mathbf{C}^{r_i p\times p}$ by

$$\begin{aligned} W_i^{r_i} &= \operatorname{col}(W_{i0}, W_{i1}, \ldots, W_{i,r_i-1}) && (5.8)\\ F^{r_i} &= \operatorname{col}(I, 0, \ldots, 0) && (5.9) \end{aligned}$$

and let $N_i^{r_i} = \operatorname{col}(N_{i0}, N_{i1}, \ldots, N_{i,r_i-1})$, $D_i^{r_i} = \operatorname{col}(D_{i0}, D_{i1}, \ldots, D_{i,r_i-1})$ be defined by

$$Mi \begin{bmatrix} N_i^{r_i} \\ D_i^{r_i} \end{bmatrix} = (\Theta^{[r_i]}(\lambda_i))^{-1} Mi \begin{bmatrix} W_i^{r_i} \\ F^{r_i} \end{bmatrix}. \tag{5.10}$$

Then the interpolation conditions on $G(s)$ are given by

$$G^{[r_i]}(\lambda_i) D_i^{r_i} = N_i^{r_i}. \tag{5.11}$$

Finding an RH_∞ matrix $G(s)$ that satisfies the interpolation conditions (5.11) and the norm constraint $\|G(s)\|_\infty < 1$ is a boundary Nevanlinna-Pick problem [2, 3]. It is shown in [9, Lemma 3.5] that under the conditions of Lemma 5.1 a solution always exists. Finally, $W(s)$ is calculated from (5.7). The fact that the boundary Nevanlinna-Pick problem comes up in connection with regulation constraints in an H_∞ context has been recognized before in [28].

The main result of the present paper is stated in the theorem below which shows that the smaller of the upper bounds given by (5.2) and Lemma 4.1 determines the achievable robustness of stability.

Theorem 5.2 *Define the subspace-valued function $\mathcal{P}(s)$ by (2.13) and $\mathcal{K}$ by (3.6). Let $\tilde{P}(s)$ be a normalized kernel representation for $\mathcal{P}(s)$. Under the assumptions (A1-A4), the problem RRRS(γ) is solvable if and only if*

$$\gamma < \min\{ \min_{\lambda \in \sigma(A_{22})} \sin\phi(\mathcal{P}^{\perp}(\lambda), \mathcal{K}^{\perp}), \ \sqrt{1 - \|\Gamma_{\tilde{P}^*}\|^2} \}. \tag{5.12}$$

Proof The necessity of the condition follows from the upper bounds on robustness of stability derived in Lemma 4.1 and given by (5.2). So assume that the condition holds. Write $R(s) = (1-\gamma^2)^{-1/2}\tilde{P}^*(s)$; then $R(s)$ is normalized such that $\|\Gamma_R\| < 1$. Let Π be the orthogonal projection on $\mathcal{K}^{\perp}$. For every eigenvalue λ of A_{22} of multiplicity r define matrices $W^0, \ldots, W^{r-1}$ of size $(m+p) \times p$ by

$$\begin{aligned} W^0 &= \Pi R(\lambda) && (5.13)\\ W^k &= R^{(k)}(\lambda) \quad k = 1, \ldots, r-1. && (5.14) \end{aligned}$$

Note that $\|W^0\| = (1-\gamma^2)^{-1/2}\|\Pi\tilde{P}^*(\lambda)\| = (1-\gamma^2)^{-1/2}\cos\phi(\mathcal{K}^{\perp}, \mathcal{P}^{\perp}(\lambda)) < 1$. Hence by Lemma 5.1 there exists $W(s) \in RL_{\infty}^{(m+p)\times p}$ such that $W(s) - R(s) \in RH_{\infty}^{(m+p)\times p}$, $\|W(s)\|_{\infty} < 1$ and $W^{(k)}(\lambda) = W^k$ for $k = 0, \ldots, r-1$. Let $C(s) = \sqrt{1-\gamma^2}(R(s) - W(s))$. Then $\|\tilde{P}^*(s) - C(s)\|_{\infty} = \sqrt{1-\gamma^2}\|W(s)\|_{\infty} < \sqrt{1-\gamma^2}$. Hence $C(s)$ is an image representation of a solution for the robust stabilization problem with margin γ [19]. On the other hand,

$$\begin{aligned} C(\lambda) &= (I - \Pi)\tilde{P}^*(\lambda) && (5.15)\\ C^{(k)}(\lambda) &= 0 \qquad\qquad (k = 1, \ldots, r-1). && (5.16) \end{aligned}$$

Hence $\operatorname{im} C^{[r]}(\lambda) \subset \mathcal{K}^{[r]}$ and it follows from Thm. 3.3 that $C(s)$ is also a solution of RRIS. Therefore $C(s)$ is an image representation of a solution for RRRS(γ). □

The sufficiency part of the proof is constructive; we illustrate the computational procedure by an example in section 4. State space parameters for the compensator can be obtained from an image representation as in [15] for instance.

Remark **5.3** An estimate of the order of the compensator designed by the method of Thm. 5.2 can be obtained as follows. Let n_1 denote the order of the plant and n_2 be the degree of the minimal polynomial of A_{22}. Comparing the state space formulas given in [2, pp. 410–411] with the construction in [9] (formulas (A.1–3)), one finds that there is an $(m+p) \times p$ matrix $G(s)$ of order at most pn_2 that has norm less than one and that satisfies interpolation conditions at the exosystem poles with total multiplicity n_2. The order of the matrix $\Theta(s)$ generating all solutions of the Nehari problem (5.1) is $2n_1$. Thus $W(s)$ calculated by the linear fractional transformation (5.7) has order at most $2n_1 + pn_2$. The matrix $(1-\gamma^2)^{-1/2}\tilde{P}^*(s)$ that we use in the role of $R(s)$ is antistable and has order n_1. So the image representation of the compensator is given by the stable part of $W(s)$ (see (5.4)), and its order, which is equal to the McMillan degree of the controller, is at most $n_1 + pn_2$. We note that this order does not exceed the sum of the orders required for the internal model and robust stabilization.

The angle appearing in the upper bound (5.12) can also be expressed in terms of coprime factorizations, as follows.

Proposition 5.4 *Let $N(s)D^{-1}(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$ be normalized left and right factorizations respectively, and write $P(s) = [\begin{smallmatrix} N(s) \\ D(s) \end{smallmatrix}]$. For all $\lambda \in i\mathbb{R}$, the following holds:*

$$\sin\phi((\operatorname{im} P(\lambda))^{\perp}, \operatorname{im}[\begin{smallmatrix} I \\ 0 \end{smallmatrix}]) = \sqrt{1 - \|\tilde{D}(\lambda)\|^2} = \sigma_{\min}(N(\lambda)) \tag{5.17}$$

where $\sigma_{\min}(N(\lambda))$ is the smallest singular value of $N(\lambda)$ if $N(\lambda)$ has full row rank, and is 0 if $N(\lambda)$ does not have full row rank.

Proof Since $\left[\begin{smallmatrix} \tilde{D}^*(\lambda) \\ -\tilde{N}^*(\lambda) \end{smallmatrix}\right] [\tilde{D}(\lambda) \;\; -\tilde{N}(\lambda)]^*$ is a normalized image representation of $(\operatorname{im} P(\lambda))^\perp$, it follows from (2.21) that $\cos\phi((\operatorname{im} P(\lambda)^\perp, \operatorname{im}\left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]) = \|\tilde{D}^*(\lambda)\| = \|\tilde{D}(\lambda)\|$. For the second equality one can employ a standard argument: since the matrix

$$\begin{bmatrix} \tilde{D}(\lambda) & -\tilde{N}(\lambda) \\ N^*(\lambda) & D^*(\lambda) \end{bmatrix}$$

is unitary, one has in particular $\|\tilde{D}(\lambda)\xi\|^2 + \|N^*(\lambda)\xi\|^2 = 1$ for all ξ with $\|\xi\| = 1$, so that

$$\|\tilde{D}(\lambda)\|^2 \;=\; \max_{\|\xi\|=1} \|\tilde{D}(\lambda)\xi\|^2 \;=\; 1 - \min_{\|\xi\|=1} \|N^*(\lambda)\xi\|^2 \;=\; 1 - \sigma^2_{\min}(N(\lambda)).$$

□

6. Modifying a given solution

The problem RRRS(γ) only calls for solutions achieving a certain degree of robustness of stability, and so solutions are not expected to be unique. The freedom that remains may be used to achieve additional design goals. We shall assume here that this design freedom will take the form of 'fine-tuning' around a given compensator which already satisfies the robust regulation and robust stabilization requirements. Our task is then to describe the changes that may be made in the compensator without impairing these requirements. For this purpose we first show how compensators can be parametrized in such a way that the robust regulation property is guaranteed. Then a sufficient condition for robust stabilization can be stated in terms of the parameter. It is possible to give a complete parametrization of the solutions to RRRS(γ) via a reduction to a boundary Nevanlinna-Pick problem, but as this would require a substantial further development we shall not pursue this line here.

Let $P(s)$ be an image representation for the subspace-valued function associated to the plant and $C_0(s)$ be an image representation for a particular solution of RRIS. Image representations of all stabilizing controllers are given by the Kučera-Youla parametrization [22, 34]

$$C(s) = C_0(s) - P(s)Q(s) \tag{6.1}$$

where $Q(s) \in RH_\infty^{m\times p}$ is a free parameter. It is clear from (3.9) that the compensator given by (6.1) will also satisfy the robust regulation requirement if and only if

$$\tilde{K}^{[r]}P^{[r]}(\lambda)Q^{[r]}(\lambda) = 0 \quad \text{for all } \lambda \in \sigma(A_{22}) \text{ of multiplicity } r \tag{6.2}$$

where $\tilde{K} = [I \;\; 0]$ (cf. (3.7). We can relate this to a divisibility property via the following result [8, Prop. 3.5].

Lemma 6.1 *Let $Q(s) \in RH_\infty^{m\times p}$ and $H(s) \in RH_\infty^{m\times m}$ and suppose that $H(s)$ is nonsingular. Under these conditions, there exists $\Psi(s) \in RH_\infty^{m\times p}$ such that $Q(s) = H(s)\Psi(s)$ if and only if $\operatorname{im} Q^{[r]}(\lambda) \subset \operatorname{im} H^{[r]}(\lambda)$ for each zero λ of $H(s)$ with multiplicity r.*

Write $\tilde{K}P(s) = N(s)$, to emphasize that this matrix appears as the numerator matrix in a right coprime factorization of the plant transfer matrix. An equivalent formulation of the solvability condition (4.3) of the robust regulation problem with internal stability is then that $N(\lambda)$ is of full row rank for all $\lambda \in \sigma(A_{22})$. Let $V(s)$ be a unimodular matrix such that

$$N(s)V(s) = [M(s) \;\; 0] \tag{6.3}$$

where $M(s)$ is a $p \times p$ square matrix. Let $h(s)$ be a biproper RH_∞ function whose numerator is the minimal polynomial of A_{22} and define

$$H(s) = V(s) \begin{bmatrix} h(s)I_p & 0 \\ 0 & I_{m-p} \end{bmatrix}. \tag{6.4}$$

Then it can be easily verified that $H(s)$ is nonsingular and $\operatorname{im} H^{[r]}(\lambda) = \ker N^{[r]}(\lambda)$ for all zeros λ of $H(s)$ of multiplicity r which coincide with the eigenvalues of A_{22}; moreover $H(s)$ has no other zeros in $\mathbb{C}^+$. Hence by Lemma 6.1 the general solution of (6.2) can be written as

$$Q(s) = H(s)\Psi(s) \tag{6.5}$$

for some arbitrary $RH_\infty^{m\times p}$ matrix $\Psi(s)$. Combining (6.1) and (6.5) we obtain a parametrization of all solutions of RRIS.

Proposition 6.2 *Consider the system (2.1-2.3) under the assumptions (A1-A4). Let $P(s)$ be an image representation for the subspace-valued function $\mathcal{P}(s)$ associated to the plant as defined by (2.13). Assume that the robust regulation problem with internal stability is solvable and let $C_0(s)$ be an image representation of the function $\mathcal{C}_0(s)$ associated to a particular solution. Let $H(s)$ be defined as in (6.4). Under these conditions, the general form of an image representation $C(s)$ of a solution of the robust regulation problem with internal stability is given by*

$$C(s) = C_0(s) - P(s)H(s)\Psi(s) \tag{6.6}$$

where $\Psi(s)$ is an arbitrary element of $RH_\infty^{m\times p}$.

This parametrization explicitly shows the constraints one has to impose on the Kučera-Youla parametrization of all stabilizing controllers in order to obtain controllers that achieve robust regulation: the 'central' controller $C_0(s)$ is a solution of RRIS, and the free parameter is left divisible by $H(s)$ which contains the minimal polynomial of A_{22} in its p nontrivial invariant factors. The appearance of the minimal polynomial of the exosystem may be viewed as another manifestation of the internal model principle already referred to in Remark 3.4. A similar parametrization of solutions in terms of a divisibility property has been given by Vidyasagar [30, Thm.7.5.2]; since Vidyasagar's problem formulation is different from ours the two parametrizations are however not directly comparable.

Suppose now that the central controller $C_0(s)$ is chosen such that the robust stability condition (2.22) is satisfied. Then robust stability will be preserved by any controller that is obtained from (6.6) by choosing a 'sufficiently small' $\Psi(s)$.

Proposition 6.3 *In the situation of the previous proposition, let $C_0(s)$ be a normalized image representation of a particular solution to the robust regulator problem with robust stability margin γ. Write $\gamma = \sin\phi_{\mathrm{tol}}$ with $0 \le \phi_{\mathrm{tol}} \le \frac{1}{2}\pi$. If the parameter $\Psi(s)$ is chosen such that*

$$\|\Psi(\lambda)\| \;<\; \frac{1}{\|P(\lambda)H(\lambda)\|}\sin(\phi(\mathcal{P}(\lambda), \mathcal{C}_0(\lambda)) - \phi_{\mathrm{tol}}) \tag{6.7}$$

for all $\lambda \in i\mathbb{R}$, then the controller $C(s)$ obtained from (6.6) will also be a solution to RRRS(γ).

The proof of the proposition is immediate from the following geometric lemma.

Lemma 6.4 *Let $\mathcal{P}$ and $\mathcal{C}$ be complementary subspaces of a unitary space, and suppose that the matrix C is a normalized image representation of $\mathcal{C}$ (i.e. $\operatorname{im} C = \mathcal{C}$ and $C^*C = I$). If C' is a matrix such that $\|C - C'\| < \sin\phi(\mathcal{P}, \mathcal{C})$, then the subspace $\mathcal{C}'$ defined by $\mathcal{C}' = \operatorname{im} C'$ is also complementary to $\mathcal{P}$, and we have*

$$\phi(\mathcal{P}, \mathcal{C}') \;\ge\; \phi(\mathcal{P}, \mathcal{C}) - \arcsin\|C - C'\|. \tag{6.8}$$

Proof According to Lemma 2.2 in [11], we have

$$\phi(\mathcal{P}, \mathcal{C}) - \phi(\mathcal{P}, \mathcal{C}') \;\le\; \arcsin\delta(\mathcal{C}, \mathcal{C}') \tag{6.9}$$

where $\delta(\mathcal{C}, \mathcal{C}')$ denotes the gap between the subspaces $\mathcal{C}$ and $\mathcal{C}'$ (i.e. the norm of the difference of the orthogonal projections on the two subspaces, see for instance [17, Ch. IV.7]). Moreover, Lemma 8 of [10] (applied in dual form, and with use of the assumption that C is normalized) implies that

$$\delta(\mathcal{C}, \mathcal{C}') \;\le\; \|C - C'\|. \tag{6.10}$$

The statement of the lemma now follows immediately. □

The design freedom that is reflected here can be used to achieve objectives other than robust regulation and robust stability. The bound (6.7) may for instance be used to determine the step size in an iterative optimization routine. After a step has been taken, the modified controller may be taken as a new center and the iteration step may be repeated.

7. Example

Consider the system described by

$$\begin{aligned} \dot{x}_1(t) &= ax_1(t) + x_2(t) + u(t) && (7.1)\\ \dot{x}_2(t) &= -\omega x_3(t) && (7.2)\\ \dot{x}_3(t) &= \omega x_2(t) && (7.3)\\ y(t) &= x_1(t). && (7.4) \end{aligned}$$

The equations represent a first-order plant subject to a sinusoidal disturbance $x_2(t)$ of frequency ω. The objective is to find a controller of the form (2.4–2.5) such that the variable $y(t)$ converges to zero in the presence of uncertainties in (7.1) and the robust stability requirement (2.22) is fulfilled. As a typical example of a system subject to a sinusoidal disturbance, consider the tracking mechanism of a compact disk player. If the hole in the CD that is being played is slightly off-center, there will be a periodic disturbance at the frequency given by the rotation speed of the disk. If the rotation frequency is known, it is a natural objective to use this information in a controller design.

The subspace-valued function associated to the plant is given by

$$\mathcal{P}(s) = \ker[s - a \quad -1]. \tag{7.5}$$

Normalized kernel and image representations for $\mathcal{P}(s)$ are

$$\tilde{P}(s) = \frac{1}{s + \sqrt{a^2+1}}[s - a \ \ -1], \qquad P(s) = \frac{1}{s + \sqrt{a^2+1}}\begin{bmatrix} 1 \\ s - a \end{bmatrix}. \tag{7.6}$$

Note that the exosystem poles are at $s = \pm i\omega$ so the upper bound on the robustness of stability imposed by the robust regulation constraint is computed from (5.17) as

$$\sin\phi(\mathcal{P}^{\perp}(i\omega), \mathcal{K}^{\perp}) = \frac{1}{\sqrt{\omega^2 + a^2 + 1}}. \tag{7.7}$$

Since we are working with real systems it is sufficient to do the computations for only one of the complex conjugate poles. Note that the above expression tends to zero as ω tends to infinity; so the constraint imposed by the regulation requirement becomes more severe as the frequency of the disturbance signal goes up. The achievable robustness of stability, not taking into account the robust regulation constraint, is calculated from (5.2):

$$(1 - \|\Gamma_{\tilde{P}^*}\|^2)^{\frac{1}{2}} = \frac{\sqrt{2}}{2}(1 - \frac{a}{\sqrt{a^2+1}})^{\frac{1}{2}}. \tag{7.8}$$

So the problem RRRS(γ) is solvable if and only if γ is less than the smaller of the two bounds given by (7.7) and (7.8). Since we are in the SISO case, the bound imposed by robust regulation is the same as the bound imposed by plain regulation (cf. Remark 4.2).

The robust regulation constraint is restrictive with respect to robustness of stability when the bound given in (7.7) is less than the one given by (7.8). One easily verifies that this occurs when

$$\omega^2 \ > \ a^2 + 1 + 2a\sqrt{a^2+1}. \tag{7.9}$$

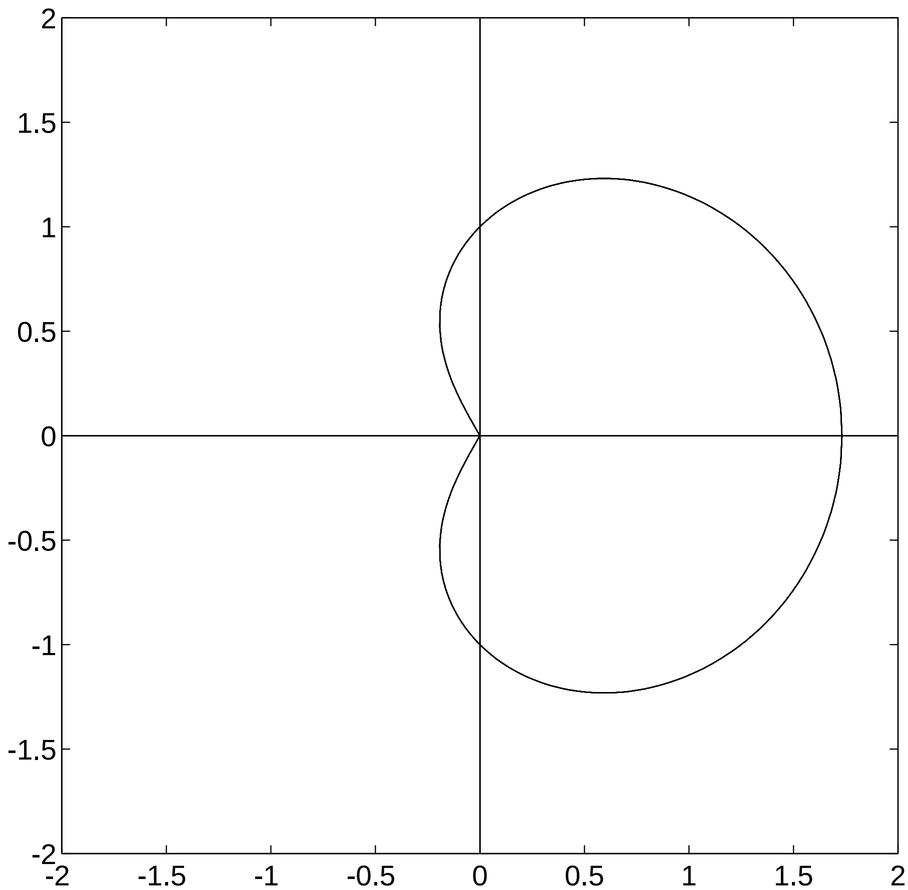

Figure 0.1: 'Critical curve' for first-order systems. Sinusoidal disturbances at frequencies corresponding to points on the Nyquist curve inside the area enclosed by the curve can not be regulated against without compromising the achievable robustness of closed-loop stability.

In particular, the regulation constraint is restrictive at all frequencies for open-loop stable first-order plants with dc gains less than $\sqrt{3}$; this ties in with the result found in [9] for regulation against constant disturbances. For other first-order systems, the 'critical' frequency can be found as the frequency corresponding to the point where the Nyquist curve of the plant intersects the curve given in Fig. 0.1.

To simplify the calculations we shall explain the construction of an actual controller for the specific values of the parameters given by

$$a = 0, \quad \omega = 2. \tag{7.10}$$

In this case the upper bound on the achievable stability margin calculated from (7.8) is $\sqrt{2}/2$ whereas the one due to robust regulation constraint is $\sqrt{5}/5$. So we should be able to find a controller achieving a robustness margin of at least γ, where γ is any number less than $\sqrt{5}/5 = 0.4472$. Let us take for instance $\gamma = 0.4$. Following the sufficiency proof of Thm. 5.2 define $\alpha = (1-\gamma^2)^{-1/2} = 1.0911$ and write

$$R(s) = \alpha \tilde{P}^*(s) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{\alpha}{s-1} + \begin{bmatrix} \alpha \\ 0 \end{bmatrix}. \tag{7.11}$$

Let $\Pi R(s)$ denote the orthogonal projection of $R(s)$ on $\mathcal{K}^\perp = \mathrm{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. We are looking for a Nehari extension $W(s)$ of $R(s)$ which satisfies the norm bound $\|W(s)\|_\infty < 1$ and the interpolation condition

$$W(i\omega) = \Pi R(i\omega) = \frac{\alpha}{i\omega - 1}\begin{bmatrix} i\omega \\ 0 \end{bmatrix}. \tag{7.12}$$

Obviously the second interpolation condition at $s = -i\omega$ will be automatically satisfied by requiring that $W(s)$ be real. All rational Nehari extensions of $R(s)$ satisfying the norm bound $\|W(s)\|_\infty < 1$ are given by

$$W = (\Theta_{11}G + \Theta_{12})(\Theta_{21}G + \Theta_{22})^{-1} \tag{7.13}$$

where $G(s)$ is an arbitrary function in $RH_\infty^{2\times 1}$ of norm less than one and the matrix $\Theta(s)$ is computed from the formulas in [5] as

$$\Theta(s) = \frac{\alpha}{(2-\alpha^2)(s^2-1)}\begin{bmatrix} -\alpha(s+1) & -\alpha(s+1) & 2(s+1) \\ -\alpha(s+1) & -\alpha(s+1) & 2(s+1) \\ -2(s-1) & -2(s-1) & 2\alpha(s-1) \end{bmatrix} + I. \tag{7.14}$$

Next, we translate the interpolation condition (7.12) into the free parameter $G(s)$. For this, we solve the equation

$$\Theta(i\omega)\begin{bmatrix} N(\omega) \\ D(\omega) \end{bmatrix} = \begin{bmatrix} W(i\omega) \\ 1 \end{bmatrix} \tag{7.15}$$

and then set $G(i\omega) = N(\omega)D^{-1}(\omega)$. This gives

$$G(i\omega) = \frac{\alpha}{(2-\alpha^2)\omega^2 + \alpha^2 + 2}\begin{bmatrix} (2-\alpha^2)\omega^2 + 2 \\ 2 + i(2-\alpha^2)\omega \end{bmatrix}. \tag{7.16}$$

Now we need to solve the boundary Nevanlinna-Pick problem of finding an RH_∞ function $G(s)$ of norm less than one that satisfies (7.16). Note that $\|G(i\omega)\| < 1$ so a solution always exists [9, Lemma 3.5]. The general procedure of solving boundary Nevanlinna-Pick problems is described in [2, 3]. For the assumed values of the parameters a first-order solution (in four decimals) is

$$G(s) \;=\; \begin{bmatrix} 0.8890 \\ 0.4367\,\psi(s) \end{bmatrix}, \qquad \psi(s) = \frac{s - 0.7081}{s + 0.7081}. \tag{7.17}$$

Note that the first component of $G(s)$ is constant and the second component is taken as a multiple of an inner function to satisfy the interpolation condition (7.16) and the norm constraint $\|G(s)\|_\infty < 1$. Substituting (7.17) in (7.13) we get

$$W(s) = \frac{s+1}{(s-1)(0.8095s^2 + 0.8706s + 1.5602)}\begin{bmatrix} 0.7197s^2 + 0.3938s + 0.6543 \\ 0.3535s^2 + 1.4142 \end{bmatrix}. \tag{7.18}$$

The image representation of the controller is computed from

$$C(s) = R(s) - W(s) = \frac{1}{0.8095s^2 + 0.8706s + 1.5602}\begin{bmatrix} 0.1636s^2 + 0.6543 \\ -0.3535s^2 + 0.1762s - 0.2881 \end{bmatrix}. \tag{7.19}$$

The controller transfer function is given by

$$c(s) = \frac{-2.1615s^2 + 1.0770s - 1.7615}{s^2 + 4}. \tag{7.20}$$

The controller has poles at $s = \pm 2i$ as it should have to satisfy the robust regulation requirement. The eigenvalues of the closed-loop system are placed at $\lambda_{1,2} = -0.5807 \pm 1.1934i$, $\lambda_3 = -1$ and the actual robustness of stability achieved by this controller is $\sin\phi(\mathcal{P}, \mathcal{C}) = 0.4030$ which is indeed better than the required margin $\gamma = 0.4$.

8. Conclusions

A basic result of regulator theory is that, for robust regulation to be possible, the numerator matrix in a right coprime factorization of the plant has to have full row rank at the exosystem poles. As a quantitative measure of how close the full rank condition is to being not satisfied, one might take the minimum of the singular values of the evaluations of the numerator matrix at the exosystem poles. In this paper we have found that the number obtained this way characterizes the achievable robustness of closed-loop stability under the restriction of robust regulation, if robustness of stability is understood in the sense of normalized coprime factorizations. To be precise, any degree of robustness of closed-loop stability can be obtained that is both less than the cited number (which might be called the 'servo bound') and less than the least upper bound of robustness degrees achievable by arbitrary linear compensators (which might be called the 'overall bound'). Moreover, we have shown how to construct robust regulators that achieve such a degree of robustness of stability and how to modify a given controller in order to accommodate other design purposes without sacrificing the robustness properties.

Although the solution of the problem presented in the paper is constructive, computation by hand quickly becomes laborious as was already clear in the simple example that was presented above. There is a need for sophisticated software that is able to deal efficiently with the required interpolations, in one form or another. We have suggested in section 6 how the design methodology proposed here could be incorporated in a more general controller design procedure, but an actual implementation of this of course requires much more work; this would include of the study of the effect of various design parameters, such as the choice of a metric on input/output space and more general plant weightings. A theoretical problem that we have left open is the parametrization of all solutions to RRRS. Also it would be of interest to consider the trade-offs of other performance requirements with robustness of stability. In the case of disturbance decoupling it has been found in [6] that the trade-off is not as benign as in the case of robust regulation that was considered in this paper.

Appendix

The following lemmas will be needed in the proof of Thm. 3.3. The first lemma is a standard result; see for instance Lemma A in [28]. The second lemma is immediate for instance from [35, Lemma 2.15].

Lemma A.1 *Let $V \in \mathbf{C}^{k\times\ell}$ and $Z \in \mathbf{C}^{j\times\ell}$ be given matrices such that* $\operatorname{rank} V = \ell$*. There exists $A \in \mathbf{C}^{j\times k}$ such that $\|A\| \le \epsilon$ and $AV = Z$ if and only if $Z^*Z \le \epsilon^2 V^*V$.*

Lemma A.2 *Let A, B be Hermitian matrices such that $B > 0$, $A \ge 0$. Then there exists $\alpha > 0$ such that $\alpha A - B \le 0$.*

We want to show that if the subspace-valued function of the controller satisfies (3.8) and (2.20) but does not satisfy (3.9), then an arbitrarily small perturbation in A_{12} violates condition (3.8) and the controller fails to satisfy the regulation requirement. In order to state and prove this result it is convenient to introduce some notation. The perturbed value of A_{12} is denoted by $\hat{A}_{12}$ and the subspace-valued function associated to the plant and exosystem corresponding to the perturbed values is denoted by $\hat{\mathcal{M}}(s)$. Let $v^r = \operatorname{col}(v_0, \ldots, v_{r-1})$ denote a generalized eigenvector chain of A_{22} associated with the eigenvalue λ of multiplicity r, i.e. $A_{22}v_0 = \lambda v_0$, $A_{22}v_i = \lambda v_i + v_{i-1}$ for $i = 1, \ldots, r-1$, or equivalently

$$(sI - A_{22})^{[r]}(\lambda)v^r = 0. \tag{A.1}$$

Define

$$\Sigma(s) = \begin{bmatrix} sI - A_{11} & 0 & -B_1 \\ C_1 & -I & 0 \end{bmatrix} \tag{A.2}$$

and consider the condition

$$\Sigma^{[r]}(\lambda) Mi \begin{bmatrix} x^r \\ y^r \\ u^r \end{bmatrix} \in \begin{bmatrix} A_{12} \\ -C_2 \end{bmatrix}^{[r]} \operatorname{span}(v^r) \tag{A.3}$$

where $y^r = \operatorname{col}(y_0, \ldots, y_{r-1}) \in \mathcal{Y}^r$ and x^r, u^r are defined likewise. For $\lambda \in \sigma(A_{22})$ of multiplicity r the subspace $\mathcal{M}^{[r]}(\lambda)$ can be written as

$$\mathcal{M}^{[r]}(\lambda) = \left\{ Mi \begin{bmatrix} y^r \\ u^r \end{bmatrix} \in (\mathcal{Y} \times \mathcal{U})^r \mid \exists x^r, v^r \text{ s.t. (A.1) and (A.3) hold} \right\}. \tag{A.4}$$

Under the conditions of Thm. 3.2 the subspaces $\mathcal{P}(\lambda)$ and $\mathcal{C}(\lambda)$ are complementary for $\lambda \in \sigma(A_{22})$, and $\mathcal{M}(\lambda)$ contains $\mathcal{P}(\lambda)$ as a proper subspace (this follows from our detectability assumption (A2)). So we have $\mathcal{C}(\lambda) \cap \mathcal{M}(\lambda) \neq \{0\}$. In the sequel we need a somewhat stronger result, as given in the next lemma.

Lemma A.3 *In the setting of Section 2, let λ be an eigenvalue of A_{22} of multiplicity r and $\Pi : (\mathcal{Y} \times \mathcal{U})^r \mapsto \mathcal{Y} \times \mathcal{U}$ be the projection defined by $\Pi \operatorname{col}(y_0, u_0, \ldots, y_{r-1}, u_{r-1}) = \operatorname{col}(y_0, u_0)$. Assume that the subspace-valued function of the controller satisfies $\mathcal{C}(\lambda) \oplus \mathcal{P}(\lambda) = \mathcal{Y} \times \mathcal{U}$. Under these conditions, we have $\Pi(\mathcal{C}^{[r]}(\lambda) \cap \mathcal{M}^{[r]}(\lambda)) \neq \{0\}$.*

Proof Let $\tilde{P}(s)$ be a kernel representation for the subspace valued function of the plant and $\tilde{M}(s)$ be a kernel representation for the sequence of subspace valued functions $\mathcal{M}^{[r]}(s)$ defined by (3.5). It has been shown in [8, Lemma 5.4] that $\tilde{M}(s)$ can be written in the form $\tilde{M}(s) = \tilde{H}(s)\tilde{P}(s)$ where $\tilde{H}(s)$ is a square and nonsingular RH_∞ matrix such that the nontrivial elementary divisors of $\tilde{H}(s)$ are the same as those of $sI - A_{22}$. Let $C(s)$ be an image representation for the subspace-valued function of the controller. Since the subspaces $\mathcal{P}(\lambda)$ and $\mathcal{C}(\lambda)$ are complementary, $\tilde{P}(s)C(s)$ has no zeros or poles at λ so that $\tilde{H}(s)\tilde{P}(s)C(s) = \tilde{M}(s)C(s)$ has a zero of order r at λ. Therefore there exists a vector function $y(s)$ analytic in a neighborhood of λ such that $y(\lambda) \neq 0$ and the first r coefficients in the Taylor series expansion of $\tilde{M}(s)C(s)y(s)$ are zero, i.e., $C^{[r]}(\lambda)\operatorname{col}(y(\lambda), \ldots, \frac{1}{(r-1)!} y^{(r-1)}(\lambda)) \in \mathcal{M}^{[r]}(\lambda)$. Noting that $C(\lambda)$ has full column rank, we see that $0 \neq C(\lambda)y(\lambda) \in \Pi(\mathcal{C}^{[r]}(\lambda) \cap \mathcal{M}^{[r]}(\lambda))$. □

Lemma A.4 *Assume that for $\lambda \in \sigma(A_{22})$ of multiplicity r the subspace-valued function $\mathcal{C}(s)$ of the controller satisfies $\mathcal{C}^{[r]}(\lambda) \cap \mathcal{M}^{[r]}(\lambda) \subset \mathcal{K}^{[r]}$, $\mathcal{C}(\lambda) \oplus \mathcal{P}(\lambda) = \mathcal{Y} \times \mathcal{U}$ and $\mathcal{C}^{[r]}(\lambda) \not\subset \mathcal{K}^{[r]}$. Under these conditions, there exists for any $\epsilon > 0$ a matrix $\hat{A}_{12}$ such that $\|\hat{A}_{12} - A_{12}\| \leq \epsilon$ and $\mathcal{C}^{[r]}(\lambda) \cap \hat{\mathcal{M}}^{[r]}(\lambda) \not\subset \mathcal{K}^{[r]}$.*

Proof Let $0 \neq Mi \operatorname{col}(0^r, u^r) \in \mathcal{C}^{[r]}(\lambda) \cap \mathcal{M}^{[r]}(\lambda)$. By the previous lemma, we may assume without loss of generality that $u_0 \neq 0$. There exist x^r and a generalized eigenvector chain v^r of A_{22} such that

$$\Sigma^{[r]}(\lambda) Mi \begin{bmatrix} x^r \\ 0^r \\ u^r \end{bmatrix} = \begin{bmatrix} A_{12} \\ -C_2 \end{bmatrix}^{[r]} v^r. \tag{A.5}$$

Note that $v_0 \neq 0$ because $\operatorname{col}(0, u_0) \notin \mathcal{P}(\lambda)$. Since $\mathcal{C}^{[r]}(\lambda) \not\subset \mathcal{K}^{[r]}$ there exist $y_c^r \in \mathcal{Y}^r$, $u_c^r \in \mathcal{U}^r$ such that $y_c^r \neq 0$, $Mi \operatorname{col}(y_c^r, u_c^r) \in \mathcal{C}^{[r]}(\lambda)$. Because C_1 has full row rank by assumption (A4), there exist $\xi_0, \ldots, \xi_{r-1}$ such that $C_1[\xi_0, \ldots, \xi_{r-1}] = [y_{c,0}, \ldots, y_{c,r-1}]$. Define $z^r = \operatorname{col}(z_0, \ldots, z_{r-1})$ as

$$\Sigma^{[r]}(\lambda) Mi \begin{bmatrix} \xi^r \\ y_c^r \\ u_c^r \end{bmatrix} = Mi \begin{bmatrix} z^r \\ 0^r \end{bmatrix}. \tag{A.6}$$

Because of the independence property of generalized eigenvectors, the matrix $[v_0, \ldots, v_{r-1}]$ has full column rank. So, by Lemma A.2, there exists $\alpha > 0$ such that

$$\alpha^2 [z_0, \ldots, z_{r-1}]^* [z_0, \ldots, z_{r-1}] \leq \epsilon^2 [v_0, \ldots, v_{r-1}]^* [v_0, \ldots, v_{r-1}]. \tag{A.7}$$

Then by Lemma 3.2 there exists A_ϵ such that $\|A_\epsilon\| \le \epsilon$ and $A_\epsilon[v_0,\ldots,v_{r-1}] = \alpha[z_0,\ldots,z_{r-1}]$. Take $\hat{A}_{12} = A_{12} + A_\epsilon$; then $\hat{A}_{12}$ satisfies the norm constraint in the statement of the lemma. Moreover,

$$\begin{aligned}\Sigma^{[r]}(\lambda)\left(Mi\begin{bmatrix} x^r \\ 0^r \\ u^r \end{bmatrix} + \alpha Mi \begin{bmatrix} \xi^r \\ y_c^r \\ u_c^r \end{bmatrix}\right) &= \begin{bmatrix} A_{12} \\ -C_2 \end{bmatrix}^{[r]} v^r + \alpha Mi \begin{bmatrix} z^r \\ 0 \end{bmatrix} = \\ &= \begin{bmatrix} A_{12} \\ -C_2 \end{bmatrix}^{[r]} v^r + \begin{bmatrix} A_\epsilon \\ 0 \end{bmatrix}^{[r]} v^r = \begin{bmatrix} \hat{A}_{12} \\ -C_2 \end{bmatrix}^{[r]} v^r. \end{aligned} \tag{A.8}$$

Thus $Mi\begin{bmatrix} 0^r \\ u^r \end{bmatrix} + \alpha Mi \begin{bmatrix} y_c^r \\ u_c^r \end{bmatrix} \in \hat{\mathcal{M}}^{[r]}(\lambda) \cap \mathcal{C}^{[r]}(\lambda)$ which proves that $\hat{\mathcal{M}}^{[r]}(\lambda) \cap \mathcal{C}^{[r]}(\lambda) \not\subset \mathcal{K}^{[r]}$. □

Finally we are able to prove the theorem.

Proof (of Thm. 3.3) Assume that conditions (2.20) and (3.9) are satisfied. Since eigenvalues of a matrix are continuous functions of its elements, the internal stability requirement (2.20) is satisfied in a neighborhood of (A_{11}, A_{12}, B_1) and (3.9) is satisfied for all values of the plant parameters. Therefore the controller is a solution of the RRIS. Conversely, if (3.9) is not satisfied it follows from Lemma A.4 and Thm. 3.2 that the controller cannot be a solution of RRIS. □

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